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Revisiting the Brans solutions of scalar-tensor gravity

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Motivated by statements in the literature which contradict two general theorems, the static and spherically symmetric Brans solutions of scalar-tensor gravity are analyzed explicitly in both the Jordan and the Einstein conformal frames. Depending on the parameter range, these solutions describe wormholes or naked singularities but not black holes.

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I. INTRODUCTION

Brans-Dicke theory [1] is the prototypical theory of gravity alternative to Einstein's general relativity (GR). Not long after its introduction, it was generalized to scalar-tensor theories [2] and, with the advent of string theories, new interest was generated by the fact that the simple bosonic string theory reduces to a Brans-Dicke theory with coupling parameter $\omega = -1$ [3]. The original Brans-Dicke theory contains a massless scalar field ϕ (acting approximately as the inverse of the gravitational coupling strength $\phi = G_{\text{eff}}^{-1}$) and a dimensionless parameter ω which would naturally be of order unity, but is constrained by Solar System experiments to satisfy $|\omega| > 40000$ [4]. For this reason, theorists have moved on to more sophisticated versions of Brans-Dicke theory, such as scalar-tensor gravities [2] in which ω becomes a function of the Brans-Dicke scalar field, which also acquires a mass or a self-interaction potential. In cosmology,[1] $f(\mathcal{R})$ theories of gravity, which are ultimately classes of scalar-tensor theories with Brans-Dicke-like scalar degree of freedom $\phi = f'(\mathcal{R})$, have become extremely popular for explaining the current acceleration of the Universe without invoking an *ad hoc* dark energy (see the reviews [5–7]). It is natural, in this context, to search for analogs of the Schwarzschild solution of GR. Shortly after Brans-Dicke theory was introduced [1], Brans presented four families of geometries which are static, spherically symmetric, vacuum solutions of the Brans-Dicke field equations [8]. Although there is legitimate suspicion that these solutions may not be very significant from the physical point of view (but the literature has contradictory statements about this point), it is often necessary to pick some simple (i.e., static, spherical, and asymptotically flat) solutions of an alternative theory of gravity as toy models for theoretical purposes or as physical solutions to test a theory experimentally. Currently a large amount of work is devoted to testing deviations from GR in black hole environments (see, e.g., [9]). The Brans solutions, being the first of their kind discovered in scalar-tensor or dilaton gravity, are a natural choice. However, they are surrounded by some ambiguity. According to a theorem by Agnese and La Camera [10], all static and spherically symmetric solutions of the (Jordan frame) Brans-Dicke theory are either naked singularities if the post-Newtonian parameter

$$\gamma = \frac{\omega + 1}{\omega + 2} \tag{1.1}$$

satisfies $\gamma < 1$, or wormholes if $\gamma > 1$. The Brans classes I–IV solutions fall into this category and, therefore, they can only describe naked singularieties or wormholes. This result seems to be missed by several authors since there are claims in the literature that certain static, spherical classes of solutions of Brans-Dicke theory describe black holes, which would contradict the Agnese-La Camera theorem. For example, the Campanelli-Lousto solutions [11] have been believed to be black holes for a long time until it was shown recently that they indeed describe either wormholes or naked singularities [12]. Similarly, reading the existing literature, because of explicit or implicit statements one is left with the impression that Brans solutions can describe black holes for some range of their parameters [13–16]. Similar statements about "cold black holes" similar to the Campanelli-Lousto solutions are found in the literature [14,15,17,18]. If Brans geometries were black hole ones, they would also contradict a theorem by Hawking [19] (recently extended to general scalar-tensor gravity [20,21]) stating that all Brans-Dicke black holes are the same as in GR.

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[1]Here $\mathcal{R}$ is the Ricci scalar associated with the connection of the spacetime metric g_{ab}.

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Naked singularities are of little interest from the physical point of view because they correspond to the breakdown of the Cauchy problem. Wormholes are completely speculative objects [22], but there is plenty of astrophysical evidence for, and interest in, black holes. It is of some interest, therefore, to clarify the confusion existing in the literature about the Brans geometries, which we set out to do.

The Brans-Dicke action in the absence of matter is

$$S_{\rm BD} = \int d^4x \frac{\sqrt{-g}}{16\pi}\left(\phi\mathcal{R} - \frac{\omega}{\phi}\nabla^a\phi\nabla_a\phi\right), \quad (1.2)$$

where ϕ is the Brans-Dicke scalar field (approximately equivalent to the inverse of the gravitational coupling), $\mathcal{R}$ is the Ricci scalar, and g is the determinant of the spacetime metric g_{ab}. We follow the notation of Ref. [23]. The Brans-Dicke field equations *in vacuo* derived from the action (1.2) are [1]

$$\mathcal{R}_{ab} - \frac{\mathcal{R}}{2}g_{ab} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}\nabla_a\nabla_b\phi, \quad (1.3)$$

$$\Box\phi = 0. \quad (1.4)$$

By performing the conformal transformation of the metric

$$g_{ab} \to \tilde{g}_{ab} = \phi g_{ab}, \quad (1.5)$$

and the scalar field redefinition

$$\phi \to \tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi G}}\ln\left(\frac{\phi}{\phi_*}\right), \quad (1.6)$$

where ϕ_* is a constant (Einstein frame quantities are denoted by a tilde), the Brans-Dicke action (1.2) assumes its Einstein frame form

$$S_{\rm BD} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{\mathcal{R}}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\nabla_a\tilde{\phi}\nabla_b\tilde{\phi}\right]. \quad (1.7)$$

This action formally looks like the Einstein-Hilbert action of GR in the presence of a matter scalar field endowed with canonical kinetic energy. The Einstein frame vacuum field equations are

$$\tilde{\mathcal{R}}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{\mathcal{R}} = 8\pi G\left(\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\nabla_c\tilde{\phi}\nabla_d\tilde{\phi}\right), \quad (1.8)$$

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} = 0. \quad (1.9)$$

We now proceed to analyze the four classes of Brans solutions in the Jordan and in the Einstein conformal frames. We will examine the behavior of the areal radius R, which is a geometric (i.e. coordinate-invariant) quantity in spherical symmetry. The horizons correspond to the roots of the equation

$$\nabla^c R\nabla_c R = 0 \quad (1.10)$$

when they exist. If there is a single (real and positive) root $R_{\rm H}$ of this equation, then the areal radius is spacelike outside the horizon and becomes a timelike coordinate inside of it (i.e., for $R < R_{\rm H}$), with $\nabla^c R$ becoming null at the horizon. This situation is familiar from the study of Schwarzschild space and a single root of Eq. (1.10) corresponds to a black hole horizon. If instead there is a double (real and positive) root $R_{\rm H}$ of Eq. (1.10) (or a root of even order $n = 4, 6, \ldots$), then the areal radius is always a spacelike coordinate outside of the horizon, with $\nabla^c R$ becoming null at $R_{\rm H}$. The areal radius increases as one moves away from this horizon, in both directions, i.e., it is always $R > R_{\rm H}$ at points which do not lie on the surface $R = R_{\rm H}$ itself. In this case two spacetime regions join at the horizon and the areal radius cannot assume values $R < R_{\rm H}$. This situation describes a wormhole throat.

II. BRANS CLASS I SOLUTIONS

Class I Brans solutions have been discussed in several papers [13,15,24–31]. It is found that these metrics can describe wormholes, which is not surprising since a Brans-Dicke-like scalar field in scalar-tensor gravity has a noncanonical kinetic energy and its effective stress-energy tensor on the right-hand side of Eq. (1.3) can violate all of the energy conditions. More recent solutions proposed in the literature [32] have been identified as special limits of Brans I solutions [15,33].

A. Jordan frame

In the Jordan frame representation, the Brans class I line element and scalar field are, respectively,

$$ds^2_{(\rm I)} = -\left(\frac{1-B/r}{1+B/r}\right)^{2/\lambda}dt^2 + \left(1+\frac{B}{r}\right)^4\left(\frac{1-B/r}{1+B/r}\right)^{\frac{2(\lambda-C-1)}{\lambda}}(dr^2 + r^2 d\Omega^2_{(2)}), \quad (2.1)$$

$$\phi_{(\rm I)} = \phi_0\left(\frac{1-B/r}{1+B/r}\right)^{C/\lambda}, \quad (2.2)$$

in polar coordinates (t, r, θ, φ), where r is an isotropic radius and $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere. It must be $r \geq 0$ and [8]

$$\lambda^2 = (C+1)^2 - C\left(1 - \frac{\omega C}{2}\right) > 0. \qquad (2.3)$$

Here ω is a parameter of the theory and B, C, and λ are parameters of this family of solutions. B plays the role of a mass parameter and, in analogy with the Schwarzschild geometry of GR, it makes sense to consider only non-negative values of this parameter. There are actually two other reasons why we restrict our study here to non-negative values of B. The first reason is that one can still include the case $B = 0$, but then class I solutions simply reduce to the trivial Minkowski space. The second reason is that one can also include the case $B < 0$, but then, as we will shortly see, one just recovers the case of positive B by taking the mass parameter of the theory to be $-B$ instead of B. Therefore, we shall hereafter assume $B > 0$.

Once the Brans-Dicke coupling parameter ω of the theory is fixed, one has two independent parameters (B, C) or (B, λ), since Eq. (2.3) relates λ and C. It is useful to restrict the parameter space and, to this end, we note that, according to our assumption that $B > 0$, only positive values of λ will be relevant here. In fact, consider as an example the simple case $C = 0$, in which the Brans-Dicke scalar ϕ reduces to a constant and Eq. (2.3) yields $\lambda = \pm 1$. When ϕ is constant, Brans-Dicke theory reduces to GR and the Schwarzschild solution, which is the unique vacuum, static, and spherically symmetric solution of the Einstein equations must be recovered. By setting $\lambda = 1$ the line element (2.1) reduces to

$$ds^2_{(+1)} = -\left(\frac{1 - B/r}{1 + B/r}\right)^2 dt^2 + \left(1 + \frac{B}{r}\right)^4 (dr^2 + r^2 d\Omega^2_{(2)}) \qquad (2.4)$$

which is the Schwarzschild metric in isotropic coordinates [13]. If instead $\lambda = -1$, one obtains

$$ds^2_{(-1)} = -\left(\frac{1 + B/r}{1 - B/r}\right)^2 dt^2 \qquad (2.5)$$

$$+ \left(1 - \frac{B}{r}\right)^4 (dr^2 + r^2 d\Omega^2_{(2)}). \qquad (2.6)$$

This is again just the Schwarzschild solution provided that, either $r \to -r$ which we do not consider here [13], or that $B < 0$ and, hence, as alluded to above, one then has just to interpret $-B$ as the mass parameter instead of B; a case we have already chosen to exclude. Therefore, we also assume $\lambda > 0$ in the following.

We can now study the limit of Brans I solutions to GR. When $\omega \to \infty$, it is $\lambda^2 \simeq \omega C^2/2 \to \infty$ and, if $C \neq 0$ (the case $C = 0$ having already been discussed), the line element (2.1) reduces to

$$ds^2_{(\infty)} = -dt^2 + \left(1 - \frac{B^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega^2_{(2)}), \qquad (2.7)$$

while the Brans-Dicke scalar becomes constant. If $C \neq 0$, the corresponding solution of GR is not recovered from Brans I solutions in the $\omega \to \infty$ limit. Instances in which solutions of scalar-tensor theories do not reduce to the corresponding GR limit have been discussed in [34] and possible reasons for this behavior have been identified in the anomalous asymptotic dependence of ϕ on ω as $\omega \to \infty$ [34–36].

The condition (2.3) amounts to imposing that C is such that points on the parabola of equation $\lambda^2(C) = (\frac{\omega}{2} + 1)C^2 + C + 1$ lie in the $\lambda^2 > 0$ half-plane. The roots of the equation $\lambda^2(C) = 0$ are

$$C_\pm = \frac{-1 \pm \sqrt{-(2\omega + 3)}}{\omega + 2} \qquad (2.8)$$

and they are real only if $\omega \leq -3/2$. By looking at the sign of the coefficient $(1 + \omega/2)$ of this parabola, it is easy to establish the following:

- (i) If $\omega < -2$, the parabola has concavity facing downward and intersects the C axis at $C_\pm > 0$. It must be $C_- < C < C_+$.
- (ii) If $\omega = -2$, then the parabola degenerates into the straight line $\lambda^2 = C + 1$ and it must be $C > -1$.
- (iii) If $-2 < \omega < -3/2$, the parabola has concavity facing upward and it must be $C < C_-$ or $C > C_+$, where $C_- < C_+ < 0$.
- (iv) If $\omega = -3/2$, then $\lambda^2 = [(C+2)/2]^2$ and the parabola has concavity facing upward and touches the C axis only at $C = -2$; therefore the only restriction is $C \neq -2$.
- (v) If $\omega > -3/2$, the concavity still faces upward but there are no intersections between the C axis and the parabola, which always lies above it. There is no restriction on the values of C.

Let us consider now the Ricci scalar $\mathcal{R}$: by contracting the Brans-Dicke field equations (1.3) and using Eq. (2.2), one obtains

$$\mathcal{R} = \frac{\omega}{\phi^2} \nabla^c \phi \nabla_c \phi = \frac{4\omega B^2 C^2}{\lambda^2 r^4} \left[\left(1 + \frac{B}{r}\right)^{-2 - \frac{(C+1)}{\lambda}} \left(1 - \frac{B}{r}\right)^{-2 + \frac{(C+1)}{\lambda}}\right]^2. \qquad (2.9)$$

If $\omega \neq 0$ and $C \neq 0$, then the Ricci scalar is singular at $r = B$ when $(C+1)/\lambda < 2$. Whether this value of the isotropic radius is physically significant is discussed case-by-case below.

The areal radius is read off the line element (2.1) and is

$$R(r) = \left(1+\frac{B}{r}\right)^{1+\frac{(C+1)}{\lambda}}\left(1-\frac{B}{r}\right)^{1-\frac{(C+1)}{\lambda}} r, \qquad (2.10)$$

and its derivative is

$$\frac{dR}{dr} = \left(\frac{1+B/r}{1-B/r}\right)^{\frac{C+1}{\lambda}}\left[r^2 - \frac{2B(C+1)}{\lambda}r + B^2\right]\frac{1}{r^2}. \qquad (2.11)$$

In the following it is useful to know the roots of the equation $dR/dr = 0$, which are

$$r_{(\pm)} = \frac{B(C+1)}{\lambda}\left(1 \pm \sqrt{1-\left(\frac{\lambda}{C+1}\right)^2}\right). \qquad (2.12)$$

In order to make our discussion of the various regions of the parameter space more compact, we focus on the possible values of the parameter combination $(C+1)/\lambda$, which is relevant for both the roots of the equation $dR/dr = 0$ and in the search for horizons. The horizons (which, when existing, are both apparent and event horizons), are located by the roots of the equation [37,38]

$$\nabla^c R \nabla_c R = 0, \qquad (2.13)$$

which is equivalent to

$$\left[r^2 - 2B\frac{(C+1)}{\lambda}r + B^2\right]^2 = 0. \qquad (2.14)$$

Its roots coincide with those of the equation $dR/dr = 0$ and, when they exist in the real domain, they are always double roots. Let us consider separately the various relevant cases.

1. Parameter range $(C+1)/\lambda < 1$

In this case

$$\frac{dR}{dr} = \left(\frac{1+B/r}{1-B/r}\right)^{\frac{C+1}{\lambda}}\left[r^2 - 2B\frac{(C+1)}{\lambda}r + B^2\right]\frac{1}{r^2}$$
$$> \left(1+\frac{B}{r}\right)^{\frac{C+1}{\lambda}}\left(1-\frac{B}{r}\right)^{-\frac{C+1}{\lambda}}(r-B)^2 > 0, \qquad (2.15)$$

for all values of $r > B$. Moreover,

$$R(r) = \left(1+\frac{B}{r}\right)^{1+\frac{C+1}{r}}\left(1-\frac{B}{r}\right)^{|1-\frac{C+1}{\lambda}|} r, \qquad (2.16)$$

shows that $r = B$ corresponds to areal radius $R = 0$, hence the range $0 < r < B$ is unphysical. The Ricci scalar (2.9) is singular at $R = 0$. In this parameter range the spacetime always hosts a naked central singularity if $\omega \neq 0$. The details of the geometry near this singularity vary with the value of $(C+1)/\lambda$ as described below.

(i) If $0 < (C+1)/\lambda < 1$ then $dR/dr \to +\infty$ as the spacetime singularity is approached ($R \to 0^+$ or $r \to B^+$).

(ii) If $(C+1)/\lambda = 0$, then

$$R(r) = \left(1-\frac{B^2}{r^2}\right)r \to 0, \qquad (2.17)$$

$$\frac{dR}{dr} = 1 + \frac{B^2}{r^2} \to 2, \qquad (2.18)$$

as the singularity at $R = 0$ is approached.

(iii) If $(C+1)/\lambda < 0$, then

$$\frac{dR}{dr} = \frac{(1-B/r)^{|\frac{C+1}{\lambda}|}}{(1+B/r)^{|\frac{C+1}{\lambda}|}}\left[r^2 - 2B\frac{C+1}{\lambda}r + B^2\right] \qquad (2.19)$$

tends to zero at the singularity $R = 0$ or $r = B$.

2. Parameter range $(C+1)/\lambda = 1$

We have $R(r) = r(1+B/r)^2$ and $R(B) = 4B > 0$; therefore the range $0 < r < B$ of the isotropic radius is now physically meaningful. Note that $R(r) \to +\infty$ as $r \to 0^+$ and that

$$\frac{dR}{dr} = \left(1+\frac{B}{r}\right)\left(1-\frac{B}{r}\right); \qquad (2.20)$$

therefore the function $R(r)$ decreases if $0 < r < B$, has the absolute minimum $R(B) = 4B > 0$, and increases for $r > B$. The equation $\nabla^c R \nabla_c R = 0$ locating the horizons is equivalent to $(1 - B/r)^2 = 0$, with $r = B$ a double root. If $\omega \neq 0$, there is a would-be wormhole throat at $r = B$ (or, at $R = 4B$) where, however, the Ricci scalar is singular. This finite radius singularity separates two disconnected spacetimes.

If $\omega = 0$, $\lambda > 0$, and $0 < C < 1$, also the Brans-Dicke scalar diverges at $r = B$, which means that the effective gravitational constant vanishes. If $C < 0$ or $C \geq 1$, then ϕ vanishes and the gravitational coupling strength diverges. The case $C = 0$ has already been discussed for all values of ω. In the context of black holes, the divergence or the vanishing of the Brans-Dicke scalar denotes "maverick" black holes which are contrived, unstable, or pathological and are usually discarded as unphysical (e.g., [20]) and the same criterion should be adopted for wormholes (naked singularities are already unphysical).

3. Parameter range $(C+1)/\lambda > 1$

In this case the equation

$$\frac{dR}{dr} = \left(\frac{1+B/r}{1-B/r}\right)^{|\frac{C+1}{\lambda}|} \left[r^2 - \frac{2B(C+1)}{\lambda} r + B^2\right] \cdot \frac{1}{r^2} = 0 \tag{2.21}$$

has the two roots (2.12), which are both positive. It is straightforward to see also that

$$0 < r_{(-)} < B < r_{(+)}. \tag{2.22}$$

The areal radius

$$R(r) = \frac{(1+B/r)^{|1+\frac{C+1}{\lambda}|} r}{(1-B/r)^{|\frac{C+1}{\lambda}-1|}} \to +\infty \tag{2.23}$$

as $r \to B^+$, hence the range $r < B$ of the isotropic radius is unphysical and we ignore the root $r_{(-)} < B$. The apparent horizons are located at the roots of the equation

$$\left(1 - \frac{B}{r}\right)^{-2} \left[r^2 - \frac{2B(C+1)}{\lambda} r + B^2\right]^2 = 0. \tag{2.24}$$

Ignoring the root $r = B$, which corresponds to $R = +\infty$, $r = r_{(+)} > B$ is a double root and we have a wormhole throat at $r_{(+)}$. As seen earlier, the Ricci scalar (2.9) is singular at $r = B$ if $(C+1)/\lambda < 2$ and $\omega \neq 0$, but this singularity is actually pushed to infinity since $r \to B^+$ corresponds to infinite physical radius R; hence this is an acceptable solution. The Ricci scalar is regular for $(C+1)/\lambda \geq 2$.

B. Einstein frame class I solutions

The Einstein frame metric and free scalar field are

$$d\tilde{s}^2_{(\mathrm{I})} = \phi_{(\mathrm{I})} ds^2_{(\mathrm{I})} = -\left(\frac{1-B/r}{1+B/r}\right)^{\frac{C+2}{\lambda}} dt^2 + \left(1+\frac{B}{r}\right)^{2+\frac{C+2}{\lambda}} \left(1-\frac{B}{r}\right)^{2-\frac{(C+2)}{\lambda}} (dr^2 + r^2 d\Omega^2_{(2)}) \tag{2.25}$$

$$\tilde{\phi}_{(\mathrm{I})} = \sqrt{\frac{|2\omega+3|}{16\pi G}} \frac{C}{\lambda} \ln\left(\frac{1-B/r}{1+B/r}\right) + \text{const.} \tag{2.26}$$

The areal radius and its derivative are

$$\tilde{R}(r) = \phi R = \left(1+\frac{B}{r}\right)^{1+\frac{C+2}{2\lambda}} \left(1-\frac{B}{r}\right)^{1-\frac{C+2}{2\lambda}} r, \tag{2.27}$$

$$\frac{d\tilde{R}}{dr} = \left(\frac{1+B/r}{1-B/r}\right)^{\frac{C+2}{2\lambda}} \left[1 - \left(\frac{C+2}{\lambda}\right)\frac{B}{r} + \frac{B^2}{r^2}\right], \tag{2.28}$$

while the Einstein frame Ricci scalar [obtained by contracting the field equations (1.8)] is

$$\tilde{\mathcal{R}} = 8\pi G \tilde{g}^{rr} \left(\frac{d\tilde{\phi}}{dr}\right)^2 = \frac{2B^2C^2|2\omega+3|}{\lambda^2 r^4} \left(1+\frac{B}{r}\right)^{-4-\frac{C+2}{\lambda}} \left(1-\frac{B}{r}\right)^{\frac{C+2}{2\lambda}-4}. \tag{2.29}$$

If $(C+2)/\lambda < 4$, the Ricci scalar is singular at $r = B$ (and it is always singular at $r = 0$ unless $C = 0$, in which case it is $\tilde{\mathcal{R}} = 0$).

The equation $\nabla^c \tilde{R} \nabla_c \tilde{R} = 0$ locating the horizons is

$$\left(1 - \frac{B^2}{r^2}\right)^{-2} \left[1 - \left(\frac{C+2}{\lambda}\right)\frac{B}{r} + \frac{B^2}{r^2}\right]^2 = 0 \tag{2.30}$$

and has the same roots

$$r_{(\pm)} = \frac{B(C+2)}{2\lambda}\left(1 \pm \sqrt{1 - \frac{4\lambda^2}{(C+2)^2}}\right) \tag{2.31}$$

as the equation $d\tilde{R}/dr = 0$. When these roots exist and are real and positive, they are always double roots and, therefore, the solutions always contain either wormhole throats or naked singularities. Assuming that $B > 0$ and $\lambda > 0$ as in the Jordan frame, if $[(C+2)/\lambda]^2 < 4$ there are no real roots and no horizons. If $(C+2)/\lambda = \pm 2$ there is a quadruple root $r_0 = B$. If instead $[(C+2)/\lambda]^2 > 4$, there are two double roots $r_{(\pm)}$. Further, if $C > -2$ the two double roots $r_{(\pm)}$ are both positive; if $C = -2$ the only (quadruple) root vanishes and, if $C < -2$, there are no horizons. Let us examine the situation in more detail.

1. Parameter range $(C+2)/\lambda < -2$

In this case it is

$$\tilde{R}(r) = \left(1+\frac{B}{r}\right)^{1+\frac{C+2}{2\lambda}} \left(1-\frac{B}{r}\right)^{|1+\frac{C+2}{2\lambda}|} r \tag{2.32}$$

and $\tilde{R}(r) \to 0^+$ as $r \to B$; hence the range $0 < r < B$ is unphysical, while $\tilde{R}(r) \to 0^+$ as $r \to +\infty$. The roots $r_{(\pm)}$ are negative and the Ricci scalar diverges at $\tilde{R} = 0$, where there is a naked singularity.

2. Parameter range $(C+2)/\lambda = -2$

In this case

$$\tilde{R}(r) = \frac{(r-B)^2}{r} \tag{2.33}$$

vanishes as $r \to B$ and diverges in both limits $r \to 0^+$ and $r \to +\infty$. It could seem that there is a wormhole throat at $r = B$ but the Ricci scalar diverges there. Also this geometry hosts a naked singularity at $\tilde{R} = 0$.

3. Parameter range $-2 < (C+2)/\lambda < 2$

Then

$$\tilde{R}(r) = \left(1 + \frac{B}{r}\right)^{1+\frac{C+2}{2\lambda}} \left(1 - \frac{B}{r}\right)^{|1-\frac{C+2}{2\lambda}|} \frac{1}{r} \tag{2.34}$$

and $\tilde{R}(r) \to 0^+$ as $r \to B$ while $\tilde{R}(r) \to +\infty$ as $r \to +\infty$. There are no real roots $r_{(\pm)}$ and no horizons. The Ricci scalar diverges at $\tilde{R} = 0$, where we have again a naked singularity.

4. Parameter range $(C+2)/\lambda = 2$

$r = B$ is a quadruple root and

$$\tilde{R}(r) = \left(1 + \frac{B}{r}\right)^2 r \tag{2.35}$$

has the limits $\tilde{R} \to +\infty$ as $r \to 0^+$ and $\tilde{R} \to +\infty$ as $r \to +\infty$. There is a wormhole throat at $\tilde{R} = 4B$, the minimum value of $\tilde{R}$.

5. Parameter range $(C+2)/\lambda > 2$

Both double roots $r_{(\pm)}$ are positive and

$$\tilde{R}(r) = \left(1 + \frac{B}{r}\right)^{1+\frac{C+2}{2\lambda}} \left(1 - \frac{B}{r}\right)^{-|1-\frac{C+2}{2\lambda}|} r \tag{2.36}$$

diverges in both limits $r \to B^+$ and $r \to +\infty$; hence the range $0 < r < B$ is unphysical. In this parameter range it is $0 < r_{(-)} < B < r_{(+)}$ and there is a wormhole throat at $r_{(+)}$.

III. BRANS CLASS II SOLUTIONS

A. Jordan frame class II solutions

There is a duality relating class II and class I solutions [14,15], so these two classes are not independent. We shall come back to this duality in Sec. VI below. The Jordan frame Brans class II line element and scalar field are

$$ds^2_{(\mathrm{II})} = -e^{\frac{4}{\Lambda}\arctan(r/B)} dt^2 + e^{\frac{-4(C+1)}{\Lambda}\arctan(r/B)} \left(1 + \frac{B^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega^2_{(2)}), \tag{3.1}$$

$$\phi_{(\mathrm{II})} = \phi_0 e^{\frac{2C}{\Lambda}\arctan(r/B)}, \tag{3.2}$$

where

$$\Lambda^2 = C\left(1 - \frac{\omega C}{2}\right) - (C+1)^2 > 0. \tag{3.3}$$

This implies that $C \neq 0$; hence this value of the parameter C will not be considered in the following even though it is clear that it would play a role if the inequality (3.3) is forgotten. Indeed, note that if Λ and B are allowed to take simultaneously imaginary values, then setting $C = 0$ will just turn the metric (3.1) into the Schwarzschild metric (2.4) written in isotropic coordinates. We shall come back to this remark in Sec. VI.

Let us examine the possible range of the parameters B, C, and Λ. The points of the parabola $\Lambda^2(C) = -(\frac{\omega}{2}+1)C^2 - C - 1$ must lie in the positive Λ^2 half-plane. This parabola has concavity facing downwards and it intersects the C axis at

$$C_\pm = \frac{-1 \mp \sqrt{-(2\omega+3)}}{\omega+2}. \tag{3.4}$$

There are no such intersections if $\omega > -3/2$ and two coincident intersections if $\omega = -3/2$; we conclude that it must be $\omega < -3/2$ for class II solutions to exist in the Jordan frame. Assuming this condition, it is easy to see that:

(i) If $-2 < \omega < -3/2$ [corresponding to $-(1+\omega/2) < 0$], the parameter C must lie in the range $C_- < C < C_+$.

(ii) If $\omega = -2$, the parabola degenerates into the straight line $\Lambda^2(C) = -(C+1)$ and it must be $C < -1$.

(iii) If $\omega < -2$, it must be $C < C_-$ or $C > C_+$.

The Ricci scalar is

$$\mathcal{R} = \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi = \frac{4\omega B^2 C^2 r^4 e^{\frac{4(C+1)}{\Lambda}\arctan(r/B)}}{\Lambda^2(r^2+B^2)^4} = \frac{4\omega B^2 C^2}{\Lambda^2} \frac{e^{\frac{4(C+1)}{\Lambda}\arctan(r/B)}}{R^4}. \tag{3.5}$$

The only possible singularity of the Ricci scalar $\mathcal{R}$ can occur as $R \to 0$. The areal radius is

$$R(r) = \left(1 + \frac{B^2}{r^2}\right) e^{\frac{-2(C+1)}{\Lambda}\arctan(r/B)} r \tag{3.6}$$

and its derivative is

$$\frac{dR}{dr} = e^{\frac{-2(C+1)}{\Lambda}\arctan(r/B)} \left[r^2 - 2B\frac{(C+1)}{\Lambda} r + B^2\right]. \tag{3.7}$$

Note that $R > 0$ for all values of r and that $R \to +\infty$ as $r \to +\infty$ and also as $r \to 0^+$. Since the Ricci scalar (3.5) can only diverge as $R \to 0^+$, there are no singularities of the Ricci scalar in Brans class II spacetimes.

The roots of the equation $dR/dr = 0$ are

$$r_{(\pm)} = \frac{B}{\Lambda}\left(C + 1 \pm \sqrt{(C+1)^2 + \Lambda^2}\right). \tag{3.8}$$

Let us examine their sign, keeping in mind that

$$\sqrt{(C+1)^2 + \Lambda^2} + C + 1 > 0, \tag{3.9}$$

$$C + 1 - \sqrt{(C+1)^2 + \Lambda^2} < 0. \tag{3.10}$$

(i) If $\Lambda B > 0$, the parabola $\psi(r) \equiv r^2 - 2B\frac{(C+1)}{\Lambda} r - B^2$ has concavity facing upwards and crosses the r axis at $r_{(-)}$ and $r_{(+)}$, with $r_{(-)} < 0 < r_{(+)}$. Therefore $dR/dr < 0$ and the function $R(r)$ decreases if $0 < r < r_{(+)}$, it has an absolute minimum at $r_{(+)}$, and increases for $r > r_{(+)}$.

(ii) If $\Lambda B < 0$, the parabola $\psi(r)$ still has concavity facing upward but now $r_{(+)} < 0 < r_{(-)}$ and the discussion is the same as in the previous case provided that the switch $r_{(+)} \leftrightarrow r_{(-)}$ is made.

The equation $\nabla^c R \nabla_c R = 0$ locating the horizons becomes

$$\left[1 - \frac{B^2}{r^2} - \frac{2B(C+1)}{\Lambda} r + B^2\right]^2 = 0. \tag{3.11}$$

The roots are the same as for the equation $dR/dr = 0$ and, when they are real and positive, they are always double roots. This fact implies that there are no black hole horizons and that class II solutions do not describe black holes but only wormhole throats or naked singularities. Further, the roots $r_{(\pm)}$ can be written as

$$r_{(\pm)} = \frac{B}{\Lambda}\left(C + 1 \pm \sqrt{C\left(1 - \frac{\omega C}{2}\right)}\right) \tag{3.12}$$

and the inequality (3.3) implies that $C(1-\frac{\omega C}{2}) > (C+1)^2 \geq 0$; hence there are always two real roots $r_{(\pm)}$ of the equation $\nabla^c R \nabla_c R = 0$ locating the horizons in the allowed range of parameters. Are these roots positive? In order to answer this question, note that $\sqrt{(C+1)^2 + \Lambda^2} + C + 1 > |C+1| + C + 1 \geq 0$; hence

$$\mathrm{sign}(r_{(+)}) = \mathrm{sign}(\Lambda B), \tag{3.13}$$

while $C + 1 - \sqrt{(C+1)^2 + \Lambda^2} < C + 1 - |C+1| \leq 0$ yields

$$\mathrm{sign}(r_{(-)}) = -\mathrm{sign}(\Lambda B). \tag{3.14}$$

We can now analyze all the possibilities for the two parameters B and Λ.

1. Parameter range $B > 0$, $\Lambda > 0$

When $B > 0$ and $\Lambda > 0$, it is $r_{(-)} < 0 < r_{(+)}$ and there is a double root $r_{(+)}$ marking the location of a wormhole throat. The same situation occurs when $B < 0$ and $\Lambda < 0$.

2. Parameter range $B > 0$, $\Lambda < 0$

When $B > 0$ and $\Lambda < 0$, it is $r_{(+)} < 0 < r_{(-)}$ and there is a wormhole throat at $r_{(-)}$. The same situation occurs when $B < 0$ and $\Lambda > 0$.

3. Limit to GR

Finally, let us consider the limit to GR of Brans II solutions. Since $\omega < -3/2$, the limit should be $\omega \to -\infty$, which implies that $\Lambda^2 \approx -\omega C^2/2 \to +\infty$ (remember that $C \neq 0$). In this limit the Brans-Dicke scalar (3.2) becomes constant but the line element reduces to

$$ds^2_{(\infty)} = -dt^2 + \left(1 + \frac{B^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega^2_{(2)}). \tag{3.15}$$

The areal radius is

$$R(r) \approx r + \frac{B^2}{r}; \tag{3.16}$$

by inverting this relation one obtains $r^2 - Rr + B^2 = 0$ and there are the two values of the isotropic radius

$$r_{1,2} = \frac{1}{2}\left(R \pm \sqrt{R^2 - 4B^2}\right) \tag{3.17}$$

for each value of the physical areal radius R, which implies that it must be $R \geq 2|B|$. The equation locating the apparent horizons

$$\nabla^c R \nabla_c R = g^{rr}\left(\frac{dR}{dr}\right)^2 = \left(\frac{1 - B^2/r^2}{1 + B^2/r^2}\right)^2 = 0 \tag{3.18}$$

has the double root $r = |B|$ (corresponding to $R = 2|B|$). There is always a wormhole throat in this spacetime, which is not the spherically symmetric, static, asymptotically flat, vacuum solution of GR (i.e., Schwarzschild space). Therefore, the limit $\omega \to -\infty$ fails to reproduce the GR limit even though the Brans-Dicke scalar becomes constant.

B. Einstein frame class II solutions

The Einstein frame class II line element and scalar field are

$$d\tilde{s}^2_{(\mathrm{II})} = \phi_{(\mathrm{II})} ds^2_{(\mathrm{II})} = -\mathrm{e}^{\frac{6C}{\Lambda}\arctan(r/B)} dt^2 + \mathrm{e}^{\frac{-2(C+2)}{\Lambda}\arctan(r/B)} \left(1+\frac{B^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega^2_{(2)}), \tag{3.19}$$

$$\tilde{\phi}_{(\mathrm{II})} = \sqrt{\frac{|2\omega+3|}{16\pi G}} \frac{2C}{\Lambda} \arctan\left(\frac{r}{B}\right) + \mathrm{const.} \tag{3.20}$$

The areal radius and its derivative are

$$\tilde{R}(r) = \left(1+\frac{B^2}{r^2}\right)^2 \mathrm{e}^{\frac{-(C+2)}{\Lambda}\arctan(r/B)} r, \tag{3.21}$$

$$\frac{d\tilde{R}}{dr} = \mathrm{e}^{\frac{-(C+2)}{\Lambda}\arctan(r/B)} \left[1 - \frac{(C+2)}{\Lambda}\frac{B}{r} - \frac{B^2}{r^2}\right], \tag{3.22}$$

while the Ricci scalar is

$$\tilde{\mathcal{R}} = \frac{2B^2C^2|2\omega+3|}{\Lambda^2 r^4 (1+B^2/r^2)^4} \mathrm{e}^{\frac{2(C+2)}{\Lambda}\arctan(r/B)} \tag{3.23}$$

and is never singular. The equation $\nabla^c \tilde{R} \nabla_c \tilde{R} = 0$ locating the horizons is

$$\left[r^2 - \frac{B(C+2)}{\Lambda} r - B^2\right]^2 = 0 \tag{3.24}$$

and, when its roots

$$r_{(\pm)} = \frac{B(C+2)}{\Lambda}\left[1 \pm \sqrt{1+\left(\frac{2\Lambda}{C+2}\right)^2}\right] \tag{3.25}$$

are real and positive they are always double roots; hence there can only be either wormhole throats or naked singularities.

1. *Parameter range $B(C+2)/\Lambda < 0$*

$r_{(+)} < 0 < r_{(-)}$ and $\tilde{R}(r) \to +\infty$ as $r \to 0^+$, while $\tilde{R}(r) \to +\infty$ as $r \to +\infty$. There is a wormhole throat at $r_{(-)}$.

2. *Case $C = -2$*

$r_{(\pm)} = \pm B$ and $\tilde{R}(r) = (1 + B^2/r^2) r$ has the limits $\tilde{R}(r) \to +\infty$ as $r \to 0^+$ and $\tilde{R}(r) \to +\infty$ as $r \to +\infty$. There is a wormhole throat at $r = |B|$ (corresponding to physical radius $\tilde{R} = 2|B|$).

3. *Parameter range $B(C+2)/\Lambda > 0$*

In this case we have $r_{(-)} < 0 < r_{(+)}$ and $\tilde{R}(r) \to +\infty$ in both limits $r \to 0^+$ and $r \to +\infty$. There is a wormhole throat at $r_{(+)}$.

IV. BRANS CLASS III SOLUTIONS

Although it is claimed that the class III family does not admit wormholes [24], this is not the case, as shown below.

A. Jordan frame class III solutions

The line element and Brans-Dicke scalar of Jordan frame class III Brans solutions are, respectively,

$$ds^2_{(\mathrm{III})} = -\mathrm{e}^{-\frac{2r}{B}} dt^2 + \frac{B^4}{r^4} \mathrm{e}^{\frac{2(C+1)}{B} r} (dr^2 + r^2 d\Omega^2_{(2)}), \tag{4.1}$$

$$\phi_{(\mathrm{III})} = \phi_0 \mathrm{e}^{-Cr/B}, \tag{4.2}$$

where

$$C = \frac{-1 \pm \sqrt{-(2\omega+3)}}{\omega+2} \tag{4.3}$$

and, clearly $B \neq 0$, $\omega \leq -3/2$, $\omega \neq -2$. The areal radius and its derivative are

$$R(r) = \frac{B^2}{r} \mathrm{e}^{\frac{(C+1)}{B} r}, \tag{4.4}$$

$$\frac{dR}{dr} = \frac{B(C+1)}{r^2} \mathrm{e}^{\frac{(C+1)}{B} r} \left(r - \frac{B}{C+1}\right). \tag{4.5}$$

We can rewrite the line element (4.1) using the areal radius R instead of the isotropic radius r by means of the substitution

$$dr = \frac{r^2}{B(C+1)(r - \frac{B}{C+1})} \mathrm{e}^{-\frac{(C+1)}{B} r} dR, \tag{4.6}$$

which yields

$$ds^2_{(\mathrm{III})} = -\mathrm{e}^{-\frac{2r}{B}} dt^2 + \frac{B^2}{(C+1)^2 (r - \frac{B}{C+1})^2} dR^2 + R^2 d\Omega^2_{(2)}. \tag{4.7}$$

The horizons, when they exist, are located by the equation $\nabla^c R \nabla_c R = 0$, which becomes simply $g^{RR} = 0$, or

$$\left(\frac{C+1}{B}\right)^2 \left(r - \frac{B}{C+1}\right)^2 = 0. \tag{4.8}$$

There is a double root

$$r_{\rm H} = \frac{B}{C+1} \tag{4.9}$$

when this quantity is positive, with corresponding areal radius

$$R_{\rm H} = {\rm e}B(C+1). \tag{4.10}$$

Therefore, there are either zero or two coincident real roots and there cannot be black holes: Brans class III solutions always describe naked singularities or wormholes.

The Ricci scalar is

$$\mathcal{R} = \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi = \frac{\omega C^2}{B^6}{\rm e}^{-\frac{2(C+1)}{B}r}r^4. \tag{4.11}$$

Let us examine the various possibilities for the range of parameters B and C.

1. Parameter range $C < -1$, $B > 0$

In this case we have

$$R(r) = \frac{B^2}{r}{\rm e}^{-|\frac{C+1}{B}|r}, \tag{4.12}$$

$$\frac{dR}{dr} = -\frac{B^2}{r^2}{\rm e}^{-|\frac{C+1}{B}|r}\left(1+\left|\frac{B}{C+1}\right|r\right), \tag{4.13}$$

and the function $R(r)$ is monotonically decreasing with $R(r) \to 0$ as $r \to +\infty$ and $R(r) \to +\infty$ as $r \to 0^+$. Since $R_{\rm H} = {\rm e}B(C+1) < 0$ there are no horizons and there are no wormholes. The Brans-Dicke scalar $\phi = \phi_0 {\rm e}^{|\frac{C}{B}|r} \to 0$ as $R \to 0$ (corresponding to $r \to +\infty$) and the Ricci scalar

$$\mathcal{R} = \frac{\omega C^2}{B^6}{\rm e}^{2|\frac{C+1}{B}|r}r^4 \to +\infty \tag{4.14}$$

as $R \to 0^+$. Therefore, there is a naked singularity at $R = 0$.

2. Parameter range $C > -1$, $B > 0$

In this case the areal radius $R(r) \to +\infty$ as $r \to +\infty$ and $R(r) \to +\infty$ as $r \to 0^+$. Its derivative dR/dr is negative, and $R(r)$ decreases, for $0 < r < r_{\rm H}$. $R(r)$ has the absolute minimum $R_{\rm H} = {\rm e}B(C+1) > 0$ at $r_{\rm H}$, and increases for $r > r_{\rm H}$. The double root $r_{\rm H}$ of the equation $\nabla^c R\nabla_c R = 0$ is positive and there is a wormhole throat at $r_{\rm H}$, where the Brans-Dicke field (4.2) assumes the finite value $\phi_{\rm H} = \phi_0 {\rm e}^{-\frac{C}{C+1}}$ and it becomes constant if $C = 0$ (the $C = 0$ solution is treated below in the discussion of the limit to GR).

Special subcases are as follows:

(i) $C > 0$, $B > 0$, in which the Brans-Dicke scalar is a finite and decreasing function of r for all values of this coordinate. Its derivative with respect to the areal radius is

$$\frac{d\phi}{dR} = \frac{d\phi}{dr}\frac{dr}{dR} = \frac{d\phi}{dr}\left(\frac{dR}{dr}\right)^{-1} = \frac{-\phi_0 C r^2}{B^2(C+1)(r-r_{\rm H})}{\rm e}^{-\frac{(2C+1)}{B}r} \to \infty \tag{4.15}$$

as $r \to r_{\rm H}$. Therefore, for $r < r_{\rm H}$ (or $R > R_{\rm H}$ in the "left branch" of R), it is $d\phi/dR > 0$, with $d\phi/dR \to +\infty$ as $r \to r_{\rm H}^-$. For $r > r_{\rm H}$ (or $R > R_{\rm H}$ in its "right branch"), instead, it is $d\phi/dR < 0$ with $d\phi/dR \to -\infty$ as $r \to r_{\rm H}^+$. The Brans-Dicke scalar has a cusp, but remains finite, at the horizon $R_{\rm H}$ where it attains its maximum value, which means that the effective gravitational coupling $G_{\rm eff} \sim \phi^{-1}$ is maximum there.

(ii) $-1 < C < 0$, $B > 0$: in this case the Brans-Dicke scalar

$$\phi = \phi_0 {\rm e}^{|\frac{C}{B}|r} \tag{4.16}$$

is an increasing function of the isotropic radius r and its derivative with respect to the areal radius is

$$\frac{d\phi}{dR} = \frac{|C|r^2}{B^2(C+1)(r-r_{\rm H})}{\rm e}^{\frac{-2(C+1)}{B}r}. \tag{4.17}$$

We need to further distinguish the situation $-1 < C \le -1/2$, in which

$$\frac{d\phi}{dR} = \frac{|C|r^2}{B^2|C+1|(r-r_{\rm H})}{\rm e}^{|\frac{2(C+1)}{B}|r} \to \infty \tag{4.18}$$

as the wormhole throat is approached when $r \to r_{\rm H}$. In this case $d\phi/dR$ is negative for $0 < r < r_{\rm H}$, vanishes at $r_{\rm H}$, and is positive for $r > r_{\rm H}$. The Brans-Dicke scalar is minimum and finite but has a cusp (and $G_{\rm eff}$ is maximum) at the wormhole throat.

3. Parameter range $C > -1$, $B < 0$

In this case it is $R_{\rm H} = {\rm e}B(C+1) < 0$ and there are no horizons and no wormhole throats. Since dR/dr is always negative the areal radius is the decreasing function of the isotropic radius

$$R(r) = \frac{B^2}{r}{\rm e}^{-|\frac{C+1}{B}|r}. \tag{4.19}$$

The limit $r \to 0^+$ corresponds to $R \to +\infty$, while $r \to +\infty$ corresponds to $R \to 0^+$. The Ricci scalar is

$$\mathcal{R} = \frac{\omega C^2}{B^6} e^{2|\frac{C+1}{B}|r} \to +\infty \tag{4.20}$$

as $R \to 0^+$. Therefore, there is a central naked singularity for these parameter values.

4. Parameter range $C = -1$, $B \neq 0$

In this case we have $R = B^2/r$ and the line element becomes

$$ds^2_{(\mathrm{III})} = -e^{-2r/B}dt^2 + \frac{B^4}{r^4}(dr^2 + r^2 d\Omega^2_{(2)})$$
$$= -e^{-2r/B}dt^2 + dR^2 + R^2 d\Omega^2_{(2)}. \tag{4.21}$$

The spatial sections are flat and there are no horizons. The limits $r \to 0^+$ and $r \to +\infty$ correspond to $R \to +\infty$ and $R \to 0^+$, respectively. Both the Ricci scalar and the Brans-Dicke scalar field

$$\mathcal{R} = \frac{\omega B^2}{R^4}, \tag{4.22}$$

$$\phi = \phi_0 e^{B/R}, \tag{4.23}$$

diverge as $R \to 0^+$: there is a naked singularity at $R = 0$.

5. Parameter range $C < -1$, $B < 0$

This situation is identical to the case $C > -1$, $B > 0$.

6. Limit to GR

Finally, let us discuss the limit to GR $\omega \to -\infty$, which yields $C \to 0$. In this limit the Brans-Dicke scalar (4.2) becomes constant and the line element reduces to

$$ds^2_{(\infty)} = -e^{-2r/B}dt^2 + \left(\frac{B}{r-B}\right)^2 dR^2 + R^2 d\Omega^2_{(2)}. \tag{4.24}$$

There is a wormhole throat at the horizon $R = R_{\mathrm{H}} = eB$. Also for Brans III solutions, the limit in which ϕ becomes constant does not reproduce the corresponding solution of GR.

B. Einstein frame class III solutions

In the Einstein frame the line element and scalar of class III solutions are, respectively,

$$d\tilde{s}^2_{(\mathrm{III})} = \phi_{(\mathrm{III})} ds^2_{(\mathrm{III})}$$
$$= -e^{-\frac{(C+2)r}{B}}dt^2 + \frac{B^4}{r^4} e^{\frac{(C+2)r}{B}}(dr^2 + r^2 d\Omega^2_{(2)}), \tag{4.25}$$

$$\tilde{\phi}_{(\mathrm{III})} = -\sqrt{\frac{|2\omega+3|}{16\pi G}} \frac{Cr}{B} + \mathrm{const.} \tag{4.26}$$

The areal radius and its derivative are

$$\tilde{R}(r) = \frac{B^2}{r} e^{\frac{(C+2)r}{2B}}, \tag{4.27}$$

$$\frac{d\tilde{R}}{dr} = \frac{B^2}{r^2} e^{\frac{(C+2)r}{2B}} \left(\frac{C+2}{2B}\right)\left(r - \frac{2B}{C+2}\right), \tag{4.28}$$

while the Einstein frame Ricci scalar is

$$\tilde{\mathcal{R}} = \frac{|2\omega+3|C^2 r^4}{2B^6} e^{-\frac{(C+2)r}{B}}. \tag{4.29}$$

The equation $\nabla^c \tilde{R} \nabla_c \tilde{R} = 0$ becomes

$$(C+2)^2\left(r - \frac{2B}{C+2}\right)^2 = 0 \tag{4.30}$$

and has the double root

$$r_* = \frac{2B}{C+2} \tag{4.31}$$

(if $C = -2$ there are no roots).

1. Parameter range $B/(C+2) > 0$

In this range of parameters the double root r_* is positive and the areal radius $\tilde{R}(r)$ diverges in both limits $r \to 0^+$ and $r \to +\infty$. There is a wormhole throat at r_*, corresponding to $\tilde{R}_* = eB(C+2)/2$.

2. Parameter range $B/(C+2) < 0$

In this case there are no horizons, the areal radius $\tilde{R}(r)$ tends to zero value as $r \to +\infty$, where the Ricci scalar diverges, and to infinity as $r \to 0^+$. There is a naked central singularity.

3. Case $C = -2$

In this case there are no horizons and the areal radius $\tilde{R}(r) = B^2/r$ behaves as in the previous case. The Ricci scalar diverges again at $\tilde{R} = 0$ and there is a naked central singularity.

V. BRANS CLASS IV SOLUTIONS

There is another duality relating class III and class IV solutions [15]. We shall come back again to this duality in Sec. VI. Brans IV solutions were examined, for a restricted range of parameters,[2] in the recent Ref. [39] in both the Jordan and Einstein frames. There it is shown that, for a certain range of parameters, the formal solution is a wormhole in the Jordan frame and a naked singularity in

[2]Specifically, for the situations $B > 0$; and $B > 0$, $C > -1$.

the Einstein frame, and the detailed reason why this happens was pointed out [39]. For completeness, we briefly revisit also those cases.

A. Jordan frame class IV solutions

The Jordan frame line element and Brans-Dicke scalar field for Brans class IV solutions are, respectively,[3]

$$ds^2_{(\mathrm{IV})} = -\mathrm{e}^{-\frac{2B}{r}}dt^2 + \mathrm{e}^{\frac{2B(C+1)}{r}}(dr^2 + r^2 d\Omega^2_{(2)}) \quad (5.1)$$

$$\phi_{(\mathrm{IV})} = \phi_0 \mathrm{e}^{-\frac{BC}{r}}, \quad (5.2)$$

where

$$C = \frac{-1 \pm \sqrt{-(2\omega+3)}}{\omega+2}. \quad (5.3)$$

Clearly, the Brans-Dicke parameter is limited to the range $\omega \neq -2$, $\omega < -3/2$. The parameter B has the dimensions of a length and $r > 0$. The areal radius and its derivative are

$$R(r) = \mathrm{e}^{\frac{B(C+1)}{r}} r, \quad (5.4)$$

$$\frac{dR}{dr} = \mathrm{e}^{\frac{r_\mathrm{H}}{r}}\left(1 - \frac{r_\mathrm{H}}{r}\right), \quad (5.5)$$

where

$$r_\mathrm{H} = B(C+1) \quad (5.6)$$

is the root of the equation $dR/dr = 0$ and $R_\mathrm{H} = \mathrm{e} r_\mathrm{H}$ is the corresponding value of the areal radius. The isotropic radius (5.6) is also the double root of the equation locating the horizons $\nabla^c R \nabla_c R = 0$, which becomes $(1 - r_\mathrm{H}/r)^2 = 0$. When r_H is real and positive the solution describes a wormhole, otherwise there are no horizons and no black holes. The Ricci scalar is

$$\mathcal{R} = \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi = \frac{\omega B^2 C^2}{r^4}\mathrm{e}^{-\frac{2B(C+1)}{r}}. \quad (5.7)$$

[3] Note that in the original Brans class IV metric and the corresponding scalar field, introduced in Ref. [8], the parameter B appears in the denominator of the exponents and, hence, has there the dimensions of inverse length. We chose here to put B in the numerators in order for it to have the same dimensions of length as it does within the other three classes. Furthermore, it is only under these forms of the metric and the scalar field that the dualities we are going to discuss in Sec. VI appear to be more than just mathematical transformations of the label r.

1. Parameter range $B > 0$, $C > -1$

In this case $r_\mathrm{H} = |B(C+1)| > 0$ and the areal radius $R(r) = \mathrm{e}^{|\frac{B(C+1)}{r}|} r$ diverges as $r \to 0^+$ and as $r \to +\infty$; it decreases for $0 < r < r_\mathrm{H}$, assumes its minimum value at r_H and increases for $r > r_\mathrm{H}$. We have a wormhole throat at $R_\mathrm{H} = \mathrm{e}B(C+1)$, at which ϕ and $\mathcal{R}$ are finite (see [39] for further discussion).

2. Parameter range $B < 0$, $C > -1$

The root $r_\mathrm{H} = -|B(C+1)|$ is negative and there are no horizons. The areal radius $R(r)$ increases monotonically from zero value as $r \to 0^+$ reaching infinity as $r \to +\infty$. The Ricci scalar

$$\mathcal{R} = \frac{\omega B^2 C^2}{r^4}\mathrm{e}^{2|\frac{B(C+1)}{r}|} \quad (5.8)$$

diverges as $r \to 0+$ (or $R \to 0^+$), signaling a central naked singularity.

3. Parameter range $B < 0$, $C < -1$

The double root $r_\mathrm{H} = |B(C+1)|$ is positive, $R(r) \to +\infty$ as $r \to 0^+$ and as $r \to +\infty$. There is a wormhole throat at r_H, with ϕ and $\mathcal{R}$ finite.

4. Parameter range $B > 0$, $C < -1$

It is $r_\mathrm{H} = -|B(C+1)| < 0$ and there are no horizons. The areal radius $R(r)$ increases monotonically from zero value at $r = 0$ to infinity as $r \to +\infty$. The Ricci scalar

$$\mathcal{R} = \frac{\omega B^2 C^2}{r^4}\mathrm{e}^{2|\frac{B(C+1)}{r}|} \quad (5.9)$$

diverges as $r \to 0^+$ (i.e., as $R \to 0^+$), signaling again a central naked singularity.

5. Parameter range $C = -1$, $B \neq 0$

This situation corresponds to $\omega = -2$, which is excluded by Eq. (5.3). However, one could think of considering the formal line element (5.1) without reference to its derivation in [8], that is,

$$ds^2_{(\mathrm{IV})} = -\mathrm{e}^{-\frac{2B}{r}}dt^2 + dr^2 + r^2 d\Omega^2_{(2)}, \quad (5.10)$$

for which areal and isotropic radius coincide, and which has flat spatial sections. The Ricci scalar

$$\mathcal{R} = \frac{\omega B^2 C^2}{R^4} \quad (5.11)$$

diverges as $R \to 0$ and there is a naked singularity.

6. The GR limit

The GR limit should correspond to $\omega \to -\infty$, which implies that $C \to 0$ and $r_{\rm H} \to B$. The scalar field becomes constant in this limit, the line element reduces to

$$ds^2_{(\infty)} = -e^{\frac{-2B}{r}} dt^2 + e^{\frac{2B}{r}} (dr^2 + r^2 d\Omega^2_{(2)}), \tag{5.12}$$

and the areal radius is $R(r) = r e^{\frac{B}{r}}$. The Ricci scalar is

$$\mathcal{R} = \frac{\omega B^2 C^2}{r^4} e^{-\frac{2B(C+1)}{r}} \approx \frac{-2B^2}{r^4} e^{-\frac{2B}{r}}. \tag{5.13}$$

If $B > 0$, it is $r_{\rm H} > 0$, $R(r) \to +\infty$ as $r \to 0^+$ and as $r \to +\infty$. The Ricci scalar is finite at $R = 0$ and the solution describes a wormhole.

If $B < 0$, it is $r_{\rm H} < 0$ and there are no horizons. The areal radius $R(r) \to 0$ as $r \to 0^+$ and $R(r) \to +\infty$ as $r \to +\infty$. The Ricci scalar diverges at $R = 0$: there is a central naked singularity.

In either case the limit in which ϕ becomes constant fails to reproduce the corresponding GR solution.

B. Einstein frame class IV solutions

The Einstein frame line element and scalar field for class IV solutions are

$$d\tilde{s}^2_{(\rm IV)} = \phi_{(\rm IV)} ds^2_{(\rm IV)}$$
$$= -e^{-\frac{B(C+2)}{r}} dt^2 + e^{\frac{B(C+2)}{r}} (dr^2 + r^2 d\Omega^2_{(2)}), \tag{5.14}$$

$$\tilde{\phi}_{(\rm IV)} = \sqrt{\frac{|2\omega+3|}{16\pi G}} \frac{BC}{r} + \text{const.} \tag{5.15}$$

The areal radius is simply

$$\tilde{R} = e^{\frac{B(C+2)}{2r}} \tag{5.16}$$

and the Ricci scalar is

$$\tilde{\mathcal{R}} = \frac{|2\omega+3| B^2 C^2}{2r^4} e^{\frac{-B(C+2)}{r}}. \tag{5.17}$$

The equation $\nabla^c \tilde{R} \nabla_c \tilde{R} = 0$ becomes $(r - r_*)^2 = 0$, where

$$r_* = \frac{B(C+2)}{2} \tag{5.18}$$

is the only root and a double root.

1. Parameter range $B(C+2) < 0$

In this case $r_* < 0$ and there are no horizons. The areal radius has the limits $\tilde{R}(r) \to 0^+$ as $r \to 0^+$, where the Ricci scalar diverges, and $\tilde{R}(r) \to +\infty$ as $r \to +\infty$. There is a naked central singularity.

2. Parameter range $B(C+2) > 0$

In this case $r_* > 0$, and $\tilde{R}(r) \to +\infty$ in both limits $r \to 0^+$ and $r \to +\infty$. There is a wormhole throat at physical radius $\tilde{R}_* = eB(C+2)/2$.

3. Case $C = -2$

In this case areal radius and isotropic radius coincide and the Ricci scalar diverges as $\tilde{R} \to 0^+$. There is a naked central singularity.

VI. THE DUALITIES

As mentioned above, Brans solutions are not actually all independent as there are dualities relating pairs of the solution classes [14,15]. There is a duality relating classes I and II and there is another duality relating classes III and VI. It is therefore not surprising to find the same pattern concerning the existence of wormholes and/or naked singularities within a pair of solutions related by such a duality. It is also not a coincidence that all the Brans classes fail to recover the GR limit as $\omega \to \infty$.

Furthermore, as we shall see shortly, these dualities are akin to the duality one finds for the Schwarzschild black hole solution when the latter is written in isotropic coordinates. Indeed, it is well known that the Schwarzschild metric in isotropic coordinates is self-dual under the inversion $r \leftrightarrow B^2/r$, as it can easily be verified using the metric (2.4). This fact might actually have been expected as the Schwarzschild solution in isotropic coordinates is recovered either from class I or class II when the parameter C vanishes as we saw in Eq. (2.4) and in the remark below Eq. (3.3), respectively, for then the Brans-Dicke field ϕ becomes a constant. The same observation also applies to the case of classes III and IV, as the latter reduces to the Minkowski spacetime for $B = 0$ while the former reduces, for $B \to \infty$ (C cannot vanish for these classes), to a Minkowski spacetime whose radial coordinate r has been inverted to $1/r$. All these observations remain valid when going to the Einstein frame.

The duality transformation that relates class III to class IV is the following inversion:

$$r \leftrightarrow \frac{B^2}{r}. \tag{6.1}$$

Notice that the form of the duality transformation displayed here is slightly different from that given in Ref. [15]. The dimensions here are correct as the parameter B has the same dimensions of length as r. In fact, by a straightforward substitution, one easily recovers in the Jordan frame the metric (4.1) from the metric (5.1), and vice versa, thanks to this transformation of the coordinate r. In the Einstein frame one also recovers the metric (5.14) from the metric (4.25) using such an inversion. The effect of this inversion is easily seen by comparing the location of the would-be wormholes' throats (4.9) and (5.6) in the Jordan frame and (5.18) and (4.31) in the Einstein frame; one just being the inverse of the other up to the factor B^2 as dictated by the inversion (6.1).

The duality transformation that relates class I to class II is

$$r \leftrightarrow \frac{B^2}{r}, \qquad \lambda \leftrightarrow -i\Lambda, \qquad B \leftrightarrow iB. \qquad (6.2)$$

Here also, our notation differs from that of Refs. [14,15] in that we used B^2 for the r inversion in order to get the dimensions right. In fact, by a straightforward substitution, one easily recovers in the Jordan frame the metric (3.1) from the metric (2.1), and vice versa, thanks to these three transformations. The same applies to the metrics (2.25) and (3.19) in the Einstein frame.

In contrast to the case of classes III and IV, however, the effect of the duality transformation (6.2) is not to make the radii (2.12) and (3.8) of the would-be wormhole throats in the Jordan frame inverse of each other. The same applies to the case of the radii (2.31) and (3.25) of the would-be wormholes in the Einstein frame. In both cases, one is recovered from the other just by making the substitutions (6.2) on Λ and B but the resulting radii are not inverse of each other. This could easily be understood by the fact that, in contrast to classes III and IV which admit as a limit the Minkowski spacetime, classes I and II admit as a limit the Schwarzschild metric which is already self-dual under the inversion $r \leftrightarrow B^2/r$ in isotropic coordinates.

Now, since the parameter B has the dimensions of length and the parameters λ and Λ both appear in exponents inside the metrics (3.1) and (2.1), it might seem unphysical to change these parameters into imaginary entities. Note, however, that the meaning we give to the parameter B cannot be taken independently from the parameter λ in class I or independently from the parameter Λ in class II. In fact, as we saw in Eq. (2.4), for $C = 0$ the parameter B becomes the mass parameter of the Schwarzschild solution if $\lambda = 1$, whereas for $\lambda = -1$ it is the parameter $-B$ that should be interpreted as the Schwarzschild mass parameter in Eq. (2.6). The same applies for class II whose metric (3.1) reproduces the Schwarzschild metric in isotropic coordinates only when both Λ and B become imaginary.

Finally, one might wonder at this point if there still exists another duality transformation that might relate one pair of classes to another pair. The answer is no and the reason is the following. The fact that one pair of solutions (classes I and II) reduces to the Schwarzschild metric for a specific value of the parameter C and the other pair (classes III and IV) reduces to the Minkowski spacetime for a specific value of the parameter B is what forbids the existence of any duality between the two pairs. In other words, as there is no duality between the curved Schwarzschild spacetime and the flat Minkowski spacetime, no duality could exist between the first pair and the second pair either.

VII. CONCLUSIONS

We have verified explicitly that the Brans classes I–IV of solutions [8] of Brans-Dicke theory [1] always describe either wormholes or else horizonless geometries containing naked singularities, and they never describe black holes, in agreement with the Agnese-La Camera theorem [10] and with Hawking's theorem on Brans-Dicke black holes [19–21].

Our conclusions are relevant also for $f(\mathcal{R})$ theories of gravity which, in their metric formulation, are equivalent to an $\omega = 0$ Brans-Dicke theory which, contrary to our situation, has a scalar field potential (a variant of O'Hanlon theory [40]) [5–7]. In the Palatini formalism, instead, $f(\mathcal{R})$ gravity reduces to an $\omega = -3/2$ Brans-Dicke theory (again, with a potential) [5–7]. The conclusions reached here are also consistent with the failure of the Jebsen-Birkhoff theorem in scalar-tensor gravity. In general, this theorem does not hold and only a weak version of it is valid which states that if, *in vacuo*, the Brans-Dicke scalar ϕ is static, then the solution is static (but not necessarily Schwarzschild or Schwarzschild-de Sitter). Again, this result is not in contradiction with the Hawking theorem on Brans-Dicke black holes, as discussed in detail in Ref. [41] specifically for Brans solutions. Regarding $f(\mathcal{R})$ gravity, the Jebsen-Birkhoff theorem does not hold in the metric formalism, as is well known, but holds in Palatini $f(\mathcal{R})$ gravity with a static matter distribution because in this case the Brans-Dicke scalar $\phi = f'(\mathcal{R})$ is not a dynamical degree of freedom [41].

Ambiguity and confusion lingering in the literature about the nature of Brans solutions and apparent contradictions with the theorems mentioned above are thus eliminated once and for all. Spacetimes harboring naked singularities are unphysical since one cannot prescribe regular initial data in the presence of a naked singularity and the initial value problem fails, leaving the theory void of predictability and wormholes as the only remaining Brans solutions. Wormholes are exotic objects which require the energy conditions to be violated. The Brans solutions are vacuum solutions and the Brans-Dicke scalar acts as the only form of effective "matter" once the field equations are written as effective Einstein equations, as in Eq. (1.3). It is well known that a nonminimally coupled scalar field and the Brans-Dicke scalar can violate all of the energy conditions; therefore it is no surprise that one can obtain wormholes as solutions of vacuum Brans-Dicke theory, as has already been remarked in the literature.

No Jordan frame solution of any Brans class has the correct limit to GR, which lends a word of caution on using these solutions as toy models, since they are rather pathological. The reason for their failure to reproduce the corresponding GR solution (i.e., the Schwarzschild geometry) is by now well understood, as recalled above.

The Einstein frame versions of the Brans solutions, naturally, describe only spacetime geometries containing either wormholes or naked singularities. It is interesting that a wormhole can sometimes be conformally transformed to a naked singularity or vice versa. These instances and the general reasons why they occur were analyzed in detail in the recent Ref. [39], using Brans IV solutions as an example. The present work provides more examples.

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