Gravitational scattering, post-Minkowskian approximation, and effective-one-body theory

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(Received 1 September 2016; published 7 November 2016)

A novel approach to the effective-one-body description of gravitationally interacting two-body systems is introduced. This approach is based on the *post-Minkowskian* approximation scheme (perturbation theory in G, without assuming small velocities) and employs a new dictionary focussing on the functional dependence of the scattering angle on the total energy and the total angular momentum of the system. Using this approach, we prove *to all orders in* v/c two results that were previously known to hold only to a limited post-Newtonian accuracy: (i) the relativistic gravitational dynamics of a two-body system is equivalent, at first post-Minkowskian order, to the relativistic dynamics of an effective test particle moving in a Schwarzschild metric, and (ii) this equivalence requires the existence of an *exactly quadratic map* between the real (relativistic) two-body energy and the (relativistic) energy of the effective particle. The same energy map is also shown to apply to the effective-one-body description of two masses interacting via tensor-scalar gravity.

DOI: 10.1103/PhysRevD.94.104015

I. INTRODUCTION

The effective-one-body (EOB) formalism was conceived [1–4] with the aim of analytically describing both the last few orbits of, and the complete gravitational-wave signal emitted by, coalescing binary black holes. The predictions, made as early as 2000 [2], by the EOB formalism have been broadly confirmed by subsequent numerical simulations [5–8]. This then led to the development of numerical-relativity-improved versions of the EOB dynamics and waveform (see, e.g., Refs. [9–15]), which have helped the recent discovery, interpretation, and data analysis of the first gravitational-wave signals by the Laser Interferometer Gravitational-Wave Observatory [16–18].

The aim of the present paper is to introduce a novel theoretical approach to some of the basic structures of EOB theory. The hope is that this new approach could lead to theoretically improved versions of the EOB conservative dynamics, which might be useful in the upcoming era of high signal-to-noise-ratio gravitational-wave observations. In this work, we shall only consider the interaction of nonspinning bodies at the first order in G. Our strategy is, however, generalizable to higher orders in G and to spinning bodies.

The EOB conservative dynamics is a relativistic generalization of the well-known Newtonian-mechanics fact that the relative dynamics of a two-body system (with masses m_1 and m_2) is equivalent to the motion of an effective particle of mass

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}, \tag{1}$$

submitted to the original two-body potential $V(|\mathbf{R}_1 - \mathbf{R}_2|)$. In the case of the Newtonian gravitational interaction [i.e., $V(|\mathbf{R}_1 - \mathbf{R}_2|) = -G m_1 m_2 / |\mathbf{R}_1 - \mathbf{R}_2|$], the identity $m_1 m_2 = \mu(m_1 + m_2)$ implies that the effective particle of mass μ moves in the gravitational potential of a central mass equal to

$$M \equiv m_1 + m_2. \tag{2}$$

The historical approach to defining the EOB (conservative) dynamics [1,3,4] has been, so far, based on two basic ingredients:

(1) a post-Newtonian (PN) description of the two-body dynamics (the limit of which when $c \to \infty$ is the Newtonian result just recalled),

(2) a dictionary between the PN-expanded knowledge of the two-body *bound states* (ellipticlike motions) and the bound states of a test particle moving in some external metric.

Here, these two historical ingredients will be replaced by two other ones:

(a) The PN approximation method (which is a combined expansion in $\frac{G}{c^2}$ and in $\frac{1}{c^2}$) will be replaced by the *post-Minkowskian* (PM) approximation method, i.e., by an expansion in the gravitational constant G, which never assumes that the velocities are small compared to the velocity of light c. After some pioneering work in the late 1950s [19,20], the PM approach to gravitational motion played a useful role around the 1980s in clarifying the computation of retarded interactions (and associated radiation reaction) in binary systems [21–24].

(b) The bound states of two-body systems will be replaced by *scattering states*. The replacement of bound states by scattering states will oblige us to replace the usual dictionary of EOB theory by a new one (that we shall prove to be physically equivalent). This will allow us to exploit the fully relativistic results on gravitational scattering motions [22,23,25–29] obtained by PM calculations.

Before expounding our new strategy, we shall briefly recall some of the basic features of the current approach to EOB theory.

II. REMINDER OF THE EOB DICTIONARY FOR BOUND STATES

The construction of the EOB dynamics [1–4,30] has been so far based on a dictionary between the bound states (ellipticlike motions) of a gravitationally interacting two-body system, considered in the c.m. frame, and the bound states of an effective particle moving in an effective metric $g^{\rm eff}_{\mu\nu}$. Inspired by the Bohr-Sommerfeld quantization conditions of bound states ($I_i = n_i \hbar$), the latter dictionary requires the identification between the action integrals $I_i = \frac{1}{2\pi}\oint P_i dQ^i$ (no sum on i) of the real and effective dynamics, i.e., (considering, for simplicity, the reduction of the dynamics to the plane of the motion)

$$I_R^{\rm eff} = I_R^{\rm real}; \qquad I_\varphi^{\rm eff} = I_\varphi^{\rm real}. \tag{3}$$

On the other hand, EOB theory allows for an arbitrary (*a priori* undetermined) *energy map* f between the real c.m. energy $\mathcal{E}_{\rm real}$ and the effective one $\mathcal{E}_{\rm eff}$:

$$\mathcal{E}_{\rm real} \overset{f}{\to} \mathcal{E}_{\rm eff}. \tag{4}$$

When dealing with bound states (which can be treated within PN theory, i.e., at successive orders in an expansion in $\frac{1}{c^2}$), one parametrizes the unknown energy map f by a PN expansion of the type

$$\frac{\mathcal{E}_{\rm eff}}{m_0 c^2} = 1 + \frac{E_{\rm real}}{\mu c^2}\left(1 + \alpha_1 \frac{E_{\rm real}}{\mu c^2} + \alpha_2 \left(\frac{E_{\rm real}}{\mu c^2}\right)^2 + \alpha_3 \left(\frac{E_{\rm real}}{\mu c^2}\right)^3 + \alpha_4 \left(\frac{E_{\rm real}}{\mu c^2}\right)^4 + \cdots\right), \tag{5}$$

where m_0 denotes the mass of the effective particle and where $E_{\rm real}$ denotes the "nonrelativistic" real energy, i.e., the difference

$$E_{\rm real} \equiv \mathcal{E}_{\rm real} - (m_1 + m_2)c^2. \tag{6}$$

In EOB theory, the energy map f is determined, *at any given PN approximation*, by the basic EOB bound-state requirement

$$\mathcal{E}_{\rm eff}(I_R, I_\varphi) = f[\mathcal{E}_{\rm real}(I_R, I_\varphi)], \tag{7}$$

in which both energies are expressed in terms of the *common* values of the action variables $I_i = I_i^{\rm eff} = I_i^{\rm real}$.

The so-determined energy map turns out to exhibit, so far, a very simple (and natural [31,32]) structure, described, at the presently known order, by the following coefficients in the PN expansion (5),

$$\alpha_1 = \frac{\nu}{2}; \qquad 0 = \alpha_2 = \alpha_3 = \alpha_4, \tag{8}$$

where ν denotes the symmetric mass ratio,

$$\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \tag{9}$$

More precisely, the first two results $\alpha_1 = \frac{\nu}{2}$;, $\alpha_2 = 0$, which refer to the first post-Newtonian (1PN) and second post-Newtonian (2PN) levels, respectively, were derived in Ref. [1]. At the third post-Newtonian (3PN) level, and under the assumption already made in Ref. [1] that the effective metric coincides, at linear order in Newton's constant G, with the Schwarzschild metric, Ref. [3] derived the next term in the energy map and found it simply to be $\alpha_3 = 0$. Recently, the extension of the EOB formalism to the fourth post-Newtonian (4PN) level [30] again found an uncorrected energy map, i.e., $\alpha_4 = 0$. These results suggest, without, however, proving it, that the energy map f will remain uncorrected by higher-order PN corrections and is exactly quadratic.

One of the main motivations for the present extension of EOB theory to scattering states (i.e., hyperboliclike motions) is to bypass the need to determine the energy map in the form of the PN expansion (5). This will allow us to prove, for the first time, that all the higher-order PN coefficients α_n in the PN expansion (5) actually vanish, so that the energy map $\mathcal{E}_{\rm eff} = f(\mathcal{E}_{\rm real})$ is exactly quadratic. We shall also prove what was assumed before, namely that the effective metric must coincide, at linear order in G, with the Schwarzschild metric. [Reference [1] explicitly assumed that the coefficient b_1 parametrizing the spatial part of the effective metric to order G^1 was equal to its Schwarzschild counterpart. The attempt of Ref. [3] (see Appendix A there) to relax this assumption did not lead to a convincing result, so Ref. [3] finally kept this assumption on b_1.]

III. DICTIONARY FOR SCATTERING STATES

When considering scattering states (i.e., hyperboliclike motions), we lose the possibility of uniquely parametrizing two-body bound states by means of the two action variables I_R, I_φ. Actually, we still can use the second action variable, because

$$I_\varphi \equiv \frac{1}{2\pi}\oint P_\varphi d\varphi = P_\varphi \equiv J \tag{10}$$

is simply the total (c.m.) angular momentum of the binary system (considered as moving in the equatorial plane $\theta = \frac{\pi}{2}$). (As we restrict our attention here to nonspinning systems, we could indifferently denote P_φ, which is the total *orbital* angular momentum, by J or L.) We then evidently keep the second half of the bound-state dictionary (3), namely

$$J_{\rm eff} = J_{\rm real} = J, \tag{11}$$

within our new scattering-state dictionary.

We need, however, a replacement for the first, radial half of the bound-state dictionary (3). This replacement is easily found if we recall that the radial action variable I_R is linked to the evolution of the polar angle φ within the plane of the motion. Indeed, Hamilton-Jacobi theory tells us to differentiate the energy-reduced action $S_0(R,\varphi;\mathcal{E},J) = J\varphi + \int dR P_R(R;\mathcal{E},J)$ with respect to J to obtain the functional link between φ and R, namely

$$\varphi = -\int dR \frac{\partial P_R(R;\mathcal{E},J)}{\partial J} + \text{const.} \tag{12}$$

In the bound-state case, the latter equation (which is gauge dependent) yields, upon integration over a radial period, the gauge-independent link

$$\Phi_{\rm bound} = -\oint dR \frac{\partial P_R(R;\mathcal{E},J)}{\partial J} = -2\pi \frac{\partial I_R(\mathcal{E},J)}{\partial J}, \tag{13}$$

between the total periastron precession per orbit, $\Phi_{\rm bound}$, and the J-derivative of the radial action, $I_R = \frac{1}{2\pi}\oint dR P_R$.

In the scattering case, it is very natural to replace the consideration of $\Phi_{\rm bound}$, Eq. (13), by that of the total angular change during scattering, i.e.,

$$\Phi_{\rm scatt} = -\int_{-\infty}^{+\infty} dR \frac{\partial P_R(R;\mathcal{E},J)}{\partial J}, \tag{14}$$

where the label $-\infty$ refers to the incoming state (at $-\infty$ in time and $+\infty$ in R), while the label $+\infty$ refers to the outgoing state (at $+\infty$ in time and $+\infty$ in R). The total change $\Phi_{\rm scatt}$ in polar angle is usually parametrized in terms of the corresponding "scattering angle" χ defined simply as

$$\chi \equiv \Phi_{\rm scatt} - \pi, \tag{15}$$

so that it vanishes for free motions.

In view of the links we just recalled, it is very natural, when considering scattering states, to replace the bound-state dictionary (3) by the two conditions: Eq. (11) together with

$$\chi_{\rm eff} = \chi_{\rm real}. \tag{16}$$

This leads us to replace the basic bound-state requirement (7) by the following scattering-state requirement,

$$\mathcal{E}_{\rm eff}(\chi,J) = f[\mathcal{E}_{\rm real}(\chi,J)] \tag{17}$$

or, equivalently,

$$\chi_{\rm eff}(\mathcal{E}_{\rm eff},J) = \chi_{\rm real}(\mathcal{E}_{\rm real},J), \tag{18}$$

where it is understood that $\mathcal{E}_{\rm eff}$ (on the left-hand side) and $\mathcal{E}_{\rm real}$ (on the right-hand side) must be related by the (looked-for, exact) energy map f, i.e.,

$$\mathcal{E}_{\rm eff} = f(\mathcal{E}_{\rm real}). \tag{19}$$

Some comments on this scattering dictionary are in order. First, let us note that $\chi_{\rm real}$ measures the gravitational two-body scattering *in the c.m. frame*; it is the common value of the scattering angles of each particle in the latter frame (where $\mathbf{P}_1 = -\mathbf{P}_2 = \mathbf{P}$), as well as the scattering angle of the relative motion dynamics $\mathbf{Q} = \mathbf{R}_1 - \mathbf{R}_2$, which is the dynamics we consider. We assume that we study the real relative motion in a class of coordinate gauges which is regular enough at infinity to ensure the gauge invariance of $\chi_{\rm real}$. The requirement (16) then amounts to assuming that the canonical transformation (with generating function $\mathcal{G}$) linking the real phase-space (relative) variables $\mathbf{Q}$, $\mathbf{P}$ to the effective ones, say $\mathbf{Q}'$, $\mathbf{P}'$, is such that asymptotically $\mathbf{P}' \propto \mathbf{P}$. It is easily seen that this is actually a consequence of the general form of the generating function

$$\mathcal{G}(\mathbf{Q},\mathbf{P}') = \mathbf{Q}\cdot\mathbf{P}'\left[1 + c_{11}\left(\frac{\mathbf{P}'}{\mu c}\right)^2 + c_{12}\frac{GM}{c^2 R} + \cdots\right], \tag{20}$$

which was assumed (and found consistent) in previous (PN-based) EOB works. Finally, let us note that the usefulness of considering the asymptotically defined, gauge-invariant functional link $\chi_{\rm real}(\mathcal{E}_{\rm real},J)$ between the scattering angle and the (asymptotic, conserved.[1]) total energy and angular momentum was emphasized in Ref. [33], within the context of self-force studies, and was studied in full numerical relativity in Ref. [34].

IV. GENERAL STRUCTURE OF THE POST-MINKOWSKIAN EXPANSION OF THE SCATTERING FUNCTION

As a warmup, and for orientation, let us recall the value of the scattering function $\chi_{\rm real}(\mathcal{E}_{\rm real},J)$ in the Newtonian approximation (Rutherford scattering). It is most simply obtained by starting from the polar-coordinate equation of a conic, namely $R = p/(1 + e\cos\varphi)$ (valid for all three types of conics). In the hyperbolic case (i.e., when

[1]We recall that we focus here on the conservative dynamics of two-body systems.

$e > 1$), the branches at infinity correspond to the roots of $1 + e\cos\varphi = 0$. This leads to $\Phi_{\rm scatt} = 2\arccos(-1/e)$ or, equivalently,

$$\chi^{\rm Newton} = 2\arctan\left(\frac{1}{\sqrt{e^2-1}}\right). \tag{21}$$

In the Newtonian approximation, the eccentricity e is linked to the nonrelativistic energy $E = \mathcal{E} - (m_1 + m_2)c^2$ and the angular momentum J via

$$e^2 = 1 + 2\frac{E}{\mu c^2}\left(\frac{cJ}{Gm_1m_2}\right)^2 \equiv 1 + 2\hat{E}j^2, \tag{22}$$

where we have introduced the dimensionless versions of E and J used in most PN works, namely

$$\hat{E} \equiv \frac{E}{\mu c^2}; \qquad j \equiv \frac{cJ}{Gm_1m_2} = \frac{cJ}{G\mu M}. \tag{23}$$

This leads to the following explicit form of the Newtonian scattering function,

$$\chi^{\rm Newton}(E,J) = 2\arctan\left(\frac{1}{\sqrt{2\hat{E}j^2}}\right) = 2\arctan\left(\frac{Gm_1m_2}{cJ}\sqrt{\frac{\mu c^2}{2E}}\right), \tag{24}$$

or

$$\tan\left(\frac{1}{2}\chi^{\rm Newton}(E,J)\right) = \frac{Gm_1m_2}{cJ}\sqrt{\frac{\mu c^2}{2E}}. \tag{25}$$

Note that the velocity of light c cancels between $\hat{E}$ and j^2, as expected for a Newtonian-level result.

While the PN expansion is an expansion in powers of $\frac{1}{c^2}$, the PM expansion is an expansion in powers of G. We see on Eq. (24) or (25) that χ starts (as expected) at order G in a PM expansion. As χ is a dimensionless quantity that is a function of the two dimensionless quantities $\hat{E}$ and j (and of the symmetric mass ratio, $\nu \equiv \mu/M$) and as the definitions (23) show that G enters only via $j = cJ/(Gm_1m_2) \propto c/G$, we see that the PM expansion of (half) the scattering function will be equivalent to an expansion in powers of $1/j \propto G$, say

$$\frac{1}{2}\chi(E,J) = \frac{1}{j}\chi_1(\hat{E},\nu) + \frac{1}{j^2}\chi_2(\hat{E},\nu) + \frac{1}{j^3}\chi_3(\hat{E},\nu) + \cdots \tag{26}$$

Here, $\chi_1(\hat{E},\nu)/j$ is the first post-Minkowskian (1PM) approximation of (half) the scattering function, $\chi_2(\hat{E},\nu)/j^2$ is its second post-Minkowskian (2PM) approximation, etc.

Note that each term in the PM expansion of the scattering function is a function of the energy (defined for $E > 0$). Our main tool here will be to compute and exploit the *exact form* of the 1PM function $\chi_1(\hat{E},\nu)$. As the latter function is, again, a dimensionless function of the dimensionless quantity $\hat{E} \propto \frac{1}{c^2}$, we see that the reexpansion of $\chi_1(\hat{E},\nu)$ in powers of $\hat{E}$ will correspond to the part of the PN expansion of the scattering function that is proportional to $1/J$. More precisely, remembering that $\frac{1}{j} \propto \frac{G}{c}$, and parametrizing the nonrelativistic energy $E > 0$ by a corresponding squared velocity v_E, defined via

$$E \equiv \frac{1}{2}\mu v_E^2; \qquad v_E \equiv \sqrt{\frac{2E}{\mu}}, \tag{27}$$

so that $\hat{E} = \frac{v_E^2}{2c^2}$, we see that the structure of the PN expansion of (half) the scattering function must be of the form

$$\begin{aligned}\frac{1}{2}\chi(E,J) \sim\ & \frac{Gm_1m_2}{cJ}\left(\frac{c}{v_E} + \frac{v_E}{c} + \left(\frac{v_E}{c}\right)^3 + \cdots\right) \\ & + \left(\frac{Gm_1m_2}{cJ}\right)^2\left(0\left(\frac{c}{v_E}\right)^2 + 1 + \left(\frac{v_E}{c}\right)^2 + \cdots\right) \\ & + \left(\frac{Gm_1m_2}{cJ}\right)^3\left(\left(\frac{c}{v_E}\right)^3 + \frac{c}{v_E} + \frac{v_E}{c} + \cdots\right) \\ & + \cdots \end{aligned} \tag{28}$$

where the first line sketches the form of the PN expansion of the 1PM term $\chi_1(\hat{E},\nu)$, say (now with coefficients)

$$\chi_1(\hat{E},\nu) = 1\frac{c}{v_E} + \chi_{11}(\nu)\frac{v_E}{c} + \chi_{13}(\nu)\left(\frac{v_E}{c}\right)^3 + \cdots; \tag{29}$$

the second line [where, as indicated, the term $\propto (c/v_E)^2$ is missing] sketches the form of the PN expansion of the 2PM term $\chi_2(\hat{E},\nu)$, say

$$\chi_2(\hat{E},\nu) = \chi_{20}(\nu) + \chi_{22}(\nu)\left(\frac{v_E}{c}\right)^2 + \chi_{24}(\nu)\left(\frac{v_E}{c}\right)^4 + \cdots; \tag{30}$$

and the third line sketches the form of the PN expansion of the third post-Minkowskian (3PM) term $\chi_3(\hat{E},\nu)$, say

$$\chi_3(\hat{E},\nu) = \chi_{3,-3}(\nu)\left(\frac{c}{v_E}\right)^3 + \chi_{3,-1}(\nu)\frac{c}{v_E} + \chi_{31}(\nu)\frac{v_E}{c} + \cdots \tag{31}$$

Note the presence of *inverse powers* of the velocity v_E, especially in the terms odd in J (and in G). [Such terms will also appear in the even contributions $\chi_{2n}(\hat{E},\nu)$, where they start at order $(c/v_E)^{2n-2}$.] Note also that the coefficients of all the contributions $\propto (Gm_1m_2/cJ)^n(c/v_E)^n$ will vanish for $n > 1$ if one considers the function $\tan(\chi/2)$ instead of $\chi/2$.

Each term in the usual PN expansion of $\chi/2$ is obtained from the expansion (28) above by collecting all the contributions $\sim(Gm_1m_2/cJ)^n(v_E/c)^k$ carrying a given power of $\frac{1}{c}$, i.e., having a given value of the PN order $\frac{1}{2}(n+k)$. The presence of negative powers of v_E (up to $\sim v_E^{-n}$ or $\sim v_E^{-n+2}$ in χ_n) implies that each term of the PN expansion (generally) collects an infinite number of contributions of the PM expansion. [This infinite number of contributions corresponds to the eccentricity dependence of a given PN contribution to $\chi/2$. Indeed, in view of Eq. (22), the eccentricity is of order zero in $\frac{1}{c^2}$ but of order j^1 when considering the large j expansion (26) which defines the PM expansion of the scattering function.] For instance, the 1PN [i.e., $O(\frac{1}{c^2})$] contribution to the scattering function would read

$$\frac{1}{2}\chi(E,J)_{1PN} = \chi_{11}(\nu)\frac{Gm_1m_2}{cJ}\frac{v_E}{c} + \chi_{20}(\nu)\left(\frac{Gm_1m_2}{cJ}\right)^2 + \chi_{3,-1}(\nu)\left(\frac{Gm_1m_2}{cJ}\right)^3\frac{c}{v_E} + \cdots \tag{32}$$

The coefficients $\chi_{nk}(\nu)$ are polynomials in ν of which the degrees linearly increase with (and seem to be generally equal to) the PN order $\frac{1}{2}(n+k)$ of $(Gm_1m_2/cJ)^n(v_E/c)^k$. (See below for examples of these polynomials.)

Let us mention in this respect that the PN expansion of the scattering function has been computed to 2PN accuracy in Sec. V.C of Ref. [35]. From the results there, one can then deduce [by collecting the coefficient of the first power of $1/j$ in $\frac{1}{2}\chi(E,J)_{1PN}$] the beginning of the v_E expansion of the 1PM term $\chi_1(\hat{E},\nu)$. One then finds

$$\chi_1(\hat{E}_{\rm real},\nu) = \frac{c}{v_E^{\rm real}} + \frac{15-\nu}{8}\frac{v_E^{\rm real}}{c} + \frac{35+30\nu+3\nu^2}{128}\left(\frac{v_E^{\rm real}}{c}\right)^3 + O\left(\left(\frac{v_E^{\rm real}}{c}\right)^5\right). \tag{33}$$

Here, we have specified that the energy used in this result to express the PN-expanded 1PM contribution to the scattering angle is the dimensionless μc^2-rescaled *real* nonrelativistic energy

$$\hat{E}_{\rm real} = \frac{E_{\rm real}}{\mu c^2} = \frac{\mathcal{E}_{\rm real} - Mc^2}{\mu c^2}, \tag{34}$$

the auxiliary "velocity" entering (33) denoting simply

$$v_E^{\rm real} \equiv c\sqrt{2\hat{E}_{\rm real}} \equiv \sqrt{\frac{2E_{\rm real}}{\mu}}. \tag{35}$$

[In the above discussion, we had left unspecified whether we were dealing with $\chi_{\rm real}$ as a function of $E_{\rm real}$ or with $\chi_{\rm eff}$ as a function of $E_{\rm eff}$.]

The result (33) illustrates in advance the *complementarity* between the PM-expansion approach used in the present paper and the PN expansions used in previous works. Each term of the PM expansion collects an infinite number of contributions of the PN expansion[2]. Therefore, the exact computation of a certain term in the PM expansion of χ (as we shall report below) represents information about the two-body dynamics which goes beyond all the present PN knowledge (which is limited to the 4PN level [30,36]), though it evidently misses some of the information contained in the PN computations of χ.

V. REAL, TWO-BODY SCATTERING FUNCTION AT THE FIRST POST-MINKOWSKIAN APPROXIMATION

The relativistic gravitational two-body scattering function $\frac{1}{2}\chi_{\rm real}(\mathcal{E}_{\rm real},J;m_1,m_2)$ [now expressed in terms of the total relativistic energy, $\mathcal{E}_{\rm real} = (m_1+m_2)c^2 + E_{\rm real}$] has been computed at the first post-Minkowskian approximation (i.e., first order in G) by several authors [25–29]. We recall that $\mathcal{E}_{\rm real}$ and J are both evaluated in the *c.m. frame* of the two-body system. In the following, it will often be convenient to use units where $c=1$.

The results for χ_{1PM} given in the articles cited above are not written in a way which highlights the basic physics underlying the two-body scattering. We shall therefore give a novel, more transparent derivation of χ_{1PM}, which also exhibits the link between the classical scattering function $\chi_{\rm real}(\mathcal{E}_{\rm real},J;m_1,m_2)$ and the quantum scattering two-body amplitude $\langle p_1'p_2'|p_1p_2\rangle$. Both results will appear to be directly deducible from the diagram displayed in Fig. 1.

The gravitational equation of motion of, say, particle 1 is most simply written as

$$\frac{dp_{1\mu}}{d\sigma_1} = \frac{1}{2}\partial_\mu g_{\alpha\beta}(x_1)p_1^\alpha p_1^\beta. \tag{36}$$

Here, $p_{1\mu} = g_{\mu\nu}(x_1)p_1^\nu$ denotes the (curved spacetime) covariant components of $p_1^\mu = m_1 dx_1^\mu/ds_1$, where ds_1 denotes the proper time along the worldline $x_1^\mu = x_1^\mu(s_1)$ of m_1. (We use a mostly positive signature with, e.g., $g^{\mu\nu}p_{1\mu}p_{1\nu} = -m_1^2$.) We have also introduced the rescaled proper time σ_1 defined as $\sigma_1 = s_1/m_1$ so that $p_1^\mu = dx_1^\mu/d\sigma_1$. Integrating (36) with respect to σ_1 (between $-\infty$ and $+\infty$) yields the total change $\Delta p_{1\mu} \equiv p'_{1\mu} - p_{1\mu}$ in the asymptotic 4-momentum of particle 1,

[2]This is the reciprocal of the fact mentioned above that each term of the PN expansion collects an infinite number of contributions of the PM expansion.

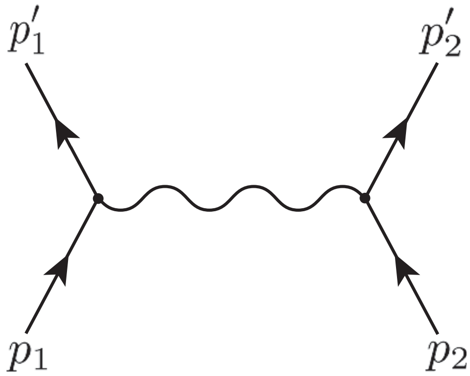

FIG. 1. Diagram displaying the physical ingredients of both the classical and the quantum two-body scattering.

$$\Delta p_{1\mu} = \int_{-\infty}^{+\infty} d\sigma_1 \frac{1}{2} p_1^\alpha p_1^\beta \partial_\mu h_{\alpha\beta}(x_1), \qquad (37)$$

where $h_{\alpha\beta} \equiv g_{\alpha\beta} - \eta_{\alpha\beta}$.

At linearized order in G, p_1^α on the rhs of (37) can be replaced by the constant incoming 4-momentum of particle 1, while the metric perturbation can be replaced by the solution of the linearized Einstein equations, namely (in four spacetime dimensions and in harmonic gauge)

$$\Box h_{\alpha\beta} = -16\pi G\left(T_{\alpha\beta} - \frac{1}{2} T \eta_{\alpha\beta}\right). \qquad (38)$$

Here and below, all index operations are performed in the flat Minkowski background (e.g., $T = \eta_{\alpha\beta} T^{\alpha\beta}$). At order G, only the metric perturbation generated by the (flat spacetime) stress-energy tensor of particle 2,

$$T_2^{\alpha\beta}(x) = \int_{-\infty}^{+\infty} d\sigma_2 p_2^\alpha p_2^\beta \delta^4(x - x_2(\sigma_2)), \qquad (39)$$

needs to be inserted in the computation (37) of $\Delta p_{1\mu}$.

In order to exhibit the link between the classical scattering and the usual Feynman diagram corresponding to Fig. 1, let us work in (four-dimensional) Fourier space ($k.x \equiv k_\mu x^\mu \equiv \eta_{\mu\nu} k^\mu x^\nu$),

$$h_{\alpha\beta}(x) = \int \frac{d^4k}{(2\pi)^4} h_{\alpha\beta}(k) e^{ik.x}. \qquad (40)$$

The m_2-generated metric perturbation reads

$$h_{2\alpha\beta}(x) = 16\pi G \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} T_2^{\alpha'\beta'}(k), \qquad (41)$$

where $P_{\alpha\beta;\alpha'\beta'}/k^2$, with $P_{\alpha\beta;\alpha'\beta'} \equiv \eta_{\alpha\alpha'}\eta_{\beta\beta'} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\alpha'\beta'}$ and $k^2 \equiv \eta^{\mu\nu} k_\mu k_\nu$, is the Fourier-space gravitational propagator. [At this order, it does not matter whether one considers a retarded or a time-symmetric propagator (with $1/k^2$ then denoting a principal value kernel).]

The k-space stress-energy tensor of m_2 reads

$$T_2^{\alpha'\beta'}(k) = \int d^4x e^{-ik.x} T_2^{\alpha'\beta'}(x)$$
$$= \int_{-\infty}^{+\infty} d\sigma_2 e^{-ik.x_2(\sigma_2)} p_2^{\alpha'} p_2^{\beta'}. \qquad (42)$$

Inserting, successively, Eq. (42) into (41) and Eq. (41) into (37) leads to the following explicit expression for the total change of the 4-momentum of particle 1:

$$\Delta p_{1\mu} = 8\pi G \int \frac{d^4k}{(2\pi)^4} ik_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'}$$
$$\times \int d\sigma_1 \int d\sigma_2 e^{ik.(x_1(\sigma_1) - x_2(\sigma_2))}. \qquad (43)$$

(By changing $p_1 \to p_2, x_1 \to x_2, k \to -k$, one immediately sees that one would have $\Delta p_{2\mu} = -\Delta p_{1\mu}$.) On the first line, we recognize all the ingredients of the quantum scattering amplitude of Fig. 1: the two matter-gravity vertices $p_1^\alpha p_1^\beta$ and $p_2^{\alpha'} p_2^{\beta'}$ (computed in the approximation $p_1' \approx p_1$, $p_2' \approx p_2$), connected by the gravitational propagator $P_{\alpha\beta;\alpha'\beta'}/k^2$. At first order in G, all the ingredients entering the rhs of (43) can be replaced by their zeroth-order (free motion) approximations, i.e., constant momenta (as already used) and straight (incoming) worldlines:

$$x_1^\mu(\sigma_1) = x_1^\mu(0) + p_1^\mu \sigma_1;$$
$$x_2^\mu(\sigma_2) = x_2^\mu(0) + p_2^\mu \sigma_2. \qquad (44)$$

Inserting the straight-worldlines expressions (44) into Eq. (43) allows one to explicitly compute the σ_1 and σ_2 integrals on the second line, with the result

$$(2\pi)^2 e^{ik.(x_1(0) - x_2(0))} \delta(k.p_1) \delta(k.p_2). \qquad (45)$$

The crucial point is that the σ_1 and σ_2 integrals have generated two (one-dimensional) delta functions involving two different linear combinations of the four Fourier-space variables k_μ. This restricts the four-dimensional integral over k_μ appearing on the first line of Eq. (43),

$$\int \frac{d^4k}{(2\pi)^4} \frac{ik_\mu}{k^2} (\cdots), \qquad (46)$$

to a two-dimensional linear subspace of k-space.

We can explicitly deal with this two-dimensional reduction of the k-integral by choosing an adapted (Lorentz) coordinate frame. More precisely, it is convenient to choose four (Lorentz-orthonormal) 4-vectors e_0, e_1, e_2, e_3 such that, say, e_0 and e_3 span the (timelike) 2-plane defined by the two 4-vectors p_1 and p_2. This will be, in particular, the case if we work in a c.m. frame with the time axis defined by $e_0 \propto p_1 + p_2$ and a third axis e_3 in the common direction of the c.m. spatial momenta, say $\mathbf{p}_{\rm c.m.} \equiv \mathbf{p}_1{}^{\rm c.m.} = -\mathbf{p}_2{}^{\rm c.m.}$. In this frame, the incoming 4-momenta have the following components,

$$\begin{aligned} p_1 &= \mathcal{E}_1^{\rm c.m.} e_0 + p_{\rm c.m.} e_3; \\ p_2 &= \mathcal{E}_2^{\rm c.m.} e_0 - p_{\rm c.m.} e_3, \end{aligned} \tag{47}$$

where $p_{\rm c.m.}$ is the magnitude of $\mathbf{p}_{\rm c.m.}$ and where $\mathcal{E}_1^{\rm c.m.} = \sqrt{m_1^2 + p_{\rm c.m.}^2}$ and $\mathcal{E}_2^{\rm c.m.} = \sqrt{m_2^2 + p_{\rm c.m.}^2}$ are the relativistic c.m. energies of the two (incoming) particles. In terms of these quantities, the total relativistic (incoming, center-of-mass) energy reads

$$\begin{aligned} \mathcal{E}_{\rm real} &= \mathcal{E}_1^{\rm c.m.} + \mathcal{E}_2^{\rm c.m.} \\ &= \sqrt{m_1^2 + p_{\rm c.m.}^2} + \sqrt{m_2^2 + p_{\rm c.m.}^2}. \end{aligned} \tag{48}$$

Among the four components of k in the frame e_0, e_1, e_2, e_3, only k^0 and k^3 appear in the two delta functions of Eq. (45). More precisely, they appear in the combinations

$$\begin{aligned} k.p_1 &= -k^0 \mathcal{E}_1^{\rm c.m.} + k^3 p_{\rm c.m.}; \\ k.p_2 &= -k^0 \mathcal{E}_2^{\rm c.m.} - k^3 p_{\rm c.m.}. \end{aligned} \tag{49}$$

As a consequence, we can write

$$\delta(k.p_1)\delta(k.p_2) = \frac{\delta(k^0)\delta(k^3)}{\mathcal{D}}, \tag{50}$$

where $\mathcal{D}$ is the absolute value of the Jacobian $\partial(k.p_1, k.p_2)/\partial(k^0, k^3)$. We can immediately give two different expressions for the Jacobian $\mathcal{D}$. First, from Eq. (49), we get

$$\mathcal{D} = (\mathcal{E}_1^{\rm c.m.} + \mathcal{E}_2^{\rm c.m.}) p_{\rm c.m.} = \mathcal{E}_{\rm real} p_{\rm c.m.}. \tag{51}$$

Second, we can write a covariant expression for $\mathcal{D}$ by thinking geometrically within the two-dimensional Lorentzian space spanned by p_1 and p_2 and realizing that $\mathcal{D}$ is simply the magnitude of the wedge product (antisymmetric bivector) $p_1 \wedge p_2$ of p_1 and p_2. This yields the manifestly covariant expression (with a minus sign linked to the timelike character of the $p_1 - p_2$ plane)

$$\begin{aligned} \mathcal{D}^2 = |p_1 \wedge p_2|^2 &= -\frac{1}{2}(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)(p_{1\mu} p_{2\nu} - p_{1\nu} p_{2\mu}) \\ &= (p_1.p_2)^2 - p_1^2 p_2^2. \end{aligned} \tag{52}$$

Note in passing that the (so-proven) equality between $\mathcal{E}_{\rm real} p_{\rm c.m.}$ and $\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}$ is not evident when using the explicit c.m. expression (48) of the c.m. energy $\mathcal{E}_{\rm real}$ in terms of $p_{\rm c.m.}$.

Inserting our various partial results in the integral expression (43), we get a result of the form

$$\Delta p_{1a} \propto G \frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}} \int d^2k \frac{ik_a}{\mathbf{k}^2} e^{i\mathbf{k}.\mathbf{b}}, \tag{53}$$

where $\mathbf{k}$ and $\mathbf{b}$ denote the projections, onto the two-dimensional (Euclidean-signature) space spanned by e_1 and e_2, of k and $x_1(0) - x_2(0)$. (The index $a = 1, 2$ on Δp_{1a} and k_a spans this two-dimensional space, e.g., $k^a e_a = k^1 e_1 + k^2 e_2$.) Clearly, $\mathbf{b}$ represents the vectorial c.m. impact parameter of the two incoming worldlines, and its Euclidean magnitude $b \equiv |\mathbf{b}|$ measures the scalar c.m. impact parameter. As a consequence, the c.m. total angular momentum is simply given by

$$J = b p_{\rm c.m.}. \tag{54}$$

From dimensional analysis and symmetry considerations (or by an explicit calculation, using a frame where, say, $b_2 = 0$), the remaining two-dimensional integral in (53) is simply found to be proportional to $-\mathbf{b}/b^2$. [The vector $\mathbf{b} = \mathbf{x}_1(0) - \mathbf{x}_2(0)$ is directed from 2 toward 1.]

Putting together all the numerical factors, one finally gets a vectorial deflection (within the e_1-e_2 2-plane, i.e., the orthogonal complement of the p_1-p_2 2-plane) given by

$$\Delta \mathbf{p}_1 = -4G \frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}} \frac{\mathbf{b}}{b^2}. \tag{55}$$

The (absolute value of the) corresponding c.m. scattering angle χ is related to the magnitude of $\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$ via

$$\sin\frac{\chi}{2} = \frac{|\Delta \mathbf{p}_1|}{2|\mathbf{p}_1|} = \frac{|\Delta \mathbf{p}_1|}{2 p_{\rm c.m.}}. \tag{56}$$

As we work to first order in G, we have $\sin\frac{\chi}{2} \approx \frac{\chi}{2}$ so that

$$\begin{aligned} \frac{1}{2}\chi_{1PM}^{\rm real} &= 2 \frac{G}{b p_{\rm c.m.}} \frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}} \\ &= 2 \frac{G}{J} \frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}}. \end{aligned} \tag{57}$$

In the nonrelativistic limit, $p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'} = (p_1.p_2)^2 - \frac{1}{2} p_1^2 p_2^2$ tends to $\frac{1}{2} m_1^2 m_2^2$, where the factor $\frac{1}{2}$ cancels the prefactor 2 on the rhs of (57). [In agreement with the Newtonian-level result (24).]

The expression (57) (and its derivation above) exhibits a simple, and manifestly covariant, link between the (Fourier-space) quantum scattering amplitude $\mathcal{M} = G p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}/k^2$ and the classical scattering angle.

The explicit form of Eq. (57) reads

$$\frac{1}{2}\chi^{\rm real}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}. \tag{58}$$

This result naturally features the following dimensionless energy variable (already used in Ref. [32]):

$$\epsilon(s) \equiv \frac{s - m_1^2 - m_2^2}{2m_1 m_2} = -\frac{p_1.p_2}{m_1 m_2}. \tag{59}$$

Here, s denotes the usual Mandelstam variable

$$s \equiv \mathcal{E}^2_{\rm real} = -(p_1 + p_2)^2, \tag{60}$$

where we recall that $\mathcal{E}_{\rm real}$ is evaluated in the c.m.

In terms of $\epsilon(s)$, the result (58) reads

$$\frac{1}{2}\chi^{\rm real}_{1PM}(s, J) = \frac{G m_1 m_2}{J}\frac{2\epsilon^2(s) - 1}{\sqrt{\epsilon^2(s) - 1}}. \tag{61}$$

Our explicit final result (58) [or (61)] is simpler than the corresponding results derived (in x-space) in Refs. [25–29]. It is, however, equivalent to them. [We note in passing that the equivalence with Eq. (12) in Ref. [29] is rather hidden.] The expression (61) is also simpler than the corresponding 2PN-expanded result (33), but is straightforwardly checked to be consistent with it, when remembering the definitions (34), (35).

VI. EFFECTIVE ONE-BODY SCATTERING FUNCTION AT THE FIRST POST-MINKOWSKIAN APPROXIMATION

In this section, we shall consider the (effective) scattering angle $\chi_{\rm eff}$ computed from the dynamics of one particle of mass m_0 moving in some effective metric $g^{\rm eff}_{\mu\nu}$. At linear order in G (and still setting $c = 1$), we parametrize the looked-for spherically symmetric effective metric as

$$g^{\rm eff}_{\mu\nu}(M_0, \beta_1) dx^\mu dx^\nu = -\left(1 - \frac{R_g}{R}\right) dt^2 + \left(1 + \beta_1 \frac{R_g}{R}\right) dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{62}$$

where the dimensionless parameter β_1 enters in the radial component of the metric and where

$$R_g \equiv 2GM_0 \tag{63}$$

denotes the (gravitational) mass of the effective background. Below, we shall set m_0 to the reduced mass μ and M_0 to the total mass $M = m_1 + m_2$, but it is useful not to assume it from the start.

Let us compute the scattering angle $\chi_{\rm eff}$ from the Hamilton-Jacobi equation for the geodesic dynamics in $g^{\rm eff}_{\mu\nu}(M_0)$, namely

$$g^{\mu\nu}_{\rm eff}\partial_\mu S_{\rm eff}\partial_\nu S_{\rm eff} = -m_0^2, \tag{64}$$

with

$$S_{\rm eff} = -\mathcal{E}_0 t + J_0 \varphi + S^{\rm eff}_R(R). \tag{65}$$

For simplicity, we denote the effective energy and angular momentum $\mathcal{E}_{\rm eff}$ and $J_{\rm eff}$ as $\mathcal{E}_0$ and J_0, respectively.

To linear order in G, the Hamilton-Jacobi equation reads

$$-\left(1 + \frac{R_g}{R}\right)\mathcal{E}_0^2 + \frac{J_0^2}{R^2} + \left(1 - \beta_1 \frac{R_g}{R}\right)\left(\frac{dS^{\rm eff}_R}{dR}\right)^2 = -m_0^2. \tag{66}$$

Computing $P_R = dS^{\rm eff}_R/dR$ from this equation, we obtain (still to linear order in G) the radial effective action in the form

$$S^{\rm eff}_R(R; \mathcal{E}_0, J_0) = \int dR P_R(R; \mathcal{E}_0, J_0), \tag{67}$$

with

$$P_R(R; \mathcal{E}_0, J_0) = \pm\left(1 + \frac{1}{2}\beta_1 \frac{R_g}{R}\right) \times \sqrt{\mathcal{E}_0^2\left(1 + \frac{R_g}{R}\right) - \left(m_0^2 + \frac{J_0^2}{R^2}\right)}, \tag{68}$$

where the sign $\pm$ is $-$ (respectively, $+$) during the incoming (respectively, outgoing) part of the scattering motion.

As already explained in Sec. III, the J_0 derivative of the radial action directly yields the scattering angle as

$$\pi + \chi_{\rm eff} = -\int_{-\infty}^{+\infty} dR \frac{\partial P_R(R; \mathcal{E}_0, J_0)}{\partial J_0}. \tag{69}$$

Let us formally expand P_R in powers of G, i.e., of R_g:

$$P_R(R; \mathcal{E}_0, J_0; R_g) = P^{(0)}_R(R; \mathcal{E}_0, J_0) + P^{(1)}_R(R; \mathcal{E}_0, J_0) + O(R_g^2). \tag{70}$$

Here, $P^{(0)}_R = \sqrt{\mathcal{E}_0^2 - (m_0^2 + \frac{J_0^2}{R^2})}$ corresponds to a free motion and therefore contributes the no-scattering value π to the integral (69). We then get the following expression for the (effective) scattering angle:

$$\chi_{\rm eff} = -\int_{-\infty}^{+\infty} dR \frac{\partial P^{(1)}_R(R; \mathcal{E}_0, J_0)}{\partial J_0}. \tag{71}$$

Actually, this expression is formal because, when explicitly written, it involves divergent integrals linked to the singular nature of the expansion of $\sqrt{\mathcal{E}_0^2 - (m_0^2 + \frac{J_0^2}{R^2}) + \varepsilon}$ in powers of ε at the turning point where P_R, i.e., the square root, vanishes. However, as explained in Ref. [37], the correct result is obtained by taking the Hadamard partie finie (Pf), at the unperturbed turning point, of these formally divergent integrals. This yields

$$\chi_{\rm eff} = \frac{R_g J_0}{2} \mathrm{Pf}\left[\int \pm \frac{dR}{R^3}(\beta_1 A^{-1/2} - \mathcal{E}_0^2 A^{-3/2})\right], \qquad (72)$$

where $A \equiv \mathcal{E}_0^2 - (m_0^2 + J_0^2/R^2)$.

The integrals entering this expression of $\chi_{\rm eff}$ are elementary to compute when replacing R by A as an integration variable [using $\mathrm{Pf}\int_{\rm in}^{\rm out} \pm dA A^p = \frac{1}{p+1}(A_{\rm out}^{p+1} + A_{\rm in}^{p+1})$ because of the two signs of the square root of A]. A simple computation then yields

$$\frac{1}{2}\chi_{1PM}^{\rm eff}(\mathcal{E}_0, J_0) = \frac{GM_0}{J_0}\left(\beta_1\sqrt{\mathcal{E}_0^2 - m_0^2} + \frac{\mathcal{E}_0^2}{\sqrt{\mathcal{E}_0^2 - m_0^2}}\right)$$
$$= \frac{GM_0}{J_0}\frac{(1+\beta_1)\mathcal{E}_0^2 - \beta_1 m_0^2}{\sqrt{\mathcal{E}_0^2 - m_0^2}}. \qquad (73)$$

Factoring m_0 out of the effective energy $\mathcal{E}_0$ yields the equivalent expression

$$\frac{1}{2}\chi_{1PM}^{\rm eff}(\mathcal{E}_0, J_0) = \frac{GM_0 m_0}{J_0}\frac{(1+\beta_1)(\mathcal{E}_0/m_0)^2 - \beta_1}{\sqrt{(\mathcal{E}_0/m_0)^2 - 1}}. \qquad (74)$$

We recall that, in the expressions above, $\mathcal{E}_0$ denotes the relativistic effective energy $\mathcal{E}_{\rm eff}$ and J_0 denotes the effective angular momentum $J_{\rm eff}$.

VII. COMPARING THE REAL AND THE EFFECTIVE SCATTERING FUNCTIONS

We have recalled above that the EOB formalism is a relativistic generalization of the Newtonian fact that the relative dynamics of a two-body system is (after separation of the center-of-mass motion) equivalent to the motion of a particle of mass μ submitted to the original two-body potential $V(|\mathbf{R}_1 - \mathbf{R}_2|)$. Therefore, in the nonrelativistic limit $c \to \infty$, we require that the effective mass m_0 coincides with μ and that, as already indicated in Eq. (5), the nonrelativistic effective energy $E_{\rm eff} = \mathcal{E}_{\rm eff} - m_0 c^2$ coincides with the nonrelativistic real c.m. energy $E_{\rm real} = \mathcal{E}_{\rm real} - (m_1 + m_2)c^2$. In this nonrelativistic limit, the real and effective scattering functions, evaluated for the same values of their arguments, $E_{\rm real} = E_{\rm eff}$, and $J_{\rm real} = J_{\rm eff}$ [see Eq. (11)], must coincide: $\chi_{\rm eff}(E,J) = \chi_{\rm real}(E,J) + O(\frac{1}{c^2})$. Applying this requirement to Eqs. (61) and (74) simply tells us that

$$M_0 m_0 = m_1 m_2 + O\left(\frac{1}{c^2}\right). \qquad (75)$$

Though other possibilities have been explored in Ref. [1], it was concluded that it is most natural to work with effective masses m_0 and M_0 which are energy independent. The requirements $m_0 = \mu + O(\frac{1}{c^2})$ and (75) then imply that

$$m_0 = \mu \equiv \frac{m_1 m_2}{m_1 + m_2}; \qquad M_0 = M \equiv m_1 + m_2. \qquad (76)$$

Then, the application of our basic scattering-state dictionary (17) [or (18)] to our scattering results (61), (74) implies that the *a priori* unknown energy map f, Eq. (4) [such that $\mathcal{E}_0 = f(\mathcal{E}_{\rm real}) \equiv f(s)$], as well as the *a priori* unknown effective metric parameter β_1 [see Eq. (62)] should be such that

$$\frac{(1+\beta_1)(f(s)/m_0)^2 - \beta_1}{\sqrt{(f(s)/m_0)^2 - 1}} = \frac{2\epsilon^2(s) - 1}{\sqrt{\epsilon^2(s) - 1}}, \qquad (77)$$

where the function $\epsilon(s)$ of the real c.m. energy was defined in Eq. (59).

The requirement (77) contains both an unknown parameter, β_1, and an unknown function, $f(s)$. It would *a priori* seem that this is not enough to determine all the unknowns. One could think that one can choose an arbitrary value of β_1 and then determine the corresponding energy map $f(s)$ by solving Eq. (77) for $f(s)$. Let us show, however, that the exact, relativistic structure of (77) is such that it uniquely determines *both* the value of β_1 and that of the energy map $f(s)$.

Indeed, denoting $u_{\rm eff} \equiv \sqrt{(f(s)/m_0)^2 - 1}$ and, correspondingly, $u_{\rm real} \equiv \sqrt{\epsilon^2(s) - 1}$, the left-hand side of Eq. (77) has the structure

$$\frac{(1+\beta_1)(u_{\rm eff}^2 + 1) - \beta_1}{u_{\rm eff}} = (1+\beta_1)u_{\rm eff} + \frac{1}{u_{\rm eff}}, \qquad (78)$$

while its rhs has the structure

$$\frac{2(u_{\rm real}^2 + 1) - 1}{u_{\rm real}} = 2u_{\rm real} + \frac{1}{u_{\rm real}}. \qquad (79)$$

However, the variable $u_{\rm real}$ runs over the full half-line $0 \le u_{\rm real} \le +\infty$ (with the limit $u_{\rm real} \to 0$ corresponding to the nonrelativistic limit), while $u_{\rm eff} \ge 0$ must also start at zero in the nonrelativistic limit. Now, a remarkable feature of the function $g_{\rm real}(u_{\rm real}) = 2u_{\rm real} + \frac{1}{u_{\rm real}}$ (which describes the energy dependence of the product χJ) is that, after *initially decreasing* with increasing energy in the nonrelativistic regime ($u_{\rm real} \approx v_E \ll 1$ implying $g_{\rm real} \approx 1/v_E = 1/\sqrt{2\hat{E}}$; Rutherford scattering), it ultimately starts to *increase* with increasing energy in the relativistic energy domain ($g_{\rm real} \sim u_{\rm real} \propto s = \mathcal{E}_{\rm real}^2$). Therefore, the function $g_{\rm real}(u_{\rm real})$ is easily found to have a (unique) minimum. The minimum value of $g_{\rm real}(u_{\rm real}) = 2u_{\rm real} + \frac{1}{u_{\rm real}}$ is reached for $u_{\rm real} = \frac{1}{\sqrt{2}}$ and is equal to $2\sqrt{2}$.

By comparison, a general function of the type $(1+\beta_1)u_{\rm eff} + \frac{1}{u_{\rm eff}}$, where the variable $u_{\rm eff}$ lives on the positive half-real-line, must have $1+\beta_1 \geq 0$ to remain always positive and will then reach the minimal value $2\sqrt{1+\beta_1}$ for $u_{\rm eff} = 1/\sqrt{1+\beta_1}$.

In order for the requirement (77) to be globally satisfied, in the relativistic regime, we must identify the two minimum values $2\sqrt{2}$ and $2\sqrt{1+\beta_1}$. We therefore conclude that the value of β_1 is uniquely determined to be

$$\beta_1 = 1, \tag{80}$$

which corresponds to the linearized Schwarzschild metric.

Inserting the information (80) in the requirement (77), one can now conclude that there exists a *unique* energy map which is positive, continuous, and monotonic and that it is given by

$$\frac{f(s)}{m_0} = \epsilon(s) \equiv \frac{s - m_1^2 - m_2^2}{2m_1 m_2}, \tag{81}$$

i.e., using (76),

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2m_1 m_2} \tag{82}$$

or, equivalently,

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}. \tag{83}$$

The PN expansion (5) of this map yields $\alpha_1 = \frac{\nu}{2}$ and $\alpha_n \equiv 0$ for all higher n's.

VIII. $O(G)$ TENSOR-SCALAR GENERALIZATION

To further illustrate the usefulness of the post-Minkowskian scattering approach, let us briefly consider the modifications of the EOB results brought by considering a gravitational interaction combining massless spin-2 exchange and massless spin-zero exchange. In diagrammatic language, this amounts to considering the sum of two one-particle-exchange diagrams of the form of Fig. 1. At first order in the interaction (and therefore neglecting self-field effects), the scattering angle is then simply the sum of two contributions:

$$\chi = \chi_2 + \chi_0. \tag{84}$$

The spin-2 contribution χ_2 can be written in terms of the 4-velocities $u_1 \equiv p_1/m_1$ and $u_2 \equiv p_2/m_2$ [with $\hat{\mathcal{D}} \equiv \mathcal{D}/(m_1 m_2) = \sqrt{(u_1.u_2)^2 - 1}$] as

$$\frac{1}{2}\chi_2 = \frac{G_2 m_1 m_2}{J} \frac{2u_1^\alpha u_1^\beta P_{\alpha\beta;\alpha'\beta'} u_2^{\alpha'} u_2^{\beta'}}{\hat{\mathcal{D}}}. \tag{85}$$

The spin-zero contribution χ_0 is then simply obtained by omitting the (Newtonian-limit normalized) spin-2 vertex contraction factor, $2u_1^\alpha u_1^\beta P_{\alpha\beta;\alpha'\beta'} u_2^{\alpha'} u_2^{\beta'}$, so that

$$\frac{1}{2}\chi_0 = \frac{G_0 m_1 m_2}{J} \frac{1}{\hat{\mathcal{D}}}. \tag{86}$$

Here, G_2 denotes the spin-2 contribution to the Cavendish constant, and G_0 denotes its spin-zero contribution. Let us parametrize (in keeping with the notation of Ref. [38]) the admixtion of scalar exchange to the gravitational interaction by the fraction

$$\alpha^2 \equiv \frac{G_0}{G_2}. \tag{87}$$

This notation means that each scalar-matter vertex carries an extra factor α, with respect to a tensor-matter one.

The total observable Cavendish constant then reads

$$G = G_2 + G_0 = (1+\alpha^2)G_2, \tag{88}$$

while the total scattering function $\chi = \chi_2 + \chi_0$ reads [using Eqs. (52), (58), and (59), i.e., $\epsilon(s) = -u_1.u_2$]

$$\begin{aligned} \frac{1}{2}\chi(s,J) &= \frac{m_1 m_2}{J\hat{\mathcal{D}}}[G_2(2(u_1.u_2)^2 - 1) + G_0] \\ &= \frac{G m_1 m_2}{J} \frac{(1+\gamma)\epsilon^2(s) - \gamma}{\sqrt{\epsilon^2(s) - 1}}, \end{aligned} \tag{89}$$

where $1+\gamma = 2G_2/G = 2/(1+\alpha^2)$, i.e.,

$$\gamma = \frac{1-\alpha^2}{1+\alpha^2}. \tag{90}$$

Comparing with Eq. (74) and going through the reasoning used in the previous section, we can then conclude that, to first post-Minkowskian order, the two-body interaction in a theory of gravity combining (massless) tensor and scalar fields can be described by the following modifications of the usual EOB theory: (i) the energy map f is (to all orders in v/c) again given by the result (82), while (ii) the (linearized) effective metric differs from the (linearized) Schwarzschild metric by the presence of a coefficient $\beta_1 = \gamma$, Eq. (90), in the spatial metric, i.e.,

$$\begin{aligned} g_{\mu\nu}^{\rm eff} dx^\mu dx^\nu = &-\left(1 - \frac{2GM}{R}\right)dt^2 + \left(1 + \gamma\frac{2GM}{R}\right)dR^2 \\ &+ R^2(d\theta^2 + \sin^2\theta d\varphi^2) + O(G^2). \end{aligned} \tag{91}$$

This modification of the linearized Schwarzschild metric is equivalent [modulo the first-order coordinate change $R = R' + \gamma GM + O(G^2)$] to the G-linearization of the usual 1PN-accurate Eddington, parametrized-post-Newtonian metric

$$ds^2_{\mathrm{PPN}} = -\left(1 - 2\frac{GM}{R'} + 2\beta\left(\frac{GM}{R'}\right)^2\right)dt^2 + \left(1 + 2\gamma\frac{GM}{R'}\right)(dR'^2 + R'^2(d\theta^2 + \sin^2\theta d\varphi^2)). \tag{92}$$

The result (90) then agrees with the expression of the Eddington parameter γ in terms of the linear scalar coupling α, often also written as (see, e.g., Eq. (4.12c) in Ref. [38])

$$\bar{\gamma} \equiv \gamma - 1 = -2\frac{\alpha^2}{1+\alpha^2}. \tag{93}$$

To see the $O(G^2)$ Eddington parameter $\beta \equiv 1 + \bar{\beta}$, one would have to introduce a nonlinear vertex $-\frac{1}{2}\beta_0 \int m_a \phi^2(x_a) ds_a$ coupling the scalar field $\phi(x)$ to the worldlines ($a = 1, 2$) and to go beyond the G-linear approximation. (In the 1PN approximation, this yields the link $\bar{\beta} = +\frac{1}{2}\beta_0\alpha^2/(1+\alpha^2)^2$ [39].)

IX. CONCLUSIONS

A new approach to the effective-one-body description of gravitationally interacting two-body systems has been introduced. This approach is not based, as was previous work, on the post-Newtonian approximation scheme (combined expansion in $\frac{G}{c^2}$ and in $\frac{1}{c^2}$) but rather on the post-Minkowskian approximation scheme (perturbation theory in G, without assuming small velocities). It uses a new dictionary based on considering the functional dependence of the scattering angle χ on the total energy and total angular momentum of the system (all quantities being considered in the center-of-mass frame). By explicitly calculating (in a novel way, analogous to quantum-scattering-amplitude computations) the first post-Minkowskian $[O(G)]$ scattering function, we have proven *to all orders in* v/c two results that were previously only known to a limited post-Newtonian accuracy: (i) to order G^1, the relativistic dynamics of a two-body system (of masses m_1, m_2) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a *Schwarzschild metric of mass* $M = m_1 + m_2$, and (ii) this equivalence requires the existence of an *exactly quadratic map* f between the real (relativistic) two-body energy $\mathcal{E}_{\mathrm{real}}$ and the (relativistic) energy $\mathcal{E}_{\mathrm{eff}}$ of the effective particle given (to all orders in $1/c$) by

$$\mathcal{E}_{\mathrm{eff}} = \frac{(\mathcal{E}_{\mathrm{real}})^2 - m_1^2c^4 - m_2^2c^4}{2(m_1+m_2)c^2}. \tag{94}$$

The latter energy map was also proven to apply to the effective-one-body description of two masses interacting in tensor-scalar gravity [for which one must use an $O(G)$ effective metric modified by Eddington's γ parameter].

We leave to future work the generalization of our approach to higher orders in G and to spinning bodies.

ACKNOWLEDGMENTS

I am grateful to Ulysse Schildge for his help during many years.

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