# Smooth magnetogenesis

L. Campanelli[1,*] and A. Marrone[1,2,†]

[1]*Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italy*
[2]*INFN—Sezione di Bari, I-70126 Bari, Italy*

In the Ratra scenario of inflationary magnetogenesis, the kinematic coupling between the photon and the inflaton undergoes a nonanalytical jump at the end of inflation. Using smooth interpolating analytical forms of the coupling function, we show that such unphysical jump does not invalidate the main prediction of the model, which still represents a viable mechanism for explaining cosmic magnetization. Nevertheless, there is a spurious result associated with the nonanaliticity of the coupling, to wit, the prediction that the spectrum of created photons has a power-law decay in the ultraviolet regime. This issue is discussed using both semiclassical approximation and smooth coupling functions.

## I. INTRODUCTION

The ubiquitous presence of large-scale cosmological magnetic fields is still un unexplained feature of the Universe (for reviews on cosmic magnetic fields, see [1–6]).

One possibility is that such fields are relics from inflation, as first suggested by Turner and Widrow a long time ago [7]. The idea that a spontaneous inflationary magnetogenesis could be at the origin of cosmic magnetic fields has been since then deeply investigated. After the seminal works by Ratra [8] and Dolgov [9], numerous suggestions have been put forward in an attempt to explain cosmic magnetization [10–38], and to overcome a problem associated with inflationary magnetogenesis, known as the "strong coupling" problem [39].

Recently enough [40,41], we have reconsidered the Ratra model and showed that it is a viable magnetogenesis mechanism able to explain the presence of galactic and extragalactic magnetic fields without suffering from the aforementioned problem.

A necessary condition to creating large-scale magnetic fields at inflation is to break the conformal invariance of standard electromagnetism. In the Ratra model, this is attained by kinematically coupling the photon field $A_\mu$ to the inflaton field $\phi$. The modified electromagnetic Lagrangian turns to be [8]

$$\mathcal{L}_{\text{Ratra}} = -\frac{1}{4} f(\phi) F_{\mu\nu} F^{\mu\nu}, \qquad (1)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor and, assuming homogeneity and isotropy, the coupling function $f(\phi)$ depends only on the (conformal) time. In real models of magnetogenesis, such a coupling must (i) be a smooth function of time,[1] (ii) approach unity after reheating, so to recover standard electromagnetism in radiation and matter eras, and (iii) be greater than unity otherwise, so to avoid the strong coupling problem.

In the original work by Ratra, and in its following extensions, however, a simplified discontinuous form of $f(\phi)$ was assumed. This is because of a complete lack of any theoretical and/or phenomenological particle physics model supporting the Ratra scenario, which could have eventually justified and/or suggested a continuous form of the coupling.

In Sec. IV, we argue, by using particular smooth forms of $f(\phi)$, that the assumption of a discontinuous coupling does not invalidate the main predictions of the model. Needless to say, such a conclusion does not represent a general proof of this point, proof that could eventually be given only if the exact form of the Ratra coupling were known.

Finally, we show that there are spurious results associated with a discontinuous coupling, such as the fact that the spectrum of produced photons decays as a power law in the ultraviolet regime, instead of falling off exponentially as generally predicted by the standard theory of quantum theory in curved spacetime. This latter feature of the Ratra model will be discussed in Sec. III by using the Wentzel-Kramers-Brillouin (WKB) approximation. In the next section, we set our notations and we briefly review the Ratra scenario in the light of the results obtained in [40].

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*leonardo.campanelli@ba.infn.it
†antonio.marrone@ba.infn.it

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[1]Here, we are not considering any possible discontinuous photon-inflaton coupling which could arise if the inflaton undergoes a first-order phase transition during inflation and/or reheating, namely we stay close to the original Ratra model (and its extensions) where inflation and reheating are assumed to proceed smoothly without discontinuous jumps.

## II. CREATION OF MAGNETIC FIELDS IN THE RATRA MODEL

The creation of inflationary magnetic fields in the Ratra model is a pure quantum effect [40]. It is understood as the creation of photons out of the vacuum by the changing gravitational field between two temporal regions, the "in-region" and the "out-region," possessing two different, inequivalent vacua. For the sake of convenience, we may define these regions by $\eta \to \mp \infty$, respectively, where $\eta$ is the conformal time. Physically, the former represent the temporal region where inflation starts and the latter any temporal region after reheating.

Working in Fourier space, let us rescale the photon wave functions $A_k^{(\text{in,out})}$ in the in- and out-regions as

$$\psi_k^{(\text{in,out})} = \sqrt{f/2k}\, A_k^{(\text{in,out})}. \qquad (2)$$

The in- and out-$\psi$ modes reduce to plane waves in the corresponding in- and out regions, $\psi_k^{(\text{in,out})} = e^{-ik\eta}/\sqrt{2k}$ for $\eta \to \mp \infty$, and satisfy the equation of motion [40]

$$\ddot{\psi}_k = U_k \psi_k. \qquad (3)$$

Here,

$$U_k(\eta) = -k^2 + U_0(\eta), \qquad (4)$$

$k$ is the comoving wave number, and

$$U_0(\eta) = \frac{\ddot{\sqrt{f}}}{\sqrt{f}} \qquad (5)$$

will be referred to as the "Ratra potential." The in- and out-modes are not independent but related trough the Bogoliubov transformation [40]

$$\psi_k^{(\text{in})} = \alpha_k \psi_k^{(\text{out})} + \beta_k \psi_k^{(\text{out})*}, \qquad (6)$$

where the Bogoliubov coefficients $\alpha_k$ and $\beta_k$ satisfy the Bogoliubov condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \qquad (7)$$

The in-vacuum state will contain out-particles so long as $\beta_k \neq 0$. It can be easily shown that the number of particles, in this case, is [40,42]

$$n_{\mathbf{k}} = |\beta_k|^2. \qquad (8)$$

For a conformal theory, as the standard Maxwell theory in a Friedmann-Robertson-Walker spacetime, the in- and out-vacua are equivalent and one has $\beta_k = 0$, so that no particle creation occurs (this is the content of the well-known "Parker theorem" [42]).

Neglecting any possible effect of magnetohydrodynamic turbulence [43–50] after reheating, the actual magnetic field on a scale $\lambda = 1/k$ is [40] (see also [41,51])

$$B_k \sim k^2 \sqrt{n_{\mathbf{k}}}, \qquad (9)$$

in the (quasiclassical) limit $n_{\mathbf{k}} \gg 1$. The actual magnetic field spectrum on large (cosmological) scales is then completely determined by the density number of photons with wave number $k$ created at inflation.

In the (original) Ratra model, the coupling function is

$$f(\eta) = \begin{cases} f_*(\eta_e/\eta)^{2p}, & \eta \leq \eta_e, \\ 1, & \eta > \eta_e, \end{cases} \qquad (10)$$

where $f_* > 1$, $\eta_e$ is the conformal time at the end of inflation,[2] and $p$ is a real parameter which we assume, for the sake of simplicity, to be less than $-1$. This, in turns, gives for the Ratra potential the simple expression

$$U_0(\eta) = \begin{cases} \frac{p(p+1)}{\eta^2}, & \eta \leq \eta_e, \\ 0, & \eta > \eta_e. \end{cases} \qquad (11)$$

Modes well outside the horizon at the end of inflation, $-k\eta_e \ll 1$, are efficiently produced [40,41],

$$n_{\mathbf{k}} \simeq \frac{[2^{-p}(p+1)\Gamma(-p-1/2)]^2}{16\pi} (-k\eta_e)^{2p}, \qquad (12)$$

where $\Gamma(x)$ is the gamma function. Conversely, the production of photons with wavelength much smaller than the horizon length at the end of inflation, $-k\eta_e \gg 1$, is suppressed as [40]

$$n_{\mathbf{k}} \simeq \left[\frac{p(p+1)}{4}\right]^2 (-k\eta_e)^{-4}. \qquad (13)$$

Such an unphysical power-law decay of the photon spectrum in the ultraviolet regime can be attributed to the nonanalyticity of the Ratra potential at the point $\eta_e$. Indeed, for smooth potentials, the general expectation is that of an exponential decay (see Sec. III B and the discussion at the beginning of Sec. IV).

## III. PARTICLE CREATION ANALOGY WITH QUANTUM POTENTIAL SCATTERING

The equation of motion (3) of the $\psi$ modes is formally equal to the one-dimensional Schrodinger equation with zero energy and potential energy $U_k$, with $\eta$ taking the place of the spatial coordinate and $k$ being an external parameter. Continuing the analogy, Eq. (6) connecting the $\psi_k^{(\text{in})}$

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[2]Without loss of generality, we can assume that $\eta_e$ is negative-defined as in the case of pure de Sitter inflation.
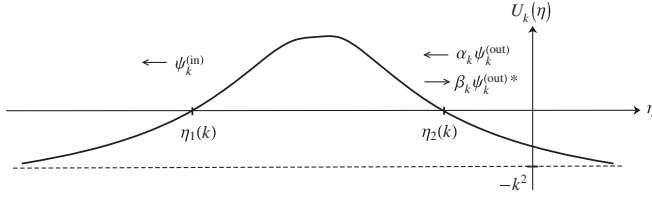
FIG. 1. Schematic illustration of the equivalence of the problem of potential scattering in quantum mechanics and the creation of particles in a curved background. The formal relations between the spectrum of particles created out from the vacuum $n_{\mathbf{k}}$ and the reflection and transmission coefficients through the barrier $U_k(\eta)$ is given in Eqs. (21)–(22). The points $\eta_{1,2}(k)$ are the classical turning points for the quantum-mechanical problem of penetration through the potential barrier.

and $\psi_k^{(\text{out})}$ modes describes the scattering of $\psi$-waves off the potential $U_k$, the incident, reflected, and transmitted waves being $\alpha_k \psi_k^{(\text{out})}$, $\beta_k \psi_k^{(\text{out})*}$, and $\alpha_k \psi_k^{(\text{in})}$, respectively. This is schematically illustrated in Fig. 1.

Moreover, one can define a density current associated to any $\psi$-mode as

$$j_k = \langle \psi_k | \psi_k^* \rangle, \tag{14}$$

where the inner product of any two solutions $\psi_k^{(1)}$ and $\psi_k^{(2)}$ of Eq. (6) is defined by [40]

$$\langle \psi_k^{(1)} | \psi_k^{(2)} \rangle = -i(\psi_k^{(1)} \dot{\psi}_k^{(2)} - \dot{\psi}_k^{(1)} \psi_k^{(2)}). \tag{15}$$

The conservation of the current (14), $\dot{j}_k = 0$, follows directly from the conservation of the inner product. The incident, reflected, and transmitted currents are then

$$j_k^{(\text{inc})} = \langle \alpha_k \psi_k^{(\text{out})} | \alpha_k^* \psi_k^{(\text{out})*} \rangle = |\alpha_k|^2, \tag{16}$$

$$j_k^{(\text{ref})} = \langle \beta_k \psi_k^{(\text{out})*} | \beta_k^* \psi_k^{(\text{out})} \rangle = -|\beta_k|^2, \tag{17}$$

$$j_k^{(\text{tr})} = \langle \psi_k^{(\text{in})} | \psi_k^{(\text{in})*} \rangle = 1, \tag{18}$$

where we used the fact that [40]

$$\alpha_k = \langle \psi_k^{(\text{in})} | \psi_k^{(\text{out})*} \rangle, \tag{19}$$

$$\beta_k = -\langle \psi_k^{(\text{in})} | \psi_k^{(\text{out})} \rangle, \tag{20}$$

and $\langle \psi_k^{(\text{in})} | \psi_k^{(\text{in})*} \rangle = \langle \psi_k^{(\text{out})} | \psi_k^{(\text{out})*} \rangle = 1$.

Taking into account Eq. (8) and the Bogoliubov condition (7), we find the reflection and transmission coefficients

$$R_k = -\frac{j_k^{(\text{ref})}}{j_k^{(\text{inc})}} = \frac{n_{\mathbf{k}}}{1 + n_{\mathbf{k}}}, \tag{21}$$

$$T_k = \frac{j_k^{(\text{tr})}}{j_k^{(\text{inc})}} = \frac{1}{1 + n_{\mathbf{k}}}, \tag{22}$$

from which the unitarity condition $R_k + T_k = 1$ directly follows.

Let us assume, for the sake of simplicity, that the Ratra potential $U_0(\eta)$ is positive and vanishes in the in- and out regions $\eta \to \mp \infty$. It is clear that if $k^2 < \max U_0(\eta)$, then the "particle" described by the wave function $\alpha_k \psi_k^{(\text{out})}$ will penetrate through the potential barrier $U_k(\eta)$. To "particles" which deeply penetrate into the barrier, $k^2 \ll \max U_0(\eta)$, there will correspond a large reflection coefficient and, in turn, by Eq. (21), a large particle number $n_{\mathbf{k}}$. On the other hand, if $k^2 > \max U_0(\eta)$, the "particle" is reflected above the barrier. For $k^2 \gg \max U_0(\eta)$, the reflection coefficient for scattering above the barrier will be small. To this case, there will correspond a small production of particles, $n_{\mathbf{k}} \ll 1$.

The usefulness of Eqs. (21)–(22) resides in the fact that if the potential $U_k$ is slowly varying, in the sense specified below, one can apply the standard semiclassical (WKB) results for the reflection and transmission coefficients. Using the formal equivalence of the two problems of potential scattering in quantum mechanics and the creation of particle out from the vacuum in a curved spacetime, one can then find the expression for the particle number $n_{\mathbf{k}}$.

### A. Particle number in WKB approximation

The WKB approximation is valid whenever the potential $U_k$ satisfies the semiclassical condition [52]

$$\left| \frac{\dot{U}_k}{2U_k^{3/2}} \right| \ll 1. \tag{23}$$

If this is the case, the reflection coefficient is either small or large, and the reflection takes place either deeply through the barrier or above it. Therefore, according to the above discussion, the WKB approximation will provide us with the interesting cases of large and small $n_{\mathbf{k}}$.

#### 1. Large particle number

Let us assume, for the sake of simplicity, that the barrier has a bell shape form, as in Fig. 1. Accordingly, there will be two classical turning points, $\eta_1(k) < \eta_2(k)$, for a deep penetration through the potential barrier. Since in this case $R_k \simeq 1$, and then $T_k \ll 1$, we have from Eq. (22), $n_{\mathbf{k}} \simeq T_k^{-1} \gg 1$. Using the standard result for the expression of the transmission coefficient in WKB approximation [52], we find

$$n_{\mathbf{k}} \simeq e^{2S_k}, \tag{24}$$

where[3]

$$S_k = \int_{\eta_1(k)}^{\eta_2(k)} d\eta \sqrt{U_k(\eta)}. \tag{25}$$

The magnetic field spectrum is proportional to the square root of the particle number [see Eq. (9)], the latter representing an amplification factor of the in-vacuum magnetic fluctuations. As first noticed by Giovannini [2], the amount of such an amplification depends, for a given mode, on the "time spent" under the potential barrier. From Eqs. (24)–(25), we see that this "time" is indeed the area enclosed by the square root of the potential between the two classical turning points.

### 2. Small particle number

The probability that a "particle" is scattered above the potential barrier is small for large values of $k^2$ compared to the height of the Ratra barrier. Using Eq. (21), we then have $n_\mathbf{k} \simeq R_k \ll 1$. Using the standard result for the expression of the reflection coefficient in WKB approximation [52], we find

$$n_\mathbf{k} \simeq e^{-4\sigma_k}, \tag{26}$$

where

$$\sigma_k = \text{Im} \int_{\eta_R}^{\eta_I(k)} d\eta \sqrt{-U_k(\eta)}. \tag{27}$$

Here, $\eta_I(k)$ is the so-called imaginary turning point, the complex solution of the equation $U_k(\eta) = 0$ for $k^2 > \max U_0(\eta)$, and $\eta_R$ is an arbitrary and inessential real parameter. The integration in Eq. (27) has to be performed in the complex upper half-plane $(\text{Im}[\eta_I(k)] > 0)$. If the equation for the imaginary turning point admits more than one solution, one must select the one for which $\sigma_k$ is smallest [52].

Let us finally observe that, for $k \to \infty$ [namely for sufficiently large wave numbers, $k^2 \gg \max(U_0, |\dot{U}_0|^{2/3})$], condition (23) is satisfied whatever is the form of the Ratra potential and the WKB approximation is then applicable. Moreover, if there exists the limit $\lim_{k\to\infty}\eta_I(k) = \eta_I(\infty)$, Eq. (26) takes the simple form

$$n_\mathbf{k} \simeq e^{-4k\text{Im}[\eta_I(\infty)]}, \tag{28}$$

at the leading order in $k \to \infty$. In the case of multiple imaginary turning points, one has to take the one which has the smallest (positive) imaginary part.

### B. Ratra model in WKB approximation

As an application of the above results, let us consider in semiclassical approximation the exactly solvable Ratra model defined by the potential (11). The WKB applicability condition (23) in this case reads

$$q|1 - (k\eta/q)^2|^{3/2} \gg 1, \tag{29}$$

where we have defined $q = \sqrt{p(p+1)}$. The classical turning points are easily found,

$$\eta_1(k) = -\frac{q}{k}, \qquad \eta_2(k) = \eta_e, \tag{30}$$

for $-k\eta_e < q$. At large scales, $-k\eta \ll q$, far from the turning points (where the WKB approximation cannot be trustful), condition (29) is satisfied for $q \gg 1$, namely for $p \ll -1$. Using Eqs. (24)–(25), we find

$$n_\mathbf{k}^{(\text{WKB})} = (2q/e)^{2q}(-k\eta_e)^{-2q}. \tag{31}$$

In the limit $p \to -\infty$, this approximate result and the exact result (12) coincide.

However, the WKB expression (31) is not a good approximation for values of $|p|$ of order unity.[4] This is due to the fact that $-k\eta_e$ is a very small quantity, so that even a small change in the value of its exponent produces a large deviation of the particle number and, in turn, of the actual magnetic field. To see this, let us use Eq. (9) and take into account that [41]

$$-k\eta_e \simeq 10^{-21} \left(\frac{M}{10^{16} \text{ GeV}}\right)^{-2/3} \left(\frac{T_{\text{RH}}}{10^{10} \text{ GeV}}\right)^{-1/3} \frac{k}{\text{Mpc}^{-1}}, \tag{32}$$

where $M$ is the scale of inflation and $T_{\text{RH}}$ the reheat temperature. We have $B_k^{(\text{WKB})}/B_k \sim (-k\eta_e)^{q-|p|}$ which, for example, results in $B_k^{(\text{WKB})}/B_k \sim 10^{12}$ for $p = -2$, $M = 10^{16}$ GeV, $T_{\text{RH}} = 10^{10}$ GeV, and $\lambda = 1/k = $ Mpc.

At small scales, $-k\eta \gg q$, the WKB approximation is valid. Nevertheless, we cannot apply Eqs. (26)–(27) in the model at hand, since they are valid only for smooth potentials $U_k(\eta)$. If there is a discontinuity in the Ratra potential $U_0(\eta)$, as in Eq. (5), the reflection coefficient is determined mainly by the wave function at that point [52]. In this case, the perturbation theory can be used to calculate $R_k$ in the case of a quasiclassical barrier. Using the standard results in [52], we find that

$$n_\mathbf{k} \simeq R_k \simeq \frac{|U_0(\eta_\star)|^2}{16k^4}, \tag{33}$$

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[3]As it is straightforward to check, the quantity $S_k$ gives, in the out-region, the so-called "squeezing" parameter $r_k(\eta)$ defined in [40], $\lim_{\eta\to+\infty} r_k(\eta) = S_k + \ln 2$. Large values of the squeezing parameter indicate that the system (the magnetic field) is in a (quasi-)classical state.

[4]In order to avoid the so-called backreaction problem of inflationary magnetogenesis [39], one must take $|p| \leq 2$ [40].

where $\eta_\star$ is the discontinuity point. Applying this result to the Ratra model, the discontinuity point being $\eta_e$, we find that the perturbation theory gives the exact result for $n_\mathbf{k}$ in the limit $-k\eta_e \gg q$, namely Eq. (13).

We stress the fact that the perturbation theory is always valid for sufficiently large wave numbers (as already observed above). Hence, any smooth form of Ratra coupling $f(\eta)$ would lead to an exponential decay of the particle number in the ultraviolet regime [see Eq. (28)], in contrast to the $k^{-4}$ law predicted by the discontinuous coupling.

## IV. SMOOTH COUPLING IN THE RATRA MODEL

As already noticed, the coupling $f(\eta)$ must be a smooth function of time in any real model of inflationary magnetogenesis. If, moreover, such a coupling function is slowly varying in the in- and out-regions, in the sense that

$$\lim_{\eta\to\pm\infty} \frac{d^n}{d\eta^n} \frac{\dot{f}}{f} = 0, \qquad (34)$$

for all $n \in \mathbb{N}$, then the in- and out- electromagnetic vacuum states are vacua of infinite adiabatic order [40].[5] This, in turns, implies that the number of produced particle must go to zero exponentially or, more precisely, faster than any power of $k$ in the limit $k \to +\infty$ [42],

$$\lim_{k\to\infty} \frac{n_\mathbf{k}}{k^{n+1}} = 0, \qquad (35)$$

for all $n \in \mathbb{N}$. In this section, we consider examples of coupling functions which smoothly interpolate the non-analytical profile $f(\eta)$ of the original Ratra model. We will see that an exponential decay of the particle number at large wave numbers will replace the (nonphysical) power-law decay discussed in Secs. II and III B. At small wave numbers, instead, all the results obtained in the discontinuous case will be confirmed. This is to be expected, since long wavelength modes should not be strongly affected by variations of the coupling function on small (time)scales.

### A. Analytical example

As a first example, let us consider the following coupling,

$$f(\eta) = (1 + e^{\eta/\eta_*})^{2b} [_2F_1(b, b, 1, -e^{\eta/\eta_*})]^2, \qquad (36)$$

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[5]It is also assumed that the (spatially flat, Friedmann-Robertson-Walker) spacetime is slowly varying in the in- and out-regions, $\lim_{\eta\to\pm\infty} \frac{d^n}{d\eta^n} \frac{\dot{a}}{a} = 0$, where $a(\eta)$ is the expansion parameter. This condition is automatically satisfied in (de Sitter or power-law) inflation as well as in radiation and matter dominated eras.
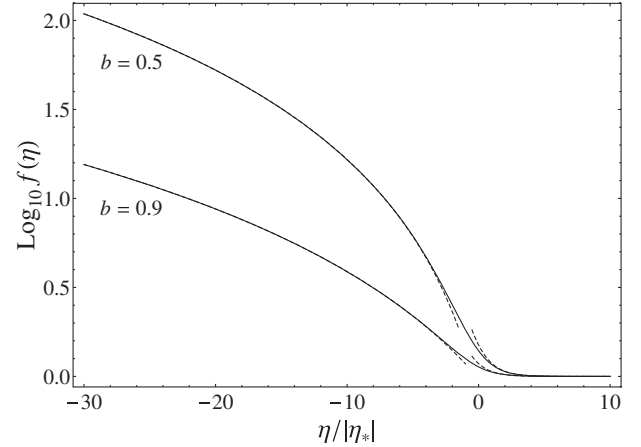


FIG. 2. The smooth coupling $f(\eta)$ in Eq. (36) as a function of the conformal time for two different values of $b$. Dashed lines are the corresponding asymptotic expansions in Eq. (37).

where $\eta_* < 0$ is a free parameter (that could represent, for example, the time at the end of inflation $\eta_e$), $_2F_1(a, b, c, z)$ is the Gauss hypergeometric function [53], and we assume that the positive parameter $b$ is in the range $b \in [1/2, 1]$. A plot of the coupling function, for two different values of $b$, is given in Fig. 2. The asymptotic expansions of $f(\eta)$ are

$$f(\eta) = \begin{cases} \left(\frac{\sin(\pi b)\frac{\eta}{\eta_*}+\gamma_b}{\pi}\right)^2 + \mathcal{O}(e^{-\eta/\eta_*}), & \eta \to -\infty, \\ 1 + 2b(1-b)e^{\eta/\eta_*} + \mathcal{O}(e^{2\eta/\eta_*}), & \eta \to +\infty, \end{cases}$$
$$(37)$$

where $\gamma_b = -\sin(\pi b)[\gamma + H_{-b} + \psi(b)]$ is a real increasing function of $b$ such that $\gamma_{1/2} = 4\ln 2$ and $\gamma_1 = \pi$. Here, $\gamma$ is the Euler's constant, $H_n$ is the harmonic number of order $n$, and $\psi(x)$ is the digamma function [53]. Therefore, $f(\eta)$ smoothly interpolates between the behaviors $f(\eta) \propto \eta^2$ for $\eta \ll \eta_*$ and $f(\eta) = 1$ for $\eta \gg \eta_*$ (see Fig. 2). This analytical case, then, represents a smooth interpolation of the Ratra discontinuous power-law case for $p \simeq -1$. Accordingly, we expect for the particle number the law $n_\mathbf{k} \sim k^{-2}$ for small values of $k$ and an "exponential" decay for large wave numbers.

The shape of the Ratra potential follows from Eq. (36),

$$U_0(\eta) = \frac{k_*^2}{1 + \cosh(\eta/\eta_*)}, \qquad (38)$$

where we have defined $k_* = \sqrt{\frac{1}{2}b(1-b)}/|\eta_*|$.

A standard calculation gives, for the Bunch-Davies-normalized out-$\psi$ modes,

$$\psi_k^{(\text{out})}(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}(1 + e^{\eta/\eta_*})^b$$
$$\times {}_2F_1(b, b - 2ik\eta_*, 1 - 2ik\eta_*, -e^{\eta/\eta_*}) \qquad (39)$$

and, for the Bunch-Davies-normalized in-$\psi$ modes, the expression in Eq. (6) with

$$\alpha_k = \frac{\cosh(4\pi k\eta_*) - \cos(2\pi b)}{2\pi^2}\Gamma(2ik\eta_*)\Gamma(1 + 2ik\eta_*)$$
$$\times \Gamma(b - 2ik\eta_*)\Gamma(1 - b - 2ik\eta_*), \quad (40)$$

$$\beta_k = i\frac{\sin(\pi b)}{\sinh(2\pi k\eta_*)}. \quad (41)$$

Equations (40) and (41) are precisely the Bogoliubov coefficients defining the in-$\psi$ modes in terms of the out-$\psi$ modes [a straightforward calculation shows that the Bogoliubov condition (7) is satisfied]. Taking the square modulus of $\beta_k$ we obtain the spectrum of the photons created out from the vacuum,

$$n_{\mathbf{k}} = \frac{\sin^2(\pi b)}{\sinh^2(2\pi k\eta_*)}. \quad (42)$$

For $b = 1$ we have $k_* = 0$ so that $U_0(\eta) = 0$, $f(\eta) = 1$, $\psi_k^{(\text{in,out})}(\eta) = e^{-ik\eta}/\sqrt{2k}$, and in turn $n_{\mathbf{k}} = 0$.

For small and large wave numbers, $-k\eta_* \ll 1$ and $-k\eta_* \gg 1$, respectively, we find $n_{\mathbf{k}} \sim (-k\eta_*)^{-2}$ and $n_{\mathbf{k}} \sim e^{-4\pi k|\eta_*|}$, as we expected.

### 1. WKB approximation

Let us now apply the WKB approximation, which is always valid for sufficiently large wave numbers, to the case under consideration. The imaginary turning points in the complex upper half-plane for the potential (38) are easily found,

$$\eta_I(k) = |\eta_*|\left[\operatorname{arccosh}\left(\frac{k_*^2}{k^2} - 1\right) + 2\pi in\right], \quad (43)$$

for $k > k_*/\sqrt{2}$ and $n \in \mathbb{N}$. In the limit $k \to \infty$, we have $\eta_I(\infty) = \pi i(1 + 2n)|\eta_*|$. According to the general prescription discussed in Sec. III A, we must take $n = 0$, so that, using Eq. (28), we obtain

$$n_{\mathbf{k}}^{(\text{WKB})} \simeq e^{-4\pi k|\eta_*|}, \quad (44)$$

at the leading order in $k/k_* \sim -k\eta_* \gg 1$. The above WKB result is in perfect agreement with the exact result (42) in the large-wavenumber regime.

### B. Numerical example

The analytical case discussed in the previous section gives an example of smooth interpolation of the Ratra discontinuous power-law case $p \simeq -1$. The most important case, that of scaling-invariant actual magnetic fields, corresponds, however, to case $p = -2$ (see [40]). We then generalize the previous example by considering the coupling function
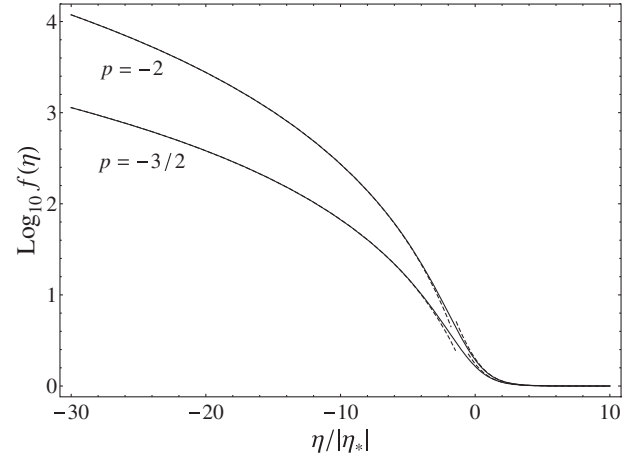


FIG. 3. The smooth coupling $f(\eta)$ in Eq. (45) as a function of the conformal time for two different values of $p$. Dashed lines are the corresponding asymptotic expansions in Eq. (46).

$$f(\eta) = \left[\frac{2}{\pi}\sqrt{1 + e^{\eta/\eta_*}}K(-e^{\eta/\eta_*})\right]^{-2p}, \quad (45)$$

where, as before, $\eta_* < 0$ is a free parameter, and $K(m)$ is the complete elliptic integral of the first kind [53]. Since

$$f(\eta) = \begin{cases} \left(\frac{\frac{\eta}{\eta_*} + \ln 16}{\pi}\right)^{-2p} + \mathcal{O}(e^{-\eta/\eta_*}), & \eta \to -\infty, \\ 1 - \frac{p}{2}e^{\eta/\eta_*} + \mathcal{O}(e^{2\eta/\eta_*}), & \eta \to +\infty, \end{cases} \quad (46)$$

$f(\eta)$ smoothly interpolates between the behaviors $f(\eta) \propto \eta^{-2p}$ for $\eta \ll \eta_*$ and $f(\eta) = 1$ for $\eta \gg \eta_*$ (see Fig. 3).

The Ratra potential $U_0(\eta)$ reads

$$U_0(\eta) = \frac{k_*^2}{1 + \cosh(\eta/\eta_*)}u_p(\eta), \quad (47)$$

where

$$u_p(\eta) = 1 - (p + 1)e^{-\eta/\eta_*}\left[1 - \frac{E(-e^{\eta/\eta_*})}{K(-e^{\eta/\eta_*})}\right]^2, \quad (48)$$

$k_* = \sqrt{\frac{|p|}{8}}/|\eta_*|$, and $E(m)$ is the complete elliptic integral of the second kind [53]. The case $p = -1$ reduces to the (analytically solvable) case (with $b = 1/2$) of Sec. IV A. The asymptotic expansions of the Ratra potential are

$$U_0(\eta) = \begin{cases} \frac{p(p+1)}{(\eta + \eta_* \ln 16)^2} + \mathcal{O}(e^{-\eta/\eta_*}), & \eta \to -\infty, \\ -\frac{p}{4\eta_*^2}e^{\eta/\eta_*} + \mathcal{O}(e^{\hat{s}\eta/\eta_*}), & \eta \to +\infty. \end{cases} \quad (49)$$

The equation of motion (3) for the $\psi$ modes cannot be integrated analytically for the general Ratra potential (47). We then numerically integrate it by imposing the Bunch-Davies boundary condition that the in-$\psi$ mode reduces to
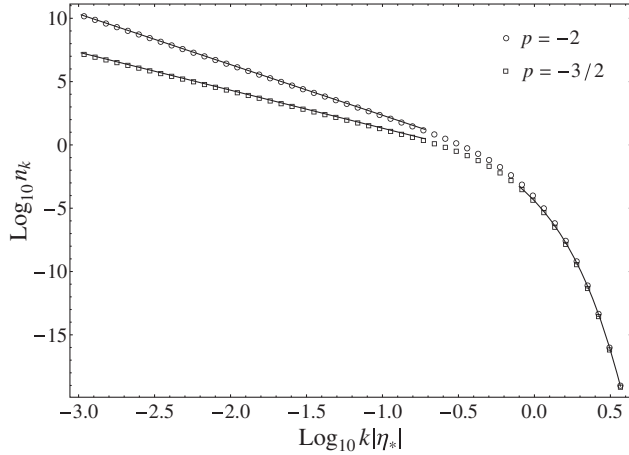
FIG. 4. The particle number $n_\mathbf{k}$ as a function of $k|\eta_*|$ in the Ratra model described by the coupling function (45). Continuous lines refer to the asymptotic numerical solutions described in the text [see Eqs. (52), (53), and (54)].

the plane-wave solution $e^{-ik\eta}/\sqrt{2k}$ in the in-region $\eta \to -\infty$. In the same way, we may impose the Bunch-Davies boundary condition in the out-region $\eta \to +\infty$ to find the out-$\psi$ mode and then, using Eq. (6), we may obtain the Bogoliubov coefficients. However, it is easier from a numerical point of view to calculate the Bogoliubov coefficients starting from the expressions

$$\alpha_k = \lim_{\eta \to +\infty} \frac{e^{ik\eta}}{\sqrt{2k}} \left( k + i\frac{\partial}{\partial \eta} \right) \psi_k^{(\text{in})}, \qquad (50)$$

$$\beta_k = \lim_{\eta \to +\infty} \frac{e^{-ik\eta}}{\sqrt{2k}} \left( k - i\frac{\partial}{\partial \eta} \right) \psi_k^{(\text{in})}, \qquad (51)$$

which make use only of the in-$\psi$ modes. Equations (50) and (51) come from Eqs. (19) and (20), respectively, taking the limit $\eta \to +\infty$ and observing that in the out-region $\psi_k^{(\text{out})} = e^{-ik\eta}/\sqrt{2k}$.

In Fig. 4, we show the particle number $n_\mathbf{k} = |\beta_k|^2$ for two different values of $p$ [we checked that the Bogoliubov condition (7) holds]. Continuous lines in the small wave number regime $(-k\eta_* \ll 1)$ correspond to

$$n_\mathbf{k} \simeq 0.021(-k\eta_*)^{-3}, \qquad (52)$$

$$n_\mathbf{k} \simeq 0.022(-k\eta_*)^{-4}, \qquad (53)$$

for $p = -3/2$ and $p = -2$, respectively. The exponents of these power laws agree with the results obtained in Ratra discontinuous model [see, in particular, Eq. (12)]. For large wave numbers $(-k\eta_* \gg 1)$, instead, the continuous line is

$$n_\mathbf{k} \simeq 11 e^{-4\pi k|\eta_*|}. \qquad (54)$$

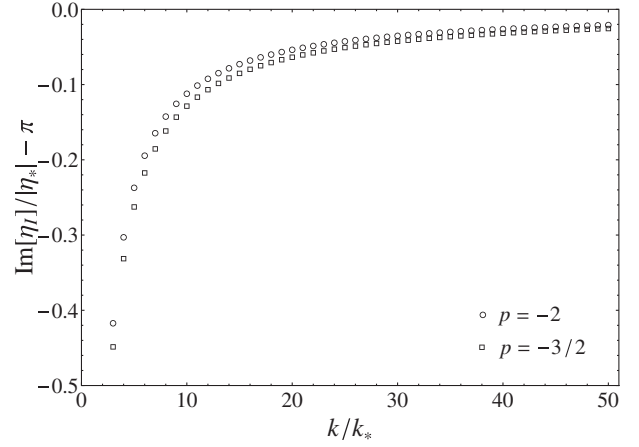Such an exponential decay of the particle number agrees well with the WKB solution discussed below.



FIG. 5. Imaginary part of the imaginary turning point $\eta_I(k)$ as a function of the wave number $k$ in the model described by the Ratra potential (47) for two different values of $p$.

### 1. WKB approximation

The imaginary turning points $\eta_I(\infty) = \pi i(1 + 2n)|\eta_*|$ $(n \in \mathbb{N})$ of the Ratra potential (38) are also imaginary turning points of the potential (47) for $p \neq -2$. This is because the function $u_p(\eta)$ tends to a nonvanishing constant when $\eta$ approaches $\eta_I(\infty)$, $\lim_{\eta \to \eta_I(\infty)} u_p(\eta) = 2 + p$. In the case $p = -2$, instead, such a constant is zero and one needs the next-to-leading order term in the expansion of the imaginary turning points of (38), $\eta_I(k) \simeq \eta_I(\infty) - i\sqrt{2}k_*/k$. Inserting the above expression in Eq. (48), we find that $u_{-2}$ vanishes logarithmically as $u_{-2}(\eta_I(k)) \simeq 8/\ln(k/k_*)$ in the limit $k \to \infty$. The function which multiplies $u_p(\eta)$ in Eq. (47), instead, diverges quadratically in $k$ in the same limit. Accordingly, the imaginary turning points of the Ratra potentials (38) and (47) coincide in the limit $k \to \infty$ also in the case $p = -2$.

This result is confirmed by a direct numerical integration of the equation defining the imaginary turning points of the Ratra potential (47), $U_0(\eta_I(k)) = k^2$.

In Fig. 5, we show the imaginary part of the imaginary turning point (with smallest positive imaginary part) for two different values of $p$. For large values of $k$, the imaginary part of $\eta_I(k)$ approaches the value

$$\text{Im}[\eta_I(\infty)] \simeq \pi|\eta_*|. \qquad (55)$$

Using Eq. (28), then, we find that the particle number in WKB approximation is given by Eq. (44) in the limit $k/k_* \sim -k\eta_* \gg 1$, as in the (analytical) case of Sec. IV A.

### V. CONCLUDING REMARKS

The Ratra model for the creation of magnetic fields during inflation has received much interest in the last years. This is primarily due to its simplicity and to its potential

feature of explaining the origin of the observed galactic and extragalactic large-scale magnetic fields.

In the original model, as well as in all subsequent variants of it, the form of coupling function between the photon and the inflaton plays the key role in the genesis of cosmological magnetic fields. Such a nonstandard time-dependent coupling, which is responsible for the breaking of the electromagnetic conformal invariance, has been always assumed to instantaneously reduce to its standard constant value in radiation era. Such a nonanalytical coupling could arise only in particular inflationary models, where the inflaton, or other background fields coupled to it, evolve discontinuously, as for example in a first-order phase transition. However, this in not the case for the Ratra and Ratra-like models, since they are constructed and framed in a smoothly evolving inflationary and reheated universe.

In this paper, we have investigated the consequences of taking an analytical coupling function that smoothly interpolates the (unphysical) discontinuous coupling usually used in the Ratra and Ratra-like models. Both using

particular smooth forms of the coupling and working in WKB approximation, we have studied the behavior of the inflation-produced photon spectrum at small and large scales. We have found an exponential decay in the ultraviolet part of such a spectrum, in contrast with the (unphysical) power-law fall-off predicted in the discontinuous case. At large cosmological scales, however, the main results of the model remain unchanged in the considered cases. Since only these scales are important for cosmic magnetic fields, we conclude that the Ratra model is still a viable mechanism of inflationary magnetogenesis.

Nevertheless, it should be said that the form of the coupling function used in this investigation as well as those assumed in both the original work by Ratra and in its developments do not have any solid theoretical and/or phenomenological justification. Hence, as in the case of (scalar field) inflation, the Ratra mechanism is just a paradigm, although very elegant and attractive, in search of a background particle physics model.

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[1] L. M. Widrow, Origin of galactic and extragalactic magnetic fields, Rev. Mod. Phys. **74**, 775 (2002).

[2] M. Giovannini, The magnetized universe, Int. J. Mod. Phys. D **13**, 391 (2004).

[3] A. Kandus, K. E. Kunze, and C. G. Tsagas, Primordial magnetogenesis, Phys. Rep. **505**, 1 (2011).

[4] D. Ryu, D. R. G. Schleicher, R. A. Treumann, C. G. Tsagas, and L. M. Widrow, Magnetic fields in the large-scale structure of the Universe, Space Sci. Rev. **166**, 1 (2012); L. M. Widrow, D. Ryu, D. R. G. Schleicher, K. Subramanian, C. G. Tsagas, and R. A. Treumann, The first magnetic fields, Space Sci. Rev. **166**, 37 (2012).

[5] R. Durrer and A. Neronov, Cosmological magnetic fields: Their generation, evolution and observation, Astron. Astrophys. Rev. **21**, 62 (2013).

[6] K. Subramanian, The origin, evolution and signatures of primordial magnetic fields, Rep. Prog. Phys. **79**, 076901 (2016).

[7] M. S. Turner and L. M. Widrow, Inflation produced, large scale magnetic fields, Phys. Rev. D **37**, 2743 (1988).

[8] B. Ratra, Cosmological 'seed' magnetic field from inflation, Astrophys. J. **391**, L1 (1992).

[9] A. D. Dolgov, Breaking of conformal invariance and electromagnetic field generation in the Universe, Phys. Rev. D **48**, 2499 (1993).

[10] M. Giovannini, Magnetogenesis and the dynamics of internal dimensions, Phys. Rev. D **62**, 123505 (2000); Backgrounds of squeezed relic photons and their spatial correlations, Phys. Rev. D **61**, 087306 (2000); Inflationary magnetogenesis from dynamical gauge couplings, arXiv: hep-ph/0104214 (unpublished); Magnetogenesis, spectator fields and CMB signatures, Phys. Lett. B **659**,

661 (2008); Inflationary susceptibilities, duality, and large-scale magnetic field generation, Phys. Rev. D **88**, 083533 (2013); Inflationary magnetogenesis, derivative couplings and relativistic Van der Waals interactions, Phys. Rev. D **92**, 043521 (2015).

[11] G. Lambiase and A. R. Prasanna, Gauge invariant wave equations in curved space-times and primordial magnetic fields, Phys. Rev. D **70**, 063502 (2004).

[12] G. Lambiase, S. Mohanty, and G. Scarpetta, Magnetic field amplification in f(R) theories of gravity, J. Cosmol. Astropart. Phys. 07 (2008) 019.

[13] H. J. Mosquera Cuesta and G. Lambiase, Primordial magnetic fields and gravitational baryogenesis in nonlinear electrodynamics, Phys. Rev. D **80**, 023013 (2009).

[14] L. Campanelli and M. Giannotti, Magnetic helicity generation from the cosmic axion field, Phys. Rev. D **72**, 123001 (2005).

[15] L. Campanelli, P. Cea, G. L. Fogli, and L. Tedesco, Inflation-produced magnetic fields in nonlinear electrodynamics, Phys. Rev. D **77**, 043001 (2008); Inflation-produced magnetic fields in $R^n F^2$ and $IF^2$ models, Phys. Rev. D **77**, 123002 (2008); L. Campanelli, P. Cea, and G. L. Fogli, Lorentz symmetry violation and galactic magnetism, Phys. Lett. B **680**, 125 (2009); L. Campanelli and P. Cea, Maxwell-Kostelecký electromagnetism and cosmic magnetization, Phys. Lett. B **675**, 155 (2009).

[16] L. Campanelli, Helical magnetic fields from inflation, Int. J. Mod. Phys. D **18**, 1395 (2009); Model of universe anisotropization, Phys. Rev. D **80**, 063006 (2009); Origin of Cosmic Magnetic Fields, Phys. Rev. Lett. **111**, 061301 (2013); Electromagnetism with dimension-five operators,

Phys. Rev. D **90**, 105014 (2014); Lorentz-violating inflationary magnetogenesis, Eur. Phys. J. C **75**, 278 (2015).

[17] M. M. Anber and L. Sorbo, N-flationary magnetic fields, J. Cosmol. Astropart. Phys. 10 (2006) 018.

[18] K. Bamba and M. Sasaki, Large-scale magnetic fields in the inflationary universe, J. Cosmol. Astropart. Phys. 02 (2007) 030.

[19] K. Bamba, N. Ohta, and S. Tsujikawa, Generic estimates for magnetic fields generated during inflation including Dirac-Born-Infeld theories, Phys. Rev. D **78**, 043524 (2008).

[20] K. Bamba, C. Q. Geng, and S. H. Ho, Large-scale magnetic fields from inflation due to Chern-Simons-like effective interaction, J. Cosmol. Astropart. Phys. 11 (2008) 013.

[21] K. Bamba, Generation of large-scale magnetic fields, non-Gaussianity, and primordial gravitational waves in inflationary cosmology, Phys. Rev. D **91**, 043509 (2015).

[22] J. D. Barrow and C. G. Tsagas, Slow decay of magnetic fields in open Friedmann universes, Phys. Rev. D **77**, 107302 (2008); Publisher's Note, Phys. Rev. D **77**, 109904(E) (2008).

[23] K. E. Kunze, Primordial magnetic fields and nonlinear electrodynamics, Phys. Rev. D **77**, 023530 (2008).

[24] C. T. Byrnes, L. Hollenstein, R. K. Jain, and F. R. Urban, Resonant magnetic fields from inflation, J. Cosmol. Astropart. Phys. 03 (2012) 009.

[25] R. J. Z. Ferreira, R. K. Jain, and M. S. Sloth, Inflationary magnetogenesis without the strong coupling problem, J. Cosmol. Astropart. Phys. 10 (2013) 004.

[26] F. A. Membiela, Primordial magnetic fields from a non-singular bouncing cosmology, Nucl. Phys. **B885**, 196 (2014).

[27] C. Caprini and L. Sorbo, Adding helicity to inflationary magnetogenesis, J. Cosmol. Astropart. Phys. 10 (2014) 056.

[28] A. P. Kouretsis, Cosmic magnetization in curved and Lorentz violating space-times, Eur. Phys. J. C **74**, 2879 (2014).

[29] K. Atmjeet, I. Pahwa, T. R. Seshadri, and K. Subramanian, Cosmological magnetogenesis from extra-dimensional Gauss Bonnet gravity, Phys. Rev. D **89**, 063002 (2014).

[30] K. Atmjeet, T. R. Seshadri, and K. Subramanian, Helical cosmological magnetic fields from extra-dimensions, Phys. Rev. D **91**, 103006 (2015).

[31] G. Tasinato, A scenario for inflationary magnetogenesis without strong coupling problem, J. Cosmol. Astropart. Phys. 03 (2015) 040.

[32] B. K. El-Menoufi, Inflationary magnetogenesis and non-local actions: The conformal anomaly, J. Cosmol. Astropart. Phys. 02 (2016) 055.

[33] P. Qian and Z. K. Guo, Model of inflationary magnetogenesis, Phys. Rev. D **93**, 043541 (2016).

[34] G. Domènech, C. Lin, and M. Sasaki, Inflationary magnetogenesis with broken local U(1) symmetry, Europhys. Lett. **115**, 19001 (2016).

[35] T. Fujita and R. Namba, Pre-reheating magnetogenesis in the kinetic coupling model, Phys. Rev. D **94**, 043523 (2016).

[36] D. Chowdhury, L. Sriramkumar, and R. K. Jain, Duality and scale invariant magnetic fields from bouncing universes, Phys. Rev. D **94**, 083512 (2016).

[37] P. Adshead, J. T. Giblin, T. R. Scully, and E. I. Sfakianakis, Magnetogenesis from axion inflation, arXiv:1606.08474.

[38] P. Qian, Y. F. Cai, D. A. Easson, and Z. K. Guo, Magnetogenesis in bouncing cosmology, Phys. Rev. D **94**, 083524 (2016).

[39] V. Demozzi, V. Mukhanov, and H. Rubinstein, Magnetic fields from inflation?, J. Cosmol. Astropart. Phys. 08 (2009) 025.

[40] L. Campanelli, Cosmic magnetization out from the vacuum, Mod. Phys. Lett. A **31**, 1650165 (2016).

[41] L. Campanelli, Superhorizon magnetic fields, Phys. Rev. D **93**, 063501 (2016).

[42] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982); L. Parker and D. J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, England, 2009).

[43] P. Olesen, On inverse cascades in astrophysics, Phys. Lett. B **398**, 321 (1997); Inverse transfer of self-similar decaying turbulent non-helical magnetic field, arXiv:1509.08962; Dimensional reduction in freely decaying turbulent non-helical magnetic fields, arXiv:1511.05007.

[44] R. Banerjee and K. Jedamzik, Are Cluster Magnetic Fields Primordial?, Phys. Rev. Lett. **91**, 251301 (2003); Erratum, Phys. Rev. Lett. **93**, 179901(E) (2004); The evolution of cosmic magnetic fields: From the very early universe, to recombination, to the present, Phys. Rev. D **70**, 123003 (2004).

[45] L. Campanelli, Scaling laws in MHD turbulence, Phys. Rev. D **70**, 083009 (2004); Evolution of Magnetic Fields in Freely Decaying Magnetohydrodynamic Turbulence, Phys. Rev. Lett. **98**, 251302 (2007); Evolution of primordial magnetic fields in mean-field approximation, Eur. Phys. J. C **74**, 2690 (2014); On the self-similarity of nonhelical magnetohydrodynamic turbulence, Eur. Phys. J. direct C **76**, 504 (2016);

[46] T. Kahniashvili, A. Brandenburg, A. G. Tevzadze, and B. Ratra, Numerical simulations of the decay of primordial magnetic turbulence, Phys. Rev. D **81**, 123002 (2010); T. Kahniashvili, A. Brandenburg, L. Campanelli, B. Ratra, and A. G. Tevzadze, Evolution of inflation-generated magnetic field through phase transitions, Phys. Rev. D **86**, 103005 (2012).

[47] A. Saveliev, K. Jedamzik, and G. Sigl, Time evolution of the large-scale tail of non-helical primordial magnetic fields with back-reaction of the turbulent medium, Phys. Rev. D **86**, 103010 (2012); Evolution of helical cosmic magnetic fields as predicted by magnetohydrodynamic closure theory, Phys. Rev. D **87**, 123001 (2013).

[48] M. Giovannini, Compressible hydromagnetic nonlinearities in the predecoupling plasma, Phys. Rev. D **85**, 043006 (2012); Anomalous magnetohydrodynamics, Phys. Rev. D **88**, 063536 (2013).

[49] A. Berera and M. Linkmann, Magnetic helicity and the evolution of decaying magnetohydrodynamic turbulence, Phys. Rev. E **90**, 041003 (2014); M. F. Linkmann, A. Berera, W. D. McComb, and M. E. McKay, Nonuniversality and Finite Dissipation in Decaying Magnetohydrodynamic Turbulence, Phys. Rev. Lett. **114**, 235001 (2015); M. F. Linkmann, A. Berera, M. E. McKay, and J. Jger, Helical mode interactions and spectral transfer processes in

magnetohydrodynamic turbulence, J. Fluid Mech. **791**, 61 (2016).

[50] A. Brandenburg, T. Kahniashvili, and A. G. Tevzadze, Nonhelical Inverse Transfer of a Decaying Turbulent Magnetic Field, Phys. Rev. Lett. **114**, 075001 (2015); A. Brandenburg and T. Kahniashvili, Classes of hydrodynamic and magnetohydrodynamic turbulent decay, arXiv: 1607.01360.

[51] C. G. Tsagas, On the magnetic evolution in Friedmann universes and the question of cosmic magnetogenesis, arXiv:1412.4806; Relaxing the limits on inflationary magnetogenesis, Phys. Rev. D **92**, 101301 (2015); Causality, initial conditions, and inflationary magnetogenesis, Phys. Rev. D **93**, 103529 (2016).

[52] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-relativistic Theory* (Pergamon Press, Oxford, England, 1965).

[53] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, New York, 1972).