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Matter-induced magnetic moment and neutrino helicity rotation in external fields

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The induced magnetic moment that arises due to the propagation of neutrinos in a dispersive medium can affect the dynamics of the neutrino spin in an external electromagnetic field. In particular, it can cause a helicity flip of a massive neutrino in a magnetic field. In some astrophysical media, this helicity transition mechanism could be more effective than a similar process caused by the anomalous magnetic moment of the neutrino. If the neutrino energy is sufficiently high, the two helicity transition mechanisms mentioned above can compensate each other. Then a helicity flip in an external field will not occur. Calculations are carried out using both the methods of relativistic quantum mechanics and the quasiclassical Bargmann-Michel-Telegdi equation.

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I. INTRODUCTION

Studying the electromagnetic properties of the neutrino is one of the intriguing problems of modern neutrino physics [1]. Electromagnetic interactions of neutrinos can lead to a number of physical effects that are of great importance, both for the physics of elementary particles and for astrophysics and cosmology [2–4].

The greatest interest has been traditionally attracted by the magnetic moments of neutrinos. The existence of a (diagonal) anomalous magnetic moment (AMM) can lead to a helicity flip of a Dirac neutrino moving in the strong magnetic field [5–7]. Nondiagonal transition magnetic moments interact with the magnetic field, causing a neutrino helicity flip that occurs simultaneously with the change of the neutrino flavor, and this effect can take place for both Dirac and Majorana neutrinos (spin-flavor oscillations [8–11]). Various physical processes in regard to the change of neutrino helicity in external fields were widely discussed in connection with the solar neutrino problem, the dynamics of stellar collapse, and the evolution of the early Universe [1,12].

The minimally extended Standard Model, in which neutrinos acquire Dirac masses, predicts very small values both for the diagonal AMM [6,13]

$$\mu_\nu = \frac{3eG_{\mathrm{F}}m_\nu}{8\sqrt{2}\pi^2} \simeq 3.2\times 10^{-19}\mu_B\left(\frac{m_\nu}{1\ \mathrm{eV}}\right) \qquad (1.1)$$

and for the transition magnetic moments of neutrinos [1,2,14]. In the formula (1.1), e is the absolute value of the electron charge, $G_{\mathrm{F}} = 10^{-5}m_p^{-2}$ is the Fermi constant, m_ν is the neutrino mass, and $\mu_B = e/2m_e$ is the Bohr magneton.

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Recent neutrino experiments, as well as the analysis of astrophysical observations [1], provide a limit on the neutrino magnetic moments at this level $\mu_\nu \lesssim (10^{-11}\text{–}10^{-12})\mu_B$, and it is many orders of magnitude higher than the theoretical predictions of the Standard Model. Some extended models that are widely discussed in the literature predict much larger values of the magnetic moments than the Standard Model does, thereby bringing the theoretical predictions closer to the existing experimental limitations (see a review in Ref. [1]).

Turning to the discussion of the issues related to the impact of the surrounding material medium on a neutrino, we note there is a great significance in regard to the theory of coherent interaction of a neutrino with the particles of the medium. The famous solar neutrino "puzzle" can be completely solved within this framework [15,16] on the basis of the Mikheyev-Smirnov-Wolfenstein effect (MSW effect) [17,18]. This approach predicts new phenomena such as the spin light of a neutrino in matter [19–22], which is accompanied by a neutrino helicity flip. There are suggestions that the influence of certain anisotropic media can also lead to a change of the neutrino helicity [23–26].

When the neutrino propagates in a dispersive medium, the effective vertex of the electromagnetic interaction of a neutrino is modified—there are new form factors specifying the interaction of neutrinos with real particles, which belong to the medium [27–31] (e.g., with electrons, if the neutrino propagates in the electron plasma).

The appearance of additional electromagnetic characteristics of the neutrino, which take place only in a medium, has the following physical explanation. This is mainly due to the medium polarization caused by weak interactions of a neutrino moving in this medium [32]. In other words, the neutrino motion in an electron medium leads to the appearance of some inhomogeneities in the electron density at small scales (of the order of the Debye radius in plasma) along the path of the neutrino motion. These inhomogeneities in the electron density induce an electric charge of the neutrino in a medium [33–38], while neutrino interactions due to the pseudovector currents with parity nonconservation result in the appearance of an axial magnetic form factor [27–30] and induce a magnetic moment of the neutrino [28,39,40].

The induced magnetic moment (IMM) of the neutrino plays an important role in astrophysical applications. Indeed, the IMM can reach extremely high values during the propagation of neutrinos in a highly degenerate gas of relativistic electrons in the collapsing core of a supernova or in the interior of neutron stars. Under such conditions, as was shown by Semikoz [39], the IMM of the electron neutrino is equal

$$\mu_\nu^{\text{ind}} = \frac{eG_{\text{F}}\mathrm{p}_{\text{F}}}{2\sqrt{2}\pi^2} \simeq 4.3\times 10^{-13}\mu_B\left(\frac{\mathrm{p}_{\text{F}}}{1\ \text{MeV}}\right), \qquad (1.2)$$

where p_{F} is the electron Fermi momentum given as

$$\mathrm{p}_{\text{F}} \simeq 130\times\left(\frac{n_e}{10^{37}\ \text{cm}^{-3}}\right)^{1/3}\ \text{MeV},$$

and n_e is the density of electrons of the medium (see, also, Refs. [41,42]). Note that the transition IMM for Majorana neutrinos was also discussed in the literature, and it was found that in a degenerate electron gas, they have the same order of magnitude as (1.2) [43].

The interaction of the IMM with an external electromagnetic field changes the dispersion relation of the neutrino. It was shown in Refs. [30,44–46] that in an external magnetic field (in the rest frame of the medium, as well as in the linear approximation in the field), additional energy of interaction with the field takes the form

$$V_H = 2\mu_\nu^{\text{ind}}(\mathbf{pH})/|\mathbf{p}|, \qquad (1.3)$$

where $\mathbf{H}$ is the magnetic field strength, and $\mathbf{p}$ is the neutrino momentum. The interaction (1.3) affects the neutrino oscillations, and it was taken into account in many studies that investigate the various schemes of oscillations and propagation of neutrinos in the magnetized media [41,42,47–52].

The explicitly anisotropic character of the interaction (1.3) (dependence on the angle between the directions of $\mathbf{p}$ and $\mathbf{H}$) may cause an asymmetry of the neutrino emission accompanying the collapse of the massive stellar core. On the basis of this phenomenon, Kusenko and Segrè have proposed a well-known mechanism to explain the observed high drift velocities of pulsars [53,54].

It should be emphasized that the IMM predicted by the Standard Model does not depend on the neutrino mass [in contrast to AMM; see (1.1)], and it follows in particular from the formula (1.2). Therefore, the IMM can also exist in the case of a massless neutrino. In fact, the neutrino was assumed massless in all papers cited above, where the research of various aspects of the IMM was carried out [formula (1.3) is also obtained in the massless limit]. Probably for this reason, many papers contain the statement that the IMM cannot cause the helicity flip of a neutrino in an external field[1]—the helicity of the massless neutrino is strictly defined: the particle has left-handed (negative) helicity, and the antiparticle a right-handed (positive) helicity.

In this paper, we perform a study of the IMM, assuming that the neutrino possesses a Dirac mass. Because we are primarily interested in the impact of the IMM on the neutrino spin dynamics in an external field, we will not take into account other possible contributions to the effective potential of the neutrino in a medium (in particular, the MSW potential [56]), we also will not take into account the existence of the AMM of the neutrino. Assuming that the value of the IMM is known and is defined, for instance, by formula (1.2), we will show that the IMM may cause a helicity flip of a massive Dirac neutrino in an external field and, in some cases, more effectively than the AMM.

II. QUANTUM-MECHANICAL DESCRIPTION

As it is mentioned in the Introduction, the vertex function of the effective electromagnetic interaction of the neutrino $\langle\nu(p')|J_\mu^{EM}(0)|\nu(p)\rangle = \bar{u}(p')\Gamma_\mu(p,p')u(p)$ is modified during the propagation of the neutrino in a medium. In particular, a new contribution to the vertex Γ_μ appears. This contribution is caused by the existence of pseudovector currents and is equal to [27–30]

$$\Gamma_\mu^{M'}(k,u) = i\mathrm{D}_M(\omega,k)e_{\mu\nu\alpha\beta}\gamma^\nu k^\alpha u^\beta\gamma^5, \qquad (2.1)$$

where $k^\mu = p^\mu - p'^\mu$, $u^\mu = \{\gamma_{\text{m}}, \gamma_{\text{m}}\mathbf{u}\}$ is the four-velocity of the medium, $\gamma_{\text{m}} = (1-\mathbf{u}^2)^{-1/2}$ is the Lorentz factor of the medium, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\mathrm{D}_M(\omega,k)$ is the axial magnetic form factor of the neutrino. In the static limit, $\mathrm{D}_M(0,0) = \mu_\nu^{\text{ind}}$ is the IMM of the neutrino in the medium [28,39].

The vertex (2.1) modifies the effective Lagrangian of the neutrino interacting with the medium [29], and as a result, the Dirac equation for a neutrino in the medium and in the external field will take the form

$$\{i\gamma^\mu\partial_\mu - m_\nu + \mu_\nu^{\text{ind}}\gamma^\mu(1+\gamma^5)\tilde{F}_{\mu\nu}u^\nu\}\Psi(\mathbf{r},t) = 0, \qquad (2.2)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}e_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Here, we are left *only* with the contribution of the IMM and added the projector on the left-handed chirality of the neutrino $(1+\gamma^5)$ in accordance with [28,30,32]. We emphasize, once again, that in our approach, we use only the IMM static value, and we do

[1] In Ref. [55], the possibility of a helicity flip of massive neutrinos due to the interaction of the IMM with a magnetic field is for the first time considered. The main conclusions of Ref. [55] concerning the helicity flip caused by IMM are in agreement with ours.

not consider its possible dependence on properties of the medium, on medium velocity, and also on the neutrino energy and on the intensity of the external field.

Let us consider the constant uniform magnetic field $\mathbf{H}$ directed along the z axis. Then in the rest frame of the medium $u^\mu = \{1, \mathbf{0}\}$, Eq. (2.2) takes the form

$$\mathcal{H}\psi(\mathbf{r}) = \varepsilon\psi(\mathbf{r}); \quad \mathcal{H} = \mathcal{H}_0 + \mathrm{V}; \\ \mathcal{H}_0 = (\boldsymbol{\alpha}\mathbf{p}) + \gamma^0 m_\nu, \quad \mathrm{V} = -\mu_\nu^{\mathrm{ind}} H \Sigma_3 (1+\gamma^5), \tag{2.3}$$

where $\Psi(\mathbf{r}, t) = e^{-i\varepsilon t}\psi(\mathbf{r})$, ε is the energy of neutrino, $\mathbf{p}$ is the neutrino momentum, $\mathcal{H}_0$ is the Hamiltonian of a free neutrino, $H = |\mathbf{H}|$, and V is a term describing the interaction of the IMM with the magnetic field. We use the standard representation for the Dirac matrices [57], and $\alpha_i = \gamma^0\gamma^i$, $\Sigma_i = -\gamma^5\gamma^0\gamma^i$.

The third component of the four-vector spin polarization operator T^μ, which was discussed by Bargmann and Wigner [58] (see, also, Refs. [59,60]), commutes with the Hamiltonian $\mathcal{H}$ and is in this case the exact spin integral of motion:

$$\mathrm{T}^3 = \gamma^0\Sigma_3 - \gamma^5 \frac{p_z}{m_\nu}. \tag{2.4}$$

The operator (2.4) describes the neutrino spin projection on the direction of the magnetic field.

Let us define the dispersion relation of a massive Dirac neutrino, taking into account the IMM. In our case, it is sufficient to confine ourselves to the linear approximation across the field $\mathbf{H}$. The stationary wave function of a free neutrino satisfies the conditions

$$\mathcal{H}_0\psi_0 = \varepsilon_0\psi_0, \qquad \mathrm{T}^3\psi_0 = \zeta\frac{\lambda}{m_\nu}\psi_0, \tag{2.5}$$

and has the form of a plane wave

$$\psi_0 = \frac{1}{L^{3/2}}\begin{pmatrix} C_1 \\ C_2 e^{i\varphi} \\ C_3 \\ C_4 e^{i\varphi}\end{pmatrix} \exp(i\mathbf{pr}), \tag{2.6}$$

where $\tan\varphi = p_y/p_x$. Spin factors $C_1 - C_4$ are defined from the joint solution of Eq. (2.5):

$$C_{1,4} = \frac{\zeta}{2\sqrt{2}}\sqrt{1 \pm \zeta\frac{\lambda}{\varepsilon_0}}\left(\sqrt{1+\frac{p_z}{\lambda}} + \zeta\sqrt{1-\frac{p_z}{\lambda}}\right),$$
$$C_{2,3} = \frac{1}{2\sqrt{2}}\sqrt{1 \mp \zeta\frac{\lambda}{\varepsilon_0}}\left(\sqrt{1+\frac{p_z}{\lambda}} - \zeta\sqrt{1-\frac{p_z}{\lambda}}\right),$$

where $\varepsilon_0 = \sqrt{\mathbf{p}^2 + m_\nu^2}$, $\lambda = \sqrt{m_\nu^2 + p_z^2}$. Keeping in mind (2.6) and (2.4), we obtain the energy levels of a neutrino with the IMM in an external magnetic field:

$$\varepsilon = \varepsilon_0 + \mu_\nu^{\mathrm{ind}} H\left\{\frac{p_z}{\varepsilon_0} - \zeta\frac{1}{\varepsilon_0}\sqrt{m_\nu^2 + p_z^2}\right\}. \tag{2.7}$$

As we can see from (2.7), the neutrino energy explicitly depends on the spin orientation in an external field: the case $\zeta = -1$ corresponds to a neutrino spin orientation against and $\zeta = +1$ along the magnetic field direction. Only in the case of a relativistic neutrino motion along the field $p_z \gg m_\nu$ (neglecting the mass of a neutrino) does the dispersion relation (2.7) turn into the formula

$$\varepsilon \simeq \varepsilon_0 + \mu_\nu^{\mathrm{ind}} H \frac{p_z}{\varepsilon_0}(1-\zeta), \tag{2.8}$$

which at $\zeta = -1$ corresponds to the interaction (1.3). Note that for the neutrino moving along the field, the case $\zeta = -1$ corresponds to the negative helicity of the neutrino. If the neutrino possesses a positive helicity, then it should not interact with a magnetic field in this limit [see (2.8)].

On the other hand, if a neutrino moves perpendicularly to the direction of the magnetic field ($p_z = 0$, but the neutrino at the same time can be relativistic), then, according to (1.3), it cannot interact with an external field. In fact, as can be seen from (2.7), an interaction with the field exists for such a neutrino, although it is suppressed by a small factor m_ν/ε_0 (see, also, Ref. [55]).

We notice further that in (2.7), the term proportional to the factor p_z/ε_0 comes from the term $\sim\Sigma_3\gamma^5 = -\alpha_3$ in the Hamiltonian (2.3) [or, equivalently, from the term $\sim\gamma^\mu \tilde{F}_{\mu\nu}u^\nu$ in (2.2)]. This term gives a constant contribution to the neutrino energy (2.7); it does not depend on the spin projection on the direction of the magnetic field, and it has, in essence, no relation to the IMM. Later, we will not take this term into account when analyzing the neutrino spin precession.

It is interesting to compare the dispersion law (2.7) with the dispersion relation for the Dirac neutrino with the AMM interacting with an external uniform magnetic field [the result (2.9), which was calculated earlier [61–63]]:

$$\varepsilon = \varepsilon_0 - \tilde{\zeta}\frac{\mu_\nu H}{\varepsilon_0}\sqrt{m_\nu^2 + p_\perp^2}. \tag{2.9}$$

Here, μ_ν is the AMM of neutrino, and the quantum number $\tilde{\zeta} = \pm 1$ just like ζ corresponds to the orientation of the neutrino spin along or against the magnetic field direction $\mathbf{H}$. We will notice, however, that the quantum number $\tilde{\zeta}$ is defined not by the eigenvalue of the operator T^3 [see (2.4) and (2.5)] but by the eigenvalue of the magnetic polarization operator

$$\mu_3 = \Pi^{12} = m\Sigma_3 + i\gamma^0\gamma^5[\Sigma \times \mathbf{p}]_3,$$

which is the component Π^{12} of the tensor of the spin polarization [64] (see, also, Ref. [60]). Comparing formulas (2.7) and (2.9), we can conclude that the interaction between the AMM and the external field become apparent principally for transverse neutrino motion (with respect to the field direction), while the IMM interaction is most essential in the case of longitudinal movement.

We consider further the movement of the longitudinally polarized neutrino possessing the IMM in an external magnetic field. The operator of longitudinal polarization (i.e., helicity) $(\mathbf{\Sigma p})/|\mathbf{p}|$ does not commute with the Hamiltonian (2.3) and, therefore, is not an integral of motion in these conditions. We construct the superposition of states (2.6)

$$\Psi(t) = A\psi_0(\zeta = +1)e^{-i\varepsilon_{+1}t} + B\psi_0(\zeta = -1)e^{-i\varepsilon_{-1}t}, \tag{2.10}$$

where $\varepsilon_{\pm 1}$ are energy levels determined by the formula (2.7). We choose the coefficients A and B in (2.10) so that the function $\Psi(t)$ satisfies the initial condition

$$(\mathbf{\Sigma p})\Psi(0) = -|\mathbf{p}|\Psi(0);$$

i.e., we assume that the spin of the neutrino at the initial time is directed against its momentum (negative helicity). Using the superposition (2.10), we find the average value of the projection of the neutrino spin on the direction of motion at an arbitrary time:

$$\begin{aligned}\langle(\mathbf{\Sigma p})/|\mathbf{p}|\rangle_t^{\mathrm{IMM}} &= -\left\{\frac{p_z^2}{p^2}\frac{\varepsilon_0^2}{\lambda^2} + \left(1-\frac{p_z^2}{p^2}\right)\left(1-\frac{p_z^2}{\lambda^2}\right)\cos\omega_H t\right\} \\ &= -\frac{1}{1-v^2\sin^2\theta} \\ &\quad\times\{\cos^2\theta + (1-v^2)\sin^2\theta\cos\omega_H t\},\end{aligned} \tag{2.11}$$

where $\omega_H = 2\mu_\nu^{\mathrm{ind}}H\lambda/\varepsilon_0 = 2\mu_\nu^{\mathrm{ind}}H(1-v^2\sin^2\theta)^{1/2}$, $p = |\mathbf{p}|$, and θ is the angle between the neutrino momentum $\mathbf{p}$ and the magnetic field direction $\mathbf{H}$.

It follows from (2.11) that if the neutrino with the IMM moves along the magnetic field ($\cos\theta = 1$), then the longitudinal polarization (i.e., helicity) is conserved [as in the case of the neutrino with an AMM; see below (2.13)]. When the neutrino moves perpendicularly to the field direction ($\cos\theta = 0$), we have

$$\begin{aligned}&\langle(\mathbf{\Sigma p})/|\mathbf{p}|\rangle_t^{\mathrm{IMM}} = \cos\omega_H t, \\ &\qquad \omega_H = 2\mu_\nu^{\mathrm{ind}}H\sqrt{1-v^2} = 2\mu_\nu^{\mathrm{ind}}Hm_\nu/\varepsilon_0,\end{aligned} \tag{2.12}$$

or the helicity precesses with a frequency ω_H.

It is interesting to compare the formula (2.11) that describes the time evolution of the helicity for a neutrino with an IMM, with the corresponding result describing the time evolution of the helicity of a massive Dirac neutrino due to the interaction of the AMM with the external magnetic field [61,62]:

$$\begin{aligned}&\langle(\mathbf{\Sigma p})/|\mathbf{p}|\rangle_t^{\mathrm{AMM}} \\ &\quad = -\frac{1}{1-v^2\cos^2\theta}\{(1-v^2)\cos^2\theta + \sin^2\theta\cos\tilde{\omega}_H t\},\end{aligned} \tag{2.13}$$

where the helicity precession frequency[2] is equal to $\tilde{\omega}_H = 2\mu_\nu H(1-v^2\cos^2\theta)^{1/2}$. Comparing expressions (2.13) and (2.11) and (2.12), we can conclude that in the case of a neutrino moving perpendicularly to the direction of the field, the precession frequency ω_H connected with the IMM includes (unlike the frequency $\tilde{\omega}_H$ characterizing the precession connected with the AMM) the additional small factor $m_\nu/\varepsilon_0 = \gamma^{-1}$, where γ is the Lorentz factor of the neutrino.

Thus, the characteristic time of a helicity flip[3] due to the IMM

$$T = \frac{\pi}{\omega_H} = \frac{\pi}{2\mu_\nu^{\mathrm{ind}}H}\gamma \tag{2.14}$$

proves to be very large for the ultrarelativistic neutrinos ($\gamma \gg 1$).

However, if we assume for estimates that the electron density is equal to $n_e \simeq 10^{37}$ cm^{-3} (for the internal regions of neutron stars), then the IMM of neutrino according to (1.2) will be equal to $\mu_\nu^{\mathrm{ind}} \simeq 0.6\times10^{-10}\mu_B$. If we set $m_\nu \simeq 1$ eV [70], $\varepsilon_\nu = 1$ MeV, then (for $\mathbf{p}\perp\mathbf{H}$) the frequency of the spin precession, which is caused by the IMM [see (2.12)] will be equal to $2\mu_\nu^{\mathrm{ind}}Hm_\nu/\varepsilon_\nu \simeq 1.2\times10^{-16}\mu_B H$. As can be seen from a comparison of (2.13) and (2.12), the spin precession with exactly the same frequency could be caused by the AMM being equal to $\mu_\nu \simeq 6\times10^{-17}\mu_B$. The Standard Model under these conditions gives the AMM value of the order $\mu_\nu \simeq 3.2\times10^{-19}\mu_B$ [see (1.1)]. Thus, under the conditions considered (in the sense of the impact on the spin precession), the IMM of a neutrino is about 200 times more effective than the vacuum AMM predicted by the Standard Model. It is clear that the presence of the AMM of a neutrino can be completely disregarded in this case.

As it follows from (2.14), the time of a helicity flip due to the IMM decreases with the decreasing of neutrino energy and reaches the minimum value in the nonrelativistic limit ($\gamma \to 1$). We note that the average values of the helicity of neutrinos possessing the IMM and AMM behave equally in this limit [see (2.11) and (2.13)]:

$$\langle(\mathbf{\Sigma p})/|\mathbf{p}|\rangle_t^{\mathrm{IMM,AMM}} \simeq \cos^2\theta + \sin^2\theta\cos\omega_H^{(1,2)}t,$$

[2]Equation (2.13) also describes the time evolution of the longitudinal polarization of a neutron with an AMM [65] and of an electron with an AMM [66,67] in a uniform magnetic field.

[3]In papers [68,69], the spin dynamics of a Dirac neutrino with an AMM in an external field and in a medium is considered when determining the helicity using a method different from ours.

where the precession frequency is equal to $\omega_H^{(1)} = 2\mu_\nu^{\mathrm{ind}} H$ for a neutrino with an IMM and $\omega_H^{(2)} = 2\mu_\nu H$—for a neutrino with an AMM. Considering the values given above for the IMM (1.2) and for the AMM (1.1), we come to a conclusion that under the conditions of a degenerate electron gas, the IMM can flip the helicity of nonrelativistic neutrinos in a magnetic field about 10^9 times more efficiently than the AMM can.

It is well known that calculations made within the framework of some extended theoretical models (beyond the Standard Model) give values of the AMM of a neutrino that are considerably larger than the one in formula (1.1). In such a case, the precession frequencies [see (2.11) and (2.13)] might be of the same order of magnitude. Then it is necessary to take into account that the neutrino helicity flip in a magnetic field can occur both due to the IMM and due to the AMM. Generally, it appears impossible to separate the contributions of these two mechanisms because both processes lead to the same physical consequence—to the helicity flip of a neutrino. Note that if the IMM and the AMM of a neutrino are oppositely directed (the IMM can be positive or negative depending on the neutrino flavor [41,42]), these two mechanisms can compensate each other. For example, if the AMM and the IMM have different signs, and the neutrino moves perpendicularly to the magnetic field direction, then the helicity flip will not occur under the condition

$$|\mu_\nu| = |\mu_\nu^{\mathrm{ind}}|/\gamma. \tag{2.15}$$

It is interesting to note that the condition (2.15) can also be fulfilled in the framework of the Standard Model [when the AMM is defined by the formula (1.1)], if the neutrino energy is sufficiently high. Using for estimates the data discussed earlier in this section (for the inner regions of a neutron star), we find that the condition (2.15) will be satisfied when the neutrino energy is equal to $\varepsilon_\nu \simeq 200$ MeV.

III. QUASICLASSICAL DESCRIPTION

As is known, in the framework of the quasiclassical theory of spin (see Ref. [71] and references cited therein), the evolution of the spin of a relativistic electron in an external electromagnetic field is described by the Bargmann-Michel-Telegdi equation [72] (BMT equation; see, also, Refs. [57,73]). Because the electron is a charged particle, the evolution of its spin is determined by the interaction of both the normal magnetic moment (equal to the Bohr magneton $\mu_B = e/2m_e$) and the AMM (approximately equal to the Schwinger value $\mu_B\alpha/2\pi$ [74], $\alpha \simeq 1/137$ is the fine structure constant) with the external field.

In the case of a massive Dirac neutrino, the entire magnetic moment is anomalous (1.1); therefore, the BMT equation for a neutrino will look like (see, also, Ref. [23])

$$\frac{ds^\mu}{d\tau} = 2\mu_\nu\{F^{\mu\nu}s_\nu + v^\mu(s_\alpha F^{\alpha\beta}v_\beta)\}. \tag{3.1}$$

In the formula (3.1), we use the notation τ for the proper time, μ_ν for the AMM of neutrino, $v^\mu = \{\gamma, \gamma\mathbf{v}\}$ for the four-vector of its velocity, $\gamma = \varepsilon_\nu/m_\nu$ for the Lorentz factor, $\mathbf{v} = \mathbf{p}/\varepsilon_\nu$ for the three-dimensional neutrino velocity, and s^μ for the "classical" four-vector of neutrino spin polarization having components [57,73]

$$s^\mu = \left\{\frac{\boldsymbol{\zeta}\mathbf{p}}{m_\nu}, \boldsymbol{\zeta} + \frac{\mathbf{p}(\boldsymbol{\zeta}\mathbf{p})}{m_\nu(\varepsilon_\nu + m_\nu)}\right\} = \left\{(\boldsymbol{\zeta}\mathbf{v})\gamma, \boldsymbol{\zeta} + \frac{\gamma^2}{1+\gamma}\mathbf{v}(\boldsymbol{\zeta}\mathbf{v})\right\}. \tag{3.2}$$

Here, $\boldsymbol{\zeta}$ is the unit vector in the direction of polarization in the particle rest frame ($\mathbf{p} = \mathbf{0}$). It is equal to the doubled mean value of the Pauli spin operator $\frac{1}{2}\boldsymbol{\sigma}$ (i.e., $\boldsymbol{\zeta} = \langle\boldsymbol{\sigma}\rangle_0$), and the mean value of the Pauli sigma matrices is calculated over the spin state specified by the three-dimensional spinor φ in the bispinor of a free neutrino [57]

$$u(p) = \frac{1}{\sqrt{2m_\nu}}\begin{pmatrix} \sqrt{\varepsilon_\nu + m_\nu}\varphi \\ \sqrt{\varepsilon_\nu - m_\nu}(\boldsymbol{\sigma}\mathbf{n})\varphi \end{pmatrix}, \quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|, \qquad \bar{u}(p)u(p) = 1. \tag{3.3}$$

On the basis of (3.1), it is possible to obtain an equation describing the time evolution of the vector $\boldsymbol{\zeta}$, directly characterizing the polarization of the particle in its "instantaneous" rest frame [57,73]:

$$\frac{d\boldsymbol{\zeta}}{dt} = 2\mu_\nu\left\{[\boldsymbol{\zeta}\times\mathbf{H}] - [\boldsymbol{\zeta}\times[\mathbf{v}\times\mathbf{E}]] - \frac{\gamma}{1+\gamma}[\boldsymbol{\zeta}\times\mathbf{v}](\mathbf{v}\mathbf{H})\right\}, \tag{3.4}$$

where $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields in the laboratory frame.

Below, we will carry out a generalization of the BMT equation [(3.1) and (3.4)] in the case of a neutrino having the IMM. It is known that the BMT equation is the classical approximation of the general equation of spin evolution in the Heisenberg representation. The corresponding method of obtaining the BMT equation for the electron has been developed in [75,76] (this method was used also in Ref. [77]), and the general idea of this method ascends to Ref. [59].

We consider the three-dimensional vector operator of spin polarization

$$\mathbf{O} = \gamma^0\Sigma - \gamma^5\frac{\mathbf{p}}{\varepsilon_\nu} - \gamma^0\frac{\mathbf{p}(\Sigma\mathbf{p})}{\varepsilon_\nu(\varepsilon_\nu + m_\nu)} \tag{3.5}$$

introduced by Stech [78] (see, also, Ref. [59]). The operator $\mathbf{O}$ characterizes the "true" spin in the particle rest frame because the averaging over the wave functions of a free neutrino (3.3) yields $\langle \mathbf{O} \rangle = \langle \boldsymbol{\sigma} \rangle_0 = \boldsymbol{\zeta}$. Obtaining the explicit form of the quasiclassical equation of the evolution of the neutrino spin, in the case where a neutrino has the IMM, includes three stages [79].

(i) We write down the quantum equation for the time evolution of the spin operator (3.5) in the Heisenberg representation:

$$\frac{d\mathbf{O}}{dt} = i(\mathcal{H}\mathbf{O} - \mathbf{O}\mathcal{H}),$$

where the Hamiltonian $\mathcal{H}$ is given by the formula (2.3). The operator describing the interaction of the IMM with the external field in expression (2.3) in the rest frame of the medium in the presence of only a uniform magnetic field of arbitrary direction takes the form $\mathrm{V} = -\mu_\nu^{\mathrm{ind}}(\mathbf{H}\Sigma)$, where (as in Sec. II) we mean that μ_ν^{ind} is the IMM static value. Then we obtain the expression

$$\frac{d\mathbf{O}}{dt} = 2\mu_\nu^{\mathrm{ind}}\left\{\gamma^0[\Sigma \times \mathbf{H}] + \gamma^0 \frac{(\Sigma[\mathbf{p} \times \mathbf{H}])\mathbf{p}}{\varepsilon_\nu(\varepsilon_\nu + m_\nu)}\right\}. \qquad (3.6)$$

Direct physical interpretation of this equation is difficult because, as is well known (see, e.g., [80]), in the Dirac theory, the relation between the operators and classical quantities becomes complicated because of the special nature of the particle motion—a rapidly oscillating trembling, which has been called by Schrödinger *"Zitterbewegung,"* i.e., "trembling motion."

(ii) We turn, therefore, to the operators with a definite parity, assuming that the even part of any operator F is defined by the expression [81]

$$\mathrm{F}^{\mathrm{even}} = [\mathrm{F}] = \frac{1}{2\varepsilon_\nu}(\mathrm{F}\mathcal{H} + \mathcal{H}\mathrm{F}).$$

The introduction of the even operators that do not mix the states with different signs of energy allows for the exclusion of the phenomenon of *Zitterbewegung* in the one-particle quantum theory, as well as allowing for the restoration of the correct relations between the operators and the corresponding classical quantities. This opens up the possibility of an intuitive physical interpretation of the results [75,76,81]. In this way, from (3.6), we can obtain the equation for the even part of the operator $[d\mathbf{O}/dt]$. We do not write this equation here due to its cumbersome nature.

(iii) Furthermore, it is necessary to average the operator equation for $[d\mathbf{O}/dt]$ over the state of the quasiclassical wave packet [57,59,75,76]: $[d\mathbf{O}/dt] \to \langle [d\mathbf{O}/dt] \rangle$, assuming that $\langle \mathbf{O} \rangle = \boldsymbol{\zeta}$, $\langle \mathbf{p} \rangle = \mathbf{v}\varepsilon_\nu$. Note that in our case (the electric charge of the neutrino is equal to zero), the specified averaging can be performed over the solutions of the free Dirac equation (3.3). As a result, we obtain an equation describing the neutrino spin evolution, one originating from the interaction of the IMM with a homogeneous magnetic field in the rest frame of the medium:

$$\frac{d\boldsymbol{\zeta}}{dt} = 2\mu_\nu^{\mathrm{ind}}\left\{\frac{1}{\gamma}[\boldsymbol{\zeta} \times \mathbf{H}] + \frac{\gamma}{1+\gamma}[\boldsymbol{\zeta} \times \mathbf{v}](\mathbf{v}\mathbf{H})\right\}. \qquad (3.7)$$

Analyzing the resulting equation (3.7) in comparison with Eq. (3.4) for a neutrino with the AMM [in this case it is necessary to set $\mathbf{E} = 0$ in (3.4)], we note that when the neutrino is moving perpendicularly to the field direction ($\mathbf{v} \perp \mathbf{H}$), then the neutrino spin precession frequency in (3.7) contains, in contrast to (3.4), a small factor γ^{-1}, and this leads to an increase in the period of the precession of the neutrino with the IMM. We already paid attention to this fact in Sec. II; see (2.14).

If the directions of $\mathbf{v}$ and $\mathbf{H}$ coincide, the vector $\boldsymbol{\zeta}$ will precess around the common direction of $\mathbf{v}\uparrow\uparrow\mathbf{H}$, with an angular velocity equal to $2\mu_\nu^{\mathrm{ind}}H$ or $2\mu_\nu H\gamma^{-1}$ for a neutrino with the IMM or for a neutrino with the AMM, respectively. Thus, in this case, the spin precession frequency for a neutrino with an IMM *does not contain* the suppression factor γ^{-1} in contrast to the precession frequency for a neutrino with an AMM.

We now assume that in addition to the magnetic field, there exists also a uniform electric field $\mathbf{E}$. In this case, the generalization of Eq. (3.7), taking into account the nonzero velocity of the medium ($\mathbf{u} \neq 0$), leads to the equation

$$\begin{aligned}\frac{d\boldsymbol{\zeta}}{dt} = 2\mu_\nu^{\mathrm{ind}}\frac{\gamma_{\mathrm{m}}}{\gamma}\Big\{&[\boldsymbol{\zeta} \times \mathbf{H}] - [\boldsymbol{\zeta} \times [\mathbf{u} \times \mathbf{E}]] \\ &+ \frac{\gamma^2}{1+\gamma}[\boldsymbol{\zeta} \times \mathbf{v}](\mathbf{v}\mathbf{H}) - \frac{\gamma^2}{1+\gamma}[\boldsymbol{\zeta} \times \mathbf{v}](\mathbf{v}[\mathbf{u} \times \mathbf{E}])\Big\} \\ &- 2\mu_\nu^{\mathrm{ind}}\gamma_{\mathrm{m}}[\boldsymbol{\zeta} \times \mathbf{v}](\mathbf{u}\mathbf{H}),\end{aligned} \qquad (3.8)$$

where $\mathbf{u}$ is the three-dimensional speed of the medium and γ_{m} is its Lorentz factor.

Analyzing Eq. (3.8), we pay attention to two facts. First, the motion of the medium can lead to an increase in the precession frequency of the neutrino spin. Indeed, setting in (3.8) $\mathbf{E} = 0$ and $\mathbf{u} \perp \mathbf{H}$ (the medium moves perpendicularly to the magnetic field direction), we obtain the equation coinciding with (3.7) with the only difference: the multiplier $2\mu_\nu^{\mathrm{ind}}$ is replaced by $2\mu_\nu^{\mathrm{ind}}\gamma_{\mathrm{m}}$. This means that the angular velocity of precession of the vector $\boldsymbol{\zeta}$ increases by a factor γ_{m} in the case of a neutrino moving perpendicularly to the field direction, as well as in case of $\mathbf{v}\uparrow\uparrow\mathbf{H}$.

Second, if we set $\mathbf{u} = \mathbf{v}$ in Eq. (3.8) (the neutrino moves in the same direction as the medium with a speed equal to the velocity of medium), then the condition $\gamma_{\mathrm{m}} = \gamma$ will be fulfilled, and Eq. (3.8) will take the form of Eq. (3.4), describing the spin precession of a neutrino with an AMM. Thus, in this case, the impact of the IMM on the neutrino spin dynamics is indistinguishable from the influence of the AMM. The corresponding equations will differ only by a common factor on the right-hand side (μ_ν or μ_ν^{ind}).

We further obtain the covariant generalization of the BMT equation (3.1) for a neutrino possessing the IMM.

This can be done by a method that is similar to the one by which Eqs. (3.7) and (3.8) were obtained. However, now we set as basis for the calculations, not a three-dimensional vector operator $\mathbf{O}$ (3.5) but the four-vector operator of the spin polarization [58–60]

$$\mathrm{T}^{\mu} = \gamma^{5}\gamma^{\mu} - \gamma^{5} p^{\mu}/m_{\nu}. \tag{3.9}$$

Note that earlier we used the third component of the operator (3.9) when determining the dispersion relation for neutrinos with the IMM (2.7). We now should take into account that the averaging over the state (3.3) gives $\langle \mathrm{T}^{\mu} \rangle = s^{\mu}$, where the four-vector of spin polarization s^{μ} is defined by the above formula (3.2).

As a result, performing calculations, we find the covariant generalization of the BMT equation in the form

$$\frac{ds^{\mu}}{d\tau} = 2\mu_{\nu}^{\mathrm{ind}}\{F^{\mu\nu}s_{\nu}(u_{\alpha}v^{\alpha}) - F^{\mu\nu}v_{\nu}(u_{\alpha}s^{\alpha}) + u^{\mu}(s_{\alpha}F^{\alpha\beta}v_{\beta})\}. \tag{3.10}$$

It should be noted that Eq. (3.10) has an elegant covariant form which generalizes Eqs. (3.7) and (3.8) obtained under some special assumptions.

If we set $u^{\mu} = v^{\mu}$ in Eq. (3.10) (i.e., 4-velocities of the medium and of the neutrino coincide), then we will obtain that $u_{\alpha}v^{\alpha} = v_{\alpha}v^{\alpha} = 1$ and also $u_{\alpha}s^{\alpha} = v_{\alpha}s^{\alpha} = 0$ (see Refs. [57,73]). As a result, Eq. (3.10) will coincide with (3.1) exactly if one performs the redesignation $\mu_{\nu}^{\mathrm{ind}} \to \mu_{\nu}$. The coincidence of the three-dimensional equations (3.8) and (3.4) in a similar case was discussed above.

Explicitly writing out the time component in (3.10), we derive the equation for the projection of the spin polarization vector $\boldsymbol{\zeta}$ on the direction of motion (defined by the unit vector $\mathbf{n} = \mathbf{v}/|\mathbf{v}|$), for longitudinal polarization (or helicity) of neutrino ($\boldsymbol{\zeta}\mathbf{n}$):

$$\frac{d}{dt}(\boldsymbol{\zeta}\mathbf{n}) = 2\mu_{\nu}^{\mathrm{ind}}\frac{\gamma_{\mathrm{m}}}{\gamma}\{(\boldsymbol{\zeta}_{\perp}[\mathbf{H}\times\mathbf{n}]) + (\mathbf{E}[\mathbf{u}\times[\mathbf{n}\times\boldsymbol{\zeta}_{\perp}]])\}, \tag{3.11}$$

where $\boldsymbol{\zeta}_{\perp}$ is the component of the vector $\boldsymbol{\zeta}$ perpendicular to the velocity $\mathbf{v}$. According to the formula (3.11), the general scheme of the neutrino helicity evolution in a purely magnetic field ($\mathbf{E} = 0$), as well as in a static medium ($\mathbf{u} = 0$), corresponds to the result obtained in Sec. II. The motion of medium ($\mathbf{u} \neq 0$) leads to an increase in the helicity precession frequency (the frequency is multiplied by the Lorentz factor of medium γ_{m}).

It should be noted that our conclusion (see Sec. II) that two mechanisms of the neutrino helicity flip (due to the IMM and due to the AMM) can compensate each other when acting together, remains true also in the framework of the quasiclassical theory. The validity of this conclusion can be verified, in particular, by analyzing Eqs. (3.4) and (3.7) under the assumption that the AMM and the IMM are oppositely directed, and the conditions $\mathbf{E} = 0$ and $\mathbf{u} = 0$ are satisfied. Then the spin precession will be missing if the neutrino moves perpendicularly to the magnetic field direction ($\mathbf{v}\perp\mathbf{H}$), and the relation (2.15) is fulfilled. This means that under such conditions, the projection of the spin on the direction of the neutrino velocity remains constant; i.e., a helicity flip will not occur. The same conclusion can be made by comparing Eq. (3.11) with the known quasiclassical equation describing the helicity evolution caused by the AMM [57,73].

IV. CONCLUSIONS

In this paper, we considered the influence of the magnetic moment induced by the medium (IMM) on the neutrino spin dynamics in external fields. The above consideration was carried out both by methods of relativistic quantum mechanics and by a quasiclassical method using the generalized Bargmann-Michel-Telegdi equation.

We showed that the IMM interacting with the external field can cause the helicity flip of a massive neutrino and, in some cases, with higher efficiency than the AMM. In particular, the considered mechanism of the neutrino helicity flip may be important in the study of various schemes of conversion of longitudinally polarized neutrinos in the dense astrophysical media, such as a degenerate electron gas in the core of a supernova or in the interior of a neutron star in the presence of strong electromagnetic fields.

Indeed, if we use for estimates the data represented in Sec. II, which are typical for the interior regions of the neutron star, assuming that the magnetic field is equal to $H \sim 10^{14}$ G (see, e.g., [82,83]), we will obtain the characteristic length of the helicity flip of the neutrino [the so-called half rotation length; see (2.14)], equal to

$$L = cT = c\pi/\omega_{H} \simeq 5.6 \times 10^{6}\ \mathrm{cm}.$$

This length coincides in the order of magnitude with the typical values of the radii of neutron stars $R \simeq 10$–15 km (see, e.g., [83]).

Note also that the linear approximation in the magnetic field is quite adequate for the problem considered. The conditions of applicability of this approach follow from the formula (2.7). The most stringent restriction that follows from (2.7) and that is valid for nonrelativistic neutrinos, gives $\mu_{\nu}^{\mathrm{ind}} H \ll m_{\nu}$. Taking into account the data from Sec. II, we obtain the following field strength limitation:

$$H \ll 2.9 \times 10^{18}\ \mathrm{G}. \tag{4.1}$$

The value $H \sim 10^{14}$ G used for our estimates does not contradict the condition (4.1).

Further, the expression (1.2) for μ_{ν}^{ind} is true in the so-called weak field approximation, namely, when

$$eH \ll \mathrm{p}_{F}^{2}; \tag{4.2}$$

see [84]. The data used by us for numerical estimates in Secs. II and IV do not violate also the condition (4.2).

It should be stressed that in this case, the conversion of neutrinos from left handed to right handed can take place *without* the participation of the AMM of the neutrino (the AMM value predicted by the Standard Model is many orders lower than the IMM of neutrino).

With an increase in the neutrino energy, the efficiency of the helicity flip mechanism associated with the IMM decreases. When the energy of a neutrino is sufficiently high, it is necessary to take into account that the helicity flip in an external field can occur both due to the IMM and due to the AMM. It is shown that under certain conditions, these two mechanisms can compensate each other (see Secs. II and III for details). As a result, a helicity flip in an external field will not occur.

The increase in the frequency of the precession of the neutrino helicity in moving media (Sec. III) can be important when considering the interaction of neutrinos with the relativistic plasma jets, which are observed in many astrophysical objects, for example, in active galactic nuclei, in microquasars, and in cosmological gamma-ray burst sources [82,85].

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