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BRST symmetry: Boundary conditions and edge states in QED

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In manifolds with spatial boundary, BRST formalism can be used to quantize gauge theories. We show that, in a $U(1)$ gauge theory, only a subset of all the boundary conditions allowed by the self-adjointness of the Hamiltonian preserves BRST symmetry. Hence, the theory can be quantized using BRST formalism only when that subset of boundary conditions is considered. We also show that for such boundary conditions, there exist fermionic states which are localized near the boundary.

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I. INTRODUCTION

Topological insulators and their surface modes are subjects of emerging interest (for example, see Refs. [1–5]). Especially, understanding a two-dimensional topological insulator in the light of the fractional Hall effect has been *a priori*ty in the subject for the last decade [6,7]. In this context, theories of gauge fields interacting with matter, especially in two and three spatial dimensions, have gained importance. We investigate the quantization of $U(1)$ gauge theories with Dirac fermions from this perspective.

Quantization of gauge theories using BRST (where the BRST refers to Becchi, Rouet, Stora and Tyutin) formalism is conventional. It is elegant, yet simple. One introduces a ghost field for every constraint of the system. This breaks the gauge symmetry but introduces a new global symmetry (called BRST symmetry) generated by appropriate combinations of the ghosts and the constraints. The generators of this new global symmetry are fermionic and hence nilpotent, and the physical Hilbert space is identified by its cohomology.

We are interested in systems like topological insulators. All such real systems available for experiment are of finite size and hence have spatial boundaries. The presence of boundaries, in general, can reduce the symmetry of the system. As a reflection of this, all boundary conditions might not preserve the symmetry (as shown in Ref. [8]). Therefore, boundary conditions naturally assume significance in the discussion of gauge theories in manifolds with boundaries and their quantization using the BRST formalism.

We do not address the question of how to derive the boundary conditions when the system is restricted to a partial domain with boundaries. The general treatment of this problem does not seem to exist. This problem has been dealt with in some isolated cases. An important example is Halperin's work on quantum Hall edge states [9]. The general approach has been to assume certain boundary conditions and then check the predictions against experiments.

Edge states of course have come to play an important role in string theories (D branes) and related spacetime models which treat the four-dimensional spacetime as the boundary of a higher-dimensional one.

The boundary conditions cannot be chosen arbitrarily. Rather, we need to consider only those boundary conditions which define domains of self-adjointness of the Hamiltonian [10–12]. That is, we treat our system as a closed system with unitary evolution. The more general scenario in which this is not true is also important, but we do not consider it here. Even for our closed systems, all associated domains might not be preserved under BRST transformations. Therefore, in order to quantize the closed system by BRST formalism, we must choose only those boundary conditions which not only define a self-adjoint Hamiltonian but are also consistent with the BRST symmetry.

The presence of boundaries also naturally leads to the discussion of edge states, which, if extant, play an important role in the physics of the boundary in systems like topological insulators [5,13].

In this paper, we consider a $U(1)$ gauge theory with Dirac fermions on a $(d+1)$-dimensional manifold $M \times \mathbb{R}$ with spatial boundary ∂M of codimension 1. In Sec. II, we review the usual discussion of a $U(1)$ gauge theory. In Sec. III, we introduce the ghosts and invoke BRST symmetry. We obtain the set of all allowed boundary

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conditions by demanding the self-adjointness of the gauge fixed Hamiltonian. We show that, out of the set of boundary conditions on the gauge fields consistent with the self-adjointness of the Hamiltonian, only some of them are invariant under BRST transformations. This subset of boundary conditions is the same as that obtained by quantization of the system using the canonical formalism [10,14].

However, we show that there is no such constraint on the boundary conditions of the Dirac fermions. Hence, any domain of self-adjointness of the Dirac Hamiltonian is compatible with the BRST symmetry.

For a system like a topological insulator, we are further required to use physical conditions to choose the suitable boundary conditions from this set of allowed BRST-preserving boundary conditions.

Finally, we discuss the possibility of fermionic edge states in the system. In a simple $(2+1)$-dimensional geometry, we solve for the eigensates of the Hamiltonian in the limit of a small coupling constant, with boundary conditions that ensure the self-adjointness of the Hamiltonian and preserve the BRST symmetry. We show that there exist fermionic edge states (protected by a mass gap), which interact with soft photons and do not break BRST symmetry. These states should be experimentally detectable.

II. MAXWELL-DIRAC SYSTEM

Consider a gauge theory of Dirac fields [which we call the $U(1)$ Maxwell-Dirac system] on a $(d+1)$-dimensional flat manifold $M\times\mathbb{R}$ with spatial boundary ∂M of codimension 1. We choose the metric $g^{\mu\nu}=\mathrm{diag}(1,-1,\ldots,-1)$. We use the convention that Greek alphabets $(\mu,\nu,\ldots)$ range from 0 to d and indices with Latin alphabets $(i,j,\ldots)$ range from 1 to d.

The $U(1)$ gauge fields A_μ are Hermitian,

$$A_\mu^\dagger = A_\mu, \tag{2.1}$$

and the field strength is given by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{2.2}$$

The covariant derivative is

$$D_\mu=\partial_\mu-ieA_\mu, \tag{2.3}$$

with e the gauge coupling constant.

The Maxwell-Dirac action is given by

$$S=\int_{M\times\mathbb{R}} d^{d+1}x\mathcal{L}, \tag{2.4}$$

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\bar{\psi}\gamma^\mu D_\mu\psi-m\bar{\psi}\psi, \tag{2.5}$$

where m is the mass of the fermions and $\bar{\psi}=\psi^\dagger\gamma^0$. The Gamma matrices generate the Clifford algebra:

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu},\qquad \gamma^{0\dagger}=\gamma^0,\qquad \gamma^{i\dagger}=-\gamma^i. \tag{2.6}$$

The conjugate momenta to the gauge fields A_μ and the fermions $\psi,\bar{\psi}$ are given by

$$\Pi^i_{\mathrm{gauge}}\equiv\frac{\partial\mathcal{L}}{\partial\dot{A}_i}=F^{i0},\qquad \Pi^0_{\mathrm{gauge}}\equiv\frac{\partial\mathcal{L}}{\partial\dot{A}_0}=0, \tag{2.7}$$

$$\Pi_\psi\equiv\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\bar{\psi}\gamma^0=i\psi^\dagger,\qquad \Pi_{\bar{\psi}}\equiv\frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}}=0, \tag{2.8}$$

where the dot denotes derivation with respect to time. Notice that the field A_0 is not dynamical. In other words, the Lagrangian (2.5) does not depend on $\dot{A}_0$. As a consequence, the momentum Π^0_{gauge} conjugate to A_0 vanishes, and thus A_0 is arbitrary and plays the role of a Lagrange multiplier. In fact, $\Pi^0_{\mathrm{gauge}}=0$ is a primary constraint, and as such, it is part of the gauge symmetry generator.[1] The Hamiltonian is

$$H=\int_M d^dx(\Pi^i_{\mathrm{gauge}}\dot{A}_i+\Pi_\psi\dot{\psi}-\mathcal{L}) \tag{2.9}$$

$$=\int_M d^dx\left[\frac{1}{2}(\Pi^i_{\mathrm{gauge}})^2+\frac{1}{4}F_{ij}F^{ij} - \Pi_\psi\gamma^0(\gamma^iD_i+im)\psi+GA_0\right], \tag{2.10}$$

where G, the Gauss law operator, is

$$G=\partial_i\Pi^i_{\mathrm{gauge}}-ie\Pi_\psi\psi. \tag{2.11}$$

In order for this operator to generate gauge transformations infinitesimally, the correct expression for the Gauss law is not (2.11) but rather

$$G(h)\equiv\int_M d^dx[\Pi^i_{\mathrm{gauge}}\partial_i-ie\Pi_\psi\psi]h(x^0,\vec{x})=0, \tag{2.12}$$

with $h(x^0,\vec{x})$ a test function that vanishes at the spatial boundary of our manifold:

$$h(x^0,\vec{x})|_{\partial M}=0. \tag{2.13}$$

The operator $G(h)$ vanishes on quantum state vectors in the physical subspace.

This analysis must be followed by a suitable choice of boundary conditions on A_i and ψ invoking the

[1] A detailed review of primary constraints and their relation to gauge transformations can be found in Ref. [15].

self-adjointness of the Hamiltonian and subsequent canonical quantization, as in Ref. [10].

III. BRST SYMMETRY

In this section, we explore the quantization of the Maxwell-Dirac theory using BRST formalism. The BRST formalism deals with the quantization of gauge fields in a rigorous mathematical framework. This approach amounts to replacing the gauge symmetry of the theory by a global BRST symmetry, which enlarges the number of degrees of freedom in the original theory. In this enlarged Hilbert space, the usual canonical quantization can be performed. Then, restricting attention to BRST-invariant states, one recovers the Hilbert space of physical states of the original theory.

The gauge symmetry of the above Maxwell-Dirac system can be replaced by the BRST global symmetry by introducing three additional fields: an auxiliary field $\mathcal{B}$, a ghost field $\mathcal{G}$, and an antighost field $\bar{\mathcal{G}}$. This new action is given by

$$S_{\mathrm{BRST}} = \int_M d^{d+1}x \mathcal{L}_{\mathrm{BRST}}, \tag{3.1}$$

$$\mathcal{L}_{\mathrm{BRST}} = \mathcal{L} + \mathcal{B}\left(\partial^\mu A_\mu - \frac{\zeta}{2}\mathcal{B}\right) + (\partial^\mu \bar{\mathcal{G}})(\partial_\mu \mathcal{G}), \tag{3.2}$$

where ζ is a real parameter and $\mathcal{L}$ is given in (2.4).

In the presence of such new fields, the conjugate momentum Π^0_{gauge} becomes nonzero:

$$\Pi^0_{\mathrm{gauge}} = \mathcal{B}. \tag{3.3}$$

On the other hand, the conjugate momenta to the auxiliary, ghost, and antighost fields are given by

$$\Pi_{\mathcal{B}} \equiv \frac{\partial \mathcal{L}_{\mathrm{BRST}}}{\partial \dot{\mathcal{B}}} = 0, \qquad \Pi_{\mathcal{G}} \equiv \frac{\partial \mathcal{L}_{\mathrm{BRST}}}{\partial \dot{\mathcal{G}}} = \dot{\bar{\mathcal{G}}},$$
$$\Pi_{\bar{\mathcal{G}}} \equiv \frac{\partial \mathcal{L}_{\mathrm{BRST}}}{\partial \dot{\bar{\mathcal{G}}}} = \dot{\mathcal{G}}. \tag{3.4}$$

The Hamiltonian is

$$H = \int_M d^d x (\Pi^\mu_{\mathrm{gauge}} \dot{A}_\mu + \Pi_\psi \dot{\psi} + \Pi_{\mathcal{G}} \dot{\mathcal{G}} + \dot{\bar{\mathcal{G}}} \Pi_{\bar{\mathcal{G}}} - \mathcal{L}_{\mathrm{BRST}}) \tag{3.5}$$

$$= \int_M d^d x \left[\frac{\zeta - 1}{2} (\Pi^0_{\mathrm{gauge}})^2 + \frac{1}{2}(\Pi^0_{\mathrm{gauge}} - \partial^i A_i)^2 - \frac{1}{2}(\partial^i A_i)^2 + \frac{1}{2}(\Pi^i_{\mathrm{gauge}} + \partial_i A_0)^2 - \frac{1}{2}(\partial_i A_0)^2 + \frac{1}{4} F_{ij} F^{ij} - \Pi_\psi \gamma^0 (\gamma^i D_i + im - ie\gamma^0 A_0)\psi + \Pi_{\mathcal{G}} \Pi_{\bar{\mathcal{G}}} - (\partial^i \bar{\mathcal{G}})(\partial_i \mathcal{G})\right]. \tag{3.6}$$

Defining

$$\mathcal{P}^0 \equiv \Pi^0_{\mathrm{gauge}} - \partial^i A_i, \qquad \mathcal{P}^i \equiv \Pi^i_{\mathrm{gauge}} - \partial^i A_0, \tag{3.7}$$

we can rewrite the Hamiltonian as

$$H = \int_M d^d x \left[\frac{\zeta - 1}{2}(\Pi^0_{\mathrm{gauge}})^2 + \frac{1}{2}(\mathcal{P}^0)^2 + \frac{1}{2}(\mathcal{P}^i)^2 - \Pi_\psi \gamma^0 (\gamma^i D_i + im - ie\gamma^0 A_0)\psi + \Pi_{\mathcal{G}} \Pi_{\bar{\mathcal{G}}} - \bar{\mathcal{G}}(\partial_i^2 \mathcal{G}) + \frac{1}{2} A_0 (\partial_i^2 A_0) + A_i (\partial_i \partial_j A_j) - \frac{1}{2} A_i (\partial_j^2 A_i)\right] + \int_{\partial M} d^{d-1}x \left[\bar{\mathcal{G}} \partial_n \mathcal{G} - \frac{1}{2} A_0 \partial_n A_0 + \frac{1}{2} A_i \partial_n A_i - \frac{1}{2} A_n \partial_i A_i - \frac{1}{2} A_i \partial_i A_n\right], \tag{3.8}$$

where n denotes the outward-pointing unit vector of the boundary ∂M.

We can derive the equations of motion and the current conservation consistently by taking Poisson brackets of (3.8) with fields supported in the interior of M.

The boundary conditions in the classical theory are imposed to have unique solutions of the equations of motion. However, in the quantum theory, boundary conditions are imposed so that the Hamiltonian is self-adjoint. To analyze the self-adjointness of the Hamiltonian, we can ignore the boundary term in the above Hamiltonian. For a scalar field theory, this discussion can be found in Ref. [16]. Here, we adapt the same discussion to the interacting gauge theory.

Removing the boundary terms, the Hamiltonian is

$$H = \int_M d^d x \left[\frac{\zeta - 1}{2}(\Pi^0_{\rm gauge})^2 + \frac{1}{2}(\mathcal{P}^0)^2 + \frac{1}{2}(\mathcal{P}^i)^2 - \Pi_\psi \gamma^0(\gamma^i D_i + im - ie\gamma^0 A_0)\psi + \Pi_{\mathcal{G}}\Pi_{\bar{\mathcal{G}}} - \bar{\mathcal{G}}(\partial_i^2 \mathcal{G}) + \frac{1}{2}A_0(\partial_i^2 A_0) + \frac{1}{2}A_i(-\partial_j^2 A_i + 2\partial_i\partial_j A_j)\right]. \tag{3.9}$$

The fields can be expanded in the basis of the eigenfunctions of the operators

$$-\partial_j^2 A_i + 2\partial_i\partial_j A_j = \omega^2 A_i, \qquad \partial_i^2 A_0 = \omega_0^2 A_0, \qquad \partial_i^2 \mathcal{G} = \omega_g^2 \mathcal{G}, \qquad H_D\psi = E_D\psi, \tag{3.10}$$

where H_D is the Dirac Hamiltonian given by

$$H_D = i\gamma^0\gamma^\mu D_\mu - m\gamma^0 \tag{3.11}$$

and $\omega^2, \omega_0^2, \omega_g^2, E_D \geq 0$, by the requirement of positivity of the Hamiltonian.

As we show in detail in the Appendix, this requirement leads to the following most general boundary conditions on the fields:

$$(\vec{A}_\perp + i\vec{F}_{n\perp})(x)|_{\partial M} = U_\perp(x)(\vec{A}_\perp - i\vec{F}_{n\perp})(x)|_{\partial M}, \tag{3.12}$$

$$(A_n + i\partial_i A_i)(x)|_{\partial M} = U_n(x)(A_n - i\partial_i A_i)(x)|_{\partial M}, \tag{3.13}$$

$$(A_0 + i\partial_n A_0)(x)|_{\partial M} = U_0(x)(A_0 - i\partial_n A_0)(x)|_{\partial M}, \tag{3.14}$$

$$(\mathcal{G} + i\partial_n \mathcal{G})(x)|_{\partial M} = U_g(x)(\mathcal{G} - i\partial_n \mathcal{G})(x)|_{\partial M}, \tag{3.15}$$

$$\psi_+(x)|_{\partial M} = U_F(x)\gamma^0\psi_-(x)|_{\partial M}. \tag{3.16}$$

Here, $\forall x \in \partial M$, and we have defined

$$\psi_\pm \equiv \frac{1}{2}(\mathbb{I} \pm \gamma^0\vec{\gamma}\cdot\hat{n})\psi, \qquad F^{(A)}_{\rm in} \equiv \partial_i A_n - \partial_n A_i, \tag{3.17}$$

and the operators $U_\perp$, U_n, U_0, U_g, and U_F satisfy

$$U_\perp^\dagger U_\perp = \mathbb{I}, \quad U_n^\dagger U_n = \mathbb{I}, \quad U_0^\dagger U_0 = \mathbb{I}, \quad U_g^\dagger U_g = \mathbb{I}, \quad U_F^\dagger U_F = \mathbb{I}, \quad [U_F, \gamma^0\vec{\gamma}\cdot\hat{n}] = 0. \tag{3.18}$$

The ghost field can be expanded in a complete orthonormal set of functions $\{H_k(x^0, x^i)\}$ as

$$\mathcal{G}(x^0, x^i) = \sum_k \mathcal{C}_k H_k(x^0, x^i). \tag{3.19}$$

Using the Gauss law (2.12), the momenta (2.7)–(2.8) and (3.3)–(3.4) and the above ghost field expansion, the BRST charge can be written as

$$\hat{\Omega} \equiv G\left(\sum_k \mathcal{C}_k H_k\right) - i\int_M d^d x \Pi_{\bar{\mathcal{G}}}\Pi^0_{\rm gauge}, \tag{3.20}$$

where

$$G\left(\sum_k \mathcal{C}_k H_k\right) \equiv \int_M [\Pi^i_{\rm gauge}(\partial_i - ie\Pi_\psi\psi]\sum_k \mathcal{C}_k H_m(x^0, x^i). \tag{3.21}$$

This BRST charge generates the variation of the fields under which the action (3.1) remains invariant. In this work, we are only interested in the BRST variation of the gauge fields A^i and fermions ψ. Upon imposing the following canonical commutation relations,

$$[\Pi^i_{\rm gauge}(x^0, \vec{x}), A^j(x^0, \vec{y})] = -i\delta^{ij}\delta^d(\vec{x} - \vec{y}), \tag{3.22}$$

$$\{\Pi_\psi(x^0, \vec{x}), \psi(x^0, \vec{y})\} = \delta^d(\vec{x} - \vec{y}), \tag{3.23}$$

the BRST variations of our interest are

$$\delta A^0 = i\epsilon[\hat{\Omega}, A^0] = \epsilon\partial^0\mathcal{G}, \tag{3.24}$$

$$\delta A^i = i\epsilon[\hat{\Omega}, A^i] = \epsilon\partial^i\mathcal{G}, \tag{3.25}$$

$$\delta\psi = i\epsilon\{\hat{\Omega}, \psi\} = -\epsilon\mathcal{G}\psi, \tag{3.26}$$

$$\delta\mathcal{G} = i\epsilon\{\hat{\Omega}, \mathcal{G}\} = 0, \tag{3.27}$$

where ϵ is a Grassmannian number.

IV. BOUNDARY CONDITIONS

The boundary conditions (3.12)–(3.16) which preserve the self-adjointness of the Hamiltonian are not consistent with BRST symmetry. In the following, we show that only a smaller subset of these boundary conditions preserves BRST symmetry.

As we mentioned, the BRST charge $\hat{\Omega}$ in (3.20) generates a global BRST symmetry in the action (3.1). However, in order for $\hat{\Omega}$ to generate the BRST symmetry infinitesimally, all $H_k(x^0, x^i)$ in (3.19) must vanish on ∂M:

$$\mathcal{G}|_{\partial M} = \sum_k \mathcal{C}_k H_k(x^0, x^i)|_{\partial M} = 0. \tag{4.1}$$

This requirement implies that

$$\vec{\nabla}_\perp \mathcal{G}(x)|_{\partial M} = 0, \qquad \partial_0 \mathcal{G}(x)|_{\partial M} = 0. \tag{4.2}$$

Thus, the BRST transformation (3.27) enforces $U_g = -\mathbb{I}$ in (3.15).

A. Allowed boundary conditions on A_μ

From (3.24) and (4.2), it follows that

$$\delta A_0(x)|_{\partial M} = 0. \tag{4.3}$$

Using the above in (3.14), we get

$$[1 + U_0(x)]\delta(\partial_n A_0)(x)|_{\partial M} = \epsilon[1 + U_0(x)]\partial_0 \partial_n \mathcal{G}(x)|_{\partial M} = 0. \tag{4.4}$$

As $\partial_n \mathcal{G}(x)|_{\partial M} \neq 0$ in general, the above implies that BRST symmetry enforces $U_0 = -\mathbb{I}$, and hence the BRST-preserving boundary condition on A_0 is

$$A_0(x)|_{\partial M} = 0. \tag{4.5}$$

For any other boundary condition on A_0, the BRST symmetry will be broken.

From (3.25) and (4.2), it is easy to check that

$$\delta \vec{A}_\perp(x)|_{\partial M} = 0, \qquad \delta \vec{F}_{n\perp}(x)|_{\partial M} = 0. \tag{4.6}$$

Consequently, the BRST variation of (3.12) becomes trivial, and the boundary conditions (3.12) are allowed by BRST symmetry for all $U_\perp$. In a similar fashion, the boundary conditions (3.12) are also not constrained by the BRST symmetry, and any $U_n(x)$ is allowed.

These are the same set of boundary conditions (4.5) and (3.12) that one obtains if the theory is quantized using Dirac constraints in canonical formalism [10,14].

In a system like a topological insulator where boundaries play a vital role, we can further use physical conditions to constrain this allowed set of BRST-preserving boundary conditions. The surface of a topological insulator, unlike the bulk (which is an insulator), behaves like a conductor. Therefore, the tangential component of the electric field must vanish on the boundary of the topological insulator. Then, recalling that A_0 vanishes on the boundary, we need to choose

$$\vec{A}_\perp(x)|_{\partial M} = 0. \tag{4.7}$$

This is one of the allowed boundary conditions from the set (3.12) (for this case, $U_\perp = -\mathbb{I}$), and this ensures that the tangential component of the electric field $\vec{E}_\perp = \partial_0 \vec{A}_\perp - \vec{\nabla}_\perp A_0$ vanishes on the boundary. Also, this is one of the boundary conditions obtained in Ref. [14] using canonical formalism.

B. Fermionic boundary conditions

From (3.26) and (4.1), it follows that

$$\delta\psi(x)|_{\partial M} = 0. \tag{4.8}$$

Using the above in (3.17), it is easy check that

$$\delta\psi_\pm(x)|_{\partial M} = 0. \tag{4.9}$$

A BRST variation of the boundary condition (3.16) is thus trivial, and hence the boundary condition (3.16) is compatible with the BRST symmetry for any choice of U_F. Thus, the BRST symmetry constrains the boundary conditions on the gauge fields A_μ, but it does not constrain the fermionic boundary conditions.

V. (2 + 1)-DIMENSIONAL EXAMPLE

In the following, we consider the $(2+1)$-dimensional case, which is particularly relevant in the context of topological insulators. Consider the $(2+1)$-dimensional manifold

$$\tilde{M} \equiv \{x^0, x^1, x^2 \colon x^1 \leq 0\} \tag{5.1}$$

with spatial boundary

$$\partial\tilde{M} = \{x^0, x^1, x^2 \colon x^1 = 0\}. \tag{5.2}$$

We choose the representation of the Gamma matrices,

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^3, \tag{5.3}$$

with σ^i's the Pauli matrices. It follows then that $\psi_\pm$ are given by

$$\psi_+ = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \qquad \psi_- = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}. \tag{5.4}$$

It is easy to check that the matrix U_F must then take the form

$$U_F = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\tilde{\theta}} \end{pmatrix}, \qquad \theta, \tilde{\theta} \in \mathbb{R}. \tag{5.5}$$

The boundary conditions in (3.16) in this case are simply

$$\psi_1|_{x_1=0} = -ie^{i\tilde{\theta}}\psi_2|_{x_1=0}, \tag{5.6}$$

and the gauge fields satisfy the following boundary conditions:

$$A_0|_{x_1=0} = 0, \qquad A_2|_{x_1=0} = 0. \tag{5.7}$$

The vanishing of A_0 on the boundary $x_1 = 0$ is required by BRST symmetry. However, the condition $A_2|_{x_1=0} = 0$ is one of the many boundary conditions (3.12) that preserves BRST symmetry. We choose this particular boundary condition because it leads to the vanishing of the tangential component of the electric field on the boundary, as it should in a topological insulator.

It is easy to check that

$$A_0^{(k)} = 0, \qquad A_1^{(k)} = a_k k_2 \cos(k_1 x_1)\cos(k_2 x_2),$$
$$A_2^{(k)} = a_k k_1 \sin(k_1 x_1)\sin(k_2 x_2), \tag{5.8}$$

with $a_k \in \mathbb{C}$, satisfy the above boundary conditions and are solutions of the eigenvalue equations

$$-\partial_j^2 A_i^{(k)} + 2\partial_i\partial_j A_j^{(k)} = \omega_k^2 A_i^{(k)}, \qquad \partial_i^2 A_0^{(k)} = \omega_0^2 A_0^{(k)}, \tag{5.9}$$

with

$$\omega_k^2 = k_1^2 + k_2^2, \qquad \omega_0 = 0. \tag{5.10}$$

Thus, the gauge field can be expressed as

$$A_0 = 0, \qquad A_1 = \sum_{k_1,k_2} a_k k_2 \cos(k_1 x_1)\cos(k_2 x_2),$$
$$A_2 = \sum_{k_1,k_2} a_k k_1 \sin(k_1 x_1)\sin(k_2 x_2). \tag{5.11}$$

Demanding reality of the gauge fields yields

$$a_k^* = a_k \Rightarrow a_k \in \mathbb{R}. \tag{5.12}$$

The ghost field can be expanded in the eigenfunctions of the scalar Laplacian

$$H_{\tilde{k}} = e^{i\tilde{k}_2 x_2}\sin(\tilde{k}_1 x_1) \tag{5.13}$$

with eigenvalues

$$\omega_g^2 = \tilde{k}_1^2 + \tilde{k}_2^2. \tag{5.14}$$

Hence, the ghost field can be expressed as

$$\mathcal{G} = \sum_{\tilde{k}_1,\tilde{k}_2} \mathcal{C}_{\tilde{k}} H_{\tilde{k}}, \qquad \mathcal{C}_{\tilde{k}} \in \mathbb{C}. \tag{5.15}$$

A. Eigenstates of the Dirac operator

In this section, we solve for the fermionic edge states in $\tilde{M}$ when the coupling constant g is small. As we have shown in Sec. IV, the BRST symmetry and the self-adjointness of the Hamiltonian yield the same boundary conditions that are obtained from the standard Hamiltonian formalism. Consequently, the edge states that we show to exist here can be obtained using the canonical formalism as well. The existence of the edge states is a physical property of the Yang-Mills-Dirac system and is not an artifact of the BRST formalism in a manifold with boundaries.

We want to consider the interaction of the fermions with photons of very small energies. For such soft photons, we can terminate the sums in (5.11) at small values of k_1, k_2, which in turn imply a small ω_k.

For simplicity, we will assume that $\tilde{\theta} = \pi/2$ in (5.6). With this choice, the fermionic boundary condition (5.6) reduces to

$$\psi_1|_{x_1=0} = \psi_2|_{x_1=0}. \tag{5.16}$$

However, it is not difficult to generalize the analysis to arbitrary $\tilde{\theta}$.

For small gauge coupling constant e, we expand the field ψ in e as

$$\psi = \chi + e\xi + \cdots \tag{5.17}$$

The eigenvalue equation for the Dirac fermions

$$H_D\psi \equiv [i\gamma^0\gamma^i(\partial_i - ieA_i) + eA_0 + m\gamma^0]\psi = E\psi, \qquad E \in \mathbb{R}, \tag{5.18}$$

at order 1, leads to

$$i\gamma^0(\gamma^i\partial_i - im)\chi = E\chi, \tag{5.19}$$

subject to the boundary condition (5.16). It is easy to see that the above has solution

$$\chi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{mx_1 + iEx_2}. \tag{5.20}$$

At order e, the eigenvalue equation (5.18) gives

$$i\gamma^0(\gamma^i\partial_i - im)\xi + \gamma^0\gamma^i A_i\chi = E\xi. \tag{5.21}$$

To solve this, we start by rewriting A_i as

$$A_1 = \sum_{k_1,k_2} \frac{a_k}{4} k_2 (e^{ik_1x_1} + e^{-ik_1x_1})(e^{ik_2x_2} + e^{-ik_2x_2}), \tag{5.22}$$

$$A_2 = -\sum_{k_1,k_2} \frac{a_k}{4} k_1 (e^{ik_1x_1} - e^{-ik_1x_1})(e^{ik_2x_2} - e^{-ik_2x_2}). \tag{5.23}$$

Inserting the ansatz

$$\xi = \sum_{k_1,k_2} (\xi_k^{(1)} e^{(ik_1+m)x_1 + i(k_2+E)x_2} + \xi_k^{(2)} e^{(-ik_1+m)x_1 + i(k_2+E)x_2} + \xi_k^{(3)} e^{(ik_1+m)x_1 + i(-k_2+E)x_2} + \xi_k^{(4)} e^{(-ik_1+m)x_1 + i(-k_2+E)x_2}) \tag{5.24}$$

in (5.21), we obtain

$$\xi_k^{(1)} = -\frac{a_k}{4}(2Ek_2 - 2imk_1 + \omega_k^2)^{-1}\begin{pmatrix} 2Ek_1 + 2imk_2 - \omega_k^2 \\ 2Ek_1 + 2imk_2 + \omega_k^2 \end{pmatrix},$$
$$\xi_k^{(2)} = \frac{a_k}{4}(2Ek_2 + 2imk_1 + \omega_k^2)^{-1}\begin{pmatrix} 2Ek_1 - 2imk_2 + \omega_k^2 \\ 2Ek_1 - 2imk_2 - \omega_k^2 \end{pmatrix},$$
$$\xi_k^{(3)} = -\frac{a_k}{4}(2Ek_2 + 2imk_1 - \omega_k^2)^{-1}\begin{pmatrix} 2Ek_1 - 2imk_2 - \omega_k^2 \\ 2Ek_1 - 2imk_2 + \omega_k^2 \end{pmatrix},$$
$$\xi_k^{(4)} = \frac{a_k}{4}(2Ek_2 - 2imk_1 - \omega_k^2)^{-1}\begin{pmatrix} 2Ek_1 + 2imk_2 + \omega_k^2 \\ 2Ek_1 + 2imk_2 - \omega_k^2 \end{pmatrix}. \quad (5.25)$$

When ω_k is very small, we can set $\omega_k^2 \approx 0$, and hence the above reduces to

$$\xi_k^{(1)} = -\frac{a_k}{4}\frac{Ek_1 + imk_2}{Ek_2 - imk_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_k^{(2)} = \frac{a_k}{4}\frac{Ek_1 - imk_2}{Ek_2 + imk_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\xi_k^{(3)} = -\frac{a_k}{4}\frac{Ek_1 - imk_2}{Ek_2 + imk_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_k^{(4)} = \frac{a_k}{4}\frac{Ek_1 + imk_2}{Ek_2 - imk_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (5.26)$$

Therefore, in the presence of soft photons,

$$\psi = [(e^{mx_1 + iEx_2} + e\sum_{k_1,k_2}(a_k^{(1)}e^{(ik_1+m)x_1 + i(k_2+E)x_2} + a_k^{(2)}e^{(-ik_1+m)x_1 + i(k_2+E)x_2}$$
$$+ a_k^{(3)}e^{(ik_1+m)x_1 + i(-k_2+E)x_2} + a_k^{(4)}e^{(-ik_1+m)x_1 + i(-k_2+E)x_2})]\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(e^2) \quad (5.27)$$

with

$$a_k^{(1)} = -\frac{a_k}{4}\frac{Ek_1 + imk_2}{Ek_2 - imk_1}, \quad a_k^{(2)} = \frac{a_k}{4}\frac{Ek_1 - imk_2}{Ek_2 + imk_1},$$
$$a_k^{(3)} = -\frac{a_k}{4}\frac{Ek_1 - imk_2}{Ek_2 + imk_1}, \quad a_k^{(4)} = \frac{a_k}{4}\frac{Ek_1 + imk_2}{Ek_2 - imk_1}, \quad (5.28)$$

are eigenmodes of (5.18) and satisfy the boundary condition (5.16).

For a sufficiently large mass m, these eigenmodes are exponentially damped in the bulk and are localized near the edge $x_1 = 0$. In real systems, like topological insulators, these modes are presumably amenable to detection.

VI. DISCUSSIONS

The BRST formalism provides a natural framework to quantize gauge theories in the presence of spatial boundaries, which are particularly important in real systems, like topological insulators. We have shown that in a $U(1)$ gauge theory, out of the set of all local boundary conditions on the gauge fields allowed by the self-adjointness of the Hamiltonian, only some preserve BRST symmetry. These BRST-preserving boundary conditions are, in general, consistent with observations in a topological insulator.

The presence of fermionic edge states in the theory is also very interesting from the perspective of a system like a topological insulator. These edge states are expected to assume an important role in the physics at the boundary; it is possible to experimentally verify the presence of these fermions localized at the boundary.

To demonstrate the presence of edge states, in the previous section, we have considered a very simple $(2+1)$-dimensional system with flat boundaries. However, those results can be easily extended to any spacetime dimension and to any curved boundary of codimension 1. Also, we considered the fermions to be massive so that the edge states are protected by the corresponding mass gap. However, one might also consider a gapless system with time-reversal symmetry. There also, we expect to find edge-localized fermions in a similar fashion, though the details in that case will be a bit different.

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APPENDIX A: BOUNDARY CONDITIONS OF THE GAUGE FIELDS

As mentioned in Sec. III, the fields A_i can be expanded in the basis of the eigenfunctions of the operator $\hat{\mathcal{O}} \equiv (-\partial_j^2 + 2\partial_i\partial_j)$. This operator is studied in Ref. [14]. To find the domain of self-adjointness of this operator, we impose that

$$\int_M d^dx[B_i^\dagger(-\partial_j^2 A_i + 2\partial_i\partial_j A_j) - (-\partial_j^2 B_i^\dagger + 2\partial_i\partial_j B_j^\dagger)A_i], \quad \forall\ A_i \in \mathcal{D}_{\hat{\mathcal{O}}}, \qquad B_i \in \mathcal{D}_{\hat{\mathcal{O}}^\dagger} \quad (A1)$$

vanishes if and only if the same boundary conditions are imposed on both A_i and B_i. Now, Eq. (A1) leads to the boundary term

$$\int_{\partial M} d^{d-1}x[B_i^\dagger(-\partial_n A_i + \partial_i A_n) + B_n^\dagger(\partial_i A_i) - (-\partial_n B_i^\dagger + \partial_i B_n^\dagger)A_i - (\partial_i B_i^\dagger)A_n], \tag{A2}$$

which must vanish with the same conditions on A_i and B_i. The most general local boundary conditions for which the above rule is satisfied are

$$(\vec{A}_\perp + i\vec{F}_{n\perp})(x)|_{\partial M} = U_\perp(x)(\vec{A}_\perp - i\vec{F}_{n\perp})(x)|_{\partial M}, \tag{A3}$$

$$(A_n + i\partial_i A_i)(x)|_{\partial M} = U_n(x)(A_n - i\partial_i A_i)(x)|_{\partial M}, \quad x \in \partial M, \tag{A4}$$

with

$$\vec{F}_{n\perp} = \partial_n \vec{A}_\perp - \vec{\nabla}_\perp A_n, \qquad U_\perp^\dagger U_\perp = \mathbb{I} = U_n^\dagger U_n. \tag{A5}$$

Similarly, A_0 can be expanded in the eigenfunctions of $\hat{\mathcal{O}}_0 \equiv \partial_j^2$. The domain of self-adjointness of $\hat{\mathcal{O}}_0$ is obtained by demanding that

$$\int_M d^d x[B_0^\dagger(\partial_j^2 A_0) - (\partial_j^2 B_0^\dagger)A_0], \quad \forall A_0 \in \mathcal{D}_{\hat{\mathcal{O}}_0}, \quad B_0 \in \mathcal{D}_{\hat{\mathcal{O}}_0^\dagger} \tag{A6}$$

vanishes with the same boundary conditions on A_0 and B_0. The above leads to the boundary term

$$\int_{\partial M} d^{d-1}x[B_0^\dagger(\partial_n A_0) - (\partial_n B_0^\dagger)A_0], \tag{A7}$$

which must vanish with the same conditions on A_0 and B_0. It is easy to check that the most general local boundary condition which satisfies the above requirement is

$$(A_0 + i\partial_n A_0)(x)|_{\partial M} = U_0(x)(A_0 - i\partial_n A_0)(x)|_{\partial M}, \quad x \in \partial M, \tag{A8}$$

with $U_0^\dagger U_0 = \mathbb{I}$.

APPENDIX B: FERMIONIC BOUNDARY CONDITIONS

The conventional way to quantize the fermionic field is to expand it in the basis of eigenfunctions of the Dirac Hamiltonian H_D given by

$$H_D = i\gamma^0\gamma^\mu D_\mu + m\gamma^0 \tag{B1}$$

$$= i\gamma^0\gamma^i(\partial_i - ieA_i) + eA_0 + \gamma^0 m. \tag{B2}$$

The domain of self-adjointness of H_D can be obtained by demanding that

$$\int_M d^d x\chi^\dagger H_D\psi - \int_M d^d x(H_D\chi)^\dagger\psi = 0, \ \forall\psi \in \mathcal{D}_{H_D}, \ \chi \in \mathcal{D}_{H_D^\dagger} \tag{B3}$$

if and only if ψ and χ fulfill the same boundary conditions.

We assume that the photon fields are real:

$$A_\mu^\dagger = A_\mu. \tag{B4}$$

Then, Eq. (B3) reduces to

$$i\int_M d^d x[\chi^\dagger\gamma^0\gamma^i\partial_i\psi + (\partial_i\chi)^\dagger\gamma^0\gamma^i\psi] = 0, \tag{B5}$$

which leads to

$$\int_{\partial M} d^{d-1}x\chi^\dagger\gamma^0\vec{\gamma}\cdot\hat{n}\psi = 0. \tag{B6}$$

We define the operators

$$P_\pm \equiv \frac{1}{2}(\mathbb{I} \pm \gamma^0\vec{\gamma}\cdot\hat{n}). \tag{B7}$$

These are projectors, since they satisfy $(P_\pm)^2 = P_\pm$. In terms of these projectors, the above integral can be written as

$$\int_{\partial M} d^{d-1}x\chi^\dagger(P_+ - P_-)\psi = \int_{\partial M} d^{d-1}x\chi^\dagger(P_+^2 - P_-^2)\psi = 0. \tag{B8}$$

Calling $\psi_\pm \equiv P_\pm\psi$, we can further rewrite the above as

$$\int_{\partial M} d^{d-1}x(\chi_+^\dagger\psi_+ - \chi_-^\dagger\psi_-) = 0. \tag{B9}$$

This requirement leads to the domain of self-adjointness of H_D,

$$\mathcal{D}_{H_D} = \{\psi\colon \psi_+|_{\partial M} = U_F\gamma^0\psi_-|_{\partial M}\}, \tag{B10}$$

where the matrix U_F satisfies

$$U_F^\dagger U_F = \mathbb{I}. \tag{B11}$$

Also, as $P_+\gamma^0 = \gamma^0 P_-$ and $P_\pm^2 = P_\pm$, U_F must satisfy

$$[U_F, \gamma^0\vec{\gamma}\cdot\hat{n}] = 0. \tag{B12}$$

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