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Anomaly matching condition in two-dimensional systems

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Based on the Son-Yamamoto relation obtained for the transverse part of the triangle axial anomaly in QCD_4, we derive its analog in a two-dimensional system. It connects the transverse part of the mixed vector-axial current two-point function with the diagonal vector and axial current two-point functions. Being fully nonperturbative, this relation may be regarded as anomaly matching for conductivities or certain transport coefficients depending on the system. We consider the holographic renormalization group flows in holographic Yang-Mills-Chern-Simons theory via the Hamilton-Jacobi equation with respect to the radial coordinate. Within this holographic model, it is found that the renormalization group flows for the following relations are diagonal: the Son-Yamamoto relation and the left-right polarization operator. Thus, the Son-Yamamoto relation holds at a wide range of energy scales.

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I. INTRODUCTION

The usefulness of anomalies is partially related to the fact that they are exact and can be determined at strong coupling. This is a consequence of certain nonrenormalization properties and allows nonperturbative insight. Indeed, the Adler-Bell-Jackiw (ABJ) axial anomaly can be captured perturbatively by the one-loop Feynmann diagram. However, the result is nonperturbative, being exact from low to high energies since the anomaly reflects the spectral flow at all scales. Recently, Son and Yamamoto derived an anomaly matching condition which relates the $U(1)^3$ AVV triangle anomaly [1], Fig. 1, to the two-point VV and AA current functions, where V refers to the vector current and A refers to the axial current. The result was obtained via holography and can be regarded as a nonperturbative exact relation between three- and two-point current functions. They used a five-dimensional Yang-Mills action of the holographic dual of QCD and considered a holographic mechanism of chiral symmetry breaking via the boundary conditions for the gauge fields in the infrared. This class of holographic theories incorporates a bottom-up AdS/QCD inspired models and the top-down Sakai-Sugimoto models.

In this paper, we consider the holographic dual of $(1+1)$-dimensional systems given by a three-dimensional action and derive the analog of Son-Yamamoto relation, Fig. 2. Like in holographic QCD, dual action can be considered as the world volume action on the probe flavor brane, and therefore it involves the 3D Yang-Mills and Chern-Simons terms. The duals of the $(1+1)$-dimensional systems in this approach have been considered in Refs. [2,3]. We will discuss the case of the single flavor in the boundary non-Abelian gauge theory at large N. The cigar geometry implies that, like in the $4+1$ case, we have to consider left and right copies of the gauge group in the 3D bulk theory reflecting the global $U(1)_L \times U(1)_R$ symmetry at the boundary. The 2D QCD enjoys the chiral symmetry breaking [4] via chiral condensate formation. There is the pionlike degree of freedom of which the mass is related to the fermion mass via the analog of the GOR relation.

There is some important difference between the $4+1$ and $2+1$ bulk gauge theories. The Son-Yamamoto relation has been derived in five-dimensional theory taking into account that the contribution of Chern-Simons (CS) terms is suppressed by the large 't Hooft coupling. Therefore, it was possible to first consider the equation of motion without the CS term, derive the constant Wronskian condition, and then treat the CS term as a kind of perturbation. The situation in $2+1$ is different, and there is no suppression of the CS term anymore, which is crucial for imposing the self-consistent boundary conditions [2]. Therefore, we have to consider the equation of motion including the CS terms which have the opposite signs for the left and right fields. Therefore one can not obtain the constant Wronskian condition analytically in $1+1$ case.

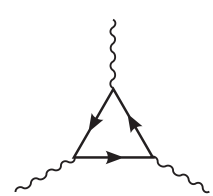

FIG. 1. Axial (ABJ) anomaly in $(3+1)$ dimensions: pion decay $\pi_0 \to 2\gamma$. Solid lines represent chiral fermions, and wavy lines represent U(1) gauge bosons.

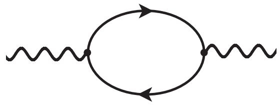

FIG. 2. Parity-violating anomaly in $(1+1)$ dimensions: mass generation $m\bar{\psi}\psi$. Solid lines represent chiral fermions, and wavy lines represent U(1) gauge bosons.

However, the numerical analysis of the equations of motion demonstrates that the Wronskian exhibits a plateau in the very wide interval of the radial holographic coordinate and the transition to the plateau is very sharp. One could also have in mind the formal regime when Yang-Mills (YM) terms dominate. Therefore, we can explore the constant Wronskian condition with some reservations in the $2+1$ case as well.

It was shown in Ref. [1] that the Son-Yamamoto (SY) relation is consistent with the Vainshtein relation [5] for the magnetic susceptibility of the quark condensate in QCD introduced in Ref. [6]. However, in two dimensions, the operator product expansion (OPE) for the vector-axial correlator trivially reduces the four-fermion operator to the square of the chiral condensate due to the 2D chiral algebra. As a result, we obtain from the Son-Yamamoto relation an estimate for the pion decay constant. We note that it is derived in the region when the application of the low-energy theory is questionable. Hence, this result should be taken with some reservation and deserves the additional study.

An additional question concerns renormalization group (RG) flow of our holographic model. This question is related to renormalization and regularization of effective theories in holography, which were solved along two avenues. First is the method of standard holographic renormalization that involves the cancellation of all cutof-related divergences from the gravity on-shell action by adding the counterterms on the cutoff boundary surface and the subsequent removal of cutoff [7]. Holographic renormalization has been used in the calculation of two-point functions in deformed conformal field theory (CFT) [7]. In parallel development, the Hamilton-Jacobi equation was used for renormalization in order to separate terms in the bulk on-shell action, which can be written as local functions of boundary data. The remaining nonlocal expression was identified, according to the AdS/CFT prescription, with the generating functional of a boundary field theory [8,9]. In the Hamilton-Jacobi equation, the bulk radial coordinate is treated as the time variable, which is consistent with holographic identification of radial coordinate with the RG energy scale. The second approach provides correct results for anomalies and gives a simple description of RG flow in deformed CFTs [10]. We apply the Hamilton-Jacobi equation in the bulk theory to the Yang-Mills-Chern-Simons holographic action similar to Ref. [11] and demonstrate that the SY relation is diagonal with respect to holographic RG flow.

The paper is organized as follows. We derive the two-dimensional Son-Yamamoto relation in Sec. II. In Sec. III, we check the Son-Yamamoto relation in the small- and large-Q^2 limits and obtain an estimate for the pion decay constant. In Sec. IV, we demonstrate using the Hamilton-Jacobi equations in the bulk theory that the Son-Yamamoto relation is diagonal under the RG flows. Section V is devoted to the comparison of our results for the $1+1$-dimensional Son-Yamamoto relation with that obtained in the $3+1$-dimensional QCD. The results are summarized in the Conclusion, and technical details are collected in Appendixes A and B.

II. MODEL AND SON-YAMAMOTO RELATION

We consider chiral dynamics in two dimensions. Chiral symmetry is $U(1)_L \times U(1)_R$, which corresponds to the conserved left- and right-handed currents. According to AdS/CFT duality, there are left- A_L and right-handed A_R gauge fields in a three-dimensional dual model. The 3D dual action involves three-dimensional Maxwell theory and the topological Chern-Simons term

$$\begin{aligned} S &= S_M + S_{\mathrm{CS}} \\ &= S_M(A_L) + S_M(A_R) + S_{\mathrm{CS}}(A_L) - S_{SC}(A_R), \end{aligned} \quad (1)$$

where S_M and S_{CS} are defined as

$$S_M(\mathcal{A}) = \int d^2x dz \left(f(z)\mathcal{F}_{z\mu}^2 - \frac{1}{2g(z)}\mathcal{F}_{\mu\nu}^2 \right) \quad (2)$$

and [2,12]

$$S_{\mathrm{CS}}(\mathcal{A}) = \kappa \int d^2x dz (\mathcal{A} * \mathcal{F}) \quad (3)$$

with [2,12]

$$\kappa = \frac{N_c}{4\pi}, \quad (4)$$

and the dual field strength is $*\mathcal{F}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\mathcal{F}^{\nu\lambda}$.

The IR brane is located at $z=0$, and the UV boundary of the asymptotic 3 dimensional anti-de Sitter space (AdS_3) is located at $z=z_0$. It is convenient to use vector V and axial A gauge fields

$$A_L = V + A, \qquad A_R = V - A \quad (5)$$

which obey Neumann and Dirichlet boundary conditions in the IR, respectively,

$$\mathrm{IR:}\ \partial_z V_\mu(z=0) = 0, \qquad A_\mu(z=0) = 0, \quad (6)$$

and $V(-z) = V(z)$ is parity even, and $A(-z) = -A(z)$ is parity odd. Making use of the decomposition (5), the Maxwell and Chern-Simons terms in the action (1) are given by

$$S_M = \int d^2xdz\left(f(z)^2(F^2_{Vz\mu}+F^2_{Az\mu})-\frac{1}{2g(z)^2}(F^2_{V\mu\nu}+F^2_{A\mu\nu})\right) \tag{7}$$

$$S_{\rm CS} = 2\kappa\int d^2xdz(V_\mu * F_{A\mu} + A_\mu * F_{V\mu}). \tag{8}$$

We will work in the radial gauge, $V_z = A_z = 0$, and assume there is a translation invariance along the boundary "UV" brane and perform the Fourier transform for gauge fields,

$$V_\mu(x,z) = \int \frac{d^2q}{(2\pi)^2}\mathrm{e}^{-iqx}V(q,z), \tag{9}$$

and the same for the axial field A_μ. Substituting these expressions into the action, we can write down the holographic Maxwell and Chern-Simons terms in 3D explicitly,

$$S_M = \int\frac{d^2q}{(2\pi)^2}dz\bigg(f(z)^2((\partial_zV_\mu)^2+(\partial_zA_\mu)^2) - \frac{1}{2g(z)^2}(F^2_{V\mu\nu}+F^2_{A\mu\nu})\bigg), \tag{10}$$

$$S_{\rm CS} = 2\kappa\int\frac{d^2q}{(2\pi)^2}dz\varepsilon^{\mu\nu}(\partial_zA_\mu V_\nu + \partial_zV_\mu A_\nu), \tag{11}$$

where we used the convention [13] $\varepsilon^{z\mu\nu}\equiv\varepsilon^{\mu\nu}$. Here, $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ is the two-dimensional antisymmetric symbol, $\varepsilon_{01} = 1 = -\varepsilon^{01}$, which obeys

$$\varepsilon^{\mu\lambda}\varepsilon_{\nu\rho} = -\delta^\mu_\nu\delta^\lambda_\rho + \delta^\mu_\rho\delta^\lambda_\nu, \qquad \varepsilon^{\mu\nu}\varepsilon_{\nu\rho} = \delta^\mu_\rho. \tag{12}$$

Following Son and Yamamoto [1], we are interested in the transversal part of the correlators. Further, for short, we will omit perpendicular projectors $P^\perp_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$ in the expressions for gauge fields V and A. However, it can be easily reinstated in the resulting formulas by substituting $\delta_{\mu\nu}\to P^\perp_{\mu\nu}$. We perform the field decomposition

$$V_\mu(q) = V(q,z)V_{0\mu}(q), \qquad A_\mu(q) = A(q,z)A_{0\mu}, \tag{13}$$

where $V_{0\mu}$ and $A_{0\mu}$ are the sources of the corresponding boundary currents. We require the Dirichlet boundary conditions in the UV,

$$\mathrm{UV:}\;\; V(q,z)|^{z=z_0} = 1, \qquad A(q,z)|^{z=z_0} = 1, \tag{14}$$

and therefore sources coincide with the bulk gauge fields at the boundary. According to the AdS/CFT prescription to obtain correlation functions for currents, we need to vary the action evaluated on the classical solutions with respect to the corresponding sources of the currents, $V_{0\mu}$ and $A_{0\mu}$.

Next, let us remind the reader of the observation made by Son and Yamamoto, which made it possible to derive the relation between three- and two-point functions [1]. In our case, we will obtain the relation between diagonal and mixed two-point functions. The linearized equation of motion in the pure Maxwell theory is

$$\partial_z(f^2\partial_zV) - \partial_\mu\left(\frac{1}{g^2}\partial_\mu V\right) = 0, \tag{15}$$

which, due to the translation invariance along the boundary direction, is

$$\partial_z(f^2(z)\partial_zV(q,z)) + \frac{q^2}{g^2(z)}V(q,z) = 0. \tag{16}$$

Here, we did not take into account the Chern-Simons term. The same equation of motion is satisfied by the axial gauge field; i.e., we have

$$V'' + \frac{\partial f^2}{f^2}V' + \frac{q^2}{g^2}V = 0 \tag{17}$$

$$A'' + \frac{\partial f^2}{f^2}A' + \frac{q^2}{g^2}A = 0. \tag{18}$$

Since V and A are linearly independent solutions of the same equation, we have [1]

$$V(q,z)\partial A(q,z) - A(q,z)\partial V(q,z) = W(q)\mathrm{e}^{-\int\partial f^2/f^2}, \tag{19}$$

which can be written as

$$f^2(z)(V(q,z)\partial_zA(q,z) - A(q,z)\partial_zV(q,z)) = W(q), \tag{20}$$

where $W(q)$ is a z-independent Wronskian. The independence on z of the combination in Eq. (20) is crucial to obtaining the Son-Yamamoto relation, and it is responsible for the unique properties of the RG equations for the diagonal and mixed correlators. The relation for Wronskian (20) is obtained in the pure Maxwell theory.

In Appendix B, we include the Chern-Simons term and regulate the Maxwell-Chern-Simons theory using dimensional regularization. We show numerically that the logarithmic divergences characteristic of the $(1+1)$-dimensional boundary theory [2] are regularized; solutions for the gauge fields $V(Q,z)$ and $A(Q,z)$ converge in the UV. It justifies the use of the finite boundary conditions in Eq. (14). Furthermore, the Wronskian for the regulated Maxwell-Chern-Simons theory has a plateau starting from some radial coordinate $z > z_p$. Therefore, it is legitimate to take the z-independent Wronskian for large Q^2 or when the coefficient in front of the Chern-Simons term is small.

A. Diagonal correlators

Now, we are ready to obtain two-point diagonal correlators for the vector $\langle V_\mu V_\nu\rangle$ and axial $\langle A_\mu A_\nu\rangle$ fields in $(1+1)$ dimensions. Varying the action twice with respect to the boundary value $V_{0\mu}$, we obtain the vector current two-point correlator

$$\langle j_\mu(q) j_\nu(-q)\rangle = \int d^2x e^{iqx} \langle j_\mu(x) j_\nu(0)\rangle \tag{21}$$

$$\frac{\delta^2 S_{\rm YM}}{\delta V_{0\mu}(q)\delta V_{0\nu}(-q)} = \langle j_\mu(q) j_\nu(-q)\rangle, \tag{22}$$

where the Yang-Mills action is given by Eq. (10). Integrating the first term of Eq. (10) by parts, we obtain

$$S_{\rm YM} = 2\int \frac{d^2q}{(2\pi)^2} f^2(z) \mathcal{A}_\mu(q,z)\partial_z \mathcal{A}^\mu(q,z)\Big|_0^{z=z_0}, \tag{23}$$

where $\mathcal{A}$ stands either for V or A, and the action is evaluated at the solutions (18). Varying Eq. (23) twice with respect to the boundary values V_0 (22),

$$\langle j_\mu j_\nu\rangle = 2f^2(z)V(q,z)V'(q,z)|^{z=z_0}\delta_{\mu\nu}, \tag{24}$$

where we introduced the notation $V' = \partial_z V$. Substituting the boundary conditions (14), we obtain for the two-point correlation functions

$$\langle j_\mu j_\nu\rangle = 2f^2 V'|^{z=z_0}\delta_{\mu\nu} \equiv \Pi_V(q)\delta_{\mu\nu}, \tag{25}$$

$$\langle j^5_\mu j^5_\nu\rangle = 2f^2 A'|^{z=z_0}\delta_{\mu\nu} \equiv \Pi_A(q)\delta_{\mu\nu}, \tag{26}$$

where we introduced (dimensionless) polarization operators Π_V and Π_A. Therefore, the polarization operators are given by

$$\Pi_V = 2f^2V'|^{z=z_0}, \qquad \Pi_A = 2f^2A'|^{z=z_0}. \tag{27}$$

Equation (27) represents the known expression for a diagonal conductivity obtained from the Kubo formula. From Eq. (23), the diagonal current-current correlator is given by

$$\langle j_i j_j\rangle = 2f^2 \frac{\partial_z \mathcal{A}_i}{\mathcal{A}_i}\Big|^{z=z_0}\delta_{ij}, \tag{28}$$

where via the Kubo formula the relation to the conductivity σ_{ij} is $\langle j_i j_j\rangle = \sigma_{ij}$.

Using the expression for the Wronskian (20) and the boundary conditions (14), we obtain the relation

$$f^2(VA' - AV') = \frac{1}{2}(\Pi_A - \Pi_V) = W(q), \tag{29}$$

which we use further. Since the combination in Eq. (20) does not depend on z, it can be estimated at any point, for example, at $z = z_0$ where the polarization operators are defined by Eq. (27). The Wronskian equation (29) is the crucial formula to establish a relation between diagonal and mixed current correlators.

B. Mixed correlator

Now, our aim is to obtain the mixed correlator for axial and vector fields $\langle V_\mu A_\nu\rangle$. It is easy to see that the only contribution to this correlator function is coming from the Chern-Simons term (11). After Fourier transformation, the Chern-Simons term (11) can be rewritten as

$$S_{\rm CS} = 2\kappa\int \frac{d^2q}{(2\pi)^2} dz\varepsilon^{\rho\sigma}(\partial_z A_\rho(-q)V_\sigma(q) - \partial_z V_\sigma(q)A_\rho(-q)). \tag{30}$$

As in Refs. [1,14], we can add the surface term to the action,

$$\delta S_{\rm CS} = -2\kappa\int \frac{d^2q}{(2\pi)^2} dz\varepsilon^{\rho\sigma}\partial_z(A_\rho(-q)V_\sigma(q)), \tag{31}$$

which is equivalent to the gauge transformation done in Ref. [1]. The Chern-Simons term becomes $S_{\rm CS} + \delta S_{\rm CS} \to S_{\rm CS}$,

$$S_{\rm CS} = -4\kappa\int \frac{d^2q}{(2\pi)^2} dz\varepsilon^{\rho\sigma}A_\rho(-q)\partial_z V_\sigma(q). \tag{32}$$

Varying twice the (new) Chern-Simons term with respect to the boundary fields

$$\frac{\delta^2 S_{\rm CS}}{\delta V_{0\mu}\delta A_{0\nu}} = \langle j_\mu j^5_\nu\rangle, \tag{33}$$

we obtain

$$\begin{aligned}\frac{\delta^2 S_{\rm CS}}{\delta V_{0\mu}\delta A_{0\nu}} &= \langle j_\mu j^5_\nu\rangle \\ &= -2\kappa\varepsilon^{\nu\mu}AV|^{z_0} + 2\kappa\varepsilon^{\nu\mu}\int^{z_0} dz(A'V - V'A) \\ &\equiv \frac{1}{2\pi}\varepsilon^{\mu\nu}w_T(q),\end{aligned} \tag{34}$$

where we introduced a (dimensional) transversal part of the vector-axial current correlator w_T. Therefore, we have

$$w_T = 4\pi\kappa - 4\pi\kappa\int^{z_0} dz(A'V - V'A). \tag{35}$$

From Eq. (29), it can be written as

$$w_T(Q^2) = 4\pi\kappa - 4\pi\kappa\int_0^{z_0}\frac{dz}{2f^2(z)}(\Pi_A(Q^2) - \Pi_V(Q^2)), \tag{36}$$

where we used and $Q^2 = -q^2$. In the context of the QCD$_2$, $4\pi\kappa = N_c$ (4). The relation between the mixed and diagonal correlators (36) is the $(1+1)$-dimensional analog of the Son-Yamamoto relation, which was originally obtained for QCD in $(3+1)$ dimensions [1]. One can think of the Son-Yamamoto relation (36) as the expansion in the large Q^2. The first term [normalized by Eq. (34) to be equal to $4\pi\kappa = N_c$] is the perturbative contribution of the axial anomaly, which is obtained from perturbation theory loop calculation. In QCD, the integral of the metric factor $\int 1/f^2$ produces $1/f_\pi^2$ where f_π is the pion decay constant [11].

We rewrite the Son-Yamamoto relation in the form

$$(\mathrm{SY}) = w_T - 4\pi\kappa + 4\pi\kappa \int^{z_0} \frac{dz}{2f^2(z)}(\Pi_A - \Pi_V) = 0. \tag{37}$$

This relation holds for any metric factors $f(z)$ and $g(z)$.

Generally, we decompose the axial anomaly as

$$\langle j_\mu j_\nu^5 \rangle = \frac{1}{2\pi} P_\mu^{\alpha\perp}(P_\nu^{\beta\perp} w_T + P_\nu^{\beta\parallel} w_L)\varepsilon_{\alpha\beta}, \tag{38}$$

where the transverse and longitudinal projection tensors are $P_\mu^{\alpha\perp} = \eta_\mu^\alpha - q_\mu q^\alpha/q^2$ and $P_\mu^{\alpha\parallel} = q_\mu q^\alpha/q^2$, respectively. In Eq. (38), the perturbative w are given by [13]

$$w_T = 4\pi\kappa = N_c, \qquad w_L = 8\pi\kappa = 2N_c. \tag{39}$$

The same relation $w_L = 2w_T$ holds in QCD_4.

We give another representation for the Son-Yamamoto relation through the left-right correlator. The left-right correlator $\langle LR \rangle$, which is the measure of the chiral symmetry breaking, can be expressed through the diagonal correlators $\langle VV \rangle$ and $\langle AA \rangle$ as

$$\Pi_{LR} = \Pi_A - \Pi_V. \tag{40}$$

Using the definition of w_T (38), we rewrite the Son-Yamamoto relation (36) as

$$\langle j_\mu^L j_\nu^R \rangle^\perp = P_\mu^{\alpha\perp} P_\nu^{\beta\perp} \varepsilon_{\alpha\beta}\left(4\kappa - 4\kappa \int_0^{z_0} \frac{dz}{2f^2}\Pi_{LR}\right), \tag{41}$$

where $j_\mu^L = \bar{\psi}_L \gamma_\mu \psi_L$ is the left-handed current and $j_\mu^R = \bar{\psi}_R \gamma_\mu \psi_R$ is the right-handed current. We rewrite Eq. (41) in more physical terms,

$$\langle j_\mu^L j_\nu^R \rangle^\perp = P_\mu^{\alpha\perp} P_\nu^{\beta\perp} \varepsilon_{\alpha\beta} \frac{N_c}{\pi}\left(1 - \frac{\Pi_{LR}}{f_\pi^2}\right), \tag{42}$$

where we introduced as in Ref. [1] the pion decay constant $\frac{1}{f_\pi^2} = \int \frac{dz}{2f^2(z)}$. The pion decay constant f_π can be obtained from the longitudinal part of the axial current correlator w_L [1]. The first term in Eq. (42) is a perturbative contribution,

$$\langle j_\mu^L j_\nu^R \rangle_{PT}^\perp = -\varepsilon_{\mu\lambda}(q_\nu q^\lambda - \eta_\nu^\lambda)\frac{N_c}{\pi q^2}, \tag{43}$$

where the pole $1/q^2$ corresponds to the physical propagation of the single effective bosonic degree of freedom which is massless in the absence of electromagnetic interaction. In the Schwinger model, the intermediate boson becomes massive due to the electric field $\tilde{F}$ which is responsible for the chiral anomaly $\partial^\mu j_\mu^5 = \frac{N_c}{\pi}\tilde{F}$. The residue of the pole in the correlation function of Eq. (43) is given by the coefficient of chiral current anomaly N_c/π. The massless pole $1/q^2$ with finite residue determined by the chiral anomaly is obtained in the Schwinger model in Ref. [13] and in the two-dimensional massless QCD_2 [4]. The second term in Eq. (42) proportional to Π_{LR} is a nonperturbative correction. This is a new relation which is not known in field theory. It shows that the transverse part of the chiral anomaly has a dynamical nature rather than a topological one. It can be regarded as an anomaly matching condition for resonances as an analog of that for the massless excitations in QCD_2 (Sec. IV).

In what follows, we use the two representations of the Son-Yamamoto relation, Eqs. (37) and (41).

III. CHECKING THE SON-YAMAMOTO RELATION. THE PION DECAY CONSTANT

Let us check if the Son-Yamamoto relation (37) is satisfied in a model-independent setting. To estimate the individual two-point current correlators, we will consider the two opposite limits of small and large momenta Q^2, where some simplifications can be done. Also, we will make an estimate for the decay constant.

A. Regime of small Q^2

First, we consider the limit of small $Q^2 \ll \Lambda^2$. In this case, we estimate the Son-Yamamoto relation at the point $z_0 \to 0$. In the next section, we associate the UV boundary cutoff z_0 with the RG scale in the Hamilton-Jacobi equation. Therefore, the limit when the UV cutoff is taken to be zero corresponds to the field theory in the regime of the low energy/momentum. In the limit $z_0 = 0$, the different boundary conditions (6) enable us to simplify the holographic action. As discussed in Ref. [11], in the Yang-Mills action, we can neglect $\partial_z V_\mu = 0$; however, we approximate $\partial_z A_\mu = \frac{A_\mu}{z_0}$. Therefore, we can write to the leading order

$$S_{\mathrm{YM}} = z_0 \int d^2x \left(\frac{f^2}{z_0^2}A_\mu^2 - \frac{1}{2g^2}F_{V\mu\nu}^2\right), \tag{44}$$

which gives

$$\Pi_A = \frac{2f^2(z_0)}{z_0}, \qquad \Pi_V = 0, \tag{45}$$

and together with the integral,

$$\int \frac{dz}{2f^2}(\Pi_A - \Pi_V) = \frac{z_0}{2f^2(z_0)} \frac{2f^2(z_0)}{z_0} = 1. \tag{46}$$

In the Chern-Simons action (8), (30) and the boundary term (31), we can neglect the term $\varepsilon^{\mu\nu\lambda} A_\mu F_{V\nu\lambda}$ but again take $\partial_z A_\mu = \frac{A_\mu}{z_0}$ in the term $\varepsilon^{\mu\nu\lambda} V_\mu F_{A\nu\lambda}$. Approximating the integral over z, we have to leading order

$$\begin{aligned} S_{\mathrm{CS}} &= 2\kappa \int d^2x \varepsilon^{\mu\nu} A_\mu V_\nu, \\ \delta S_{\mathrm{CS}} &= -2\kappa \int d^2x \varepsilon^{\mu\nu} A_\mu V_\nu; \end{aligned} \tag{47}$$

thus, from Eq. (34),

$$w_T = 0. \tag{48}$$

Combining together Eqs. (46) and (48), the Son-Yamamoto relation (37) is satisfied at $z_0 = 0$; i.e., it holds for small Q^2.

B. Regime of large Q^2

Next, we consider the opposite limit $Q^2 \gg \Lambda^2$, where we can use the operator product expansion. From the Son-Yamamoto relation, we will make an estimate for the decay constant.

In two dimensions, the field dimensions are given as follows: $[\psi] = \sqrt{E}$ for the fermion field; $[F] = E$, $[A] = 1$ for the gauge field; $[g] = E$ for the coupling between fermion and gauge fields $[f_\pi] = 1$ for the decay constant. The anomalous divergence of the axial current is

$$\partial_\mu j_5^\mu = 4\kappa g \tilde{F}, \tag{49}$$

where the dual field strength equals the 2D electric field $\tilde{F} = \frac{1}{2}\varepsilon_{\mu\nu} F^{\mu\nu} = E$ which is the pseudoscalar. The Dirac matrices in the 2×2 chiral representation are given by the Pauli matrices [13,15]

$$\gamma_0 = \sigma_1, \qquad \gamma_1 = -i\sigma_2, \qquad \gamma_5 = \gamma_0\gamma_1 = \sigma_3, \tag{50}$$

where $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ with $i, j, k = 1, 2, 3$. A special property of the gamma matrices in two dimensions is [13]

$$\gamma^\mu \gamma_5 = \gamma_\nu \varepsilon^{\nu\mu}, \tag{51}$$

which gives for the spin-operator $\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$$\sigma_{\mu\nu} = i\gamma_5 \varepsilon_{\mu\nu}. \tag{52}$$

This property enables us to make significant simplifications in the diagrams of the OPE in the two dimensions, which follows next.

We check the Son-Yamamoto relation at large virtualities and obtain the result for the decay constant. As in Ref. [1], we compare the OPE and the Son-Yamamoto relation for the two-point left-right current correlator. Diagrams contributing to the $\langle j_\mu j_\nu^5 \rangle$ in the OPE are the fermion loops, which are open on two sides and have insertions of the (chiral) scalar $\langle \bar\psi\psi \rangle$ and the spin-chiral $\langle \bar\psi \sigma_{\mu\nu} i\gamma_5 \psi \rangle$ condensates with spin operator $\sigma_{\mu\nu}$, and different arrangements of a photon line in the fermion loop are possible. Therefore, on one hand, the OPE is written as

$$\langle j_\mu^L j_\nu^R \rangle = \frac{1}{2}\langle j_\mu j_\nu^5 \rangle = P_\mu^{\alpha\perp} P_\nu^{\beta\perp} \left(4\kappa\varepsilon_{\alpha\beta} + \frac{2g^2}{Q^2} O_{\alpha\beta} \right), \tag{53}$$

where the operator

$$O_{\alpha\beta} = \frac{\langle (\bar\psi\gamma_\alpha\gamma_5\psi)(\bar\psi\gamma_\beta\psi) \rangle}{Q^2} \tag{54}$$

is the four-fermion operator. Using the Fierz transformation and in the large-N_c limit where the four-fermion operator factorizes, we have in the leading order

$$O_{\alpha\beta} = \frac{\langle \bar\psi\psi \rangle \langle \bar\psi \sigma_{\alpha\beta} i\gamma_5 \psi \rangle}{Q^2} = -\frac{\langle \bar\psi\psi \rangle^2}{Q^2} \varepsilon_{\alpha\beta}, \tag{55}$$

where we simplified the spin-chiral condensate with the help of Eq. (52). The leading-order OPE for the left-right current correlator is given by the operator dimension 2,

$$\langle j_\mu^L j_\nu^R \rangle^\perp = P_\mu^{\alpha\perp} P_\nu^{\beta\perp} \varepsilon_{\alpha\beta} \left(4\kappa - \frac{2g^2}{Q^4} \langle \bar\psi\psi \rangle^2 \right). \tag{56}$$

On the other hand, the Son-Yamamoto relation is given by Eq. (41),

$$\langle j_\mu^L j_\nu^R \rangle^\perp = P_\mu^{\alpha\perp} P_\nu^{\beta\perp} \varepsilon_{\alpha\beta} \left(4\kappa - 4\kappa \int_0^{z_0} \frac{dz}{2f^2} \Pi_{LR} \right). \tag{57}$$

The leading term in the OPE for Π_{LR} is dimension-2 operator [16]

$$\Pi_{LR} = -\frac{g^2}{Q^2} \frac{\langle (\bar\psi_L \gamma_\mu \psi_L)(\bar\psi_R \gamma_\mu \psi_R) \rangle}{Q^2} = \frac{2g^2}{Q^4} \langle \bar\psi\psi \rangle^2. \tag{58}$$

Comparing terms proportional to $\langle \bar\psi\psi \rangle^2$ in Eqs. (56) and (57), we find

$$\frac{4\kappa}{f_\pi^2} = 1, \tag{59}$$

where we made the following identification of the integral with the decay constant [1]

$$\frac{1}{f_\pi^2} = \int^{z_0} \frac{dz}{2f^2(z)}. \tag{60}$$

Equation (59) relies completely on the Son-Yamamoto relation. We consider it not as an exact result but as an estimate for the f_π, because, as discussed for QCD_4 in Ref. [1], the Son-Yamamoto equation does not provide a complete match for resonances at large virtualities. In the QCD_2, the Chern-Simons κ is proportional to N_c (4). Therefore, we have from Eq. (59)

$$f_\pi^2 \sim N_c, \tag{61}$$

which agrees with the estimate done in the weak coupling regime of the 't Hooft solution $N_c \to \infty$ and $g^2 N_c = \mathrm{const}$ for QCD_2 by Zhitnitsky in Ref. [4].

Also, we can check the Son-Yamamoto relation using the parallel component. The OPE for the parallel component is given by Fig. 1 in Ref. [4] and includes the operator of dimension 2,

$$\begin{aligned}\langle j_\mu j_\nu^5\rangle^\parallel &= P_\mu^{\alpha\perp} P_\nu^{\beta\parallel}\left(4\kappa\varepsilon_{\alpha\beta} - \frac{2m\langle\bar\psi\sigma_{\alpha\beta}i\gamma_5\psi\rangle}{Q^2}\right)\\ &= P_\mu^{\alpha\perp} P_\nu^{\beta\parallel}\varepsilon_{\alpha\beta}\left(4\kappa + \frac{2m\langle\bar\psi\psi\rangle}{Q^2}\right).\end{aligned} \tag{62}$$

On the other hand, expanding the pion pole propagator [4,13], we have

$$\frac{f_\pi^2}{Q^2+m_\pi^2} = \frac{f_\pi^2}{Q^2}\left(1-\frac{m_\pi^2}{Q^2}\right). \tag{63}$$

Comparing Eqs. (62) and (63), we obtain

$$f_\pi^2 m_\pi^2 = -2m\langle\bar\psi\psi\rangle, \tag{64}$$

which is the Gell-Mann-Oakes-Renner (GOR) relation; i.e., we trivially satisfy the Son-Yamamoto relation. Using that $m_\pi^2 \sim m + 1/N_c$ and the GOR relation, we find the behavior of the chiral $\langle\bar\psi\psi\rangle$ and the spin-chiral $\langle\bar\psi\sigma_{\mu\nu}i\gamma_5\psi\rangle$ condensates at $N_c \to \infty$,

$$\langle\bar\psi\psi\rangle \sim N_c, \qquad \langle\bar\psi\sigma_{\mu\nu}i\gamma_5\psi\rangle \sim N_c, \tag{65}$$

where we used $\langle\bar\psi\sigma_{\mu\nu}i\gamma_5\psi\rangle = -\varepsilon_{\mu\nu}\langle\bar\psi\psi\rangle$. The estimate for the chiral condensate agrees with the one for the massless QCD_2 in Ref. [4]. Due to the chiral anomaly $\partial j^5 \sim \tilde F$, the vacuum of QCD_2 exhibits chiral symmetry breaking (CSB) $\langle\bar\psi\psi\rangle \neq 0$ and there exit the massless Goldstone boson which is a pion with pion decay constant and the superfluid pairing. In the Schwinger model, this corresponds to the chiral bosonization and superfluid pairing of bosons composed of fermion pairs [13].

Our calculations for the parallel component rely on the pion pole dominance—saturation of the two-point correlators by the pion pole contribution—valid at small Q^2 [2]. Here, we analytically continue it to the large Q^2.

IV. SUM RULES FOR QCD_2

Here, we summarize the sum rules for the resonances. In the large-N_c limit, besides the pion state, a tower of resonances with the decay widths $\Gamma \sim 1/N_c$ is defined. The matrix elements of vector and axial currents between the vacuum and one-particle states (a pion π, a vector meson V_i, or an axial-vector meson A_j) are fixed by the nonperturbative correction in the Son-Yamamoto relation (37) and Eq. (42). Derivation is identical to that for QCD_4 in Ref. [1], and therefore we omit it. The matrix elements are

$$\langle 0|j_\mu(0)|\pi(q)\rangle = iq^\nu \frac{N_c}{\pi f_\pi}\varepsilon_{\mu\nu}, \tag{66}$$

$$\langle 0|j_\mu(0)|A_j(q,\epsilon)\rangle = -\epsilon^\alpha\left(\eta_\mu^\beta - \frac{q_\mu q^\beta}{m_{A_j}^2}\right)\frac{N_c}{2\pi f_\pi^2} g_{A_j}\varepsilon_{\alpha\beta}, \tag{67}$$

$$\begin{aligned}&\langle 0|j_\mu^5(0)|V_i(q,\epsilon)\rangle\\ &\quad= -\epsilon^\alpha\left[\left(\eta_\mu^\beta - \frac{q_\mu q^\beta}{m_{V_i}^2}\right)\frac{N_c}{2\pi f_\pi^2} g_{V_i} - \frac{q_\mu q^\beta}{m_{V_i}^2} f_\pi g_{V_i\pi}\right]\varepsilon_{\alpha\beta},\end{aligned} \tag{68}$$

where Eq. (66) is the longitudinal and Eqs. (67) and (68) are the transversal set of sum rules. Here, f_π is the pion decay constant; g_{V_i} and g_{A_j} are the vector and axial-vector decay constants defined as

$$\langle 0|j_\mu(0)|V_i(p,\epsilon)\rangle = g_{V_i}\epsilon_\mu, \tag{69}$$

$$\langle 0|j_\mu^5(0)|A_j(p,\epsilon)\rangle = g_{A_j}\epsilon_\mu; \tag{70}$$

and we define $V_i\pi$ and V_iA_j couplings in two-dimensional QCD_2,

$$L_{V_i\pi} = \varepsilon^{\mu\nu} g_{V_i\pi} V_{i\mu}\partial_\nu\pi, \tag{71}$$

$$L_{V_iA_j} = \varepsilon^{\mu\nu} g_{V_iA_j} V_{i\mu} A_{j\nu}. \tag{72}$$

If one replaces the vector current by an on-shell photon in Eq. (66), Eq. (66) represents the decay of the pion. Our Eq. (66) agrees with the result found for QCD_2 when $N_c \to \infty$ and $g^2 N_c = \mathrm{const}$ in Ref. [4]. However, Eqs. (67) and (68) are the new formulas involving resonances. There are also sum rules which provide stringent constraints between the resonance parameters. Again, this set of sum rules is a direct consequence of the nonperturbative correction in the Son-Yamamoto relation, and it is obtained similarly to QCD_4 in Ref. [1],

$$\sum_i \frac{g_{V_i\pi} g_{V_i}}{m_{V_i}^2} = \frac{N_c}{\pi f_\pi}, \tag{73}$$

$$\sum_j \frac{g_{V_i A_j} g_{A_j}}{m_{A_j}^2 - m_{V_i}^2} = -\frac{N_c}{2\pi f_\pi^2} g_{V_i}, \qquad i = 1, 2, \ldots, \tag{74}$$

$$\sum_i \frac{g_{V_i A_j} g_{V_i}}{m_{A_j}^2 - m_{V_i}^2} = -\frac{N_c}{2\pi f_\pi^2} g_{A_j}, \qquad j = 1, 2, \ldots, \tag{75}$$

where Eq. (73) expresses the longitudinal and Eqs. (74) and (75) express the transversal set of sum rules. Here, m_{V_i} and m_{A_j} are masses of the vector and axial mesons. The above sum rules and resultant matrix elements are generic to any theory with a Yang-Mills-Chern-Simons gravity dual in the limit of large N_c. The set of sum rules for the matrix elements and for the parameters can be checked explicitly in a specific model (for example, the "cosh" model).

V. HAMILTON-JACOBI EQUATION

As was argued in Ref. [8], the holographic renormalization group equation can be obtained as a Hamiltonian-Jacobi equation when time is identified with the radial z coordinate. According to the AdS/CFT prescription, it is written as

$$\frac{\partial S}{\partial z_0} + H(\pi_\alpha, \mathcal{A}_\alpha, z_0) = 0, \tag{76}$$

where evolution goes from the IR to the UV boundary z_0; i.e., the bulk action S and Hamiltonian H are taken at z_0. The Hamiltonian is expressed through canonical momentum π conjugated to the gauge field $\mathcal{A}_0$ at the boundary

$$\pi_\alpha = \frac{\partial L}{\partial(\partial_z \mathcal{A}_{0\alpha})} = \frac{\delta S}{\delta \mathcal{A}_{0\alpha}}. \tag{77}$$

According to the AdS/CFT prescription, because $\mathcal{A}_0$ is a source of the current, we vary once and get

$$\langle j_\alpha \rangle = \frac{\delta S}{\delta \mathcal{A}_{0\alpha}}. \tag{78}$$

From the action

$$S = \int d^2x dz \bigg(f^2((\partial_z A_\mu)^2 + (\partial_z V_\mu)^2) - \frac{1}{2g^2}(F_{A\mu\nu}^2 + F_{V\mu\nu}^2) + 4\kappa \epsilon^{\nu\sigma} \partial_z V_\nu A_\sigma \bigg), \tag{79}$$

we find the canonical momenta

$$\pi_{A\mu} = \frac{\partial L}{\partial(\partial_z A_\mu)} = 2f^2 \partial_z A_\mu,$$
$$\pi_{V\mu} = \frac{\partial L}{\partial(\partial_z V_\mu)} = 2f^2 \partial_z V_\mu + \phi_{V\mu}, \tag{80}$$

where the shift in the canonical momenta due to the Chern-Simons term is

$$\phi_{V\mu} = 4\kappa \epsilon^{\mu\sigma} A_\sigma \tag{81}$$

and the corresponding "velocities" are

$$\partial_z A_\mu = \frac{1}{2f^2} \pi_{A\mu}, \qquad \partial_z V_\mu = \frac{1}{2f^2} \tilde{\pi}_{V\mu}. \tag{82}$$

To simplify the notation, we introduced shifted momentum $\tilde{\pi}_{V\mu}$,

$$\pi_{A\mu} = \langle j_\mu^5 \rangle, \qquad \tilde{\pi}_{V\mu} = \pi_{V\mu} - \phi_{V\mu} = \langle j_\mu \rangle - \phi_{V\mu}, \tag{83}$$

where ϕ's are given by Eq. (81). This shows the mechanism of how the bulk Chern-Simons term leads to the parity breaking in $1+1$-dimensional boundary field theory. Namely, the Chern-Simons term is responsible for the shift in canonical momenta, which gives a nonzero vacuum expectation value of the current

$$\langle j_\mu \rangle = 4\kappa \epsilon^{\mu\nu} \mathcal{A}_\nu \neq 0. \tag{84}$$

Expressing velocities through momenta

$$H = \int d^2x (\pi_{A\mu} \partial_z A_\mu + \pi_{V\mu} \partial_z V_\mu - L), \tag{85}$$

we obtain the Hamiltonian at the UV boundary, at $z = z_0$,

$$\begin{aligned} H = & \int d^2x \bigg(\frac{1}{2f^2} \pi_{A\mu}^2 + \frac{1}{2f^2} \pi_{V\mu} (\pi_{V\mu} - 4\kappa \epsilon^{\mu\sigma} A_\sigma) \\ & - \bigg[f^2((\partial_z A_\mu)^2 + (\partial V_\mu)^2) - \frac{1}{2g^2}(F_{A\mu\nu}^2 + F_{V\mu\nu}^2) \\ & + 4\kappa \epsilon^{\nu\sigma} \partial_z V_\nu A_\sigma \bigg] \bigg) \\ = & \int d^2x \bigg(\frac{1}{4f^2} \pi_{A\mu}^2 + \frac{1}{4f^2} (\pi_{V\mu} - \phi_{V\mu})^2 \\ & + \frac{1}{2g^2}(F_{A\mu\nu}^2 + F_{V\mu\nu}^2) \bigg). \end{aligned} \tag{86}$$

Finally, we arrive at the Hamilton-Jacobi equation (76),

$$\frac{\partial S}{\partial z_0} + \int \frac{d^2q}{(2\pi)^2} \left(\frac{1}{4f^2} (\pi_{A\mu}^2 + \tilde{\pi}_{V\mu}^2) + \frac{1}{2g^2} (F_{A\mu\nu}^2 + F_{V\mu\nu}^2) \right) \bigg|^{z_0} = 0, \tag{87}$$

where we traded time for holographic coordinate z and the shifted momentum is given by Eq. (83). Now, to obtain the corresponding RG equations for correlators, we vary the Hamilton-Jacobi equation (87) with respect to boundary values of the gauge fields and introduce the notation for the one-point functions,

$$\pi_{A\alpha} = \frac{\delta S}{\delta A_{0\alpha}} = \langle j^5_\alpha \rangle, \qquad \pi_{V\alpha} = \frac{\delta S}{\delta V_{0\alpha}} = \langle j_\alpha \rangle, \quad (88)$$

and the two-point functions,

$$\frac{\delta^2 S}{\delta A_{0\alpha}\delta A_{0\beta}} = \langle j^5_\alpha j^5_\beta \rangle = \Pi_A \delta_{\alpha\beta}, \\ \frac{\delta^2 S}{\delta V_{0\alpha}\delta V_{0\beta}} = \langle j_\alpha j_\beta \rangle = \Pi_V \delta_{\alpha\beta}, \quad (89)$$

$$\frac{\delta^2 S}{\delta V_{0\alpha}\delta A_{0\beta}} = \langle j_\alpha j^5_\beta \rangle = \frac{1}{2\pi}\varepsilon_{\alpha\beta} w_T \equiv \varepsilon_{\alpha\beta}\tilde{w}_T, \quad (90)$$

One- and two-point functions were calculated in Sec. II. We introduced a notation $w_T/(2\pi) = \tilde{w}_T$.

A. Hamilton-Jacobi equation for diagonal correlators

First, we examine the RG equation for the diagonal correlators (26). We start off varying the Hamiltonian (86) twice with respect to the boundary value A_0,

$$\frac{\delta^2}{\delta A_{0\alpha}\delta A_{0\beta}} \left(\frac{1}{4f^2}(\pi^2_{A\mu} + \tilde{\pi}^2_{V\mu}) + \frac{1}{2g^2}(F^2_{A\mu\nu} + F^2_{V\mu\nu}) \right) \Big|^{z_0}, \quad (91)$$

and we obtain

$$\frac{1}{2f^2}\left(\left(\frac{\delta\pi_A}{\delta A_0}\right)^2 + \left(\frac{\delta\tilde{\pi}_V}{\delta A_0}\right)^2 + \pi_A \frac{\delta^2 \pi_A}{\delta A_0^2} + \tilde{\pi}_V \frac{\delta^2\tilde{\pi}_V}{\delta A_0^2} \right) \\ = \frac{1}{2f^2}\Bigg((\langle j^5 j^5 \rangle)^2 + \left(\langle j j^5 \rangle - \frac{\delta\phi_V}{\delta A_0} \right)^2 \\ + \langle j^5 \rangle \langle j^5 j^5 j^5 \rangle + (\langle j \rangle - \phi_V)\langle j j^5 j^5 \rangle \Bigg) \\ = \frac{1}{2f^2}(\Pi_A^2 + (\tilde{w}_T - 4\kappa A|^{z_0})^2)\delta^{\alpha\beta}. \quad (92)$$

Our Abelian action is quadratic in fields, and therefore we neglect three-point functions. Varying the Hamilton-Jacobi (HJ) equation (87) and using Eq. (14), we obtain the HJ equations for the diagonal correlators:

$$\frac{\partial}{\partial z_0}\Pi_A + \frac{1}{2f^2}(\Pi_A^2 + (\tilde{w}_T - 4\kappa)^2) = 0 \quad (93)$$

$$\frac{\partial}{\partial z_0}\Pi_V + \frac{1}{2f^2}(\Pi_V^2 + (\tilde{w}_T - 4\kappa)^2) = 0. \quad (94)$$

The HJ equation for the difference is given by

$$\frac{\partial}{\partial z_0}(\Pi_A - \Pi_V) + \frac{1}{2f^2}(\Pi_A^2 - \Pi_V^2) = 0, \quad (95)$$

where all quantities are taken at the point z_0, i.e., $f(z_0)$, $\Pi_A(z_0)$, and $\Pi_V(z_0)$. Using Eq. (40), we can rewrite Eq. (95) for the left-right correlator,

$$\frac{\partial}{\partial z_0}\Pi_{LR} = -\frac{1}{2f^2}(\Pi_A + \Pi_V)\Pi_{LR}. \quad (96)$$

The RG equation for the left-right correlator is diagonal; i.e., its running is expressed again through the left-right correlator. The momentum-dependent coefficient is given by the sum of the correlators $\Pi_A + \Pi_V$.

B. Hamilton-Jacobi equation for mixed correlator

Varying the Hamiltonian part in the HJ equation (87) twice with respect to the boundary values V_0 and A_0, we obtain

$$\frac{\delta^2}{\delta V_{0\alpha}\delta A_{0\beta}} \left(\frac{1}{4f^2}(\pi^2_{A\mu} + \tilde{\pi}^2_{V\mu}) + \frac{1}{2g^2}(F^2_{A\mu\nu} + F^2_{V\mu\nu}) \right) \Big|^{z_0}, \quad (97)$$

and we obtain

$$\frac{1}{2f^2}\left(\frac{\delta\pi_A}{\delta V_0}\frac{\delta\pi_A}{\delta A_0} + \frac{1}{2f^2}\frac{\delta\tilde{\pi}_V}{\delta V_0}\frac{\delta\tilde{\pi}_V}{\delta A_0} + \pi_A \frac{\delta^2\pi_A}{\delta V_0 \delta A_0} + \tilde{\pi}_V \frac{\delta^2\tilde{\pi}_V}{\delta V_0\delta A_0} \right) \\ = \frac{1}{2f^2}\Bigg(\langle j j^5 \rangle \langle j^5 j^5 \rangle + \left(\langle j j^5 \rangle - \frac{\delta\phi_V}{\delta A_0} \right) \langle j j \rangle \\ + \langle j^5 \rangle \langle j j^5 j^5 \rangle + (\langle j \rangle - \phi_V) \langle j^5 j j \rangle \Bigg) \\ = \frac{1}{2f^2}(\varepsilon^{\alpha\gamma}\tilde{w}_T \Pi_A \delta^{\gamma\beta} + (\varepsilon^{\gamma\beta}\tilde{w}_T - \varepsilon^{\gamma\beta}4\kappa A|^{z_0})\Pi_V \delta^{\gamma\alpha}) \\ = \frac{1}{2f^2}\varepsilon^{\alpha\beta}(\tilde{w}_T\Pi_A + (\tilde{w}_T - 4\kappa)\Pi_V). \quad (98)$$

Varying the HJ equation (87), we obtain the HJ equations for the mixed correlator,

$$\frac{\partial}{\partial z_0}\tilde{w}_T + \frac{1}{2f^2}(\tilde{w}_T(\Pi_A + \Pi_V) \\ + 2\kappa(\Pi_A - \Pi_V) - 2\kappa(\Pi_A + \Pi_V)) = 0, \quad (99)$$

where $f(z_0)^2$, $\tilde{w}_T(z_0)$, $\Pi_A(z_0)$, and $\Pi_V(z_0)$ are taken at z_0. As seen from Eq. (99), due to the Chern-Simons term, $\kappa \neq 0$, the RG equation for the mixed correlator $\tilde{w}_T$ is not diagonal,

$$\frac{\partial}{\partial z_0}\tilde{w}_T = -\frac{1}{2f^2}(\Pi_A+\Pi_V)\tilde{w}_T + \frac{2\kappa}{2f^2}(\Pi_A+\Pi_V) - \frac{2\kappa}{2f^2}(\Pi_A-\Pi_V). \tag{100}$$

It is remarkable that the diagonal RG flow for w_T has the same rate $(1/2f^2)(\Pi_A+\Pi_V)$ as the left-right correlator Π_{LR}.

C. Hamilton-Jacobi equation for Son-Yamamoto relation

We write the HJ equation for the Son-Yamamoto relation (37),

$$(\mathrm{SY}) = \tilde{w}_T - 2\kappa + 2\kappa\int^{z_0}\frac{dz}{2f^2}(\Pi_A-\Pi_V), \tag{101}$$

where $\tilde{w}_T = w_T/(2\pi)$. To this end, we differentiate Eq. (101) with respect to z_0 and use the HJ equations for the diagonal and mixed two-point functions, Eqs. (95) and (99),

$$\frac{\partial}{\partial z_0}\tilde{w}_T + 2\kappa\int^{z_0}\frac{dz}{2f^2}\frac{\partial}{\partial z_0}(\Pi_A-\Pi_V) + \frac{2\kappa}{2f^2}(\Pi_A-\Pi_V) + \tag{102}$$

$$+\frac{1}{2f^2}(\tilde{w}_T(\Pi_A+\Pi_V) + 2\kappa(\Pi_A-\Pi_V) - 2\kappa(\Pi_A+\Pi_V)) + 2\kappa\int^{z_0}\frac{dz}{2f^2}\frac{1}{2f^2}(\Pi_A^2-\Pi_V^2) - 2\kappa\frac{1}{2f^2}(\Pi_A-\Pi_V) = 0, \tag{103}$$

where the first and the fourth terms constitute Eq. (99) and the second and the fifth terms constitute Eq. (95). Note that the integral with the metric term $1/f^2$ in Eq. (101) is differentiated. Combining the terms, we have

$$\frac{\partial}{\partial z_0}\left(\tilde{w}_T - 2\kappa + 2\kappa\int\frac{dz}{2f^2}(\Pi_A-\Pi_V)\right) + \frac{1}{2f^2}(\Pi_A+\Pi_V)\left(\tilde{w}_T - 2\kappa + 2\kappa\int\frac{dz}{2f^2}(\Pi_A-\Pi_V)\right) = 0, \tag{104}$$

which can be written in a short form,

$$\frac{\partial}{\partial z_0}(\mathrm{SY}) = -\frac{1}{2f^2}(\Pi_A+\Pi_V)(\mathrm{SY}), \tag{105}$$

where (SY) denotes the Son-Yamamoto relation (101). The RG flow for the Son-Yamamoto relation is diagonal. It is remarkable that the Son-Yamamoto relation (SY) and the left-right correlator Π_{LR} (96) both flow with the same coefficient which is given by the sum $\sim(\Pi_A+\Pi_V)$.

In Sec. III (the regime of small momenta), we showed that the Son-Yamamoto relation (101) is satisfied at the point $z_0\to 0$. This means that, since the RG (105) is diagonal, the Son-Yamamoto relation holds for any energy scale z_0.

VI. SIMILARITY OF QCD AND TWO-DIMENSIONAL SYSTEM. DIMENSIONAL REDUCTION

In this section, we draw parallels between the four-dimensional QCD [1,11] and our two-dimensional system. We write formulas for the 2D system in the context of QCD_2. We summarize the RG equations,

$$\frac{\partial}{\partial z_0}\Pi_{LR} = -\frac{1}{2f^2}(\Pi_A+\Pi_V)\Pi_{LR} \tag{106}$$

$$\frac{\partial}{\partial z_0}(\mathrm{SY}) = -\frac{1}{2f^2}(\Pi_A+\Pi_V)(\mathrm{SY}), \tag{107}$$

which are identical in both the QCD and linear cases. It is remarkable that the two equations have the same rate of change $\frac{1}{f^2}(\Pi_A+\Pi_V)$. Further comparing QCD_4 and QCD_2, the polariztion operators Π_A and Π_V are the same; however, the Son-Yamamoto relations slightly differ. Explicitly, they are given by

$$2\mathrm{D}:\ (\mathrm{SY}) = w_T - N_c + N_c\int_0^{z_0}\frac{dz}{2f^2}(\Pi_A-\Pi_V) = 0, \tag{108}$$

$$\langle j_\mu j_\nu^5\rangle^\perp = \frac{1}{2\pi}\left(N_c - N_c\int_0^{z_0}\frac{dz}{2f^2}(\Pi_A-\Pi_V)\epsilon_{\mu\nu}\right), \tag{109}$$

where $w_T\sim\langle j_A(q)j_V(-q)\rangle^\perp$ and [1]

$$4\mathrm{D}:\ (\mathrm{SY}) = w_T - \frac{N_c}{Q^2} + \frac{N_c}{f_\pi^2}(\Pi_A-\Pi_V) = 0, \tag{110}$$

$$\langle j_\mu j_\nu^5\rangle^\perp = \frac{Q^2}{4\pi^2}\left(\frac{N_c}{Q^2} - \frac{N_c}{f_\pi^2}(\Pi_A-\Pi_V)\tilde{F}_{\mu\nu}\right). \tag{111}$$

Here the dual field strength is $\tilde{F}_{\mu\nu} = 1/2\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, and the transverse part of the vector-axial current correlator is defined as $w_T F(k)\sim\langle j_A(q)j_V(-q-k)j_V(k)\rangle^\perp$ with $k=0$, F is the field strength of the vector gauge field, $Q^2=-q^2$. The following identification is done:

$$\frac{1}{f_\pi^2} = \int_0^{z_0}\frac{dz}{2f^2}. \tag{112}$$

Note that the dimension of the pion decay constant is $[f_\pi] = 1$ in two dimensions and $[f_\pi] = E$ in four dimensions.

Next, we consider the dimensional reduction $d \to d-2$ that occurs at strong magnetic field $B \to \infty$, in order to see a connection between four-dimensional QCD and 2D systems. The Dirac action is written as

$$S_F = i \int d^4x \bar{\psi} (\Gamma_\mu D^\mu - m) \psi, \tag{113}$$

where Γ are the four-component gamma matrices and the covariant derivative contains the gauge field. We choose the gauge

$$A_y = -yB, \tag{114}$$

with $B \| z$ and where B is positive, and consider a z slice for the time being. Then, we decompose the Dirac spinor into two two-component Weyl spinors,

$$\psi = \mathrm{e}^{-i\omega t + ikx} \begin{pmatrix} \xi_1(y) \\ \xi_2(y) \end{pmatrix}, \tag{115}$$

with $k_x \equiv k$. For a new variable,

$$\eta = \sqrt{B} \left(y + \frac{k}{B} \right), \tag{116}$$

the Dirac equation for ξ_i is reduced to a harmonic oscillator, where a solution is defined in terms of the Hermite polynomials H_n,

$$\xi_1 = c(\omega, k) I_n(\eta), \qquad \xi_2 = \pm c(\omega, k) I_{n-1}(\eta)$$
$$I_n(\eta) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{-\eta^2/2} H_n(\eta), \tag{117}$$

and the energy is quantized

$$\omega = \pm \sqrt{2Bn}. \tag{118}$$

$n = 0, 1, 2, \ldots$ are Landau levels, where $\pm$ distinguishes the two solutions. Motion in the direction perpendicular to the magnetic field (x, y) is described by Larmor orbits. In the limit $B \to \infty$, only the lowest Landau level (LLL) is important. Indeed the LLL has a vanishing energy because the zero point energy $\frac{1}{2}B$ is exactly compensated by the Zeeman splitting due to spin coupling $-\frac{1}{2}B$. Therefore, the zero modes for each of the two-component spinors is given by

$$\xi_i = \mathrm{e}^{-\eta^2/2} \begin{pmatrix} 0 \\ \zeta_i \end{pmatrix}. \tag{119}$$

The fact that only one spin component is populated means that the LLL is spin polarized. Reinstating the z dependence back, the zero modes become functions $\zeta_i(t, z)$. There is one zero mode for each state of the LLL, for each k. These zero modes are described by a $1+1$-dimensional effective action for a two-component Weyl spinor $\zeta = (\zeta_1, \zeta_2)$,

$$S_{\text{eff}} = i \int d^2x \bar{\zeta} \gamma_\mu D_\mu \zeta, \tag{120}$$

where γ are given by the Pauli matrices, and the covariant derivative does not contain the gauge field now. In strong magnetic fields $B \to \infty$ where only the LLL is important, the dynamics is reduced from four dimensions to two dimensions. Since the LLL is spin polarized, the density of states for the LLL is $\frac{B}{2\pi}$. This means that in the limit $B \to \infty$, in order to get one- and two-point functions of the currents, we calculate correlators for the two-dimensional fermions and then sum over the Fermi zero modes using the density of states in the LLL,

$$\langle \bar{\psi} \Gamma \psi \rangle = \frac{B}{2\pi} \langle \bar{\zeta} \gamma \zeta \rangle, \tag{121}$$

and schematically

$$\langle J(x) J(0) \rangle_{4d} = \frac{B}{2\pi} \langle j(x) j(0) \rangle_{2d}, \tag{122}$$

where $\bar{\psi} \Gamma \psi = J$ and $\bar{\zeta} \gamma \zeta = j$ are fermion currents in four dimensions and two dimensions, respectively. Similar calculations can be done in a holographically dual theory with dual fermions and currents, where the reduction in the bulk theories from five dimensions to three dimensions occurs [17,18]. This means that at large B the dimensional reduction from four dimensions to two dimensions for the current correlators holds also nonperturbatively.

VII. CONCLUSIONS

We derived the analog of the Son-Yamamoto relation for $(1+1)$-dimensional systems. Quantum field theory does not allow us to predict and calculate the Son-Yamamoto relation between the anomalous two-point current correlation function and the nonanomalous ones. The holographic method permits us to establish this relation using the holographic dictionary. We considered the implications of the Son-Yamamoto relation in QCD_2 and obtained a new set of sum rules for the current matrix elements and for the decay parameters for the resonances. We also discussed the Son-Yamamoto relation in application to the field theory models: the Schwinger model with N fermions and the 't Hooft solution of QCD_2 in the limit of $N_c \to \infty$, $g^2 N_c = \text{const}$.

Two-dimensional systems are presently realized by organic quasi-one-dimensional metals, organic nanotubes, edge states of quantum Hall liquids, one-dimensional semiconducting structures, and edge states of topological insulators [19]. In these systems, it is believed that electron-electron interaction invalidates Landau Fermi liquid picture. Instead, a different state described approximately by Tomanaga-Luttinger theory [20] is generated. Since electronic correlations in this state are stronger than in Fermi liquid, it is interesting to calculate two-point correlations that represent conductivities or related transport coefficients and obtain relations between them. For example, it is interesting to translate our transport coefficients in terms of the Coulomb/spin drag transresistivity between two quantum wires [21] or examine transport properties in chiral edge states in the quantum Hall state and helical edge states in topological insulators/topological superconductors [22].

The Son-Yamamoto relation is based on the chiral anomaly. We show that in the transverse direction the chiral anomaly is dynamical rather than topological. As one of the consequences, the QCD_2 vacuum exhibits the chiral symmetry breaking with $\bar{\psi}\psi \neq 0$ and a pairing of the massless fermions (chiral bosonization).

We summarize the two representations of the Son-Yamamoto relation for $(1+1)$-dimensional systems. The Son-Yamamoto equation relates w_T—the mixed $\langle VA\rangle$ current correlator and the diagonal $\langle VV\rangle$ and $\langle AA\rangle$ current correlators

$$(\mathrm{SY}) = w_T - N + N\frac{\Pi_A - \Pi_V}{f_\pi^2} = 0, \tag{123}$$

Also the Son-Yamamoto equation can be written for the left-right correlator $\langle LR\rangle$

$$\langle j^L_\mu j^R_\nu\rangle^\perp = P^{\alpha\perp}_\mu P^{\beta\perp}_\nu \varepsilon_{\alpha\beta}\left(\frac{N}{\pi} - \frac{N}{\pi}\frac{\Pi_{LR}}{2f^2(z)}\right), \tag{124}$$

where j^L_μ and j^R_μ are the left- and right-handed currents, $P^\perp$ is the transverse projection operator, and N stands for N fermions in Schwinger model or for N_c number of colors in QCD_2 with f_π is the decay constant. In Eq. (124), the first term coincides with the result of the Schwinger model [13] and the QCD_2 [4]: the massless pole $1/q^2$ with finite residue N/π is determined by the chiral anomaly. The second term is purely nonperturbative. It gives a new set of sum rules. Therefore, the Son-Yamamoto relation can be viewed as an anomaly matching condition.

The key point in deriving the Son-Yamamoto relation was the independence on the radial coordinate of the Wronskian for vector and axial gauge fields. It gives the range of validity for the Son-Yamamoto relation: generally, small Chern-Simons κ $(4\pi\kappa = N)$ or large virtuality Q^2. We estimated Wronskian in the cosh and Sakai-Sugimoto models, using for parameters the momentum Q^2, the Chern-Simons κ, the ratio of holographic sources r, and ϵ in the $D = 3+\epsilon$ dimensional regularization scheme. In the dimensional regularization, there is a wide range of parameters where solutions for vector and axial gauge fields remain regular, which produce a z-independent Wronskian starting from some small z in the IR (Appendix B). Specific estimates slightly differ in the models. It would be instructive to get quantitative estimates for the parameter range where $W(z) = \mathrm{const}$.

The two-dimensional Son-Yamamoto matching condition at large virtualities provides an estimate for the decay constant

$$f_\pi^2 \sim N_c, \tag{125}$$

which was found in the limit of the weak coupling $N_c \to \infty$ and 't Hooft condition $g^2 N_c = \mathrm{const}$ by Zhitnitsky [4]. Since this estimate is done at large Q^2 where the application of the low-energy effective action is questionable, this result deserves independent derivation by other means. We also showed that the pion decay constant $f_\pi^2 \sim N_c$ is consistent with the Gell-Mann-Oakes-Renner relation and the chiral condensate $\langle\bar{\psi}\psi\rangle \sim N_c$. In QCD_4, the analog of the estimate for f_π (125) is the holographic result for magnetic susceptibility of Vainshtein [5] $\chi \sim 1/f_\pi^2$ [1].

Finally, we found that the RG flow equations for the Son-Yamamoto relations in $(1+1)$- and $(3+1)$-dimensional systems are the same and they are diagonal. Moreover, the rate of the RG flow for the SY relation and the left-right correlator is the same,

$$\frac{\partial}{\partial z_0}\Pi_{LR} = -\frac{1}{2f^2}(\Pi_A + \Pi_V)\Pi_{LR} \tag{126}$$

$$\frac{\partial}{\partial z_0}(\mathrm{SY}) = -\frac{1}{2f^2}(\Pi_A + \Pi_V)(\mathrm{SY}), \tag{127}$$

where z_0 is the UV boundary value of the radial bulk coordinate—the end point of the evolution. We believe that the diagonal form and this rate hold only for the Abelian case. We showed that the similarity between $(3+1)$- and $(1+1)$-dimensional systems can be attributed to the dimensional reduction $D \to D-2$ in strong magnetic field. However, it does not explain why the RG flows are diagonal and have the certain rate.

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APPENDIX A: CHECKING THE SON-YAMAMOTO RELATION IN A MODEL. 1 + 1 SYSTEMS IN AN ADS MODEL WITH THE CHIRAL CONDENSATE

We consider the Son-Yamamoto relation for $1+1$ systems in a gravity dual model which incorporates the chiral condensate [23]. Contrary to Ref. [23], we do not impose the hard-wall cutoff in the IR that insured confinement in 3D QCD. The metric is a slice of the AdS space

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dx^2), \tag{A1}$$

where $0 \leq z < \infty$, the AdS UV boundary is at $z = 0$, and we rescale the curvature radius of the space to unity. The action in the bulk

$$S = S_{\mathrm{YM}} + S_{\mathrm{CS}} \tag{A2}$$

includes the scalar field

$$S_{\mathrm{YM}} = \int d^3x\sqrt{g}\left(|D\Psi|^2 + M^2|\Psi|^2 - \frac{1}{4g_3^2}(F_L^2 + F_R^2)\right), \tag{A3}$$

$$S_{\mathrm{CS}} = \kappa \int d^3x(w_3(A_L) - w_3(A_R)), \tag{A4}$$

where $D\Phi = \partial\Phi - iA_L\Phi + iA_R\Phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $w(A) = A * F + \frac{2}{3}A^3$, and M^2 is specified further. From the AdS dictionary, a bulk field Θ dual to operator O behaves at the asymptotic UV boundary AdS_{d+1} as

$$\Theta(z) = \mathcal{A}z^{\Delta_-}(1+\cdots) + \mathcal{B}z^{\Delta_+}(1+\cdots), \quad z \to 0, \tag{A5}$$

where the source to O (leading term) is $\mathcal{A}$, the expectation of O (subleading term) is $\mathcal{B} = \langle O\rangle$, and the characteristic exponents $\Delta_\pm$ for scalar and vector fields are solutions to equations

$$\text{scalar:}\ \Delta(\Delta - d) = M^2, \tag{A6}$$

$$\text{vector:}\ \Delta(\Delta - d + 2) = M^2, \tag{A7}$$

where the AdS curvature radius is 1. We take the scalar mass equal to the Breitenlohner-Freedman bound $M^2 = -1$ to insure the positive energy, and the mass of the vector field is $M^2 = 0$. From Eq. (A7), for the AdS_3, $d = 2$, the characteristic exponents are $\Delta_\pm = 1$ for scalar and $\Delta = 0$ for vector fields. This implies the following behavior in the UV:

$$\text{scalar:}\ \Psi = mz\ln z + \langle q\bar{q}\rangle z, \tag{A8}$$

$$\text{vector:}\ V = A\ln z + \langle J\rangle. \tag{A9}$$

In the context of QCD, the source is the quark mass m, and the response is the chiral condensate $\langle q\bar{q}\rangle$, and the electromagnetic field A sources the $U(1)$ conserving current with expectation value $\langle J\rangle$. To check that the scalar field behaves as in Eq. (A8), we solve the equation of motion for Ψ without the gauge field

$$\partial_z\left(\frac{1}{z}\partial_z\Psi\right) - \frac{1}{z^3}M^2\Psi = 0. \tag{A10}$$

Indeed, the solution is

$$\Psi = \frac{1}{2}mz\ln z + \frac{1}{2}\sigma z, \tag{A11}$$

with $M^2 = -1$, and σ is the chiral condensate. As in Ref. [24], we parametrize the scalar field as

$$\Psi = \Psi_0 e^{i2\pi}, \qquad \Psi_0 = \frac{1}{2}v(z),$$
$$v(z) = mz\ln z + \sigma z. \tag{A12}$$

Introducing the vector and axial-vector fields, $V = (A_L + A_R)/2$ and $A = (A_L - A_R)/2$, the covariant derivative for the scalar becomes $D\Psi = 2i\Psi_0(\partial\pi - A)$. We work in the radial gauge, $V_z = A_z = 0$. We decompose the gauge fields as

$$V_\mu = V_\mu^\perp, \qquad \partial_\mu V_\mu = 0, \tag{A13}$$

$$A_\mu = A_\mu^\perp + A_\mu^\parallel, \qquad \partial_\mu A_\mu^\perp = 0. \tag{A14}$$

The action Eq. (A3) $S_{\mathrm{YM}} = S_V + S_A$ is

$$S_V = \int d^3x\left(-\frac{1}{4g_3^2}\right)2zF_V^2, \tag{A15}$$

$$S_A = \int d^3x\left[\left(-\frac{1}{4g_3^2}\right)2zF_A^2 + \frac{v^2(z)}{z}(\partial\pi - A)^2\right], \tag{A16}$$

which can be written as

$$S_V = \int d^3x\left(-\frac{1}{4g_3^2}\right)[2zF_{V\mu\nu}^{\perp 2} + 4zF_{Vz\mu}^{\perp 2}], \tag{A17}$$

$$S_A = \int d^3x\left(-\frac{1}{4g_3^2}\right)[2zF_{A\mu\nu}^{\perp 2} + 4zF_{Az\mu}^{\perp 2} + 4zF_{Az\mu}^{\parallel 2}] + \int d^3x\frac{v^2(z)}{z}[(\partial_z\pi - A_z)^2 + (\partial_\mu\pi - A_\mu^\parallel)^2 + A_\mu^{\perp 2}], \tag{A18}$$

where $v(z)$ is given by Eq. (A12). Comparing the gauge action in Eq. (23) and Eqs. (A17) and (A18), the identification of the metric factors in Eq. (23) can be made,

$$f^2(z) = \frac{1}{4g_3^2}\frac{2\sqrt{g}}{g_{zz}g_{\mu\mu}} = \frac{z}{2g_3^2},$$
$$\frac{1}{g^2} = \frac{1}{4g_3^2}\frac{2\sqrt{g}}{g_{\mu\mu}^2} = \frac{z}{2g_3^2}, \tag{A19}$$

where the determinant is $\sqrt{g} = 1/z^3$ for the AdS_3 [not to confuse factor $1/g^2$ in Eq. (23) with the metric determinant g]. Let the gauge fields be $V^\mu(q,z) = V(q,z)V_0^\mu$ and $A^\mu(q,z) = A(q,z)A_0^\mu$ with V_0, A_0 being sources of the vector and axial-vector currents and q be the Fourier transform momentum in the boundary space component x. From Eqs. (A17) and (A18), the linearized equations of motion (EOM) for the perpendicular components of the vector and axial-vector fields, $V(q,z)$ and $A(q,z)$, read

$$\partial_z(z\partial_z V) - zQ^2 V = 0, \tag{A20}$$

$$\partial_z(z\partial_z A) - zQ^2 A - \frac{g_3^2 v^2}{z}A = 0, \tag{A21}$$

where $Q^2 = -q^2$ and we omit the perpendicular sign. The boundary conditions (BCs) in the UV and IR are

$$\mathrm{UV:}\ z\partial_z V|_{z=0} = 1, \qquad z\partial_z A|_{z=0} = 1, \tag{A22}$$

$$\mathrm{IR:}\ \partial_z V|_{z_m\to\infty} = 0, \qquad \partial_z A|_{z_m\to\infty} = 0, \tag{A23}$$

where we introduced the hard-wall cutoff z_m, which we take to be infinite. The UV BC says that the source for the components $V(q,z)$ and $A(q,z)$ is unity. Indeed, the source of the vector field is given by $zV'_{z=0}$ when asymptotic behavior is as in Eq. (A9). First, we solve the EOM for the vector field

$$z^2V'' + zV' - Q^2z^2V = 0. \tag{A24}$$

The solution

$$V = c_1 I_0(Qz) + c_2 K_0(QZ) \tag{A25}$$

is expressed through the modified Bessel functions I_n, K_n with $n = 0$. Imposing BC's

$$\mathrm{IR:}\ c_1 I_1(Qz_m) - c_2 K_1(Qz_m) = 0, \qquad z_m \to \infty \tag{A26}$$

$$\mathrm{UV:}\ zQ(c_1 I_1(0) - c_2 K_1(0)) = 1, \tag{A27}$$

we obtain

$$V_\perp(Q,z) = -K_0(Qz) - \frac{K_1(Qz_m)}{I_1(Qz_m)} I_0(Qz) \overset{z_m\to\infty}{\to} -K_0(Qz). \tag{A28}$$

Using asymptotic expansion at $z = 0$ for the modified Bessel functions

$$I_0(z) \approx 1 + z^2/4 + \cdots,$$
$$K_0 \approx (-\gamma + (1-\gamma)z^2/4 + \cdots)$$
$$- \ln(z/2)(1 + z^2/4 + \cdots), \tag{A29}$$

we find that V behaves in the UV as in Eq. (A9),

$$V_\perp(Q,z) \to \ln(Qz) + \mathrm{const}, \qquad z \to 0, \tag{A30}$$

with the source being unity.

Next, we solve the EOM for the perpendicular component of the axial-vector field perturbatively for large $Q^2 \to \infty$,

$$A = A_0 + A_1 + \cdots, \tag{A31}$$

with $A_0(Q,z) = V_\perp(Q,z)$ (A28). The first correction satisfies the equation

$$x^2\partial_x^2 A_1 + x\partial_x A_1 - xA_1 = \lambda x^2 A_0, \tag{A32}$$

where we defined $x = Qz$, $\lambda = g_3^2\sigma^2/Q^2$, and $\lambda \to 0$ as $Q \to \infty$. The solution of this equation can be found by using the Green function

$$A_1 = \int dx' G(x,x')\lambda x'^2 A_0(x'), \tag{A33}$$

where the Green function is obtained from solving the homogeneous part of Eq. (A32),

$$f_1 = -K_0(x), \qquad f_2(x) = -I_0(x), \tag{A34}$$

and

$$G(x,x') = -\frac{1}{W[f_1,f_2](x')}(f_1(x)f_2(x'\Theta(x-x')$$
$$+ f_2(x)f_1(x')\Theta(x'-x)), \tag{A35}$$

with the Wronskian

$$W[f_1,f_2](x) = f_1f_2' - f_1'f_2 = 1/x, \tag{A36}$$

which is simple to estimate for $x \to 0$. We find the small-z behavior of A_1,

$$A_1(Q,z) = \frac{1}{3}\frac{g_3^2\sigma^2}{Q^2}, \tag{A37}$$

where we used the following integral:

$$\int_0^\infty dx x^3 K_0^2(x) = \frac{1}{3}. \tag{A38}$$

Summarizing, solutions near the boundary for the vector and axial-vector fields are

$$V_\perp(Q,z) = -K_0(Qz), \tag{A39}$$

$$A_\perp(Q,z) = -K_0(Qz) + \frac{1}{3}\frac{g_3^2\sigma^2}{Q^2}. \tag{A40}$$

As pointed out in Ref. [1], these solutions near the boundary are sufficient to evaluate the two-point correlation functions which are determined by the boundary values at $z=\epsilon$ or by the integrals dominated by small-z regions.

The derivation of correlation functions is similar to that in Sec. II. Therefore, we put the resulting expressions here. The transverse parts of the diagonal vector and axial current correlation functions are

$$\Pi_V(Q^2) = -\frac{1}{g_3^2} V_\perp(Q,z)|_{z=\epsilon}, \tag{A41}$$

$$\Pi_A(Q^2) = -\frac{1}{g_3^2} A_\perp(Q,z)|_{z=\epsilon}, \tag{A42}$$

where we used the boundary condition $zV'|_{z=\epsilon} = zA'|_{z=\epsilon} = 1$. We introduce a cutoff Λ as $\epsilon = 1/\Lambda$. The transverse part of the mixed vector-axial current correlator is

$$w_T = 2\kappa \int_0^{z_m\to\infty} dz (A'(Q,z)V(Q,z) - V'(Q,z)A(Q,z)). \tag{A43}$$

Here, we do not add the boundary term (31) because the gauge fields diverge on the boundary. Expanding $V_\perp$ and $A_\perp$ near the boundary,

$$V_\perp \approx \frac{1}{2}\ln(Q^2z^2) + \mathrm{const} + O(z^2), \tag{A44}$$

$$A_\perp \approx \frac{1}{2}\ln(Q^2z^2) + \frac{1}{3}\frac{g_3^2\sigma^2}{Q^2} + \mathrm{const} + O(z^2), \tag{A45}$$

with $\mathrm{const} = \gamma - \ln 2$, we obtain for the diagonal and mixed current correlators

$$\Pi_V(Q^2) = -\frac{1}{2g_3^2}\ln Q^2, \tag{A46}$$

$$\Pi_A(Q^2) = -\frac{1}{2g_3^2}\ln Q^2 - \frac{1}{3}\frac{\sigma^2}{Q^2}, \tag{A47}$$

$$w_T(Q^2) = -2\kappa\frac{1}{3}\frac{g_3^2\sigma^2}{Q^2}\ln\Lambda, \tag{A48}$$

where we used for evaluating w_T

$$\int_0^\infty dx K_0'(x) = K_0(x)|_{x=\epsilon} = -\ln\Lambda, \tag{A49}$$

with asymptotic value $K_0(x) = -\ln x$ at small x and $\epsilon = 1/\Lambda$. Combining the above results, we obtain

$$w_T(Q^2) = 2\kappa g_3^2(\Pi_A(Q^2) - \Pi_V(Q^2))\ln\Lambda. \tag{A50}$$

This expression should be compared with the Son-Yamamoto relation (37). Using Eq. (A19) for the metric factor, the integral in the Son-Yamamoto relation becomes

$$\int_0^\infty \frac{dz}{2f^2(z)} = \int_{z=\epsilon} dz \frac{g_3^2}{z} = g_3^2 \ln\Lambda, \tag{A51}$$

which has the same divergent leading-log behavior as w_T. It proves that the Son-Yamamoto relation (37) is satisfied.

Now, we calculate the left-right correlator Π_{LR} (40), to which the terms proportional to the chiral condensate and the fermion mass contribute. We consider the chiral limit with zero fermion mass $m=0$. Using Eqs. (A46) and (A47), we find that the left-right correlator is

$$\Pi_{LR} = -\frac{1}{3}\frac{\sigma^2}{Q^2} \tag{A52}$$

up to the quadratic order in the chiral condensate.

Finally, let us evaluate the chiral condensate that fixes the relation between σ and $\langle\bar\psi\psi\rangle$. In the field theory, one can evaluate the condensate as the variation of the vacuum energy with respect to the quark mass. In the dual theory, because m is the source for σ (A11), we variate action on the classical solution [24]

$$\langle\bar\psi\psi\rangle = \left.\frac{\delta S_A(\Psi_0)}{\delta m}\right|_{m=0}, \tag{A53}$$

where the action is given in Eq. (A18). The variation of the action is

$$\begin{aligned}\delta S_A &= \int d^2x\sqrt{g}4\partial_z\Psi\delta\Psi|_{z=0}\\ &= \int d^2x\frac{2}{z}(m\ln z + m + \sigma)z\delta m|_{z=0},\end{aligned} \tag{A54}$$

where we used Eq. (A12). Therefore, the chiral condensate is

$$\langle\bar\psi\psi\rangle = 2\sigma. \tag{A55}$$

APPENDIX B: NUMERICAL SOLUTION FOR THE COSH AND SAKAI-SUGIMOTO MODELS

In this Appendix, we solve equations of motion numerically and show that the Wronskian is independent on the radial coordinate in the regime of large momentum Q^2 and at large radial coordinate z. As we showed in Appendix A, the 3D bulk solutions are divergent. Using the analog of $(4-\epsilon)$-dimensional regularization, we will work in the bulk $(3+\epsilon)$ dimensions and take the limit of small ϵ at the end. Let us start with the equations of motion,

$$\begin{aligned} &\partial_z(f^2(z)\partial_z A_\mu) - \frac{Q^2}{g^2}A_\mu + 2\kappa\varepsilon_{\mu\nu}\partial_z V_\nu = 0, \\ &\partial_z(f^2(z)\partial_z V_\mu) - \frac{Q^2}{g^2}V_\mu + 2\kappa\varepsilon_{\mu\nu}\partial_z A_\nu = 0, \end{aligned} \tag{B1}$$

where $\mu, \nu = 0,\ 1$, since we work in the gauge $A_z = V_z = 0$. Again, we write down V_μ and A_μ fields through the mode and UV boundary functions $V_{0\mu}$, $A_{0\mu}$:

$$V_\mu = V(Q,z)V_{0\mu}(Q), \qquad A_\mu = A(Q,z)A_{0\mu}(Q). \tag{B2}$$

To simplify further computations, we work in the reference frame where $A_{00} = 0$. It can be easily seen that the nontrivial solution can only be found in the case when $V_{01} = 0$, $A_{01} \neq 0$, and $V_{00} \neq 0$. It means,

$$V_\mu A^\mu|_{z_0} = 0, \tag{B3}$$

that the gauge vectors V and A are perpendicular at the UV boundary. Of course, this condition does not hold in the bulk. Equation (B3) follows from the chiral algebra in $1+1$ dimensions where $j_\mu = \varepsilon_{\mu\nu}j^\nu$ for each left and right component ($j_0^L = -j_1^L$, $j_0^R = j_1^R$), and the bulk relation $VA \sim L^2 - R^2$. Using that only $A_{01} \neq 0$ and $V_{00} \neq 0$ are nonzero, we can write down explicitly a system of differential equations for the mode functions $A(Q,z)$ and $V(Q,z)$,

$$\begin{aligned} &\partial_z(f^2(z)\partial_z A) - \frac{Q^2}{g^2}A + 2\kappa r\partial_z V = 0, \\ &\partial_z(f^2(z)\partial_z V) - \frac{Q^2}{g^2}V + 2\kappa\frac{1}{r}\partial_z A = 0, \end{aligned} \tag{B4}$$

where we treat momentum Q^2, κ, and the ratio of sources $r = V_{00}/A_{01}$ as parameters. We solve this system numerically with boundary conditions

$$\text{IR brane:}\ \partial_z V(Q,z)|_{z=0} = 0, \qquad A(Q,0) = 0, \tag{B5}$$

$$\text{UV brane:}\ V(Q,z_0) = 1, \qquad A(Q,z_0) = 1, \tag{B6}$$

and $z_0 = \infty$. Further, we justify that one can choose finite UV boundary values. To regulate the divergency, we use $(3+\epsilon)$-dimensional regularization with $(2+\epsilon)$ spacial dimensions where ϵ is small. The metric factors defined in Eq. (A19) are

$$f^2(z) = \frac{1}{g_3^2}\frac{g_{\mu\mu}^{\epsilon/2}}{\sqrt{g_{zz}}}, \qquad g^2(z) = g_3^2\frac{g_{\mu\mu}^{1-\epsilon/2}}{\sqrt{g_{zz}}}, \tag{B7}$$

where we omit the factor of 2 and include it in redefining the other constant and the squared of coupling has dimension of mass $[g_3^2] = m$ in the three-dimensional theory, for the metric factors $[f^2] = [1/g^2] = L$ with L denoting the dimension of length. In $(3+\epsilon)$ dimensions, the integral defining the pion constant f_π and the susceptibility χ (which are both dimensionless in three dimensions),

$$\int_{-z_0}^{z_0}\frac{dz}{2f^2(z)}, \tag{B8}$$

becomes convergent (it will be clear in a concrete model). Let us estimate the Wronskian in the IR and see how the dimensional regularization works in this case. Using that $\partial_z V = 0$ and $A = 0$ around $z = 0$, the first equation in the system (B4) reduces to

$$\partial_z(f^2\partial_z A) = 0, \tag{B9}$$

with the solution given by

$$A(Q,z) = C\int_0^z\frac{dz}{f^2(z)}, \quad C = \left(\int_0^{z_0}\frac{dz}{f^2(z)}\right)^{-1}. \tag{B10}$$

Using the IR boundary conditions in the second equation of Eq. (B4), we have

$$-\frac{Q^2}{g^2}V + 2\kappa\frac{1}{r}\partial_z A = 0, \tag{B11}$$

Substituting the solution for A, we find

$$V(Q,z) = \frac{2\kappa C}{rQ^2}\frac{g^2(z)}{f^2(z)}. \tag{B12}$$

Using these solutions, we get for the Wronskian around $z = 0$

$$\begin{aligned} W(Q^2,z) &= f^2(VA' - AV') \to f^2VA' \\ &= \frac{2\kappa C^2}{r}\frac{1}{\Lambda^2Q^2}\frac{g^2(z)}{f^2(z)}, \end{aligned} \tag{B13}$$

where the z dependence is given by the metric factors (B7)

$$\frac{g^2(z)}{f^2(z)} = g_3^4 g_{\mu\mu}^{1-\epsilon}(z); \tag{B14}$$

Λ is introduced to make z dimensionless. We plot the Wronskian around $z = 0$ (B13) in the cosh model with $g^2/f^2 \sim (\cosh(z))^{2(1-\epsilon)}$ using different ϵ, Fig. 3. A decrease in ϵ leads to a flatter dependence for W. It is a desirable result.

In what follows, we consider the cosh and Sakai-Sugimoto models [23,25] and perform the numerical calculations of system (B4). Using Eq. (B7), we have for the cosh model in $(3+\epsilon)$ dimensions

$$f(z) = \Lambda \frac{1}{g_3} (\cosh(z))^{\delta}, \tag{B15}$$

$$g(z) = g_3 (\cosh(z))^{1-\delta}, \tag{B16}$$

where $\delta = \epsilon/2$ ($\delta = 0$ corresponds to three dimensions, and $\delta = 1$ corresponds to five dimensions), and we add the energy scale Λ to make the radial coordinate z dimensionless.

Using cosh metric factors we perform calculations numerically. We find diverging solutions in three dimensions are regulated, i.e., become converging in the $(3+\epsilon)$-dimensional Wronskian is a constant for the Maxwell case.

Using the cosh metric factors, we add the Chern-Simons term in $(3+\epsilon)$ dimensions, Fig. 3. We find that, due to the dimensional regularization, the Wronskian develops a plateau starting from some z. Also, solutions for the gauge functions converge to a finite value in the UV asymptotics. Solutions for $V(Q,z)$ and $A(Q,z)$ do not change much when the Chern-Simons term is included (with Chern-Simons, the difference between solutions becomes slightly larger in the IR). Increasing κ practically does not change the transition point at which $W(Q,z)$ tends to a plateau. We also do not see any crucial difference for the cases $Q > \Lambda$ and $Q < \Lambda$.

With cosh metric factors, decreasing ϵ, we find that the solutions $V(Q,z)$ and $A(Q,z)$ remain regular, which produce a z-independent Wronskian starting from some z. We observe numerically that the limit of small ϵ exists with regular solutions. Diverging solutions appear exactly in three dimensions. We suggest that the logarithmic divergence is an artifact of $(2+1)$-dimensional theory and it can be regulated by the dimensional regularization.

We also examine numerical solutions in the Sakai-Sugimoto model. Solutions in this model express similar behavior, although we found the cosh model is more suitable for numerical investigation.

From Eq. (B7), we have for the Sakai-Sugimoto model in $(3+\epsilon)$ dimensions

$$f(z) = \Lambda \frac{1}{g_3} (1+z^2)^{1/6+\delta}, \tag{B17}$$

$$g(z) = g_3 (1+z^2)^{1/2-\delta}, \tag{B18}$$

where $\delta = \epsilon/6$ ($\delta = 0$ is three dimensional and $\delta = 1/3$ is five dimensional) and Λ is added to make z dimensionless.

Using Sakai-Sugimoto metric factors, we find solutions in the pure Maxwell theory. We see that the dimensionally regulated solutions converge to a finite value in the UV.

With Sakai-Sugimoto metric factors, the Wronskian (left panel) and solutions (right panel) in the Maxwell-Chern-Simons theory are displayed in Fig. 4 in $(3+\epsilon)$ dimensions. The dimensionally regulated case in Fig. 4 shows that the Wronskian tends to a plateau and solutions are regular in the UV. Decreasing ϵ, we find that this trend remains, which suggests that the limit $\epsilon = 0$ exists.

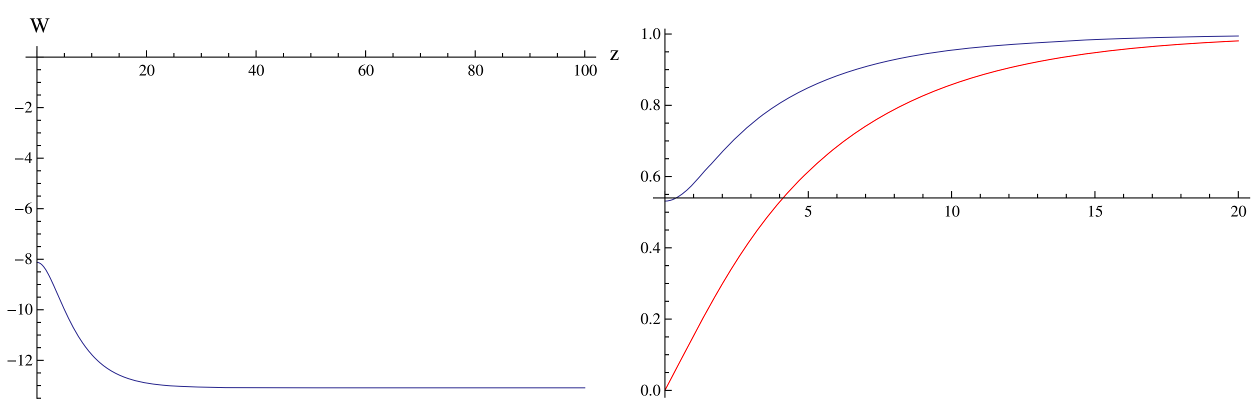

FIG. 3. Cosh model in $(3+\epsilon)$-dimensions. The Wronskian $W(Q,z)$ (left panel) and gauge functions $V(Q,z)$, $A(Q,z)$ (right panel) in the Maxwell-Chern-Simons theory. Parameters are $Q = 5$, $\Lambda = 10$, $2\kappa = 1$, $r = 10$, $\delta = 0.1$.

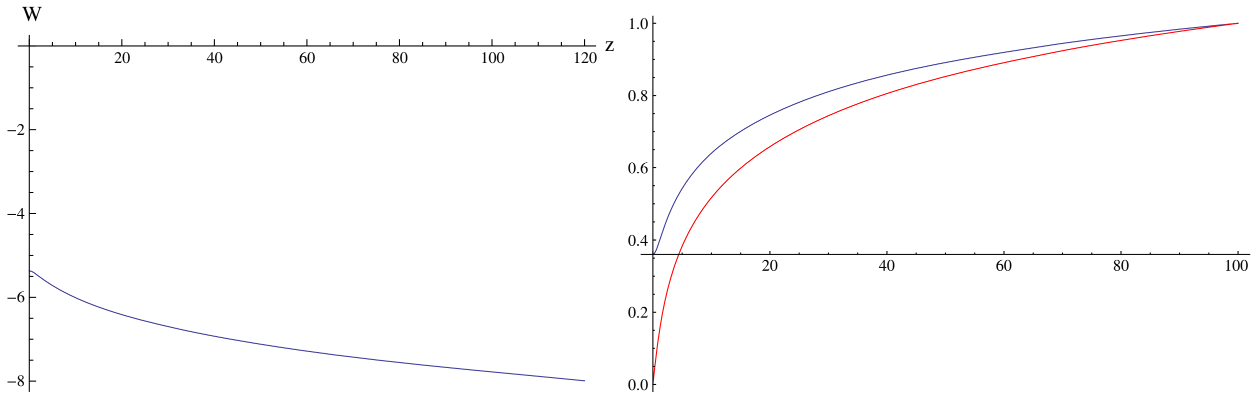

FIG. 4. Sakai-Sugimoto model in $(3+\epsilon)$ dimensions. The Wronskian $W(Q,z)$ (left panel) and gauge functions $V(Q,z)$, $A(Q,z)$ (right panel) in the Maxwell-Chern-Simons theory. Parameters are $Q=5$, $\Lambda=10$, $2\kappa=1$, $r=5$, $\delta=0.1$.

Our numerical data justify the assumption that the Wronskian is independent of the radial coordinate for $z\gg1$. We find that adding the Chern-Simons term does not solve the problem of logarithmically diverging solutions. We used the dimensional regularization in $(3+\epsilon)$ dimensions with small ϵ in order to regulate the gauge functions $V(Q,z)$ and $A(Q,z)$ which produce constant behavior for the Wronskian $W(Q)$. It also justifies the use of the finite UV boundary conditions for $V(Q,z)$ and $A(Q,z)$.

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