Bose and Fermi statistics and the regularization of the nonrelativistic Jacobian for the scale anomaly

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We regulate in Euclidean space the Jacobian under scale transformations for two-dimensional nonrelativistic fermions and bosons interacting via contact interactions and compare the resulting scaling anomalies. For fermions, Grassmannian integration inverts the Jacobian; however, this effect is canceled by the regularization procedure and a result similar to that of bosons is attained. We show the independence of the result with respect to the regulating function, and show the robustness of our methods by comparing the procedure with an effective potential method using both cutoff and ζ-function regularization.

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I. INTRODUCTION

The possibility of measuring the effects of quantum anomalies in nonrelativistic systems has prompted the development of quantum-field-theoretical approaches in the mathematical description of such anomalies [1–5]. In particular, we have recently developed a path-integral, Fujikawa approach to the calculation of anomalous corrections to virial theorems and equations of state for systems with a classical $SO(2,1)$ symmetry [6–10]. Central to this approach is the ability to calculate the Fujikawa Jacobian J for two-dimensional Bose and Fermi particles with a contact interaction:

$$\mathcal{L} = \psi^* \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{\lambda}{2} (\psi^* \psi)^2, \tag{1}$$

$$\mathcal{L} = \sum_{\sigma=\uparrow\downarrow} \psi_\sigma^* \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi_\sigma - \lambda \psi_\uparrow^* \psi_\downarrow^* \psi_\downarrow \psi_\uparrow. \tag{2}$$

In (1) and (2), the fields ψ obey Bose and Fermi statistics respectively. In [7], the path integral for the bosonic system was calculated for both zero and finite temperature, and the anomaly so calculated coincided with those obtained by other means in the literature [11,12]. These anomalies for systems with contact interactions control the anomalous sector for 2D, trapped ultracold dilute atoms, and, as shown by the author of [1], they can be interpreted as the Tan contact term, which determines much of the thermodynamics of such systems. The results and formalism of [6–10] for bosonic fields still remained to be developed for Fermi fields. We do so below, and we find similar results. In Sec. II we give a short review of the Fujikawa approach for systems with classical scale invariance [more generally $SO(2,1)$], such as the ones studied here. The details of the calculation of the Jacobian J are given in Secs. III and IV for the Fermi and Bose cases respectively. To further elucidate the consistency and robustness of our calculations, in Sec. V we compare for the fermion case the methods and results of this paper with an effective potential method using cutoff and ζ-function regularization. The selection of the regulating matrix M in both cases (fermionic and bosonic) is highlighted and the similarities and differences are commented upon in the Conclusions.

II. SCALE INVARIANCE AND FUJIKAWA

Under dilation the coordinates $x = (x_0, \vec{x})$ and fields ϕ_i transform as

$$\begin{aligned} \vec{x}' &= e^{\eta} \vec{x}, \\ x_0' &= e^{2\eta} x_0, \\ \phi_i'(x') &= e^{[\phi_i]\eta} \phi_i(x), \end{aligned} \tag{3}$$

where $[\phi_i]$ is the length dimension of ϕ_i (in units where $\hbar = m = 1$) which in two spatial dimensions is $[\phi_i] = -1$. Taking η infinitesimal

$$\begin{aligned} \delta\vec{x} &= \eta\vec{x}, \\ \delta x_0 &= 2\eta x_0, \\ \delta\phi_i &= \eta\theta\phi_i, \\ \theta &\equiv (-1 - \vec{x}\cdot\vec{\nabla} - 2x_0\partial_{x_0}). \end{aligned} \tag{4}$$

A scale-invariant Lagrangian transforms as $\mathcal{L}'(x_0', \vec{x}') = e^{[\mathcal{L}]\eta} \mathcal{L}(x_0, \vec{x})$ where in two dimensions $[\mathcal{L}] = -4$,

$$\begin{aligned} \delta\mathcal{L} &= \eta(-4 - \vec{x}\cdot\vec{\nabla} - 2x_0\partial_{x_0})\mathcal{L} = \eta\partial_\mu(\mathcal{L} f^\mu), \\ f^\mu &= (-2x_0, -\vec{x}). \end{aligned} \tag{5}$$

Under a change of variables $\phi_i'(x) = \phi_i(x) + \eta(x)\delta\phi_i(x)$ [13] the path integral becomes

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$$\begin{aligned}\int\prod_i d\phi_i e^{iS[\phi_i]} &= \int\prod_i d\phi_i' J^{\pm 1} e^{iS[\phi_i'-\eta(x)\delta\phi_i'(x)]}\\ &= \int\prod_i d\phi_i' e^{-\int d^3x\eta(x)\mathcal{A}(x)} e^{iS[\phi_i']-i\int d^3x\eta(x)\delta L - i\int d^3x\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i'}\partial_\mu\eta(x)\delta\phi_i'}\\ &= \int\prod_i d\phi_i' e^{iS[\phi_i']} e^{-\int d^3x\eta(x)\mathcal{A}(x)+i\int d^3x\eta(x)\partial_\mu(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i'}\delta\phi_i'-\mathcal{L}f^\mu)},\end{aligned} \tag{6}$$

where $+$ is for bosons, $-$ for fermions. With the far right term identified as the Noether current j^μ, and noting that the field ϕ' is a dummy variable, for this to be true for arbitrary $\eta(x)$ then:

$$\partial_\mu\langle j^\mu\rangle = -i\langle\mathcal{A}\rangle. \tag{7}$$

The anomaly $\langle\mathcal{A}\rangle$ which is given by the Jacobian in Fujikawa's method is

$$\langle\mathcal{A}\rangle = \mathrm{tr}\left[\frac{\delta\delta\phi_i(x)}{\delta\phi_j(y)}\right]_{x=y} = \pm\mathrm{tr}[\delta_{ij}\theta\delta(x-y)]_{x=y}. \tag{8}$$

III. FERMIONS

For the infinitesimal scale transformation

$$\begin{aligned}\delta\vec{x} &= \eta\vec{x},\\ \delta t &= 2\eta t,\\ \delta\psi_{\uparrow\downarrow} &= \eta\theta\psi_{\uparrow\downarrow}(\vec{x},t),\\ \delta\psi^\dagger_{\uparrow\downarrow} &= \eta\theta\psi^\dagger_{\uparrow\downarrow}(\vec{x},t),\\ \theta &\equiv (-1-\vec{x}\cdot\vec{\nabla}-2t\partial_t),\end{aligned} \tag{9}$$

we can apply Noether's theorem to the Bardeen-Cooper-Schrieffer Lagrangian $\mathcal{L}=\sum_{\sigma=\uparrow\downarrow}\psi^\dagger_\sigma(i\partial_t+\frac{\nabla^2}{2})\psi_\sigma - \lambda\psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\downarrow\psi_\uparrow$ which in two dimensions is classically scale invariant, to get a conserved charge [1]:

$$\begin{aligned}D &= \int d^2\vec{x}\,\vec{x}\cdot\vec{j} - 2tH,\\ \vec{j} &= -\frac{i}{2}(\psi^\dagger\vec{\nabla}\psi - \vec{\nabla}\psi^\dagger\psi).\end{aligned} \tag{10}$$

The classical scale invariance, hence the conservation law, is spoiled by the presence of a quantum anomaly [14]. The fermionic anomaly in Euclidean space is given by $\mathcal{A} = -\mathrm{tr}[\theta\delta^3(x)\delta_{ij}]|_{x=0}$ which differs in sign to the bosonic anomaly due to the transformation properties of Grassmann integrals [15–18]. The trace is over the internal space of the fields.

To regulate the ill-defined expression $\mathrm{tr}[\theta\delta^3(x)\delta_{ij}]|_{x=0}$, we first rewrite the Lagrangian using a constraining field ϕ:

$$\mathcal{L} = \sum_{\sigma=\uparrow\downarrow}\psi^*_\sigma\left(i\partial_t+\frac{\nabla^2}{2}\right)\psi_\sigma + \frac{\phi^*\phi}{\lambda} + (\psi_\uparrow\psi_\downarrow)\phi + (\psi^*_\downarrow\psi^*_\uparrow)\phi^*, \tag{11}$$

which has the classical solution $\phi = -\lambda\psi^*_\downarrow\psi^*_\uparrow$ [19].

The Lagrangian can be written compactly as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\psi^*_\uparrow \quad \psi_\downarrow)M\begin{pmatrix}\psi_\uparrow\\ \psi^*_\downarrow\end{pmatrix} + \frac{1}{\lambda}\phi^*\phi,\\ M &= \begin{pmatrix}-\partial_\tau+\frac{\nabla^2}{2} & -\phi^*\\ -\phi & -\partial_\tau-\frac{\nabla^2}{2}\end{pmatrix},\end{aligned} \tag{12}$$

where a transformation to Euclidean space has been made, and anticommutivity of the fields was used.

In momentum space the quadratic operator M takes for constant ϕ the form

$$M = \begin{pmatrix} i\omega-\frac{k^2}{2} & -\phi^*\\ -\phi & i\omega+\frac{k^2}{2}\end{pmatrix} \tag{13}$$

so that $M^\dagger M$ takes the form

$$M^\dagger M = \begin{pmatrix}\omega^2+\xi^2(\vec{k})+\phi\phi^* & 0\\ 0 & \omega^2+\xi^2(\vec{k})+\phi\phi^*\end{pmatrix}, \tag{14}$$

where $\xi(\vec{k}) = \frac{\vec{k}^2}{2}$. We first write $-\mathrm{tr}[\theta\delta^3(x-y)I_4]|_{x=y}$, where I_n is the $n\times n$ identity matrix, as $-2\mathrm{tr}[\theta\delta^3(x-y)I_2]|_{x=y}$. We regulate this expression by instead calculating

$$-2\mathrm{tr}\left[\theta f\left(\frac{M^\dagger M}{\Lambda^4}\right)\delta^3(x-y)I_2\right]\bigg|_{x=y}, \tag{15}$$

where f has the property that $f(\infty)=0$ and $f(0)=I_2$, and we take the limit $\Lambda\to\infty$ at the end of the calculation. This will regulate large eigenvalues of $M^\dagger M$ when $\delta^3(x-y)I_2$ is expanded via a completeness relation using the eigenbasis [20] of $M^\dagger M$ [21]. Since

$$f\left(\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\right) = \begin{pmatrix}f(\lambda_1) & 0\\ 0 & f(\lambda_2)\end{pmatrix}, \tag{16}$$

we have

$$-\mathrm{tr}\left[\theta f\left(\frac{M^\dagger M}{\Lambda^4}\right)\delta^3(x-y)I_2\right]\Big|_{x=y} = -\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}\mathrm{tr}\left[\theta f\left(\frac{M^\dagger M}{\Lambda^4}\right)\right]$$
$$= 2\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}(1+i\vec{x}\cdot\vec{k}-2ix_0\omega_0)f\left(\frac{\omega^2+\xi^2(\vec{k})+\phi\phi^*}{\Lambda^4}\right)$$
$$= 2\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}f\left(\frac{\omega^2+\xi^2(\vec{k})+\phi\phi^*}{\Lambda^4}\right)$$
$$= 2\Lambda^4\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}f\left(\omega^2+\xi^2(\vec{k})+\frac{\phi\phi^*}{\Lambda^4}\right), \tag{17}$$

where $\delta^3(x-y)$ was expanded in a Fourier transform. We Taylor expand the integrand:

$$\mathrm{tr}\left[f\left(\frac{M^\dagger M}{\Lambda^4}\right)\delta^3(x)I_2\right]\Big|_{x=0} = 2\Lambda^4\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}f(\omega^2+\xi^2(\vec{k}))+2\Lambda^4\int\frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}f'(\omega^2+\xi^2(\vec{k}))\frac{\phi\phi^*}{\Lambda^4}+O\left[\frac{(\phi\phi^*)^2}{\Lambda^4}\right]. \tag{18}$$

The first term is independent of the interaction. The second term is evaluated in (A1) of the Appendix, while higher terms vanish in the $\Lambda\to\infty$ limit.

The anomaly is therefore:

$$\mathcal{A} = -\mathrm{tr}[\delta^3(x-y)I_4]|_{x=y} = -2\mathrm{tr}[\theta\delta^3(x)I_2]|_{x=y}$$
$$= 4\Lambda^4\left(-\frac{\Omega_2^2}{4(2\pi)^3}\frac{\phi\phi^*}{\Lambda^4}\right) = -\frac{\phi\phi^*}{2\pi}. \tag{19}$$

Plugging in the classical solution for the auxiliary field:

$$\mathcal{A} = -\frac{(\lambda\psi_\downarrow^*\psi_\uparrow^*)(\lambda\psi_\downarrow^*\psi_\uparrow^*)^*}{2\pi} = -\frac{\lambda^2}{2\pi}\psi_\uparrow^*\psi_\downarrow^*\psi_\downarrow\psi_\uparrow. \tag{20}$$

IV. BOSONS

Similarly, for the boson case, a saddle point expansion of the action $\int d^3x\mathcal{L} = \int d^3x[\psi^*(i\partial_t+\frac{\nabla^2}{2})\psi-\frac{\lambda}{2}(\psi^*\psi)^2]$ about the classical solution ψ_{cl} gives for the bilinear piece:

$$\mathcal{L} = \frac{1}{2}(\psi^*\quad\psi)M\begin{pmatrix}\psi\\ \psi^*\end{pmatrix},$$
$$M = \begin{pmatrix}-\partial_\tau+\frac{\nabla^2}{2}-2\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}} & -\lambda\psi_{\mathrm{cl}}\psi_{\mathrm{cl}}\\ -\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}}^* & \partial_\tau+\frac{\nabla^2}{2}-2\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}}\end{pmatrix}. \tag{21}$$

Following the same procedure as in Sec. III, the regulating matrix M becomes in momentum space:

$$M = \begin{pmatrix}i\omega-\frac{k^2}{2}-2\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}} & -\lambda\psi_{\mathrm{cl}}\psi_{\mathrm{cl}}\\ -\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}}^* & -i\omega-\frac{k^2}{2}-2\lambda\psi_{\mathrm{cl}}^*\psi_{\mathrm{cl}}\end{pmatrix}. \tag{22}$$

The generic matrix

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{23}$$

has eigenvalues

$$\begin{pmatrix}\frac{1}{2}(A+D)+\sqrt{\frac{(A-D)^2}{4}+BC} & 0\\ 0 & \frac{1}{2}(A+D)-\sqrt{\frac{(A-D)^2}{4}+BC}\end{pmatrix}. \tag{24}$$

First multiplying $M^\dagger$ and M and then using (24) for the eigenvalues of $M^\dagger M$ [22] one gets

$$M^\dagger M = \begin{pmatrix}(\sqrt{\omega^2+\xi^2(\vec{k})+A^2}+\lambda\psi^*\psi)^2 & 0\\ 0 & (\sqrt{\omega^2+\xi^2(\vec{k})+A^2}-\lambda\psi^*\psi)^2\end{pmatrix},$$
$$A^2 = 2\lambda k^2\psi^*\psi+4\lambda^2(\psi^*\psi)^2. \tag{25}$$

Therefore $\int \frac{d\omega d^2k}{(2\pi)^3}\mathrm{tr}f(\frac{M^\dagger M}{\Lambda^4}) \equiv \int \frac{d\omega d^2k}{(2\pi)^3}\mathrm{tr}g(\sqrt{\frac{M^\dagger M}{\Lambda^4}})$

$$
\begin{aligned}
&= \Lambda^4 \int \frac{d\omega d^2k}{(2\pi)^3}\left(g\left(\sqrt{\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2}+\frac{\lambda\psi^*\psi}{\Lambda^2}}\right)+g\left(\sqrt{\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2}-\frac{\lambda\psi^*\psi}{\Lambda^2}}\right)\right)\\
&= \Lambda^4 \int \frac{d\omega d^2k}{(2\pi)^3}\left[2g\left(\sqrt{\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2}}\right)+g''\left(\sqrt{\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2}}\right)\left(\frac{\lambda\psi^*\psi}{\Lambda^2}\right)^2\right]+O\left[\frac{(\psi^*\psi)^4}{\Lambda^4}\right],\\
\tilde{A}^2 &= 2\lambda k^2\psi^*\psi+\frac{4\lambda^2(\psi^*\psi)^2}{\Lambda^2}. \qquad (26)
\end{aligned}
$$

The first term in the integrand can be rewritten as $f(\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2})$ which can be Taylor expanded:

$$
\begin{aligned}
&f(\omega^2+\xi^2(\vec{k}))+f'(\omega^2+\xi^2(\vec{k}))\frac{\tilde{A}^2}{\Lambda^2}+\frac{1}{2}f''(\omega^2+\xi^2(\vec{k}))\left(\frac{\tilde{A}^2}{\Lambda^2}\right)^2\\
&= f(\omega^2+\xi^2(\vec{k}))+f'(\omega^2+\xi^2(\vec{k}))\frac{2\lambda k^2\psi^*\psi}{\Lambda^2}+f'(\omega^2+\xi^2(\vec{k}))\frac{4\lambda^2(\psi^*\psi)^2}{\Lambda^4}\\
&+\frac{1}{2}f''(\omega^2+\xi^2(\vec{k}))\frac{(2\lambda k^2\psi^*\psi)^2}{\Lambda^4}+O\left[\frac{(\psi^*\psi)^3}{\Lambda^6}\right]. \qquad (27)
\end{aligned}
$$

The first term is independent of the interaction and can be ignored. The second term can be renormalized into a chemical potential which explicitly breaks scale invariance [23]. Using (A1) of the Appendix, the third and fourth integrals add to zero.

The integral of the last term in (26) is from (A1):

$$
\begin{aligned}
\mathrm{tr}[\theta\delta^3(x)I_2]|_{x=0} &= -\int \frac{d\omega d^2k}{(2\pi)^3}\mathrm{tr}f\left(\frac{M^\dagger M}{\Lambda^4}\right)\\
&= -\Lambda^4\int \frac{d\omega d^2k}{(2\pi)^3}g''\left(\sqrt{\omega^2+\xi^2(\vec{k})+\frac{\tilde{A}^2}{\Lambda^2}}\right)\left(\frac{\lambda\psi^*\psi}{\Lambda^2}\right)^2\\
&= -\Lambda^4\int \frac{d\omega d^2k}{(2\pi)^3}g''\left(\sqrt{\omega^2+\xi^2(\vec{k})}\right)\left(\frac{\lambda\psi^*\psi}{\Lambda^2}\right)^2+O\left[\frac{(\psi^*\psi)^3}{\Lambda^2}\right]\\
&= -\frac{\lambda^2(\psi^*\psi)^2}{4\pi}. \qquad (28)
\end{aligned}
$$

V. RELATIONSHIP WITH THE EFFECTIVE POTENTIAL METHOD

Fujikawa's method identifies the anomaly as the ill-defined expression $-\mathrm{tr}[\delta^3(x-y)I_4]|_{x=y}$, which is regulated by $-2\mathrm{tr}[f(\frac{M^\dagger M}{\Lambda^4})\delta^3(x-y)I_2]|_{x=y}$. It should be emphasized that f is arbitrary except for the reasonable boundary conditions $f(0)=1$, $f(\infty)=f'(\infty)=0$. We will now specialize to $f(X)=e^{-X}$ to make a connection between Fujikawa's method and the effective action in the fermion case. Indeed,

$$
\mathrm{tr}[e^{-\frac{M^\dagger M}{\Lambda^4}}\delta^3(x-y)I_2]|_{x=y} = \langle x|\mathrm{tr}e^{-\frac{M^\dagger M}{\Lambda^4}}|x\rangle \equiv h(x,x) \qquad (29)
$$

is the heat kernel of $M^\dagger M$, where M is the Hessian of Eq. (12), and it should be kept in mind that $h(x,x)$ depends on the "proper time" $\frac{1}{\Lambda^4}$. In what follows we will calculate Eq. (29) and use ζ regularization to derive the effective potential and from this the anomaly via the β function. With the help of an infrared regulator, we will then calculate Eq. (29) as a series similar to Eq. (18), which is analogous to a Seeley-DeWitt expansion [24,25], and with a cutoff regulator show that the anomaly is coming from the $\phi^\dagger\phi$ sector of the effective potential as indicated by Fujikawa's method. Finally, we will comment on the calculation of the determinant of $M^\dagger M$ and taking the square root (which halves the one-loop effective action up to a phase) rather than M, which unlike $M^\dagger M$, is not positive definite, a critical feature of regularization in Fujikawa's method.

Performing the path integral over the fermion fields in Eq. (12) gives

$$\int [d\psi_\sigma][d\psi^\dagger_\sigma][d\phi][d\phi^*]e^{-\int d^2xd\tau\left[\frac{1}{2}(\psi^*_\uparrow \quad \psi_\downarrow)M\begin{pmatrix}\psi_\uparrow \\ \psi^*_\downarrow\end{pmatrix}+\frac{1}{\lambda}\phi^*\phi\right]} = \int [d\phi][d\phi^*]e^{-\int d^2xd\tau V_{\rm eff}(\phi,\phi^*)}, \tag{30}$$

where $V_{\rm eff}(\phi,\phi^*) = \frac{1}{\lambda}\phi^*\phi - \frac{1}{VT}\ln(\sqrt{\det M^\dagger M})$. We will evaluate the determinant via construction of the ζ function. For constant ϕ, using Eq. (14) on Eq. (29) we get

$$h(x,x) = 2\int \frac{d\omega d^2\vec{k}}{(2\pi)^3}e^{-\frac{\omega^2+\xi^2(\vec{k})+\phi^*\phi}{\Lambda^4}} = 2\int \frac{d\omega d\xi}{(2\pi)^2}e^{-\frac{\omega^2+\xi^2+\phi^*\phi}{\Lambda^4}} = \frac{\Lambda^4}{4\pi}e^{-\frac{\phi^*\phi}{\Lambda^4}}. \tag{31}$$

We construct the ζ function for $M^\dagger M$ [26] via analytic continuation using the Mellin transform [27]:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\int d^3x h(x,x) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\frac{\mu^4}{4\pi t}e^{-\frac{\phi^*\phi}{\mu^4}t}VT = \frac{\mu^4}{4\pi}\left(\frac{\mu^4}{\phi^*\phi}\right)^{s-1}\frac{VT}{s-1}, \tag{32}$$

where VT is the volume of spacetime and $\frac{1}{\Lambda^4} = \frac{1}{\mu^4}t$ was introduced to make the proper time t dimensionless [28], and μ is a momentum scale. Therefore we have

$$\sqrt{{\rm Det}M^\dagger M} = e^{-\frac{\zeta'(0)}{2}},$$

$$V_{\rm eff} = V_0 + \frac{1}{VT}\frac{\zeta'(0)}{2} = -\frac{\phi^*\phi}{\lambda(\mu)} + \frac{\phi^*\phi}{8\pi}\left(\log\left(\frac{\phi^*\phi}{\mu^4}\right) - 1\right), \tag{33}$$

where the tree level term V_0 comes from the Hubbard-Stratonovich transformation of Eq. (12). Differentiating (33) with respect to μ, the independence of $V_{\rm eff}$ on μ gives $\beta(\lambda) = \frac{\lambda^2}{2\pi}$. The anomaly is therefore [29]

$$\mathcal{A} = \beta(\lambda)\frac{\partial\mathcal{L}}{\partial\lambda} = -\frac{\lambda^2}{2\pi}\psi^*_\uparrow\psi^*_\downarrow\psi_\downarrow\psi_\uparrow, \tag{34}$$

agreeing with Eq. (20).

For the next method, instead of calculating the entire heat kernel, we will make the expansion indicated in Eq. (18), and we are interested in the second term on the rhs which in Fujikawa's method produces the anomaly. This term is

$$h_2(x,x) = 2\Lambda^4\int \frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}e^{-(\omega^2+\xi^2(\vec{k}))}\frac{\phi\phi^*}{\Lambda^4}. \tag{35}$$

Because we are not summing the entire series, we will need to introduce an infrared regulator [23] by making the replacement $\xi(\vec{k}) \to \xi(\vec{k}) - \mu$, where μ is negative. This creates a positive gap in the spectrum which will help us avoid infrared divergences, and we will take $\mu \to 0$ at the end of the calculation

$$h_2(x,x) = 2\Lambda^4\int \frac{d\omega}{2\pi}\frac{d^2\vec{k}}{(2\pi)^2}e^{-(\omega^2+\xi^2(\vec{k})-2\frac{\mu\xi(\vec{k})}{\Lambda^2}+\frac{\mu^2}{\Lambda^4})}\frac{\phi\phi^*}{\Lambda^4} = \frac{1}{4\pi}\left(1+{\rm erf}\left[\frac{\mu}{\Lambda^2}\right]\right)\phi^*\phi, \tag{36}$$

where ${\rm erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dte^{-t^2}$ is the error function. Using the identities $\ln(M^\dagger M) = -\int_0^\infty \frac{e^{-M^\dagger M\epsilon}}{\epsilon}d\epsilon$ and $\sqrt{{\rm Det}M^\dagger M} = e^{\frac{1}{2}\int {\rm tr}\ln(M^\dagger M)d^3x}$ [30] one gets

$$V_{\rm eff} = -\frac{\phi^*\phi}{\lambda(\Lambda')} - \frac{1}{2}\int_{\frac{1}{\Lambda'^4}}^\infty \frac{h(x,x)}{1/\Lambda^4}d(1/\Lambda^4),$$

$$V^{(2)}_{\rm eff} = -\frac{\phi^*\phi}{\lambda(\Lambda')} - \frac{1}{2}\int_{\frac{1}{\Lambda'^4}}^\infty \frac{\frac{1}{4\pi}(1+{\rm erf}[\frac{\mu}{\Lambda^2}])\phi^*\phi}{1/\Lambda^4}d(1/\Lambda^4), \tag{37}$$

where Λ' was introduced to regulate the UV divergence, which along with the condition $\mu < 0$ for the IR, makes the integral convergent. Differentiating both sides of Eq. (37) with respect to Λ' and using the fundamental theorem of calculus one gets

$$0 = \frac{\lambda'}{\lambda^2} - \frac{1}{2}\frac{4}{\Lambda'^5}\frac{\frac{1}{4\pi}(1+{\rm erf}[\frac{\mu}{\Lambda'^2}])}{1/\Lambda'^4},$$

$$\Lambda'\lambda' = \beta(\lambda) = \frac{\lambda^2}{2\pi}, \tag{38}$$

where ${\rm erf}[\frac{\mu}{\Lambda'^2}] \to 0$ both as $\mu \to 0$ and $\Lambda' \to \infty$. Therefore one can see that the anomaly comes from $h_2(x,x)$ when $h(x,x)$ is Seeley-DeWitt expanded in Eq. (37). The reason that $h_2(x,x)$ [which is related to the second term on the rhs of Eq. (18) using Fujikawa's method] determines the anomaly can be seen from a comparison of Eq. (33) with the second term on the rhs of Eq. (37): while $V_0 = -\frac{\phi^*\phi}{\lambda}$ is classically conformally invariant, $\frac{\phi^*\phi}{8\pi}(\log(\frac{\phi^*\phi}{\mu^4}) - 1)$ is not due to the nonlocal term $\phi^*\phi\log(\phi^*\phi)$ [10], and the behavior of this term is related to $\phi^*\phi\log[\mu^4]$, which is provided by the $h_2(x,x)$ term by performing the integral in Eq. (37).

We now comment on setting $\sqrt{\mathrm{Det}M^\dagger M} = \mathrm{Det}M$. M is Hermitian by itself in real space, but in Euclidean space, it is not, so use of $M^\dagger M$ was required in Fujikawa's method to expand $\delta^3(x-y)I_2 = \sum_n \phi_n(x)\phi_n^\dagger(y)$ in an eigenbasis $\phi_n(x)$ of $M^\dagger M$ and to regulate the eigenvalues with $f(\frac{M^\dagger M}{\Lambda^4})$. Since $\sqrt{\mathrm{Det}M^\dagger M} = \mathrm{Det}M e^{i\theta}$, we lose the phase θ in the calculation of the effective potential, where θ is some real functional of the fields. However, in Euclidean space, this phase contributes an imaginary part to the effective action

$$e^{-S_{\mathrm{eff}}} = e^{-S_0 + \ln(\sqrt{\mathrm{Det}M^\dagger M}) - i\theta}. \tag{39}$$

While the possibility exists of complex effective potentials [31], the β function being real will not be affected by the addition of a complex part in Eqs. (33) and (37), so the argumentation leading to Eqs. (34) and (38) would still be valid. As a check on this, the one-loop contribution to the effective potential can be written as [32,33]

$$V_{\mathrm{eff}}^{(1)} = -\int \frac{d^2\vec{k}}{(2\pi)^2}[E(\vec{k}) - \xi(\vec{k})], \tag{40}$$

where $E(\vec{k}) = \sqrt{\xi^2(\vec{k}) + \phi^*\phi}$ is the single-fermion excitation energy. Performing the integral with cutoff Λ on the momentum gives

$$V_{\mathrm{eff}}^{(1)} = \frac{\phi^*\phi}{8\pi}\left(\log\left(\frac{\phi^*\phi}{\Lambda^4}\right) - 1\right) - \frac{1}{4\pi}(\phi^*\phi)\left(\frac{1}{2} + \ln 2\right), \tag{41}$$

which agrees with the result of Eq. (33) after renormalization.

VI. CONCLUSIONS

The fermion and boson anomalies for nonrelativistic scale-invariant systems such as those studied here have formally similar expressions $\pm(2)\mathrm{tr}[\theta\delta^3(x-y)I_2]|_{x=y}$, differing by a sign due to Berezin integration and a factor of 2 from the two fermion species, as expected. However, the trace is regulated with a different regulating matrix M depending on the statistics. In both cases, the real time version of M is Hermitian, but the Euclidean one is not, and hence we had to work with $M^\dagger M$ in order to assure the regulating effects of large eigenvalues of $M^\dagger M$ when $\delta^3(x-y)I_2$ is expanded via a completeness relation. Our method reproduces known results for both fermions and bosons, Eqs. (20) and (28). The robustness of our approach was studied by comparing our methods and results with an effective potential calculation using both cutoff and ζ-function regularization.

In this work, we only considered the homogeneous case, i.e., constant background fields. A heat kernel approach to consider nonhomogeneous systems (trapped systems, for instance) is currently being developed, and we hope to report on this and applications to ultracold atoms elsewhere.

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APPENDIX SOME USEFUL INTEGRALS

In this paper we make use of the following integrals:

$$\begin{aligned} \int f'(\omega^2 + \xi^2(\vec{k}))d^2\vec{k}d\omega &= -\frac{\Omega_2^2}{4}, \\ \int f''(\omega^2 + \xi^2(\vec{k}))k^4 d^2\vec{k}d\omega &= \frac{\Omega_2^2}{2}, \\ \int f''(\sqrt{\omega^2 + \xi^2(\vec{k})})d^2\vec{k}d\omega &= \frac{\Omega_2^2}{2}, \end{aligned} \tag{A1}$$

where $f(0) = 1$, $f(\infty) = f'(\infty) = 0$, and $\Omega_2 = 2\pi$ is the two-dimensional solid angle.

A derivation is as follows:

$$\begin{aligned} &\int f^{(m)}(\omega^2 + \xi^2(\vec{k}))k^{4s}d^2\vec{k}d\omega \\ &= 4^s \int f^{(m)}(\omega^2 + \xi^2(\vec{k}))\xi^{2s}d^2\vec{k}d\omega \\ &= 4^s \int f^{(m)}(\omega^2 + \xi^2(\vec{k}))\xi^{2s}d\omega\Omega_2 kdk \\ &= 4^s\Omega_2 \int f^{(m)}(\omega^2 + \xi^2)\xi^{2s}d\omega d\xi. \end{aligned} \tag{A2}$$

Due to the evenness of the integrand, we extend the integral over ξ from $[0,\infty)$ to $(-\infty,\infty)$ by including a factor of $1/2$, and then go into polar coordinates:

$$\begin{aligned} &\frac{4^s\Omega_2}{2}\int f^{(m)}(\omega^2 + \xi^2)\xi^{2s}d\omega d\xi \\ &= \frac{4^s\Omega_2}{2}\int f^{(m)}(r^2)r^{2s+1}dr \int \sin^{2s}\theta d\theta \\ &= 4^{s-1}\Omega_2 \int f^{(m)}(x)x^s dx \int \sin^{2s}\theta d\theta. \end{aligned} \tag{A3}$$

Plugging in $m = 2$ and $s = 1$, and $m = 1$ and $s = 0$, and integrating over x by parts with the specified boundary conditions on f, gives the above two integrals. The third integral of (A1) proceeds similarly.

[1] J. Hofmann, Phys. Rev. Lett. **108**, 185303 (2012).
[2] E. Braaten, Lect. Notes Phys. **836**, 193 (2012).
[3] M. Valiente, N. T. Zinner, and K. Mølmer, Phys. Rev. A **84**, 063626 (2011).
[4] E. Vogt, M. Feld, B. Fröhlich, D. Pertot, M. Koschorreck, and M. Köhl, Phys. Rev. Lett. **108**, 070404 (2012).
[5] J. Levinsen and M. M. Parish, *Annual Review of Cold Atoms and Molecules* (World Scientific, Singapore, 2015).
[6] C. R. Ordóñez, Physica (Amsterdam) **446A**, 64 (2016).
[7] C. L. Lin and C. R. Ordóñez, Phys. Rev. D **91**, 085023 (2015).
[8] C. L. Lin and C. R. Ordóñez, Adv. High Energy Phys. **2015**, 796275 (2015).
[9] C. L. Lin and C. R. Ordóñez, Phys. Rev. D **92**, 085050 (2015).
[10] C. L. Lin and C. R. Ordóñez, Adv. High Energy Phys. **2016**, 2809290 (2016).
[11] T. Haugset and F. Ravndal, Phys. Rev. D **49**, 4299 (1994).
[12] O. Bergman, Phys. Rev. D **46**, 5474 (1992).
[13] $\delta\phi_i(x)$ here does not include the parameter $\eta(x)$.
[14] R. Jackiw, in *M.A.B. Beg Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).
[15] D. Bailin and A. Love, *Introduction to Gauge Field Theory* (Institute of Physics Publishing, Bristol, England, 1993).
[16] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[17] J. Kapusta and C. Gale, *Finite Temperature Field Theory: Principles and Applications* (Cambridge University Press, Cambridge, England, 2006).
[18] F. Berezin, *Method of Second Quantization* (Academic Press, New York, 1966).
[19] This is equivalent to using the Hubbard-Stratonovich transformation in the path integral.
[20] While in this case $M^\dagger M$ is already diagonal, in general one can transform into the eigenbasis in which $M^\dagger M$ is diagonal.
[21] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).
[22] Since we are computing a trace, we can always use the eigenbasis of $M^\dagger M$ that produces Eq. (25).
[23] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[24] Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity* (Cambridge University Press, New York, 2007).
[25] R. T. Seeley, Proc. Symp. Pure Math. **10**, 288 (1967).
[26] $\zeta(s) = \sum_i \frac{1}{\lambda_i^s}$, where λ_i are the eigenvalues of $M^\dagger M$.
[27] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
[28] P. Ramond, *Field Theory A Modern Primer* (Benjamin/Cummings Publishing Company, Reading, MA, 1981).
[29] M. Peskin and D. Schroeder, *An Introduction To Quantum Field Theory* (Westview Press, Boulder, CO, 1995).
[30] D. Vassilevich, Phys. Rep. **388**, 279 (2003).
[31] E. J. Weinberg and A. Wu, Phys. Rev. D **36**, 2474 (1987).
[32] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1996), Vol. 2.
[33] A. Schakel, *Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter* (World Scientific, Singapore, 2008).