Callan-Giddings-Harvey-Strominger vacuum in loop quantum gravity and singularity resolution

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We study here a complete quantization of a Callan-Giddings-Harvey-Strominger vacuum model following loop quantum gravity techniques. Concretely, we adopt a formulation of the model in terms of a set of new variables that resemble the ones commonly employed in spherically symmetric loop quantum gravity. The classical theory consists of two pairs of canonical variables plus a scalar and diffeomorphism (first class) constraints. We consider a suitable redefinition of the Hamiltonian constraint such that the new constraint algebra (with structure constants) is well adapted to the Dirac quantization approach. For it, we adopt a polymeric representation for both the geometry and the dilaton field. On the one hand, we find a suitable invariant domain of the scalar constraint operator, and we construct explicitly its solution space. There, the eigenvalues of the dilaton and the metric operators cannot vanish locally, allowing us to conclude that singular geometries are ruled out in the quantum theory. On the other hand, the physical Hilbert space is constructed out of them, after group averaging the previous states with the diffeomorphism constraint. In turn, we identify the standard observable corresponding to the mass of the black hole at the boundary, in agreement with the classical theory. We also construct an additional observable on the bulk associated with the square of the dilaton field, with no direct classical analog.

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I. INTRODUCTION

Since the early days of the search for a quantum theory of gravity, there has always been the expectation that one of the results that such a complete theory will yield would be the resolution of the spacetime singularities. The first and simplest reason for this argument is that singularities are in a way places where general relativity and the continuous description of spacetime break down. As in other instances in the history of physics, this is the regime where one should look for a new theory. Obviously, any of those new theories should be able to produce the previously known results of the old theory and also be able to describe the physics in the regime where the old theory broke down.

Owing to the fact that working with the full theory is, so far, intractable, it has become standard practice to work with lower dimensional models or symmetry reduced ones, since generally this allows more control over the analysis. One of these systems is the well known Callan-Giddings-Harvey-Strominger (CGHS) model [1]. It is a two-dimensional dilatonic model that, in spite of being simpler and classically solvable, has nontrivial and interesting properties such as a black hole solution, Hawking radiation, etc. It has been proven to be a very convenient model for testing some of the quantum gravity ideas in the past, and it has been subject to many analyses over the past 20 years [2,3] which have shed some light on the properties of the quantum theory of the full four-dimensional theory. In particular, additional studies of the classical [4] and the semiclassical [5] regimes of this model, as well as several studies of its quantization [6–8], have yielded a deeper understanding of some of the interesting physical phenomena in this toy model that can be expected to be valid also in more realistic situations, like four-dimensional black holes. However, there are still several questions that remain unanswered, one of them being the way in which a quantum theory of gravity resolves the classical singularity.

In this article, we study the quantization of the CGHS model in a new perspective, namely within the framework of loop quantum gravity (LQG) [9–11]. This program pursues a background independent nonperturbative quantization of gravity. It provides a robust kinematical framework [12], while the dynamics has not been completely

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implemented. The application of LQG quantization techniques to simpler models, known as loop quantum cosmology (LQC), has dealt with the question of the resolution of the singularity at different levels in models similar to the one under study—see for instance Refs. [5,13–20] among others. In particular, we pay special attention to Refs. [19,20], where a complete quantization of a $3+1$ vacuum spherically symmetric spacetime has been provided, and the singularity of the model is resolved in a very specific manner. The concrete mechanisms are based on the requirement of self-adjointness of some observables of the model, and on the fact that, at the early stages of the quantization, there is a natural restriction to a subspace of the kinematical Hilbert space whose states correspond to eigenstates of the triad operators with nonvanishing eigenvalues, from which the evolution is completely determined.

The purpose of the present work is to put forward a quantization of the CGHS model, by extending the methods of [20] to the case at hand. A study of the dilatonic systems in the lines of Poisson sigma models in LQG was already carried out in Ref. [21]. The feasibility of the project we are considering rests on a classical result that allows us to cast the CGHS model in the so-called polar-type variables [22], similar to the ones used for the $3+1$ spherically symmetric case. These variables were introduced in [23,24] and were further generalized in [25]. Concretely, one introduces a triadic description of the model for the geometry, together with a canonical transformation in order to achieve a description as similar as possible to the one of Ref. [22] in $3+1$ at the kinematical level. Then, after considering some second class conditions and solving the Gauss constraint classically, one ends with a totally first class system with a Hamiltonian and a diffeomorphism constraint. Furthermore, based on a proposal in Refs. [24,25], a redefinition of the scalar constraint is made, such that this constraint admits the standard algebra with the diffeomorphism constraint, while having vanishing brackets with itself. In this situation, we can follow similar arguments to those in [20] to achieve a complete quantization of the CGHS model, showing that the quantum theory provides a description where the singularity is resolved in a certain way. Additionally some new observables emerge in the quantum regime, which have no classical analogues.

The structure of this paper is as follows: in Sec. II, we present a very brief review of the CGHS model to show that it contains a black hole solution with a singularity. Section III is dedicated to recalling how one can derive polar-type variables for the Hamiltonian formulation of the CGHS and $3+1$ spherically symmetric models from a generic two-dimensional dilatonic action, and thus showing the underlying similarity between the two models in these variables. In Sec. IV, we illustrate a way to turn the Dirac algebra of the constraint in the CGHS model into a Lie algebra, and thus prepare it for the Dirac quantization. Section V is about quantization: we first introduce the kinematical Hilbert space of the theory in V A; we then represent the Hamiltonian constraint on this space in V B, and in Sec. V C, we argue for the resolution of the singularity in the CGHS model. Then we put forward a discussion about the properties of the solutions to the Hamiltonian constraint in Sec. V D. Finally we note in Sec. V E that the same observables first derived in [19] can also be introduced here.

II. BRIEF REVIEW OF THE CGHS MODEL

The CGHS model [1] is a two-dimensional dilatonic model. It has a black hole solution, Hawking radiation, and is classically solvable. This, together with the fact that it is easier to handle than the full four-dimensional theory or many other models, makes it a powerful test bench for many of the ideas in quantum gravity. There has been extensive previous work on this model in the literature in the classical and the quantum/semiclassical regime.

The CGHS action is

$$S_{\text{g-CGHS}} = \frac{1}{2G_2}\int d^2x\sqrt{-|g|}e^{-2\phi} \times (R + 4g^{ab}\partial_a\phi\partial_b\phi + 4\lambda^2), \tag{2.1}$$

where G_2 is the two-dimensional Newton constant, ϕ is the dilaton field and λ is the cosmological constant. In double null coordinates $x^\pm = x^0 \pm x^1$ and in conformal gauge

$$g_{+-} = -\frac{1}{2}e^{2\rho}, \qquad g_{--} = g_{++} = 0, \tag{2.2}$$

the solution is

$$e^{-2\rho} = e^{-2\phi} = \frac{G_2 M}{\lambda} - \lambda^2 x^+ x^-, \tag{2.3}$$

where M is a constant of integration which can be identified as the Arnowitt-Deser-Misner (ADM) (at spatial infinity) or the Bondi (at null infinity) mass. The scalar curvature turns out to be

$$R = \frac{4G_2 M\lambda}{\frac{G_2 M}{\lambda} - \lambda^2 x^+ x^-}, \tag{2.4}$$

which corresponds to a black hole with mass M with a singularity at

$$x^+ x^- = \frac{G_2 M}{\lambda^3}. \tag{2.5}$$

The Kruskal diagram of the CGHS black hole is very similar to the four-dimensional Schwarzschild model and is depicted in Fig. 1.

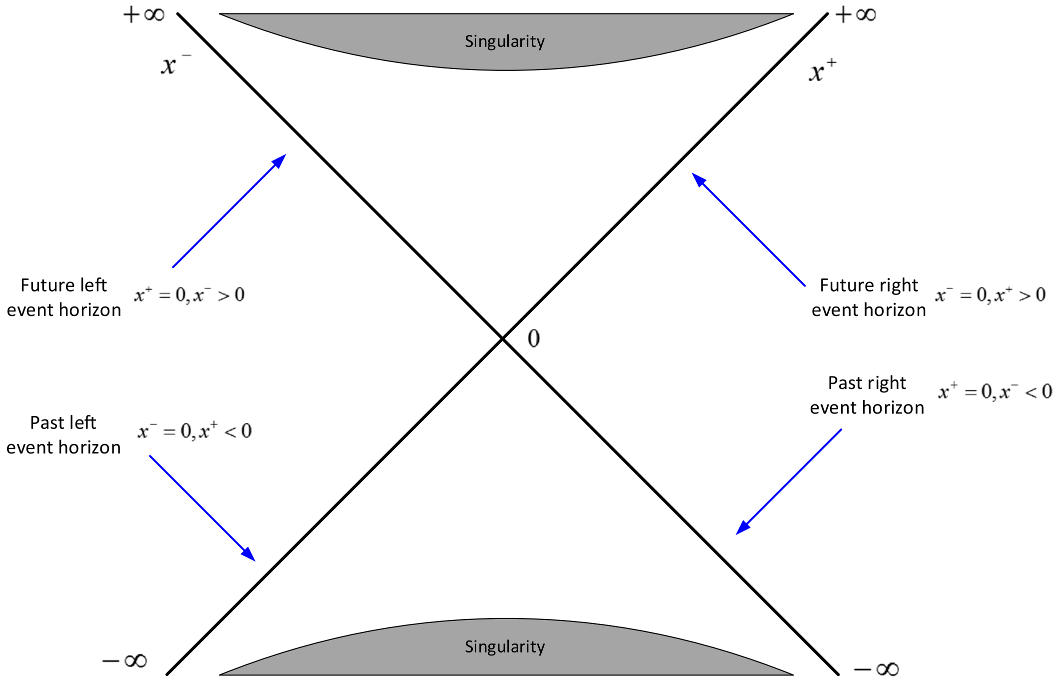

FIG. 1. The Kruskal diagram of the CGHS black hole without matter field.

III. SIMILARITY OF THE CGHS AND 3 + 1 SPHERICALLY SYMMETRIC MODELS

As we mentioned in Sec. I, the key point of the ability to extend the results of [20] to the CGHS model is writing the latter in polar-type variables [22]. This has been mainly done in [23–25]. Here, we give a brief review of this formulation and its key similarities and differences to the 3 + 1 spherically symmetric gravity.

Let us start by considering the most generic two-dimensional diffeomorphism-invariant action yielding second order differential equations for the metric g and a scalar (dilaton) field Φ [2],

$$S = \frac{1}{G_2}\int d^2x\sqrt{-|g|} \times (Y(\Phi)R(g) + V((\nabla\Phi)^2, \Phi)). \quad (3.1)$$

Within this class we choose a subclass [4,26,27] that is generic enough for our purposes,

$$S_{g\text{-dil}} = \frac{1}{G_2}\int d^2x\sqrt{-|g|} \times \left(Y(\Phi)R(g) + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + V(\Phi)\right). \quad (3.2)$$

Here, $Y(\Phi)$ is the nonminimal coupling coefficient, $V(\Phi)$ is the potential of the dilaton field, and $\frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi$ is its kinetic term. The latter can be removed at will by a conformal transformation. With the choice $Y(\Phi) = \frac{1}{8}\Phi^2$ and $V(\Phi) = \frac{1}{2}\Phi^2\lambda^2$, we obtain the CGHS model [1], which is given by the action

$$S_{\mathrm{CGHS}} = \frac{1}{G_2}\int d^2x\sqrt{-|g|} \times \left(\frac{1}{8}\Phi^2R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + \frac{1}{2}\Phi^2\lambda^2\right), \quad (3.3)$$

with λ being the cosmological constant. It coincides with Eq. (2.1) for $\Phi = 2e^{-\phi}$.

In the same way, we may notice the parallelism with 3 + 1 spherically symmetric gravity in vacuum. By using the spherical symmetry ansatz,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \Phi^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (3.4)$$

with μ, $\nu = 0$, 1, for the metric of the four-dimensional model, its action can be written as

$$S_{\mathrm{spher}} = \frac{1}{G}\int d^2x\sqrt{-|g|} \times \left(\frac{1}{4}\Phi^2R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + \frac{1}{2}\right), \quad (3.5)$$

where G is the Newton's constant in four-dimensional Einstein's theory. One can see that this is identical to (3.2) if one chooses $Y(\Phi) = \frac{1}{4}\Phi^2$, $V(\Phi) = \frac{1}{2}$, and replaces G_2 with G. Note that although the actions of both CGHS and four-dimensional models contain the variable Φ, the interpretation of this variable is different in each of these cases. In four-dimensional spherical gravity, Φ is actually a part of the metric, the coefficient multiplied by the two-sphere part of the metric as can be seen from (3.4). In the CGHS, however, it is a nongeometric degree of freedom corresponding to the scalar dilaton field.

A. Polar-type variables for the spherically symmetric model

As can be seen in detail in [23], one can write (3.5) in terms of the polar-type variables. Here we only explain the procedure briefly. In the $3+1$ spherically symmetric case, one first removes the dilaton kinetic term by a conformal transformation and then writes the theory in tetrad variables. One then adds the torsion free condition, multiplied by a Lagrange multiplier X^I, to the Lagrangian. Here I is a Lorentz internal index representing the internal local gauge group of the theory. One then makes an integration by parts such that derivatives of X^I appear in the Lagrangian. After ADM decomposition of the action and some further calculations, it turns out that the configuration variables are

$$\{{}^*X^I = \epsilon^{IJ}X_J, \omega_1\}, \qquad I,J = \{0,1\} \tag{3.6}$$

where ω_1 is the spatial part of the spin connection. The corresponding momenta are then

$$P_I = \frac{\partial\mathcal{L}}{\partial^*\dot{X}^I} = 2\sqrt{q}n_I, \tag{3.7}$$

$$P_\omega = \frac{\partial\mathcal{L}}{\partial\dot{\omega}_1} = \frac{1}{2}\Phi^2. \tag{3.8}$$

Here $n_I = n_\mu e^\mu{}_I$ is the Ith (internal) component of the normal to the spatial hypersurface, with $e^\mu{}_I$ being the tetrad, and q is the determinant of the spatial metric. Then by a Legendre transformation one can arrive at the Hamiltonian in these variables. From this Hamiltonian one can get to the Hamiltonian in polar-type variables by considering the following relation:

$$\|P\|^2 = -|P|^2 = -\eta^{IJ}P_IP_J = 4q. \tag{3.9}$$

Then we adopt the parametrization

$$q = \frac{(E^\varphi)^2}{(E^x)^{\frac{1}{2}}}, \tag{3.10}$$

based on the form of the $3+1$ metric in terms of the polar-type variables. Equation (3.9) leads to the following canonical transformation to polar-type variables:

$$P_\omega = E^x, \tag{3.11}$$

$$\|P\| = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}, \tag{3.12}$$

$$P_0 = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta), \tag{3.13}$$

$$P_1 = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta). \tag{3.14}$$

The first equation above is just a renaming, and the rest of them follow naturally from (3.9). By finding a generating function for this canonical transformation, one can find the corresponding canonical variables $\{K_x, K_\varphi, Q_\eta\}$ to the above momenta $\{E^x, E^\varphi, \eta\}$ and then write the Hamiltonian in these variables. The Hamiltonian is the sum of three constraints as expected,

$$H = \frac{1}{G}\int dx(N\mathcal{H} + N^1\mathcal{D} + \omega_0\mathcal{G}), \tag{3.15}$$

where N and N^1 are lapse and shift, respectively, ω_0 is the "time component" of the spin connection which is another Lagrange multiplier, and $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{G}$ are Hamiltonian, diffeomorphism and Gauss constraints, respectively. In order to make things simpler, Gambini *et al.* in [20] take $\eta = 1$ and since this is second class with the Gauss constraint, they can be solved to yield the final Hamiltonian

$$H = \frac{1}{G}\int dx\Bigg[N\Bigg(\frac{[(E^x)']^2}{8\sqrt{E^x}E^\varphi} - \frac{E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''}{2E^\varphi}\Bigg) + N^1(E^\varphi K_\varphi' - (E^x)'K_x)\Bigg]. \tag{3.16}$$

B. Polar-type variables for the CGHS model

By guidance from the procedure done in the spherically symmetric case, one can arrive at similar variables for the CGHS model. Most of the steps are, in principle, similar, but there are also some important differences. The details can be found in [24] and, again, we describe the process in a brief manner. First, we should mention that, although almost all of the studies of the CGHS model have utilized a conformal transformation to remove the dilaton kinetic term in an effort to render the theory as a first class system, we proceeded instead with that term present. The main reason was that, in this way, the variables admit a natural geometrical interpretation, and the quantization of the model can be carried out following the ideas of loop quantum gravity. The geometric implications can be read more easily and directly. In any case, this is just a choice and it is not of crucial importance.

It turns out that, by following the same procedure of adding the torsion free condition, writing in tetrad variables and adopting an ADM decomposition, and because the kinetic term (and hence the time derivative) of the dilaton is present, the configuration variables are

$$\{{}^*X^I, \omega_1, \Phi\} \qquad I, J = \{0, 1\} \tag{3.17}$$

with the corresponding momenta

$$P_I = \frac{\partial \mathcal{L}}{\partial^* \dot{X}^I} = 2\sqrt{q} n_I, \tag{3.18}$$

$$P_\omega = \frac{\partial \mathcal{L}}{\partial \dot{\omega}_1} = \frac{1}{4}\Phi^2, \tag{3.19}$$

$$P_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{\sqrt{q}}{N}(N^1 \Phi' - \dot{\Phi}), \tag{3.20}$$

where again N and N^1 are lapse and shift, respectively. An important consequence of these is that (3.19) is now a new primary constraint

$$\mu = P_\omega - \frac{1}{4}\Phi^2 \approx 0. \tag{3.21}$$

In the next step, by making a Legendre transformation, we get to a Hamiltonian which now should also contain the new primary constraint (3.21). To obtain the polar-type variables we use a similar relation to (3.9) which, in the case of the CGHS model, reads

$$\|P\|^2 = 4q = 4(E^\varphi)^2 \tag{3.22}$$

where we have used again a natural parametrization for q in terms of E^φ for the CGHS model. Then, we get the new variables

$$P_\omega = E^x, \tag{3.23}$$

$$\|P\| = 2E^\varphi, \tag{3.24}$$

$$P_0 = 2\cosh(\eta) E^\varphi, \tag{3.25}$$

$$P_1 = 2\sinh(\eta) E^\varphi. \tag{3.26}$$

Again, they follow naturally from (3.22) with a bit of an educated guess. These transformations do not affect the pair $\{\Phi, P_\Phi\}$. Once again, by finding a generating function for this canonical transformation, we can find the corresponding conjugate variables $\{K_x, K_\varphi, Q_\eta, \Phi\}$ to the above momenta $\{E^x, E^\varphi, \eta, P_\Phi\}$ and then write the Hamiltonian in these variables. The Hamiltonian is the sum of four constraints

$$H = \frac{1}{G_2}\int dx (N\mathcal{H} + N^1 \mathcal{D} + \omega_0 \mathcal{G} + B\mu) \tag{3.27}$$

with B being another Lagrange multiplier. Note that, in this case, unlike the spherically symmetric case, we have

$$K_x = \omega_1. \tag{3.28}$$

Also note that there is an important difference here between the $3+1$ spherically symmetric case and the CGHS model: as a consequence of what we also mentioned in the beginning of Sec. III and due to (3.19) and (3.21), one can see that E^x is classically associated to the dilaton field in the CGHS model. It has nothing to do with the metric and is a truly distinct degree of freedom. Although, as we mentioned, it is a component of the metric in the $3+1$ spherically symmetric case.

Continuing with the Dirac procedure, since we have a new primary constraint μ, we need to check its consistency under the evolution. This leads to a new secondary constraint α, namely

$$\dot{\mu} \approx 0 \Rightarrow \alpha = K_\varphi + \frac{1}{2}\frac{P_\Phi \Phi}{E^\varphi} \approx 0. \tag{3.29}$$

Preservation of α then leads to no new constraint. It turns out that these two new constraints are second class together,

$$\{\mu, \alpha\} \approx 0, \tag{3.30}$$

and thus we need to follow the second class Dirac procedure for this case. So we solve them to get

$$\mu = 0 \Rightarrow \Phi = 2\sqrt{E^x}, \tag{3.31}$$

$$\alpha = 0 \Rightarrow P_\Phi = -\frac{K_\varphi E^\varphi}{\sqrt{E^x}}. \tag{3.32}$$

This eliminates the pair $\{\Phi, P_\Phi\}$ in the Hamiltonian. In order to simplify the process of quantization, we introduce the new variable

$$A_x = K_x - \eta', \tag{3.33}$$

and choose $\eta = 1$ which is again second class with the Gauss constraint. Then, solving these second class constraints together yields an expression for Q_η in terms of the remaining variables. In this way, the pair $\{Q_\eta, \eta\}$ is also eliminated from the Hamiltonian. A similar procedure has also been done in the spherically symmetric case in [20]. Note that we now have

$$A_x = \omega_1. \tag{3.34}$$

The Dirac brackets now become

$$\begin{aligned}\{K_x(x), E^x(y)\}_D &= \{K_\varphi(x), E^\varphi(y)\}_D \\ &= G_2 \delta(x-y),\end{aligned} \tag{3.35}$$

$$\{K_x(x), K_\varphi(y)\}_D = G_2 \frac{K_\varphi}{E^x}\delta(x-y), \tag{3.36}$$

$$\{K_x(x), E^\varphi(y)\}_D = -G_2 \frac{E^\varphi}{E^x}\delta(x-y), \tag{3.37}$$

with any other brackets vanishing. These brackets can be brought to the canonical from

$$\{U_x(x), E^x(y)\}_D = \{K_\varphi(x), E^\varphi(y)\}_D = G_2\delta(x-y), \tag{3.38}$$

by introducing the redefinition

$$U_x = K_x + \frac{E^\varphi K_\varphi}{E^x}. \tag{3.39}$$

Finally, we are left with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{G_2}\int dx[N\mathcal{H} + N^1\mathcal{D}] \\ &= \frac{1}{G_2}\int dx\left[N\left(-K_\varphi U_x - \frac{E^{\varphi\prime}E^{x\prime}}{E^{\varphi 2}} - \frac{1}{2}\frac{E^{x\prime 2}}{E^\varphi E^x} + \frac{E^{x\prime\prime}}{E^\varphi} \right.\right. \\ &\quad \left. + \frac{1}{2}\frac{K_\varphi^2 E^\varphi}{E^x} - 2E^\varphi E^x\lambda^2\right) \\ &\quad \left. + N^1(-U_x E^{x\prime} + E^\varphi K_\varphi')\right]. \end{aligned} \tag{3.40}$$

IV. PREPARING THE CGHS HAMILTONIAN FOR QUANTIZATION

At this point, and in order to proceed with the Dirac quantization of the system, we adopt an Abelianization of the scalar constraint algebra. The reason is the following: the Dirac quantization approach involves several consistency conditions. For instance, the constraint algebra at the quantum level must agree with the classical one. It is well known that anomalies in the algebra can emerge, and spoil the final quantization. Usually, this situation is more likely to be satisfied if the constraints fulfil a Lie algebra (with structure constants instead of structure functions of phase-space variables). An even more favorable situation is when part of the algebra is strongly Abelian. We already know that the brackets $\{\mathcal{H}(N), \mathcal{D}(N^1)\}$ and $\{\mathcal{D}(N^1), \mathcal{D}(M^1)\}$ involve structure constants and close under the bracket. But this is not the case for $\{\mathcal{H}(N), \mathcal{H}(M)\}$. Although, in principle, nothing prevents us carrying on with the study in this situation, we adopt a strategy based on strong Abelianization that allows us to complete the quantization, since other choices are either not fully understood or not considerably developed. This strategy consists in a redefinition of the shift function

$$\overline{N}^1 = N^1 + \frac{NK_\varphi}{(E^x)'}, \tag{4.1}$$

followed by a redefinition the lapse function as

$$\overline{N} = N\frac{E^\varphi E^x}{(E^x)'}. \tag{4.2}$$

These yield

$$\begin{aligned} H &= \frac{1}{G_2}\int dx[\overline{N}\mathcal{H} + \overline{N}^1\mathcal{D}] \\ &= \frac{1}{G_2}\int dx\overline{N}\left[\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{E^{x\prime 2}}{E^{\varphi 2}E^x} - 2E^x\lambda^2 - \frac{1}{2}\frac{K_\varphi^2}{E^x}\right)\right] \\ &\quad + \overline{N}^1(-U_x(E^x)' + E^\varphi K_\varphi'). \end{aligned} \tag{4.3}$$

One can check that now

$$\{\mathcal{H}(\bar{N}), \mathcal{H}(\bar{N}')\}_D = 0, \tag{4.4}$$

and thus the Dirac quantization, particularly the loop quantization strategy, is expected to be simpler and potentially successful with respect to other choices considered so far.

We can take advantage of this form of the Hamiltonian constraint and, by making an integration by parts,[1] write Eq. (4.3) as

$$\begin{aligned} H &= \frac{1}{G_2}\int dx\overline{N}' \\ &\quad \times\left[\frac{1}{2}\frac{(E^{x\prime})^2}{E^{\varphi 2}E^x} - 2E^x\lambda^2 - \frac{1}{2}\frac{K_\varphi^2}{E^x} + \lambda G_2 M\right] \\ &\quad + \overline{N}^1(-U_x(E^x)' + E^\varphi K_\varphi'), \end{aligned} \tag{4.5}$$

where M is the ADM mass of the CGHS black hole and G_2 is the dimensionless Newton's constant in two-dimensional spacetimes.

At this point we first consider the Hamiltonian constraint and prepare it for representation on the kinematical Hilbert space. Regarding the diffeomorphism constraint, we adopt the group averaging technique, since, as is well known in loop quantum gravity, only finite spatial diffeomorphisms are well-defined unitary operators on the Hilbert space.

If we rename $\overline{N}' \to N$, the Hamiltonian constraint can now be written as

$$\begin{aligned} \mathcal{H}(N) &= \frac{1}{G_2}\int dxN \\ &\quad \times\left[\frac{1}{2}\frac{[(E^x)']^2}{E^{\varphi 2}E^x} - 2E^x\lambda^2 - \frac{1}{2}\frac{K_\varphi^2}{E^x} + \lambda G_2 M\right]. \end{aligned} \tag{4.6}$$

Our final step, before quantization, is to bring the above constraint in a form that admits a natural representation on a suitable Hilbert space. This is achieved by rescaling the lapse function $N \to 2NE^\varphi(E^x)^2$ such that

$$\begin{aligned} \mathcal{H}(N) &= \frac{1}{G_2}\int dxNE^x\left[4(E^x)^2E^\varphi\lambda^2 + K_\varphi^2E^\varphi \right. \\ &\quad \left. - 2\lambda G_2 ME^\varphi E^x - \frac{[(E^x)']^2}{E^\varphi}\right]. \end{aligned} \tag{4.7}$$

[1] In this work, we do not provide details about the boundary terms. A detailed analysis can be found in Ref. [7].

V. QUANTIZATION

A. The kinematical Hilbert space

To quantize the theory, we first need an auxiliary (or kinematical) vector space of states. Then we should equip it with an inner product and carry out a Cauchy completion of this space. We then end up with a kinematical Hilbert space. Afterwards, we need to find a representation of the phase-space variables as operators acting on this Hilbert space. In order to study the dynamics of the system, since we are dealing with a totally constrained theory, we follow the Dirac quantization approach. Here, one identifies those quantum structures that are invariant under the gauge symmetries generated by the constraints. In this particular model, we have the group of spatial diffeomorphisms (generated by the diffeomorphism constraint) and the set of time reparametrizations (associated with the Hamiltonian constraint). In the loop representation, only the spatial diffeomorphisms are well understood. Then, we must look for a suitable representation of the Hamiltonian constraint (4.7) as a quantum operator, and look for its kernel which yields a space of states that is invariant under this constraint. Finally, one should endow this space of solutions with a Hilbert space structure and suitable observables acting on it.

Here, we adhere to a loop representation for the kinematical variables, except the mass, for which a standard Fock quantization is adopted. Our full kinematical Hilbert space is the direct product of two parts,

$$\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{M} \otimes (\bigoplus_{g} \mathcal{H}_{\text{kin-spin}}^{g}). \tag{5.1}$$

One part, $\mathcal{H}_{\text{kin}}^{M} = L^2(\mathbb{R}, dM)$, is associated to the global degree of freedom of the mass of the black hole M. The other part, associated to the gravitational sector, is the direct sum of the spaces, $\mathcal{H}_{\text{kin-spin}}^{g}$, each corresponding to a given graph (spin network) g for which we use the polymer quantization. This choice seems to be natural in $3+1$ spherically symmetric models for the geometrical variables, and due to the parallelism between that model and the CGHS model, we adopt a similar representation here.

To construct $\mathcal{H}_{\text{kin-spin}}^{g}$, we first take the vector space Cyl_g, of all the functions of holonomies along the edges of a graph g, and the point holonomies "around" its vertices, and equip this vector space with the Haar measure to get the gravitational part of the kinematical Hilbert space of the given graph g. In our case these states are

$$\langle U_x, K_\varphi | g, \vec{k}, \vec{\mu} \rangle = \prod_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} dx U_x(x)\right) \times \prod_{v_j \in g} \exp\left(\frac{i}{2} \mu_j K_\varphi(v_j)\right). \tag{5.2}$$

Here e_j are the edges of the graph, v_j are its vertices, $k_j \in \mathbb{Z}$ is the edge color, and $\mu_j \in \mathbb{R}$ is the vertex color. We indicate the order (i.e. number of the vertices) of the graph g by V. Since $\mu_j \in \mathbb{R}$, the above belongs to the space of almost-periodic functions and the associated Hilbert space is nonseparable.

It is evident that this Hilbert space, $\mathcal{H}_{\text{kin-spin}}^{g}$, can be decomposed into a part associated with the normal holonomies along the edges, which is the space of square summable functions ℓ^2, and another part associated to the point holonomies, which is the space of square integrable functions over the Bohr-compactified real line with the associated Haar measure, $L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Haar}})$. The construction for the mass degree of freedom is similar and well known, and we do not give additional details here. Thus, the full kinematical Hilbert space can be written as

$$\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{M} \otimes \left(\bigoplus_{g} \mathcal{H}_{\text{kin-spin}}^{g}\right) = L^2(\mathbb{R}, dM) \otimes \left(\bigoplus_{g} \left[\bigotimes_{v_j \in g} \ell_j^2 \otimes L_j^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Haar}})\right]\right). \tag{5.3}$$

Let us call the kinematical Hilbert space of a single graph $\mathcal{H}_{\text{kin}}^{g} = \mathcal{H}_{\text{kin}}^{M} \otimes \mathcal{H}_{\text{kin-spin}}^{g}$ (not to be confused with $\mathcal{H}_{\text{kin-spin}}^{g}$). There is a basis of states in this Hilbert space denoted by $\{|g, \vec{k}, \vec{\mu}, M\rangle\}$. Then, since now we have a measure, and thus a Hilbert space, we can define the inner product on $\mathcal{H}_{\text{kin}}^{g}$ and thus on $\mathcal{H}_{\text{kin}}$. As usual in loop quantum gravity, a spin network defined on g can be regarded as a spin network with support on a larger graph $\bar{g} \supset g$ by assigning trivial labels to the edges and vertices which are not in g. Consequently, for any two graphs g and g', we take $\bar{g} = g \cup g'$ and the inner product of g and g' is

$$\langle g, \vec{k}, \vec{\mu}, M | g', \vec{k}', \vec{\mu}', M' \rangle = \delta(M - M') \prod_{\text{edges}} \delta_{k_j, k_j'} \prod_{\text{vertices}} \delta_{\mu_j, \mu_j'} = \delta(M - M') \delta_{\vec{k}, \vec{k}'} \delta_{\vec{\mu}, \vec{\mu}'}. \tag{5.4}$$

Obviously, the inner product can be extended to arbitrary states by superposition of the basis states.

B. Representation of operators

Now that we have a kinematical Hilbert space, the next step is to represent the phase-space variables on it as operators. We follow a similar strategy as the one of Ref. [20]. First, we choose the polymerization $K_\varphi \to \sin(\rho K_\varphi)/\rho$. Looking at (4.7), we note that we need to represent the following phase-space variables:

$$E^x, (E^x)', E^\varphi, \frac{1}{E^\varphi}, K_\varphi^2 E^\varphi, M. \tag{5.5}$$

Due to our polymerization scheme and the classical algebra (i.e. Dirac brackets), the momenta can be represented as

$$\widehat{E^\varphi}|g,\vec{k},\vec{\mu},M\rangle = \hbar G_2 \sum_{v_j \in g} \delta(x-x_j)\mu_j|g,\vec{k},\vec{\mu},M\rangle, \quad (5.6)$$

$$\widehat{E^x}|g,\vec{k},\vec{\mu},M\rangle = \hbar G_2 k_j|g,\vec{k},\vec{\mu},M\rangle, \quad (5.7)$$

where $\hbar G_2$ is the Planck number (recalling that $\hbar$ has dimensions $[LM]$). The presence of the Dirac delta function in (5.6) is due to E^φ being a density. The global degree of freedom M, corresponding to the Dirac observable on the boundary associated to the mass of the black hole, can be represented as

$$\hat{M}|g,\vec{k},\vec{\mu},M\rangle = M|g,\vec{k},\vec{\mu},M\rangle. \quad (5.8)$$

To represent the last contribution in (4.7), we combine $E^{x\prime}$ with $\frac{1}{E^\varphi}$, and use the Thiemann's trick [28] to represent it as

$$\left[\frac{\widehat{[(E^x)']^2}}{E^\varphi}\right]|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}\delta(x-x(v_j)) \times \frac{\mathrm{sgn}(\mu_j)\hbar G_2}{\rho^2}(k_j-k_{j-1})^2[|\mu_j+\rho|^{1/2} - |\mu_j-\rho|^{1/2}]^2|g,\vec{k},\vec{\mu},M\rangle. \quad (5.9)$$

This is due to the operator $\widehat{N^\varphi_{n\rho}}$ corresponding to K_φ, which is represented by the action of the point holonomies of length ρ,

$$\widehat{N^\varphi_{\pm n\rho}}(x)|g,\vec{k},\vec{\mu},M\rangle = |g,\vec{k},\vec{\mu}'_{\pm n\rho},M\rangle, \quad n\in\mathbb{N}. \quad (5.10)$$

In this expression, the new vector $\vec{\mu}'_{\pm n\rho}$ either has the same components as $\vec{\mu}$ but shifted by $\pm n\rho$, i.e. $\mu_j \to \mu_j \pm n\rho$, if x coincides with a vertex of the graph located at $x(v_j)$, or it is $\vec{\mu}$ but with a new component $\pm n\rho$, i.e. it is $\{\ldots,\mu_j,\pm n\rho,\mu_{j+1},\ldots\}$, if $x_{v_j} < x < x_{v_{j+1}}$.

The final term to be considered is $K_\varphi^2 E^\varphi$. For it, we choose the representation proposed in [29,30], that is we define this operator as

$$\hat{\Theta}(x)|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}\delta(x-x(v_j)) \times \hat{\Omega}^2_\varphi(v_j)|g,\vec{k},\vec{\mu},M\rangle, \quad (5.11)$$

where the nondiagonal operator $\hat{\Omega}_\varphi(v_j)$ is written as

$$\hat{\Omega}_\varphi(v_j) = \frac{1}{4i\rho}|\widehat{E^\varphi}|^{1/4}[\widehat{\mathrm{sgn}(E^\varphi)}(\widehat{N^\varphi_{2\rho}} - \widehat{N^\varphi_{-2\rho}}) + (\widehat{N^\varphi_{2\rho}} - \widehat{N^\varphi_{-2\rho}})\widehat{\mathrm{sgn}(E^\varphi)}]|\widehat{E^\varphi}|^{1/4}|_{v_j}. \quad (5.12)$$

This shows that we need to also represent $|E^\varphi|^{1/4}$ and $\mathrm{sgn}(E^\varphi)$. This can be achieved by means of the spectral decomposition of $\widehat{E^\varphi}$ on $\mathcal{H}_{\rm kin}$ as

$$|\widehat{E^\varphi}|^{1/4}(v_j)|g,\vec{k},\vec{\mu},M\rangle = |\mu_j|^{1/4}|g,\vec{k},\vec{\mu},M\rangle, \quad (5.13)$$

$$\mathrm{sgn}(\widehat{E^\varphi(v_j)})|g,\vec{k},\vec{\mu},M\rangle = \mathrm{sgn}(\mu_j)|g,\vec{k},\vec{\mu},M\rangle. \quad (5.14)$$

Combining these, a representation of our Hamiltonian constraint on $\mathcal{H}_{\rm kin}$ is

$$\hat{\mathcal{H}}(N) = \int dx N(x)\widehat{E^x}\left\{\hat{\Theta} + (4\lambda^2\widehat{E^\varphi}\widehat{E^x}^2 - 2\lambda G_2\hat{M}\,\widehat{E^\varphi}\,\widehat{E^x}) - \left[\frac{\widehat{[(E^x)']^2}}{E^\varphi}\right]\right\}. \quad (5.15)$$

C. Hamiltonian constraint: singularity resolution and solutions

1. Relation between volume and singularity

Our singularity resolution argument is based on having a zero volume at some point (or region) classically or having a zero volume eigenvalue for the quantum volume operator in quantum theory. In other words, a vanishing volume (spectrum) at a point or region means we have a singularity there. Here we give an argument supporting this statement for a generic two-dimensional metric (with generic lapse and shift).

A generic ADM decomposed two-dimensional metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + (N^1)^2 q_{11} & -N^1 q_{11} \\ -N^1 q_{11} & q_{11} \end{pmatrix} \quad (5.16)$$

where q_{11} is the spatial metric and N and N^1 are lapse and shift, respectively. Since we have a one-dimensional spatial hypersurface,

$$q_{11} = \det(q). \quad (5.17)$$

Classically we have for the volume of a region R in a spatial hypersurface Σ

$$V(\mathrm{R}) = \int_{\mathrm{R}} dx\sqrt{\det(q)}. \quad (5.18)$$

So if at some region we have $\det(q) = 0$, this means that we get $V(\mathrm{R}) = 0$ in that region. On the other hand, if $\det(q) = 0$, then due to (5.16) and (5.17), we have for that region a metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.19)$$

independently of the lapse and shift. It turns out that the Riemann invariants of the above metric (in that region) blow up and thus we have a singularity there. So we conclude that in two dimensions, a vanishing volume in a region means existence of singularity in that region. However, this does not happen for a generic genuine four-dimensional metric.

Now, for the quantum volume operator of the CGHS we have

$$\hat{\mathcal{V}}|g,\vec{k},\vec{\mu},M\rangle \propto \sum_{v_j\in g}|\mu_j||g,\vec{k},\vec{\mu},M\rangle, \tag{5.20}$$

which means that a vanishing volume in a region corresponds to having all the μ_j's equal to 0 for that region (and not for the whole spatial hypersurface). If we assume that the statement "$V(\mathrm{R})=0 \Rightarrow$ singularity" can be carried on to the quantum level, then we can say that a region (or hypersurface) described by a state with none of its μ_j's being 0 is a region that does not contain any singularity. This argument which to our knowledge only works generically for genuine two-dimensional spacetime metrics is the one we use to argue for singularity resolution in the next subsection.

2. Properties of the Hamiltonian constraint and singularity resolution

Keeping the argument of the previous subsection in mind and having obtained a representation of the Hamiltonian constraint (5.14) on $\mathcal{H}_{\mathrm{kin}}$, we study some interesting properties of this quantum Hamiltonian constraint. These properties facilitate the identification of the space of solutions of this constraint, and its relation with the singularity resolution it provides.

Let us consider any basis state $|g,\vec{k},\vec{\mu},M\rangle \in \mathcal{H}_{\mathrm{kin}}$. It turns out that the action of this constraint on it yields

$$\hat{\mathcal{H}}(N)|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g}(N(x_j)(\hbar G_2 k_j)\times[f_0(\mu_j,k_j,M)|g,\vec{k},\vec{\mu},M\rangle - f_+(\mu_j)|g,\vec{k},\vec{\mu}_{+4\rho_j},M\rangle - f_-(\mu_j)|g,\vec{k},\vec{\mu}_{-4\rho_j},M\rangle]), \tag{5.21}$$

where the functions f read

$$f_\pm(\mu_j) = \frac{\hbar G_2}{16\rho^2}|\mu_j|^{1/4}|\mu_j\pm 2\rho|^{1/2}|\mu_j\pm 4\rho|^{1/4}[\mathrm{sgn}(\mu_j\pm 4\rho)+\mathrm{sgn}(\mu_j\pm 2\rho)][\mathrm{sgn}(\mu_j\pm 2\rho)+\mathrm{sgn}(\mu_j)], \tag{5.22}$$

$$f_0(\mu_j,k_j,k_{j-1},M) = (\hbar G_2)^3\lambda^2\left(1-\frac{G_2\hat{M}}{2\hbar G_2 k_j\lambda}\right)\mu_j k_j^2 - \frac{\hbar G_2}{\rho^2}(|\mu_j+\rho|^{1/2}-|\mu_j-\rho|^{1/2})^2[k_j-k_{j-1}]^2 + \frac{\hbar G_2}{16\rho^2}\{|\mu_j|^{1/2}|\mu_j+2\rho|^{1/2}[\mathrm{sgn}(\mu_j)+\mathrm{sgn}(\mu_j+2\rho)]^2 + |\mu_j|^{1/2}|\mu_j-2\rho|^{1/2}[\mathrm{sgn}(\mu_j)+\mathrm{sgn}(\mu_j-2\rho)]^2\}. \tag{5.23}$$

Looking at (5.21) and the form of (5.22) and (5.23), we notice some important points.

(1) The scalar constraint admits a natural decomposition on each vertex v_j, such that it can be regarded as a sequence of quantum operators acting almost independently on them, up to the factors $\Delta k_j = k_j - k_{j-1}$. In other words, there would not be coupling among different vertices if it were not for the factor Δk_j.

(2) The number of vertices on a given graph g is preserved under the action of the Hamiltonian constraint.

(3) The constraint (5.21) leaves the sequence of integers $\{k_j\}$ of each graph g invariant. For instance, if we consider a ket $|g,\vec{k},\vec{\mu},M\rangle$, the successive action of the scalar constraint on it generates a subspace characterized by the original quantum numbers $\vec{k}$.

(4) The restriction of the constraint to any vertex v_j acts as a difference operator mixing the real numbers μ_j. In this case, this difference operator only relates those states which have μ_j's that belong to semi-lattices of step 4ρ due to the form of $f_\pm(\mu_j)$ that vanishes in the intervals $[0,\mp 2\rho]$.

(5) Starting from a state for which none of the μ_j's are 0 (i.e. a state containing no singularity), the result of the action of the constraint never leaves us in a state with any of the μ_j's being 0 (also look at the details in Sec. V D).

The point number 5, which may be the most important of these, states that the subspace of $\mathcal{H}_{\mathrm{kin}}$ containing spin networks for which no μ_j is 0 is preserved under the action of the Hamiltonian constraint. Simply put, if one originally starts with a state with no singularity (in the sense of $\mu_j=0$), then one never ends up in a state containing a singularity. Analogous arguments could be applied to the k_j quantum numbers; however as mentioned above, k_j are already preserved by the constraint (unlike the μ_j valences of the vertices).

Thus, one can restrict the study only to the subspace of $\mathcal{H}_{\text{kin}}$ for which there is no $\mu_j = 0$ and $k_j = 0$. As a result of this restriction, we expect that also in the physical Hilbert space, we will never have any state with a singularity.

D. Solutions to the Hamiltonian constraint and the physical Hilbert space

Let us consider a generic solution, $\langle\Psi_g|$, to the Hamiltonian constraint, i.e., a generic state annihilated by this constraint. Assuming $\langle\Psi_g|$ belongs to the algebraic dual of the dense subspace Cyl on the kinematical Hilbert space, and that it can be written as

$$\langle\Psi_g| = \int_0^\infty dM \sum_{\vec{k}} \sum_{\vec{\mu}} \langle g, \vec{k}, \vec{\mu}, M| \psi(M)\chi(\vec{k}) \times \phi(\vec{k}, \vec{\mu}, M), \tag{5.24}$$

the annihilation by the Hamiltonian constraint dictates that

$$\langle\Psi_g|\hat{\mathcal{H}}(N)^\dagger = \sum_{v_j \in g} \langle\Psi_g| N_j \hat{\mathfrak{H}}_j^\dagger = 0, \tag{5.25}$$

where $\hat{\mathfrak{H}}_j$ are difference operators acting on each vertex v_j and $N_j = N(x_j)$ is the lapse function evaluated on the corresponding vertex. In this case the functions $\phi(\vec{k}, \vec{\mu}, M)$ admit a natural decomposition of the form

$$\phi(\vec{k}, \vec{\mu}, M) = \prod_{j=1}^{V} \phi_j(k_j, k_{j-1}, \mu_j, M). \tag{5.26}$$

One can then easily see that the solutions must fulfil, at each v_j, a difference equation of the form

$$\begin{aligned} &- f_+(\mu_j - 4\rho)\phi_j(k_j, k_{j-1}, \mu_j - 4\rho, M) \\ &\quad - f_-(\mu_j + 4\rho)\phi_j(k_j, k_{j-1}, \mu_j + 4\rho, M) \\ &\quad + f_0(k_j, k_{j-1}, \mu_j, M)\phi_j(k_j, k_{j-1}, \mu_j, M) = 0, \end{aligned} \tag{5.27}$$

which is a set of difference equations to be solved together. We provide a partial resolution of the problem by means of analytical considerations. All the details can be found in Appendix A. Let us consider a particular vertex v_j. In the following we omit any reference to the label of the vertex. Due to the property 4, where μ belongs to the semilattices of the form $\mu = \epsilon \pm 4\rho\mathrm{n}$ where $\mathrm{n} \in \mathbb{N}$ and $\epsilon \in (0, 4\rho]$, different orientations of μ are decoupled. Without loss of generality, we restrict the study to a particular subspace labeled by ϵ, unless otherwise specified. This shows that the Hamiltonian constraint only relates states belonging to separable subspaces of the original kinematical Hilbert space.

These properties of the solutions together with their asymptotic limit $\mu \to \infty$, assuming the solutions are smooth there, allow us to understand several aspects of the geometrical operators (under some assumptions about their spectral decomposition). More concretely, the solutions for $\mu \to \infty$ satisfy, up to a global factor $[(\hbar G_2)^2 k]$, the differential equation

$$\begin{aligned} &- 4\mu\partial_\mu^2\phi - 4\partial_\mu\phi - \frac{4\Delta k^2 - 1}{4\mu}\phi \\ &\quad + \left(1 - \frac{G_2 M}{2\hbar G_2 \lambda k}\right)(\hbar G_2 \lambda)^2 k^2 \mu\phi = 0, \end{aligned} \tag{5.28}$$

in a very good approximation if they are smooth functions of μ. The last term plays the role of the square of a frequency of a harmonic oscillator. But the sign of this term depends on the concrete quantum numbers. Therefore, this equation admits both oscillatory solutions and exponentially growing or decreasing ones. More concretely, this differential equation is a modified Bessel equation if the sign of its last coefficient is positive, i.e. $k < M/2\hbar\lambda$, and a Bessel equation whenever that coefficient is negative, i.e. $k > M/2\hbar\lambda$. In Appendix A we include the details about the properties of the solutions in these two different regimes. Let us summarize the results obtained there.

(i) For $k < M/2\hbar\lambda$, the Hamiltonian constraint takes the form

$$\omega + \left(1 - \frac{G_2 \hat{M}}{2\hbar G_2 \lambda k}\right)(\hbar G_2 \lambda)^2 k^2 = 0, \tag{5.29}$$

where ω is the positive eigenvalue of the difference operator of (A28) that belongs to its continuous spectrum and which is nondegenerate. The corresponding eigenfunction $|\phi_\omega^{\text{cnt}}\rangle$ behaves as an exact standing wave in μ of frequency $\sigma(\omega)$ in the limit $\mu \to \infty$.

(ii) On the other hand, for $k > M/2\hbar\lambda$, the constraint is simply

$$\omega_n(M, k, \epsilon) - \Delta k^2 = 0, \tag{5.30}$$

where, again, ω_n is the positive eigenvalue of the difference operator defined in (A12), but this time it belongs to its discrete spectrum and is also nondegenerate. The corresponding eigenstates $|\phi_n^{\text{dsc}}\rangle$, with $n \in \mathbb{N}$, emerge out of $\mu \simeq \epsilon$, growing exponentially until they reach a stable regime, and at some $\mu \simeq \mu_r$ the eigenfunction enters a classically forbidden region and decays exponentially (see [31] for a related treatment). Besides, the eigenfunctions ω_n form a discrete sequence of real numbers, all of them depending continuously on the parameter ϵ. This dependence is crucial in order to have a consistent constraint solution, since the sequence of discrete Δk^2 is not expected to coincide with the sequence of

ω_n for a global fixed ϵ. Therefore, we expect that the parameter ϵ must be conveniently modified according to the values of M, k and the constraint equation (5.30).

These previous results have not been confirmed numerically (as well as those of [20]), though they will be a matter of future research. Let us comment, however, that they are based on very robust, previous results on different scenarios already studied in the LQC literature (see [30–34]). Therefore, unless a very subtle point comes into play, the mentioned properties are expected to be fulfilled.

In a final step, one should build the physical Hilbert space. The states belonging to this space are the ones that admit the symmetries of the model, i.e. the states which are invariant under both the Hamiltonian and diffeomorphism constraints. As usual in LQG, one applies the group averaging technique to get these states and the induced inner product on the resultant subspace that is provided by this process. One can start with the Hamiltonian constraint. Then the states averaged by members of the group associated to the Hamiltonian constraint are

$$\langle\Psi_g^{\mathcal{H}}| = \int_{-\infty}^{\infty}\prod_{n=1}^{V} d\mathbf{g}_n \int_0^{\infty} dM \sum_{\vec{k}}\sum_{\vec{\mu}}\langle g,\vec{k},\vec{\mu},M| \times \psi(M)\chi(\vec{k})\phi(\vec{k},\vec{\mu},M), \tag{5.31}$$

where

$$\mathbf{g} = e^{i\mathfrak{g}} \tag{5.32}$$

is the group member associated to the member of the Lie algebra $\mathfrak{g}$. In the case of the algebra member, being the Hamiltonian constraint $\hat{\mathcal{H}}(N_j) = \sum_{v_j} N_j \hat{\mathfrak{H}}_j$, we have

$$\mathfrak{g} = \hat{\mathcal{H}}(N_l) = \sum_{v_l} N_l \hat{\mathfrak{H}}_l. \tag{5.33}$$

Thus in this case, from (5.31) we get for the group averaged state

$$\begin{aligned}\langle\Psi_g^{\mathcal{H}}| &= \frac{1}{(2\pi)^V}\int_{-\infty}^{\infty} de^{iN_1\hat{\mathfrak{H}}_1}\ldots\int_{-\infty}^{\infty} de^{iN_V\hat{\mathfrak{H}}_V}\int_0^{\infty} dM\sum_{\vec{k}}\sum_{\vec{\mu}}\langle g,\vec{k},\vec{\mu},M|\psi(\vec{k},\vec{\mu},M)\\ &= \frac{1}{(2\pi)^V}\int_{-\infty}^{\infty} dN_1\ldots\int_{-\infty}^{\infty} dN_V \exp\left[i\sum_{n=1}^{V} N_n\hat{\mathfrak{H}}_n\right]\int_0^{\infty} dM\sum_{\vec{k}}\sum_{\vec{\mu}}\langle g,\vec{k},\vec{\mu},M|\psi(\vec{k},\vec{\mu},M).\end{aligned} \tag{5.34}$$

The final states are endowed with a suitable inner product defined as

$$\|\Psi_g^{\mathcal{H}}\|^2 = \langle\Psi_g^{\mathcal{H}}|\Psi_g\rangle, \tag{5.35}$$

where the ket belongs to the kinematical Hilbert space and the bra is the corresponding state after being averaged with the Hamiltonian constraint. In order to obtain explicitly the inner product, we may write $|\Psi_g\rangle$ in the basis of states of the geometrical operators involving the scalar constraint (see Appendix A). In this case

$$\langle\Psi_g^{\mathcal{H}}|\Psi_g\rangle = \int_0^{\infty} dM\sum_{\vec{k}}\int d\omega_1\ldots d\omega_V \times\prod_{j=1}^{V}\delta(\omega_j - F(k_j,M))|\psi(\vec{k},\vec{\omega},M)|^2, \tag{5.36}$$

where $F(k_j,M)$, at each vertex v_j, is given by the last addend in the left-hand side of Eq. (A21) or (A36) depending on whether $(k_j - M/2\hbar\lambda)$ is positive or negative, respectively, i.e.

$$\begin{aligned}F(k_j,M) &= (\Delta k_j)^2 \quad \text{if } k_j > M/2\hbar\lambda,\\ F(k_j,M) &= \left(1-\frac{G_2 M}{2\hbar G_2\lambda k_j}\right)(\hbar G_2\lambda)^2 k_j^2\end{aligned} \tag{5.37}$$

otherwise. The final step is to construct the solutions to the Hamiltonian constraint which are invariant under the spatial diffeomorphisms (generated by the diffeomorphism constraint). In this case we follow the ideas of the full theory [35]. There, one constructs a rigging map from the original Hilbert space to the space of diffeomorphism invariant states by averaging the initial states with respect to the group of finite diffeomorphisms. The resulting averaged states are a superposition of the original states but with their vertices in all possible positions in the original one-dimensional manifold, but preserving the order of the edges and vertices. So a physical state is

$$\langle\Psi^{\text{phys}}| = \sum_{g\in[g]}\langle\Psi_g^{\mathcal{H}}| \tag{5.38}$$

and the inner product is then

$$\|\Psi^{\text{phys}}\|^2 = \langle\Psi^{\text{phys}}|\Psi_g\rangle, \tag{5.39}$$

where, again, the ket belongs to the kinematical Hilbert space and the bra is the physical solution. In the last product, only a finite number of finite terms contribute, for all $|\Psi_g\rangle$ in the kinematical Hilbert space, so the inner product is finite and well defined. Let us mention that the diffeomorphism invariance of the inner product is guaranteed since if we compute Eq. (5.39) with any other state related to $|\Psi_g\rangle$ by a spatial diffeomorphism, it yields exactly the same result. For a recent discussion see [36].

At the end of this process, we are left with a vector space of states that are invariant under both constraints, and an inner product on this space, induced by the group averaging processes, rendering this vector space a Hilbert space. The resultant Hilbert space of diffeomorphism invariant states is the equivalence classes of diffeomorphism invariant graphs $[g]$, solutions to the scalar constraint.

Let us conclude with some remarks. In the classical theory, the geometry possesses a singularity whenever the determinant of the metric q vanishes at some point. In this manuscript, the vanishing of q corresponds to the vanishing of E^φ at a given region (local singularity). In the quantum theory we can find an analogous situation, for instance if a graph g has $\mu_j = 0$ at one or some given vertices. Fortunately, the quantum theory allows us to avoid these undesired divergences. The key idea consists in identifying a suitable invariant domain of the scalar constraint, free of such states with nonvanishing μ_j. In this way, the solutions to the constraints have support only on them, preventing the vanishing of μ_j (and k_j) at any vertex. It is straightforward to prove, as a direct consequence of the previous points 3 and 4, that the subspace formed by kets, such that their sequences $\{\mu_j\}$ (and $\{k_j\}$) contain no vanishing components, remains invariant under the action of the Hamiltonian constraint (5.21). In particular, point 4 tells us that we can never reach a vanishing μ_j by successive action of the scalar constraint, and point 3 tells us that any sequence of $\{k_j\}$ remains invariant. In conclusion, the restriction to this invariant domain allows us to resolve the classical singularity.

However, given that the sequences $\{k_j\}$ are unaltered by the scalar constraint, and since they apparently have no significance in singularity resolution of this model, we do not see any fundamental argument for discriminating those with vanishing $\{k_j\}$ components with respect to the remaining ones. In [19] it was suggested that the reality conditions of some observables of the model provide a quantum theory free of singularities. However, due to some important differences of that model compared with the present one, we have not been able to identify such suitable observables in our model along those lines.

E. Quantum observables

We saw in Sec. V D that the Hamiltonian constraint does not create any new vertices in the graph g on which it acts (and obviously neither does the diffeomorphism constraint). This means that there is a Dirac observable $\hat{\mathcal{N}}_v$ in the bulk corresponding to the fixed number of vertices $\mathcal{N}_v = V$ of a graph g,

$$\hat{\mathcal{N}}_v \Psi_{\text{phys}} = \mathcal{N}_v \Psi_{\text{phys}}. \tag{5.40}$$

This observable is strictly quantum and has no counterpart in the classical theory.

On the other hand, since this model has only one spatial direction, under the action of the diffeomorphism constraint the points cannot pass each other, i.e., the order of the positions of the vertices is preserved. This means that, associated to this preservation, we can identify another new strictly quantum observable in the bulk, $\hat{O}(z)$ such that

$$\hat{O}(z)\Psi_{\text{phys}} = k_{\text{Int}(z\mathcal{N}_v)}\Psi_{\text{phys}}, \qquad z \in [0,1] \tag{5.41}$$

where $\text{Int}(z\mathcal{N}_v)$ is the integer part of $z\mathcal{N}_v$. Together with them, we also have the observable corresponding to the mass $\hat{M}$, which does have an analogous classical Dirac observable.

Besides, as it was first observed by the authors of Ref. [19,20], one can construct an evolving constant associated to E^x from the above observable as

$$\widehat{E^x}(x)\Psi_{\text{phys}} = \hbar G_2 \hat{O}(z(x))\Psi_{\text{phys}}, \tag{5.42}$$

with $z(x)\colon [0,x] \to [0,1]$. Since E^x has a classical and quantum mechanically different interpretation in the CGHS model than in $3+1$ spherical symmetry, i.e., in the former it is related to the dilaton field, one should also be cautious about its interpretation.

These two observables were first introduced for the $3+1$ spherically symmetric case in [19] and due to the similarities of the two models, we can see that they exist also for the CGHS model. Particularly, the observable in (5.41) arises due to the existence of only one (radial) direction in both cases. So one can expect that such a quantum observable will exist in many genuinely two-dimensional and symmetry-reduced models with only one radial direction in which the quantum theory implements the spatial diffeomorphism symmetry as in loop quantum gravity.

It is worth commenting that one can promote the metric component $\hat{E}^\varphi$ as a parametrized observable. For it, we can choose the phase-space variable K_φ as an internal time function (or parametric function). Moreover, by means of the Hamiltonian constraint (on shell), it is possible to define the parametrized observable

$$\hat{E}^\varphi(x)\Psi_{\text{phys}} = \frac{\partial_x \widehat{E^x}(x)}{\sqrt{4[\widehat{E^x}(x)]^2\lambda^2 + \frac{\sin^2(\rho K_\varphi)}{\rho^2} - 2\lambda G_2 \hat{M}\widehat{E^x}(x)}}\Psi_{\text{phys}}, \tag{5.43}$$

which is defined in terms of the parameter functions $z(x)$ and $K_\varphi(x)$, and the observables $\hat{M}$ and $\hat{O}$ [through the definition of $\widehat{E^x}(x)$].

VI. SUMMARY AND CONCLUSION

We have shown that, with the introduction of polar-type variables for a CGHS dilatonic black hole and a rewrite of its Hamiltonian in terms of those variables, one can follow recent LQG inspired methods, first introduced in the $3+1$ spherically symmetric case—also written in polar-type variables—to remove the singularity of the CGHS model. The proposal is based on the assumption (proven here for the case of a two-dimensional generic metric) that states with zero volume are those containing a spacetime singularity. Then singularity resolution follows if one can show that if one starts from a state without a zero volume present in it, one can restrict the evolution to a subspace of the Hilbert space that contains no zero volume states. In other words, the subspace of quantum spacetime states without a singularity is preserved under the action of the quantum Hamiltonian constraint.

This analysis may be extended further when a matter field is present in the theory and one might then study the backreaction, but that analysis will certainly be more involved and is outside the scope of this paper. Although it has been shown recently [25] that even in the presence of matter (more precisely, the massless scalar field), one can get a Lie algebra of constraints by strong Abelianization of the $\{\mathcal{H}(N), \mathcal{H}(M)\}$ part of the classical constraint algebra, it is not clear whether the quantum theory is anomaly free and also whether one can get some useful information about the Hamiltonian constraint, as was possible in the present case without matter. Furthermore, the representation of this constraint on the Hilbert space is expected to be much more involved. For a minimally coupled scalar field (in the classical theory its dynamics reduces to the one of a scalar field in Minkowski) one can expect a more treatable model with respect to its analogue in $3+1$ spherically symmetric spacetimes, regarding its solubility. Nevertheless, this is an interesting future project worth pursuing, as is the study of the Hawking radiation based on these results.

In any case, the analysis here presented must be viewed as a first step that requires further understanding, analysis and level of precision. It can hopefully be further extended to give more insights on generic black hole singularity resolution and, more generally, on quantum gravity itself.

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APPENDIX: SPECTRUM OF GEOMETRICAL OPERATORS

In this appendix we discuss some properties of the Hamiltonian constraint restricted to one arbitrary vertex v_j (we omit the label j in the following). Let us recall that the local scalar constraint of this model, once promoted to a quantum operator, acts (almost) independently on each vertex. Its action on the corresponding states is

$$\begin{aligned}\hat{\mathfrak{H}}|g,k,\mu,M\rangle = (\hbar G_2 k)[f_0(\mu,k,M)|g,k,\mu,M\rangle \\ - f_+(\mu)|g,k,\mu+4\rho,M\rangle \\ - f_-(\mu)|g,k,\mu-4\rho,M\rangle],\end{aligned} \tag{A1}$$

with the functions

$$f_\pm(\mu) = \frac{\hbar G_2}{16\rho^2}|\mu|^{1/4}|\mu\pm 2\rho|^{1/2}|\mu\pm 4\rho|^{1/4}[\mathrm{sgn}(\mu\pm 4\rho)+\mathrm{sgn}(\mu\pm 2\rho)][\mathrm{sgn}(\mu\pm 2\rho)+\mathrm{sgn}(\mu)], \tag{A2}$$

$$\begin{aligned}f_0(\mu,k,M) = \lambda^2\left(1-\frac{G_2 M}{2\hbar G_2 k\lambda}\right)(\hbar G_2)^3\mu k^2 + \frac{\hbar G_2}{16\rho^2}\{|\mu|^{1/2}|\mu+2\rho|^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu+2\rho)]^2 \\ +|\mu|^{1/2}|\mu-2\rho|^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu-2\rho)]^2\} - \frac{\hbar G_2}{\rho^2}(|\mu+\rho|^{1/2}-|\mu-\rho|^{1/2})^2\Delta k^2.\end{aligned} \tag{A3}$$

Here, Δk is proportional to the eigenvalue of the operator $(\widehat{E^x(x)})'$. This operator is diagonal on the spin network basis of states and its explicit form depends on the definition of the operator $\hat{E}^x$.

The action of the scalar constraint resembles the one of a second order difference operator since it relates three consecutive points in a lattice with constant step. The consequence is that any function $\phi(k,\mu,M)$ that is a solution to the equation $(\phi|\hat{\mathfrak{H}}^\dagger = 0$ has support on lattices of step 4ρ, as we can deduce by direct inspection of Eq. (A1). Moreover, due to the functions $[\mathrm{sgn}(\mu\pm 2\rho)+\mathrm{sgn}(\mu)]$ in (A2), $f_\pm(\mu)$ vanishes on $[0,\mp 2\rho]$, respectively.

Thus different orientations of μ are decoupled by the difference operator (A1). We conclude that μ belongs to semilattices of the form $\mu = \epsilon \pm 4\rho \mathrm{n}$ where $\mathrm{n} \in \mathbb{N}$ and $\epsilon \in (0, 4\rho]$. Without loss of generality, we restrict the study to a particular subspace labeled by ϵ, unless otherwise specified. This shows that the Hamiltonian constraint only relates states belonging to separable subspaces of the original kinematical Hilbert space.

The solutions $\phi(k, \mu, M)$ fulfil the equation

$$\begin{aligned} &-f_+(\mu - 4\rho)\phi(k, \mu - 4\rho, M) \\ &\quad - f_-(\mu + 4\rho)\phi(k, \mu + 4\rho, M) \\ &\quad + f_0(k, \mu, M)\phi(k, \mu, M) = 0. \end{aligned} \tag{A4}$$

One can straightforwardly realize that, for any choice of the initial triad section $\mu = \epsilon$, they are completely determined by their initial data $\phi(k, \mu = \epsilon, M)$. In particular, our difference operator evaluated at $\mu = \epsilon$ relates the solution coefficient $\phi(k, \mu = \epsilon + 4\rho, M)$ with only the initial data $\phi(k, \mu = \epsilon, M)$, which can be solved easily. Therefore, the difference equation evaluated at the next successive lattice points can also be solved straightforwardly, once the initial data $\phi(k, \mu = \epsilon, M)$ are provided. Without loss of generality, we fix it to be real. This allows us to conclude that, since the coefficients of the corresponding difference equation (A4) are also real functions, the solutions $\phi(k, \mu, M)$ at any triad section $\mu = \epsilon \pm 4\rho \mathrm{n}$ will also be real functions.

Besides, the solutions to Eq. (A4), for constant values of the quantum numbers k, M, and the cosmological constant λ, have different asymptotic limits at $\mu \to \infty$. Concretely, if they fulfil $k < M/2\hbar\lambda$ or $k > M/2\hbar\lambda$, the physically relevant solutions either oscillate or decay exponentially, respectively, in that limit.

We focus now on the study of the solutions in the cases in which $k > M/2\hbar\lambda$. In this regime, it is more convenient to carry out a transformation in the functional space of solutions in order to achieve a suitable separable form of the constraint equation. In particular, following the ideas of Ref. [30], we introduce a bijection on the space of solutions defined by the scaling of the solutions

$$\phi^{\mathrm{dcr}}(k, \mu, M) = (\hbar G_2)^{1/2} \hat{b}(\mu) \phi(k, \mu, M), \tag{A5}$$

with

$$\hat{b}(\mu) = \frac{1}{\rho}(|\hat{\mu} + \rho|^{1/2} - |\hat{\mu} - \rho|^{1/2}). \tag{A6}$$

We might notice that the functions $\hat{b}(\mu)$ only vanish for $\mu = 0$. But this sector has been decoupled, since μ belongs to semilattices with a global minimum at $\mu = \epsilon > 0$. Therefore, the function $\hat{b}(\mu)$ never vanishes and the previous scaling is invertible. The new functions $\phi^{\mathrm{dcr}}(k, \mu, M)$ now fulfil the difference equation

$$\begin{aligned} &-f_+^{\mathrm{dcr}}(\mu - 4\rho)\phi^{\mathrm{dcr}}(k, \mu - 4\rho, M) \\ &\quad - f_-^{\mathrm{dcr}}(\mu + 4\rho)\phi^{\mathrm{dcr}}(k, \mu + 4\rho, M) \\ &\quad + f_0^{\mathrm{dcr}}(k, \mu, M)\phi^{\mathrm{dcr}}(k, \mu, M) = 0, \end{aligned} \tag{A7}$$

where the new coefficients are now

$$\begin{aligned} f_\pm^{\mathrm{dcr}}(\mu) = &\frac{1}{16\rho^2 b(\mu) b(\mu \pm 4\rho)} |\mu|^{1/4} |\mu \pm 2\rho|^{1/2} |\mu \pm 4\rho|^{1/4} \\ &\times [\mathrm{sgn}(\mu \pm 4\rho) + \mathrm{sgn}(\mu \pm 2\rho)] \\ &\times [\mathrm{sgn}(\mu \pm 2\rho) + \mathrm{sgn}(\mu)], \end{aligned} \tag{A8}$$

$$\begin{aligned} f_0^{\mathrm{dcr}}(\mu, k, M) = &\frac{\mu}{b(\mu)^2}\left(1 - \frac{G_2 M}{2\hbar G_2 \lambda k}\right)(\hbar G_2 \lambda)^2 k^2 \\ &+ \frac{1}{16\rho^2 b(\mu)^2} [(|\mu||\mu + 2\rho|)^{1/2} \\ &\times [\mathrm{sgn}(\mu) + \mathrm{sgn}(\mu + 2\rho)]^2 \\ &+ (|\mu||\mu - 2\rho|)^{1/2} [\mathrm{sgn}(\mu) + \mathrm{sgn}(\mu - 2\rho)]^2] \\ &- \Delta k^2. \end{aligned} \tag{A9}$$

This difference operator can be naively interpreted as a densitized scalar constraint, for instance, like the one emerging after choosing the lapse function $N\hat{b}(\mu)^{-2}$ (together a suitable factor ordering and a global factor $\hbar G_2$). Let us denote this scalar constraint in the original scaling by $\hat{\mathfrak{H}}$, and the corresponding scalar constraint by $\hat{\mathfrak{H}}^{\mathrm{dcr}}$ in the new one. Both are related by

$$\hat{\mathfrak{H}}^{\mathrm{dcr}} = \hat{b}(\mu)^{-1} \hat{\mathfrak{H}} \hat{b}(\mu)^{-1}. \tag{A10}$$

Now, we study the difference operator

$$\hat{\mathfrak{h}}^{\mathrm{dcr}} = \hat{\mathfrak{H}}^{\mathrm{dcr}} + \Delta k^2. \tag{A11}$$

We can deduce several properties about the spectrum of this difference operator as well as of its eigenfunctions. Let us consider, for consistency, its positive spectrum. The eigenvalue problem

$$\hat{\mathfrak{h}}^{\mathrm{dcr}} |\phi_\omega^{\mathrm{dcr}}\rangle = \omega |\phi_\omega^{\mathrm{dcr}}\rangle \tag{A12}$$

corresponds to a difference equation similar to Eq. (A7) but with functions

$$\begin{aligned} \tilde{f}_\pm^{\mathrm{dcr}}(\mu) = &\frac{1}{16\rho^2 b(\mu) b(\mu \pm 4\rho)} |\mu|^{1/4} |\mu \pm 2\rho|^{1/2} |\mu \pm 4\rho|^{1/4} \\ &\times [\mathrm{sgn}(\mu \pm 4\rho) + \mathrm{sgn}(\mu \pm 2\rho)] \\ &\times [\mathrm{sgn}(\mu \pm 2\rho) + \mathrm{sgn}(\mu)], \end{aligned} \tag{A13}$$

$$\tilde{f}_0^{\rm dcr}(\mu,k,M,\omega) = \frac{\mu}{b(\mu)^2}\left(1-\frac{G_2M}{2\hbar G_2\lambda k}\right)(\hbar G_2\lambda)^2k^2 + \frac{1}{16\rho^2 b(\mu)^2}[(|\mu||\mu+2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu+2\rho)]^2 + (|\mu||\mu-2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu-2\rho)]^2]-\omega. \tag{A14}$$

Let us assume that the solutions to this difference equation have a well-defined and smooth limit $\mu\to\infty$. For practical purposes this limit is similar to the limit $\rho\to 0$, but keeping in mind that while the former is expected to be well defined in our quantum theory, the latter is not. This assumption involves the solutions $\phi_\omega^{\rm dcr}(\mu)$ being continuous functions of μ. But this is not true for scales $\Delta\mu$ of the order of 4ρ (in the previous asymptotic limit). This must be tested carefully, but we do not deal with this question now. We assume its validity, at least for eigenvalues with typical scales much bigger than 4ρ.

Within this asymptotic regime and approximation, the solutions to the previous difference equation (A12) satisfy in a very good approximation the differential equation

$$\begin{aligned} 0 &= -\tilde{f}_+^{\rm dcr}(\mu-4\rho)\phi_\omega^{\rm dcr}(k,\mu-4\rho,M) \\ &\quad - \tilde{f}_-^{\rm dcr}(\mu+4\rho)\phi_\omega^{\rm dcr}(k,\mu+4\rho,M) \\ &\quad + \tilde{f}_0^{\rm dcr}(k,\mu,M,\omega)\phi_\omega^{\rm dcr}(k,\mu,M) \\ &= -4\mu^2\partial_\mu^2\phi_\omega^{\rm dcr}(k,\mu,M) - 8\mu\partial_\mu\phi_\omega^{\rm dcr}(k,\mu,M) \\ &\quad + \left[\left(1-\frac{G_2M}{2\hbar G_2\lambda k}\right)(\hbar G_2\lambda)^2k^2\mu^2-\gamma^2\right]\phi_\omega^{\rm dcr}(k,\mu,M) \\ &\quad + \mathcal{O}(\rho^2/\mu^2), \end{aligned} \tag{A15}$$

with $\gamma^2=\omega+3/4$. Let us recall that this differential equation can be analogously achieved if instead of adopting a loop quantization, one adheres to a Wheeler-DeWitt (WDW) representation for this setting, with a suitable factor ordering. It corresponds to a modified Bessel equation, where its solutions are combinations of modified Bessel functions of the form

$$\lim_{\mu\to\infty}\phi_\omega^{\rm dcr}(k,\mu,M) = Ax^{-1/2}\mathcal{K}_{i\gamma}(x) + Bx^{-1/2}\mathcal{I}_{i\gamma}(x), \tag{A16}$$

with

$$x=\mu\frac{\hbar G_2\lambda k}{2}\sqrt{\left(1-\frac{G_2M}{2\hbar G_2\lambda k}\right)}. \tag{A17}$$

In the limit $\mu\to\infty$, the solutions $\mathcal{I}$ and $\mathcal{K}$ grow and decay exponentially, respectively. Therefore, the latter is the only contribution to the spectral decomposition of $\hat{\mathfrak{h}}^{\rm dcr}$. In consequence, its possible (positive) eigenvalues ω are nondegenerate. Besides, the functions $\mathcal{K}_{i\gamma}(x)$ are normalized to

$$\langle\mathcal{K}_{i\gamma}|\mathcal{K}_{i\gamma'}\rangle=\delta(\gamma-\gamma'), \tag{A18}$$

in $L^2(\mathbb{R},x^{-1}dx)$, since the normalization in this case is ruled by the behavior of $\mathcal{K}_{i\gamma}(x)$ in the limit $x\to 0$, which corresponds to

$$\lim_{x\to 0}\mathcal{K}_{i\gamma}(x)\to A\cos(\gamma\ln|x|). \tag{A19}$$

For additional details see, for instance, Ref. [37]. This result is fulfilled in the continuous theory, whenever (A15) is valid globally. But let us recall that we are dealing with a difference equation possessing, in a good approximation, a continuous $\mu\to\infty$ limit, but not at all for $\mu\to 0$. Therefore, the previous normalization (A18) and the asymptotic limit (A19) have no meaning in our discrete theory. In this case, and in the absence of a meticulous numerical study of the solutions of this equation, we can only infer some properties about $\phi_\omega^{\rm dcr}(k,\mu,M)$. One can convince oneself that our difference equation is similar to the one studied in [31] for a closed Friedmann-Robertson-Walker spacetime. In particular, the eigenfunctions of such a difference operator have a similar asymptotic behavior for $v\to\infty$ (or equivalently $\mu\to\infty$ in our model). Nevertheless, the spectrum of the corresponding difference operator turns out to be discrete (instead of continuous like the corresponding differential operator) owing to the behavior of its eigenfunctions at $v\simeq\epsilon$ (i.e. $\mu\simeq\epsilon$). Therefore, we expect, following the results of Ref. [31], that the eigenvalues ω of the difference operator $\hat{\mathfrak{h}}^{\rm dcr}$ belongs to a countable set, which we call $\{\omega(n)\}$. One also expects that the possible (positive) values of $\omega(n)$ depend on $\epsilon\in(0,4\rho]$, and for a given ϵ, they also depend on k and M. Let us comment that the particular values of the sequence $\{\omega(n)\}$ as well as the explicit form of the eigenfunctions $\phi_\omega^{\rm dcr}(k,\mu,M)$, to the knowledge of the authors, can only be determined numerically by now, unless new analytical tools are developed. In addition, a second look at the difference equation (A12) tells us that the eigenfunctions are completely determined by their value at the initial data section $\phi_\omega^{\rm dcr}(k,\mu=\epsilon,M)$. Therefore, the spectrum of $\hat{\mathfrak{h}}^{\rm dcr}$ will be nondegenerate. Moreover, let us recall that, if we choose the initial data to be real, all the coefficients $\phi_\omega^{\rm dcr}(k,\mu,M)$ for any μ will also be real.

Eventually, the corresponding eigenfunctions, as functions of μ, will be square summable, fulfilling the normalization condition

$$\langle\phi_{\omega_n}^{\rm dcr}|\phi_{\omega_{n'}}^{\rm dcr}\rangle = \sum_{\rm n}\phi_{\omega_n}^{\rm dcr}(k,\epsilon+4{\rm n}\rho,M) \times\phi_{\omega_{n'}}^{\rm dcr}(k,\epsilon+4{\rm n}\rho,M)=\delta_{nn'}, \tag{A20}$$

recalling that these coefficients are real. It is worth commenting that due to the scaling (A5), the coefficients in the previous sum are weighted simply with the unit. This is not the case, for instance, in Ref. [31] where the norm of the corresponding eigenfunctions includes a weight function different from the unit, since no scalings of the solutions are considered.

The constraint in this basis takes the algebraic form

$$\omega(n) - \Delta k^2 = 0. \tag{A21}$$

Let us now study the solutions to the constraint when $k < M/2\hbar\lambda$. In this case we follow again the ideas of [30] in order to render our equation in a suitable representation where it again becomes separable. Let us comment that the solutions to the difference equation for $k < M/2\hbar\lambda$ have different asymptotic behaviors at $\mu \to \infty$ than the ones for $k > M/2\hbar\lambda$. It involves requiring that the change of representation that is considered in each case, in order to express the constraint equation in a simple separable form, must not be the same.

With all this in mind, let us consider this invertible scaling:

$$\phi_j^{\rm cnt}(k,\mu,M) = (\hbar G_2\hat{\mu})^{1/2}\phi(k,\mu,M). \tag{A22}$$

As before, $\mu = 0$ could be problematic in order to define this redefinition properly. But let us recall that this sector has been decoupled from the quantum theory. As a consequence μ has a global minimum equal to $\epsilon > 0$. Therefore, the previous scaling (A22) can be inverted and the original description recovered. The new functions $\phi^{\rm cnt}(k,\mu,M)$ now fulfil the difference equation

$$\begin{aligned} &- f_+^{\rm cnt}(\mu-4\rho)\phi^{\rm cnt}(k,\mu-4\rho,M)\\ &- f_-^{\rm cnt}(\mu+4\rho)\phi^{\rm cnt}(k,\mu+4\rho,M)^{\rm cnt}\\ &+ f_0^{\rm cnt}(k,\mu,M)\phi^{\rm cnt}(k,\mu,M) = 0, \end{aligned} \tag{A23}$$

but this time the coefficients are

$$\begin{aligned} f_\pm^{\rm cnt}(\mu) = \frac{1}{16\rho^2}&|\mu|^{-1/4}|\mu\pm 2\rho|^{1/2}|\mu\pm 4\rho|^{-1/4}\\ &\times[\mathrm{sgn}(\mu\pm 4\rho)+\mathrm{sgn}(\mu\pm 2\rho)]\\ &\times[\mathrm{sgn}(\mu\pm 2\rho)+\mathrm{sgn}(\mu)], \end{aligned} \tag{A24}$$

$$\begin{aligned} f_0^{\rm cnt}(\mu,k,M) = \frac{1}{16\rho^2\mu}&[(|\mu||\mu+2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu+2\rho)]^2\\ &+(|\mu||\mu-2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu-2\rho)]^2]\\ &-\frac{\mathrm{sgn}(\mu)}{\mu\rho^2}\Delta k^2(|\mu+\rho|^{1/2}-|\mu-\rho|^{1/2})^2. \end{aligned} \tag{A25}$$

This version of the scalar constraint, as we mentioned previously, can be naively understood as a densitized version of the original classical constraint after the choice of $N\mu^{-1}$ as the new lapse function (and an adequate factor ordering and a global factor $\hbar G_2$). Following the notation that we introduced, we denote this new scalar constraint by $\hat{\mathfrak{H}}^{\rm cnt}$. It is related with the original one by means of

$$\hat{\mathfrak{H}}^{\rm cnt} = (\hbar G_2\hat{\mu})^{-1/2}\hat{\mathfrak{H}}(\hbar G_2\hat{\mu})^{-1/2}. \tag{A26}$$

The difference operator that is studied now reads

$$\hat{\mathfrak{h}}^{\rm cnt} = \hat{\mathfrak{H}}^{\rm cnt} - \left(1-\frac{G_2 M}{2\hbar G_2\lambda k}\right)(\hbar G_2\lambda)^2 k^2. \tag{A27}$$

Therefore, we have written again the original constraint $\hat{\mathfrak{H}}$ in a suitable separable form according to the condition $k < M/2\hbar\lambda$.

We now study the spectrum of the difference operator $\hat{\mathfrak{h}}^{\rm cnt}$, by means of the eigenvalue problem

$$\hat{\mathfrak{h}}^{\rm cnt}|\phi_\omega^{\rm cnt}\rangle = \omega|\phi_\omega^{\rm cnt}\rangle, \tag{A28}$$

for $\omega \geq 0$, which are the physically interesting values. This equation can be written in the form of (A23), but with coefficients

$$\begin{aligned} \tilde{f}_\pm^{\rm cnt}(\mu) = \frac{1}{16\rho^2}&|\mu|^{-1/4}|\mu\pm 2\rho|^{1/2}|\mu\pm 4\rho|^{-1/4}\\ &\times[\mathrm{sgn}(\mu\pm 4\rho)+\mathrm{sgn}(\mu\pm 2\rho)]\\ &\times[\mathrm{sgn}(\mu\pm 2\rho)+\mathrm{sgn}(\mu)], \end{aligned} \tag{A29}$$

$$\begin{aligned} &\tilde{f}_0^{\rm cnt}(\mu,k,M,\omega)\\ &\quad= \frac{1}{16\rho^2\mu}[(|\mu||\mu+2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu+2\rho)]^2\\ &\qquad+(|\mu||\mu-2\rho|)^{1/2}[\mathrm{sgn}(\mu)+\mathrm{sgn}(\mu-2\rho)]^2]\\ &\qquad-\frac{\mathrm{sgn}(\mu)}{|\mu|\rho^2}\Delta k^2(|\mu+\rho|^{1/2}-|\mu-\rho|^{1/2})^2-\omega. \end{aligned} \tag{A30}$$

Let us recall, again, that the coefficients $\phi_\omega^{\rm cnt}(k,\mu,M)$ of these eigenstates are determined by their initial data $\phi_\omega^{\rm cnt}(k,\mu=\epsilon,M)$ through the difference equation (A28). In consequence, the (positive) spectrum of $\hat{\mathfrak{h}}^{\rm cnt}$ is nondegenerate. Moreover, the coefficients $\phi_\omega^{\rm cnt}(k,\mu,M)$ will be real if $\phi_\omega^{\rm cnt}(k,\mu=\epsilon,M)\in\mathbb{R}$, since the previous functions $\tilde{f}_0^{\rm cnt}$ and $\tilde{f}_\pm^{\rm cnt}$ are also real.

We assume, again, that these solutions have a well-defined and smooth asymptotic behavior for $\mu\to\infty$. Let us recall that this involves eigenvalues with typical scales much bigger than 4ρ. This continuity condition allows us to approximate the difference equation for those large scale eigenvalues at $\mu\to\infty$ by

$$0 = -\tilde{f}_+^{\text{cnt}}(\mu - 4\rho)\phi_\omega^{\text{cnt}}(k, \mu - 4\rho, M) - \tilde{f}_-^{\text{cnt}}(\mu + 4\rho)\phi_\omega^{\text{cnt}}(k, \mu + 4\rho, M) + \tilde{f}_0^{\text{cnt}}(k, \mu, M, \omega)\phi_\omega^{\text{cnt}}(k, \mu, M)$$
$$= -4\partial_\mu^2\phi_\omega^{\text{cnt}}(k, \mu, M) - \frac{\gamma^2}{\mu^2}\phi_\omega^{\text{cnt}}(k, \mu, M) - \omega\phi_\omega^{\text{cnt}}(k, \mu, M) + \mathcal{O}(\rho^2/\mu^2), \tag{A31}$$

but this time with $\gamma^2 = \Delta k^2 + 3/4$. Let us comment that this very same differential equation would have been obtained if we had considered a WDW representation, instead of the loop quantization, with a suitable choice in the ordering of the operators for the definition of the corresponding Hamiltonian constraint.

The solutions to this differential equation are linear combinations of Hankel functions of first $H_{i\gamma}^{(1)}(y)$ and second $H_{i\gamma}^{(2)}(y)$ kind, multiplied by a factor $y^{1/2}$, where $y = \mu\sqrt{\omega}/2$. As a consequence, the asymptotic limit of the eigenstates is

$$\lim_{\mu\to\infty}\phi_\omega^{\text{cnt}}(k, \mu, M) = Ay^{1/2}H_{i\gamma}^{(1)}(y) + By^{1/2}H_{i\gamma}^{(2)}(y). \tag{A32}$$

These functions have a well-known asymptotic limit at $y \to \infty$, corresponding to

$$H_{i\gamma}^{(1)}(y) = \sqrt{\frac{2}{\pi y}}e^{i(y-\pi/4+\gamma\pi/2)}, \tag{A33}$$

with $H_{i\gamma}^{(2)}(y) = (H_{i\gamma}^{(1)}(y))^*$. This asymptotic limit of the Hankel functions, together with (A32) and the fact that $\phi^{\text{cnt}}(k, \mu, M) \in \mathbb{R}$ at any μ, allows us to conclude that

$$\lim_{\mu\to\infty}\phi_\omega^{\text{cnt}}(k, \mu, M) = A\cos\left[\frac{\sqrt{\omega}}{2}\mu + \beta\right], \tag{A34}$$

with A being a normalization constant and β a phase that is expected to depend on Δk, and ϵ. This asymptotic behavior is radically different in comparison with the eigenstates $\phi_\omega^{\text{dcr}}(k, \mu, M)$. Instead of decaying exponentially, they simply oscillate as standing waves (up to negligible corrections) of frequency $\sqrt{\omega}/2$. Therefore, our experience in loop quantum cosmology [30,32,33] tells us that these eigenfunctions are normalizable functions of μ (in the generalized sense),

$$\langle\phi_\omega^{\text{in}}|\phi_{\omega'}^{\text{in}}\rangle = \sum_{\text{n}}\phi_\omega^{\text{cnt}}(k, \epsilon + 4\text{n}\rho, M) \times \phi_{\omega'}^{\text{cnt}}(k, \epsilon + 4\text{n}\rho, M) = \delta(\sqrt{\omega}/2 - \sqrt{\omega'}/2). \tag{A35}$$

Eventually, the constraint in the basis of states $|\phi_\omega^{\text{in}}\rangle$ takes the form

$$\omega + \left(1 - \frac{G_2 M}{2\hbar G_2\lambda k}\right)(\hbar G_2\lambda)^2 k^2 = 0. \tag{A36}$$

It is worth commenting that these results might be modified for those "high frequency" eigenvalues, where the discreteness of the lattice in μ is important. This will be a matter of future research.

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