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Following previous works on generalized Abelian Proca theory, also called vector Galileon, we investigate the massive extension of an SU(2) gauge theory, i.e., the generalized SU(2) Proca model, which could be dubbed non-Abelian vector Galileon. This particular symmetry group permits fruitful applications in cosmology such as inflation driven by gauge fields. Our approach consists in building, in an exhaustive way, all the Lagrangians containing up to six contracted Lorentz indices. For this purpose, and after identifying by group theoretical considerations all the independent Lagrangians which can be written at these orders, we consider the only linear combinations propagating 3 degrees of freedom and having healthy dynamics for their longitudinal mode, i.e., whose pure Stückelberg contribution turns into the SU(2) multi-Galileon dynamics. Finally, and after having considered the curved space-time expansion of these Lagrangians, we discuss the form of the theory at all subsequent orders.

DOI: [10.1103/PhysRevD.94.084041](https://doi.org/10.1103/PhysRevD.94.084041)**I. INTRODUCTION**

In the search for well-motivated theories that describe the primordial Universe, several attempts have been made to obtain inflationary descriptions from particle physics (the Standard Model, supersymmetry, grand unified theories, etc.; see, e.g., Refs. [1–5]), or from quantum theories of gravity such as supergravity, string theory, and loop quantum gravity (see, e.g., Refs. [6–10]). This top-down approach has been very fruitful, providing new ways to understand the structure of the high energy theories necessary to reproduce the observable properties of the Universe, ranging from cosmic microwave background radiation (CMB) to the large-scale structure (LSS). However, little is known from the observational point of view for many of these theories (those whose characteristic energy scale is much higher than the electroweak one), the CMB and LSS being, at present, the only situations in

which they would have had observable consequences and would thus leave testable signatures. Since the power of the current and proposed accelerators is not going to increase as much as would be needed to directly test these theories in the foreseeable future, we need to devise another approach to the fundamental theory that describes nature.

Such an approach already exists, and it boils down to the question of whether there is any choice in formulating the fundamental theory. This bottom-up approach consists in finding an action completely free of pathologies, the first of them being the Ostrogradski instability [11] (the Hamiltonian could be unbounded from below), and satisfying a given set of assumptions, e.g., symmetry requirements. One then needs to define the material content of the Universe (scalar fields, vector fields, ...), although, in principle, the construction itself and the stability requirements constrain some content and allow others so that the material content is, once the conditions are applied, somehow redefined. This very ambitious program is just beginning to be implemented, and interesting works have been carried out in which the extra material content (on top of gravity) is composed of one or many scalar fields. It was Horndeski

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[12] who found, for the first time, the most general action for a scalar field and gravity that produces second-order equations of motion. In general, if the Lagrangian is non-degenerate, having equations of motion of second order at most is a necessary requirement to avoid the Ostrogradski instability [13,14]. By pursuing this goal, an action is found that, however, still requires a Hamiltonian analysis in order to guarantee that the instability is not present.

Horndeski's construction was rediscovered in the context of what is nowadays called Galileons [15]. The Galileons are the scalar fields whose action, in flat spacetime, leads to equations of motion that involve only second-order derivatives. The idea has been extended by finding the so-called generalized Galileons, allowing for lower-order derivatives in the equations of motion [16,17]. The background spacetime geometry where these generalized Galileons live can be promoted to a curved one by replacing the ordinary derivatives with covariant ones and adding some counterterms that involve nonminimal couplings to the curvature [18,19]. The latter guarantees the equations of motion for both geometry and matter are still second order, so the Galileons, both generalized and covariantized, are found. This procedure is equivalent to that proposed by Horndeski for one scalar field [20], but it loses some interesting terms when more than one scalar field is present [21]. The Galileon approach for scalar fields has found multiple applications in cosmology, ranging from inflation (see, e.g., Refs. [22–32]) to dark energy (see, e.g., Refs. [33–50]).

The original proposal was based on the requirement of second-order equations of motion for all the additional degrees of freedom to gravity, all of them therefore being dynamical so that the system is nondegenerate. The generalization to the so-called extended Horndeski theories also includes nonphysical degrees of freedom and thus considers degenerate theories [51–55]. Such a construction is by now well understood, and some cosmological applications have also been considered [55–63].

However, scalar fields are not the only possibilities as the matter content of the Universe. Horndeski indeed wondered some 40 years ago what the action would be for an Abelian vector field in curved spacetime [64]. Working with curvature is a way to bypass the no-go theorem presented in Ref. [65], which states that the only possible action for an Abelian vector field in flat spacetime that leads to second-order equations of motion is the Maxwell-type one. Relaxing the gauge invariance allows for a nontrivial action in flat spacetime, in this way generalizing the Proca action [66,67]. The construction of the resulting vector Galileon action has been well investigated and discussed, so there is already a consensus about the number and type of terms in the action, even in the covariantized version [68–70]. Moreover, the analogous extended Horndeski theories have been built for a vector field [71,72], and the corresponding cosmological applications have been explored [67,73–78].

Some cosmological applications of vector fields have been investigated, and interesting scenarios, such as the  $fF^2$  model [79] and the vector curvaton [80,81], have been devised. There is, however, an obstacle when dealing with vector fields in cosmology: they produce too much anisotropy, both at the background and at the perturbation levels, well above the observable limits, unless one implements some dilution mechanism or considers only the temporal component of the vector field (which is, however, usually nondynamical). In the  $fF^2$  model, the potentially huge anisotropy is addressed by coupling the vector field to a scalar that dominates the energy density of the Universe and, therefore, dilutes the anisotropy; in contrast, in the vector curvaton scenario, the anisotropy is diluted by the very rapid oscillations of the vector curvaton around the minimum of its potential. Another dilution mechanism is to consider many randomly oriented vector fields [82]; however, this requires a large number of them, indeed hundreds, so it is difficult to justify it from a particle physics point of view. There is, nevertheless, another possibility, the so-called “cosmic triad” [82,83], a situation in which three vector fields orthogonal to each other and of the same norm can give rise to a rich phenomenology while making the background and perturbations completely isotropic [84]. A couple of very interesting models, gauge-flation [85,86] and chromonatural inflation [87], have implemented this idea by embedding it in a non-Abelian framework and exploiting the local isomorphism between the  $SO(3)$  and  $SU(2)$  groups of transformations. At first sight, the cosmic triad configuration looks very unnatural, but dynamical system studies have shown that it represents an attractor configuration [88]. Unfortunately, although the background dynamics of these two models is successful, their perturbative dynamics makes them incompatible with the latest Planck observations [89,90]. Despite this failure, such models have shown the applicability that non-Abelian gauge fields can have in cosmological scenarios.

Having in mind the above motivations, the purpose of this paper is to build the first-order terms of the generalized  $SU(2)$  Proca theory and to discuss the general form of the complete theory. For the most part, we focus on those Lagrangians containing up to six contracted Lorentz indices, which we obtain exhaustively. To ensure that we do not forget some terms, we first construct from group theoretical considerations all possible Lagrangians at these orders, before imposing the standard dynamical condition, i.e., that only 3 degrees of freedom propagate. Then, after identifying all the Lagrangians that imply the same dynamics, e.g., those related by a conserved current, we verify that the pure Stückelberg part of the Lagrangians is healthy, i.e., that it implies the  $SU(2)$  multi-Galileon dynamics. To this end, it is useful to derive all the equivalent formulations of the  $SU(2)$  adjoint multi-Galileon model, which we provide in Appendix A. Then, after computing the relevant curved space-time extension of our Lagrangians, we conclude

about the status of the complete formulation of the theory, i.e., that containing the higher order terms we did not consider in this work.

The layout of this paper is the following. In Sec. II, the generalized non-Abelian Proca theory is introduced, and some technical aspects needed for later sections are laid out; the procedure to build the theory is also described. In Sec. III, the building blocks of the Lagrangian are systematically obtained. Section IV deals with the right number of propagating degrees of freedom and the consistency of the obtained Lagrangian with the scalar Galileon nature of its longitudinal part. The covariantization of the theory is performed in Sec. V, and the final model, together with a discussion and comparison with the Abelian case, is presented in Sec. VI. Appendix A presents the construction of the multi-Galileon scalar Lagrangian in the three-dimensional representation of SU(2) and its equivalent formulations. Throughout this paper, we have employed the mostly plus signature, i.e.,  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ , and set  $\hbar = c = 1$ .

## II. GENERALIZED NON-ABELIAN PROCA THEORY

Our aim is to generalize the non-Abelian Proca theory, described below, to include all possible second-order ghost-free terms propagating only 3 degrees of freedom. After discussing the general symmetry case, we concentrate on the SU(2) symmetry, which is particularly interesting in a cosmological perspective, as discussed in the Introduction, and we roughly present the procedure, which will be thoroughly explained below.

### A. Non-Abelian Proca theory

Let us first present the nowadays standard non-Abelian Proca theory. Also called a massive Yang-Mills model, this theory had been extensively studied in the past, such as, e.g., in Refs. [91–95], with a Hamiltonian formulation detailed in Refs. [96,97]. Our starting point Lagrangian, including the mass term, reads

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \frac{1}{2}m^2A_a^\mu A_\mu^a, \quad (1)$$

with the non-Abelian Faraday tensor given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^a_{bc}A_\mu^b A_\nu^c, \quad (2)$$

with  $g$  being the coupling constant and  $f^a_{bc}$  the structure constants of the symmetry group under consideration. This can be considered as the limit of a valid particle physics model based on a Higgs condensate whose corresponding degree of freedom is assumed to be frozen, hence breaking the relevant symmetry [73,74].

Let us emphasize a technical point at this stage: one could work with the vector field assumed as an operator, namely,

$$A_\mu(x) = A_\mu^a(x)T_a, \quad (3)$$

with  $T_a$  representing the operators associated with corresponding elements of the underlying group in a given representation. We then have, by definition of the algebra, the commutation relations

$$[T_a, T_b] = if_{ab}{}^c T_c. \quad (4)$$

Since this work concentrates on the vector fields themselves and not on their action on other fields, it is simpler to restrict our attention to the fields themselves, i.e.,

$$A_\mu(x) = \{A_\mu^a(x)\}, \quad (5)$$

which are in the Lie algebra of the symmetry group under consideration. These two ways of writing the field operators are, of course, strictly equivalent, but the latter formalism, with group indices attached to the vectors themselves, merely does not need the introduction of the algebra operators themselves and is thus more appropriate for our purpose.

Any action needs to be a scalar, and this includes not only the Lorentz group but also any internal symmetry, such as that stemming from the algebra in Eq. (4). If the relevant symmetry is of the local type, and for an infinitesimal transformation, the vectors transform through

$$\delta A_\mu^a = -\frac{1}{g}\partial_\mu\alpha^a(x) + f^a_{bc}\alpha^b(x)A_\mu^c, \quad (6)$$

which leaves invariant only the kinetic term  $F_{\mu\nu}^a F_a^{\mu\nu}$ , but of course not even a mass term  $A_\mu^a A_a^\mu$ , much less any extension such as those we want to consider below. This is merely a restatement of the well-known fact that mass breaks gauge symmetry. We therefore restrict our attention to global transformations of the kind

$$\partial_\mu\alpha^a = 0 \Rightarrow \delta A_\mu^a = f^a_{bc}\alpha^b A_\mu^c; \quad (7)$$

i.e., we assume the vector field itself transforms as the adjoint representation, with dimension equal to that of the symmetry group itself. It is also profitable, and maybe more enlightening, to look at the effect of a finite local transformation of the group, still described by a set of parameters  $\alpha^a(x)$ . Under this transformation, the vector field transforms as

$$A_\mu(x) = A_\mu^a(x)T_a \mapsto U[\alpha^a(x)] \left[ -\frac{1}{g}\partial_\mu + A_\mu(x) \right] U^{-1}[\alpha^a(x)], \quad (8)$$

where  $U[\alpha^a(x)]$  describes the action of the group element labeled by  $\alpha^a(x)$ . This allows us to emphasize that in the case where the symmetry becomes global, i.e., where  $\alpha^a(x)$  no longer depends on the space-time point, the vector field transforms exactly as the adjoint representation of the symmetry group. This is indeed the symmetry assumed for the non-Abelian Proca (massive Yang-Mills) field. In

the Abelian case, this transformation is trivial because the action of the group commutes with the vector field, and the transformation in Eq. (8) thus reduces to the identity in the global symmetry case. In the non-Abelian case, however, one needs to specify how the extra indices are to be summed over in order to produce a singlet with respect to this global symmetry transformation. To relate the set of theories under considerations here with the more usual ones in particle physics involving a local symmetry broken by means of a Higgs field, one can envisage our transformation in Eq. (7) as the limit of that in Eq. (6).

With these motivating considerations, we now move on to evaluating the most general theory with a massive vector field transforming according to the adjoint representation of a given global symmetry group.

### B. Restricting attention to the SU(2) case

In view of the potentially relevant cosmological consequences, from now on we restrict our attention to the case for which the relevant symmetry group is SU(2), with dimension equal to 3, and therefore consider a vector field also of dimension 3. Since SU(2) is locally isomorphic to SO(3), one can then simply use a vector representation with group indices varying from 1 to 3 in  $A_\mu^a$ ; i.e., we restrict our attention to the fundamental representation of SO(3).

The set of SU(2) structure constants is identical to the three-dimensional Levi-Civita tensor  $\epsilon^{abc}$ , whereas the group metric  $g_{ab}$ , given by  $g_{ab} = -f_{ad}^e f_{be}^d$ , is simply the flat metric  $2\delta_{ab}$ . The only primitive invariants are  $\epsilon_{abc}$  and  $\delta_{ab}$  [98–100], and one can therefore write all possible contractions by merely contracting fields with contravariant indices with all appropriate combinations of those two primitive invariants written with covariant indices. Recall also the further simplification induced by the fact that contractions among structure constants (Levi-Civita symbols in the case at hand) leaving one, two or three free indices will, respectively, lead to a vanishing result, or terms proportional to  $\delta_{ab}$  and  $\epsilon_{abc}$  [101]; it is therefore often unnecessary to use multiple contractions.

As already alluded to earlier, choosing SU(2) is not innocuous as we aim at cosmological applications, in view, in particular, of implementing inflation driven by gauge fields (see, e.g., Refs. [85–90,102–115]): since its adjoint representation is three dimensional, SU(2) permits us to generate configurations for which all three vectors are nonvanishing while ensuring isotropy.

### C. Generalization

What follows is very similar to the generalized Abelian Proca case as discussed, e.g., in Refs. [66–70,116] (see also Refs. [117–119] for the equivalent curved space-time construction). In brief, we construct the most general action generalizing that of Proca for a massive SU(2) vector field, i.e.,

$$\mathcal{S}_{\text{Proca}} = \int \mathcal{L}_{\text{Proca}} d^4x = \int \left( -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \frac{1}{2} m_A^2 X \right) d^4x, \quad (9)$$

where  $X \equiv A_\mu^a A_a^\mu$ . To the above action (9), we add all possible terms containing not only functions of  $X$  but also derivative self-interactions. These terms will have to fulfill some conditions for the corresponding theory to make sense. We first split the vector into a scalar-pure vector decomposition

$$A_\mu^a = \partial_\mu \pi^a + \bar{A}_\mu^a, \quad (10)$$

where  $\pi^a$  is a scalar multiplet in the **3** representation of SU(2), i.e., the Stückelberg field generalized to the non-Abelian case, and  $\bar{A}_\mu^a$  is a divergence-free vector ( $\partial_\mu \bar{A}^{\mu a} = 0$ ), containing the curl part of the field, i.e., that for which the Abelian form of the Faraday tensor is non-vanishing. The conditions one then must impose on the theory in order for it to make (classical) sense are

- the equations of motion for all physical degrees of freedom, i.e., for both  $\bar{A}_\mu^a$  and  $\pi^a$ , and hence  $A_\mu^a$  and  $\pi^a$ , must be at most second order, thus ensuring stability [11,13,14],
- the action may contain at most second-order derivative terms in  $\pi^a$  and first-order derivatives for  $A_\mu^a$ ,
- each component of the SU(2) multiplet propagates only 3 degrees of freedom, the zeroth component being nondynamical.

In what follows, we apply these conditions and restrict our attention to the theories involving terms with up to six Lorentz indices contracted. From the cosmological perspective, such theories are expected to allow for a richer phenomenology since this is what happens for the Abelian Proca case [67,73–78].

### D. Procedure

We now proceed along the lines of Ref. [68]; i.e., we build, in Sec. III, a complete basis of linearly independent test Lagrangians describing all possible Lagrangians containing a given number of vector fields and their derivatives; the detailed prescription is given in Sec. III A. Next we demand only 3 degrees of freedom per multiplet component of the vector field, which translates into a condition on the Hessian [66,68], the latter being defined by

$$\mathcal{H}^{\mu\nu de} = \frac{\partial}{\partial(\partial_0 A_{\mu d})} \frac{\partial}{\partial(\partial_0 A_{\nu e})} \mathcal{L}, \quad (11)$$

for a given Lagrangian  $\mathcal{L}$ . This functional over the fields is symmetric under the index exchange  $(\mu, d) \leftrightarrow (\nu, e)$ .

In order for  $\mathcal{H}^{\mu\nu de}$  to have three vanishing eigenvalues, one for each timelike component of the three vectors  $A_{\mu d}$ , and since all the terms it is built of are *a priori* independent (up to symmetries), a necessary condition is that we

demand  $\mathcal{H}^{0vde} = 0$ ; this requirement will be explicitly checked in Sec. IV A for each test Lagrangian.

The above condition is, however, not sufficient, for it does not exhaust all the constraints and thus does not count the effectively propagating degrees of freedom. For instance, some terms inducing no dynamics for the time components of the vector fields may also yield no dynamics for some other component, or even for the overall vector field. The required analysis is tedious and must be followed step by step [96,97].

As the final step of the above analysis, we consider the scalar part associated with those linear combinations of test Lagrangians verifying the Hessian condition. One must check, which is done in Sec. IV D, that they are of two kinds: either they have no dynamics at all, being vanishing or given by a total derivative, or their dynamics is second order in the equations of motion of the scalar field; i.e., they belong to the class of generalized Galileons [100,120–124]. This will provide the most general terms that verify the requirements we demand, formulated in terms of the non-Abelian Faraday tensor; see Sec. IV E.

Before moving on, we mention that even though the procedure discussed above and applied below is allegedly tedious, it guarantees an exhaustive list of all possible terms at each order, and, in particular, all those specific to the non-Abelian case. Those terms might have been obtained by some quicker method, but we prefer to be able to produce all the theoretically acceptable terms rather than constructing a few. In view of possible cosmological applications, there is indeed no way to say which terms will be relevant and which ones will not.

### III. CONSTRUCTION OF THE TEST LAGRANGIANS

As anticipated above, our method relies heavily on the construction of a basis of test Lagrangians satisfying the symmetry requirement, on which we later apply the Hessian condition. This is the purpose of this section.

#### A. Description of the procedure

We now proceed to build the complete basis, in the sense of linear algebra, of test Lagrangians, for a given number of fields and their first derivatives. Since they are linearly independent, we will then be able to write down the most general theory at the given order as a linear combination of these Lagrangians.

In order to construct Lagrangians, i.e., scalars, we need to consider the Lorentz and group indices. The former spacetime indices run from 0 to 3 and are denoted by small Greek letters, while the latter group indices run from 1 to 3, since we assume the adjoint three-dimensional SU(2) representation, and are represented by small Latin letters from the beginning of the alphabet. We first write down all the Lorentz scalar quantities that may be formed with a

given number of fields and first derivatives, and then consider all the SU(2) index combinations leading to SU(2) scalars of these Lorentz scalars.

For the sake of simplicity, beginning with the Lorentz sector, we dismiss the group indices altogether, keeping in mind, however, that their presence might spoil some symmetry properties: contractions between symmetric and anti-symmetric (with respect to Lorentz indices only) tensors will not necessarily vanish when group indices are included, as exemplified by the starred equations in the next section. The Lorentz scalars, once formed, will then subsequently be assigned SU(2) indices following simple alphabetical order, leaving as many free SU(2) indices as there are fields in the term, to then be contracted with a relevant pure SU(2) tensor. For instance, a term like  $A^\mu A^\nu (\partial_\mu A^\nu)$  will be indexed as  $A^{\mu a} A^{\nu b} (\partial_\mu A^{\nu c})$ , demanding contraction with a structure constant  $\epsilon_{abc}$  to form a Lorentz and SU(2) scalar. This procedure can seem rather tedious, and it most definitely is, but it ensures that we construct a complete basis.

For simplicity, we restrict our attention to those Lagrangians containing up to six Lorentz indices contracted as they should be to form a scalar.

#### B. Lorentz sector

An easy way to classify the Lorentz scalars that one can form with a given number of 4-vectors consists in using the local equivalence, at the Lie-algebraic level, between SO(3,1) and SU(2)  $\times$  SU(2) (see, e.g., Ref. [125]). One obtains the following table [98,126]:

Number of vector fields $A^\mu$	1	2	3	4	5	6	7	8
Number of Lorentz scalars	0	1	0	4	0	25	0	196

These scalars can be written in terms of the primitive invariants, namely,  $g_{\mu\nu}$  and  $\epsilon_{\mu\nu\rho\sigma}$ . As shown in the table, an odd number of vector fields is impossible, as is obvious from the fact that one cannot form primitive Lorentz invariants with an odd number of indices. For two fields, the only contracting possibility is  $g_{\mu\nu}$ , while for four free Lorentz indices, the contractions with a term of the form  $A^\mu B^\nu C^\rho D^\sigma$  can be performed with any member of the list

$$\left\{ \begin{array}{l} g_{\mu\nu} g_{\rho\sigma}, \\ g_{\mu\rho} g_{\nu\sigma}, \\ g_{\mu\sigma} g_{\nu\rho}, \\ \epsilon_{\mu\nu\rho\sigma}. \end{array} \right. \quad (12)$$

For the case with six free indices of the form  $A^\mu B^\nu C^\rho D^\sigma E^\delta F^\epsilon$ , one finds the fifteen independent possibilities of combining three metrics, i.e.,  $g_{\mu\nu} g_{\rho\sigma} g_{\delta\epsilon}$  and the nonequivalent permutations of indices, as well as fifteen combinations of a metric and a Levi-Civita tensor, of which only ten are independent, which we choose to be

$$\left\{ \begin{array}{l} g_{\nu\rho}\epsilon_{\mu\sigma\delta\epsilon}, \\ g_{\nu\sigma}\epsilon_{\mu\rho\delta\epsilon}, \\ g_{\nu\delta}\epsilon_{\mu\rho\sigma\epsilon}, \\ g_{\nu\epsilon}\epsilon_{\mu\rho\sigma\delta}, \\ g_{\rho\sigma}\epsilon_{\mu\nu\delta\epsilon}, \\ g_{\rho\delta}\epsilon_{\mu\nu\sigma\epsilon}, \\ g_{\rho\epsilon}\epsilon_{\mu\nu\sigma\delta}, \\ g_{\sigma\delta}\epsilon_{\mu\nu\rho\epsilon}, \\ g_{\sigma\epsilon}\epsilon_{\mu\nu\rho\delta}, \\ g_{\epsilon\delta}\epsilon_{\mu\nu\rho\sigma}. \end{array} \right. \quad (13)$$

Now, one needs to take into account that when only one vector  $A^\mu$  and its gradient are plugged into these expressions, some terms are identical and can thus be simplified. The following table sums up the number of independent terms that can be built for a given product of vectors and gradients. Numbers in parentheses indicate those terms that would vanish if it were not for the group index; in our listings of all available Lagrangians below, we indicate these contractions with a star. Given the above discussion, we are sure that all the possible terms have been found, and they are all linearly independent.

	# $A^\rho A^\sigma$		
# $(\partial^\mu A^\nu)$	0	1	2
1	1 (0)	3 (1)	6 (4)
2	4 (0)	13 (3)	34 (23)
3	9 (2)	52 (22)	

We now discuss each case separately.

For a single derivative and no additional field, one gets the simplest combination, namely,  $(\partial \cdot A)$ . With two additional fields, one gets

$$\left\{ \begin{array}{l} (\partial \cdot A)(A \cdot A), \\ [(\partial^\mu A^\nu)A_\mu A_\nu], \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)A^\rho A^\sigma], \quad (*) \end{array} \right. \quad (14)$$

and with four additional fields, one obtains

$$\left\{ \begin{array}{l} (\partial \cdot A)(A \cdot A)(A \cdot A), \\ [(\partial^\mu A^\nu)A_\mu A_\nu](A \cdot A), \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)A^\rho A^\sigma](A \cdot A), \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\alpha)A^\nu A^\rho A^\sigma A_\alpha], \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\alpha A^\mu)A^\nu A^\rho A^\sigma A_\alpha], \quad (*) \\ (\partial \cdot A)[\epsilon_{\mu\nu\rho\sigma}A^\mu A^\nu A^\rho A^\sigma]. \quad (*) \end{array} \right. \quad (15)$$

With two derivatives and no additional field, one then finds

$$\left\{ \begin{array}{l} (\partial \cdot A)(\partial \cdot A), \\ [(\partial^\mu A^\nu)(\partial_\mu A_\nu)], \\ [(\partial^\mu A^\nu)(\partial_\nu A_\mu)], \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)(\partial^\rho A^\sigma)], \end{array} \right. \quad (16)$$

whereas with two additional fields, one finds<sup>1</sup>

$$\left\{ \begin{array}{l} (\partial \cdot A)(\partial \cdot A)(A \cdot A), \\ [(\partial^\mu A^\nu)(\partial_\mu A_\nu)](A \cdot A), \\ [(\partial^\mu A^\nu)(\partial_\nu A_\mu)](A \cdot A), \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)(\partial^\rho A^\sigma)](A \cdot A), \\ [(\partial^\mu A^\nu)A_\mu A_\nu](\partial \cdot A), \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)A^\rho A^\sigma](\partial \cdot A), \quad (*) \\ [A_\mu A_\nu(\partial^\mu A^\alpha)(\partial^\nu A_\alpha)], \\ [A_\mu A_\nu(\partial^\mu A^\alpha)(\partial_\alpha A^\nu)], \\ [A_\mu A_\nu(\partial^\alpha A^\mu)(\partial_\alpha A^\nu)], \\ [\epsilon_{\mu\nu\rho\sigma}A^\mu A^\nu(\partial^\rho A^\alpha)(\partial^\sigma A_\alpha)], \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma}A^\mu A^\nu(\partial^\rho A^\alpha)(\partial_\alpha A^\sigma)], \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma}A^\mu A^\alpha(\partial^\nu A^\rho)(\partial^\sigma A_\alpha)], \\ [\epsilon_{\mu\nu\rho\sigma}A^\mu A^\alpha(\partial^\nu A^\rho)(\partial_\alpha A^\sigma)]. \end{array} \right. \quad (18)$$

Finally, demanding three gradients of the vector field and no vector field itself, one obtains

$$\left\{ \begin{array}{l} (\partial \cdot A)(\partial \cdot A)(\partial \cdot A), \\ [(\partial^\mu A^\nu)(\partial_\mu A_\nu)](\partial \cdot A), \\ [(\partial^\mu A^\nu)(\partial_\nu A_\mu)](\partial \cdot A), \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)(\partial^\rho A^\sigma)](\partial \cdot A), \\ [(\partial^\mu A_\nu)(\partial^\nu A_\rho)(\partial^\rho A_\mu)], \\ [(\partial^\mu A_\nu)(\partial^\nu A_\rho)(\partial_\mu A^\rho)], \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\alpha)(\partial^\nu A_\alpha)(\partial^\rho A^\sigma)], \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\alpha)(\partial_\alpha A^\nu)(\partial^\rho A^\sigma)], \\ [\epsilon_{\mu\nu\rho\sigma}(\partial^\alpha A^\mu)(\partial_\alpha A^\nu)(\partial^\rho A^\sigma)]. \quad (*) \end{array} \right. \quad (19)$$

<sup>1</sup>As an example of the fact that not every reshuffling of indices is independent, let us consider the term  $\epsilon_{\mu\nu\rho\sigma}A^\mu A^\nu(\partial^\alpha A^\rho)(\partial_\alpha A^\sigma)$ , which could, in principle, have appeared in the list in Eq. (18). It is indeed not necessary because the property

$$g_{\mu\rho}\epsilon_{\nu\sigma\delta\epsilon} = g_{\nu\rho}\epsilon_{\mu\sigma\delta\epsilon} - g_{\rho\sigma}\epsilon_{\mu\nu\delta\epsilon} + g_{\rho\delta}\epsilon_{\mu\nu\sigma\epsilon} - g_{\rho\epsilon}\epsilon_{\mu\nu\sigma\delta} \quad (17)$$

allows us to write it as a linear combination of the terms in Eq. (18).

### C. Group sector

Let us now proceed with the similar procedure but now in the group sector. Since we assumed that the vector fields transform according to the representation of dimension 3 of SU(2), one can safely use known results from representation theory of compact Lie groups. The table below summarizes the different possibilities to obtain an SU(2) singlet as a function of the number of fields belonging to the **3** representation of SU(2) [98,126]:

Number vector fields in the <b>3</b> of SU(2)	1	2	3	4	5	6	7	
Number of SU(2) singlets		0	1	1	3	6	15	36

We reproduce below the procedure explained in Sec. III A, whereby one constructs the necessary products of group metric coefficients  $\delta_{ab}$  and structure constants  $\epsilon_{abc}$ . Getting as many independent terms as predicted by the representation theory (table above) ensures completeness of the basis. Similar to the Lorentz invariance discussed in the previous section, these two tensors are the only primitive invariants of the group [98–100].

To contract with two or three free SU(2) indices, the only possible choices are, respectively,  $\delta_{ab}$  and  $\epsilon_{abc}$ . With four fields, one can make use of the three combinations

$$\begin{cases} \delta_{ab}\delta_{cd}, \\ \delta_{ac}\delta_{bd}, \\ \delta_{ad}\delta_{bc}, \end{cases} \quad (20)$$

while five fields demand the following six possibilities, namely,

$$\begin{cases} \delta_{ab}\epsilon_{cde}, \\ \delta_{ac}\epsilon_{bde}, \\ \delta_{ad}\epsilon_{bce}, \\ \delta_{bc}\epsilon_{ade}, \\ \delta_{bd}\epsilon_{ace}, \\ \delta_{cd}\epsilon_{abe}. \end{cases} \quad (21)$$

As in Sec. III B, one can devise other possible formulations that apply, but they will always be expressible as linear combinations of the above. For instance, relations between the structure constants, such as

$$\epsilon_{ab}^e \epsilon_{cde} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \quad (22)$$

imply that contracting a four-index term with two structure constants is equivalent to a linear combination of the terms given in Eq. (20).

### D. Final test Lagrangians

Gathering the results and applying the procedure of Sec. III A, we are now in a position to write down our test Lagrangians, scalars under both Lorentz and SU(2) transformations. Some of these terms simplify through contractions, e.g.,  $\epsilon_{abc}(A^a \cdot A^b)(\partial \cdot A^c) = 0$ , and we are left with fewer terms than the naive multiplication of all singlet possibilities of each sector would have otherwise suggested. This is fortunate because the number of terms to be considered *a priori* is quickly increasing with the number of fields involved, as shown in the table below:

	# $A^{\rho b}$		
# $\partial^\mu A^{\nu a}$	0	2	4
1	0	3	36
2	4	42	510
3	9	312	

After simplifications, we find two terms (instead of three according to the table) containing a single derivative term and two additional vector fields,

$$\begin{cases} \mathcal{L}_1 = \epsilon_{abc}[(\partial^\mu A^{a\nu})A_\mu^b A_\nu^c], \\ \mathcal{L}_2 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})A^{b\rho} A^{c\sigma}], \end{cases} \quad (23)$$

and eight with four such fields, namely,

$$\begin{cases} \mathcal{L}_1 = \epsilon_{abc}[(\partial^\mu A^{d\nu})A_\mu^a A_\nu^b](A^c \cdot A_d), \\ \mathcal{L}_2 = \epsilon_{abc}[(\partial^\mu A^{a\nu})A_\mu^d A_\nu^b](A^c \cdot A_d), \\ \mathcal{L}_3 = \epsilon_{abc}[(\partial^\mu A^{a\nu})A_\mu^b A_\nu^d](A^c \cdot A_d), \\ \mathcal{L}_4 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{d\nu})A^{a\rho} A^{b\sigma}](A^c \cdot A_d), \\ \mathcal{L}_5 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})A^{d\rho} A^{b\sigma}](A^c \cdot A_d), \\ \mathcal{L}_6 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{d\alpha})A_\nu^d A^{a\rho} A^{b\sigma} A_\alpha^c], \\ \mathcal{L}_7 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\alpha A^{d\mu})A_\nu^d A^{a\rho} A^{b\sigma} A_\alpha^c], \\ \mathcal{L}_8 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial \cdot A_d)A^{d\mu} A^{a\nu} A^{b\rho} A^{c\sigma}]. \end{cases} \quad (24)$$

Note that one cannot build a single derivative term without an additional field, as it would otherwise belong to the **3** representation of SU(2).

For two first-order vector field derivatives without additional fields, one gets

$$\begin{cases} \mathcal{L}_1 = (\partial \cdot A^a)(\partial \cdot A_a), \\ \mathcal{L}_2 = [(\partial^\mu A_a^\nu)(\partial_\mu A_\nu^a)], \\ \mathcal{L}_3 = [(\partial^\mu A_a^\nu)(\partial_\nu A_\mu^a)], \\ \mathcal{L}_4 = [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})(\partial^\rho A_a^\sigma)], \end{cases} \quad (25)$$

whereas with two additional vector fields, one gets

$$\left\{ \begin{array}{l}
 \mathcal{L}_1 = (\partial \cdot A^a)(\partial \cdot A_a)(A^b \cdot A_b), \\
 \mathcal{L}_2 = (\partial \cdot A^a)(\partial \cdot A^b)(A_a \cdot A_b), \\
 \mathcal{L}_3 = [(\partial^\mu A_a^\nu)(\partial_\mu A_\nu^a)](A^b \cdot A_b), \\
 \mathcal{L}_4 = [(\partial^\mu A_a^\nu)(\partial_\mu A_\nu^b)](A^a \cdot A_b), \\
 \mathcal{L}_5 = [(\partial^\mu A_a^\nu)(\partial_\nu A_\mu^a)](A^b \cdot A_b), \\
 \mathcal{L}_6 = [(\partial^\mu A_a^\nu)(\partial_\nu A_\mu^b)](A^a \cdot A_b), \\
 \mathcal{L}_7 = [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})(\partial^\rho A_a^\sigma)](A^b \cdot A_b), \\
 \mathcal{L}_8 = [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})(\partial^\rho A_b^\sigma)](A_a \cdot A^b), \\
 \mathcal{L}_9 = [(\partial^\mu A_a^\nu)A_\mu^a A_\nu^b](\partial \cdot A_b), \\
 \mathcal{L}_{10} = [(\partial^\mu A_a^\nu)A_\mu^b A_\nu^a](\partial \cdot A_b), \\
 \mathcal{L}_{11} = [(\partial^\mu A_a^\nu)A_\mu^b A_{b\nu}](\partial \cdot A^a), \\
 \mathcal{L}_{12} = [\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\nu})A_a^\rho A_b^\sigma](\partial \cdot A^b), \\
 \mathcal{L}_{13} = [A_\mu^a A_{a\nu}(\partial^\mu A_b^\alpha)(\partial^\nu A_b^\alpha)], \\
 \mathcal{L}_{14} = [A_\mu^a A_\nu^b(\partial^\mu A_a^\alpha)(\partial^\nu A_{b\alpha})], \\
 \mathcal{L}_{15} = [A_\mu^a A_{a\nu}(\partial^\mu A^{b\alpha})(\partial_\alpha A_\nu^b)], \\
 \mathcal{L}_{16} = [A_\mu^a A_\nu^b(\partial^\mu A_a^\alpha)(\partial_\alpha A_b^\nu)], \\
 \mathcal{L}_{17} = [A_\mu^a A_\nu^b(\partial^\mu A_b^\alpha)(\partial_\alpha A_a^\nu)], \\
 \mathcal{L}_{18} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A^{b\nu}(\partial^\rho A_a^\alpha)(\partial^\sigma A_{b\alpha})], \\
 \mathcal{L}_{19} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A^{b\nu}(\partial^\rho A_a^\alpha)(\partial_\alpha A_b^\sigma)], \\
 \mathcal{L}_{20} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A^{b\nu}(\partial^\alpha A_a^\rho)(\partial_\alpha A_b^\sigma)], \\
 \mathcal{L}_{21} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A_a^\alpha(\partial^\nu A_b^\rho)(\partial^\sigma A_b^\alpha)], \\
 \mathcal{L}_{22} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A_b^\alpha(\partial^\nu A_a^\rho)(\partial^\sigma A_b^\alpha)], \\
 \mathcal{L}_{23} = [\epsilon_{\mu\nu\rho\sigma}A_a^\mu A_b^\alpha(\partial^\nu A^{b\rho})(\partial^\sigma A_a^\alpha)], \\
 \mathcal{L}_{24} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A_a^\alpha(\partial^\nu A^{b\rho})(\partial_\alpha A_b^\sigma)], \\
 \mathcal{L}_{25} = [\epsilon_{\mu\nu\rho\sigma}A^{a\mu}A_b^\alpha(\partial^\nu A_a^\rho)(\partial_\alpha A^{b\sigma})], \\
 \mathcal{L}_{26} = [\epsilon_{\mu\nu\rho\sigma}A_a^\mu A_b^\alpha(\partial^\nu A^{b\rho})(\partial_\alpha A^{a\sigma})], \\
 \mathcal{L}_{27} = [A_\mu^a A_\nu^b(\partial^\mu A_b^\alpha)(\partial^\nu A_{a\alpha})], \\
 \mathcal{L}_{28} = [A_\mu^a A_\nu^b(\partial^\alpha A_b^\mu)(\partial_\alpha A_a^\nu)].
 \end{array} \right. \quad (26)$$

Finally, with three derivatives, one finds

$$\left\{ \begin{array}{l}
 \mathcal{L}_1 = \epsilon_{abc}[(\partial^\mu A_\nu^a)(\partial^\nu A_\rho^b)(\partial^\rho A_\mu^c)], \\
 \mathcal{L}_2 = \epsilon_{abc}[(\partial^\mu A_\nu^a)(\partial^\nu A_\rho^b)(\partial_\mu A^{c\rho})], \\
 \mathcal{L}_3 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\alpha})(\partial^\nu A_\alpha^b)(\partial^\rho A^{c\sigma})], \\
 \mathcal{L}_4 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^{a\alpha})(\partial_\alpha A^{b\nu})(\partial^\rho A^{c\sigma})], \\
 \mathcal{L}_5 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma}(\partial^\alpha A^{a\mu})(\partial_\alpha A^{b\nu})(\partial^\rho A^{c\sigma})],
 \end{array} \right. \quad (27)$$

completing our list of test Lagrangians.

## IV. CONSTRUCTION OF THE HEALTHY TERMS

### A. Hessian condition

Let us now apply the Hessian condition, as discussed in Sec. II D. The first step is to calculate the Hessians

associated with the various test Lagrangians, defined by Eq. (11). One sees that only those terms containing at least two first-order derivatives of the vector field yield a nonvanishing value. In practice, one gets

$$\left\{ \begin{array}{l}
 \mathcal{H}_1^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}, \\
 \mathcal{H}_2^{\mu\nu de} = -2g^{\mu\nu}g^{de}, \\
 \mathcal{H}_3^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}, \\
 \mathcal{H}_4^{\mu\nu de} = 0,
 \end{array} \right. \quad (28)$$

for the terms with two first-order derivatives and no additional fields, and

$$\left\{ \begin{array}{l}
 \mathcal{H}_1^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}(A^b \cdot A_b), \\
 \mathcal{H}_2^{\mu\nu de} = 2g^{0\mu}g^{0\nu}(A^d \cdot A^e), \\
 \mathcal{H}_3^{\mu\nu de} = -2g^{\mu\nu}g^{de}(A^b \cdot A_b), \\
 \mathcal{H}_4^{\mu\nu de} = -2g^{\mu\nu}(A^d \cdot A^e), \\
 \mathcal{H}_5^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}(A^b \cdot A_b), \\
 \mathcal{H}_6^{\mu\nu de} = 2g^{0\mu}g^{0\nu}(A^d \cdot A^e), \\
 \mathcal{H}_7^{\mu\nu de} = 0, \\
 \mathcal{H}_8^{\mu\nu de} = 0, \\
 \mathcal{H}_9^{\mu\nu de} = A^{0d}A^{\mu e}g^{0\nu} + A^{0e}A^{\nu d}g^{0\mu}, \\
 \mathcal{H}_{10}^{\mu\nu de} = A^{0e}A^{\mu d}g^{0\nu} + A^{0d}A^{\nu e}g^{0\mu}, \\
 \mathcal{H}_{11}^{\mu\nu de} = A^{0b}A_b^\mu g^{0\nu}g^{de} + A^{0b}A_b^\nu g^{0\mu}g^{de}, \\
 \mathcal{H}_{12}^{\mu\nu de} = \epsilon^{0\mu}{}_{\rho\sigma}A^{\rho d}A^{\sigma e}g^{0\nu} + \epsilon^{0\nu}{}_{\rho\sigma}A^{\rho e}A^{\sigma d}g^{0\mu}, \\
 \mathcal{H}_{13}^{\mu\nu de} = 2A^{0b}A_b^\mu g^{\nu\lambda}g^{de}, \\
 \mathcal{H}_{14}^{\mu\nu de} = 2A^{0d}A^{0e}g^{\mu\nu}, \\
 \mathcal{H}_{15}^{\mu\nu de} = A^{0b}A_b^\nu g^{\mu 0}g^{de} + A^{0b}A_b^\mu g^{\nu 0}g^{de}, \\
 \mathcal{H}_{16}^{\mu\nu de} = A^{0d}A^{\nu e}g^{\mu 0} + A^{0e}A^{\mu d}g^{\nu 0}, \\
 \mathcal{H}_{17}^{\mu\nu de} = A^{0e}A^{\nu d}g^{\mu 0} + A^{0d}A^{\mu e}g^{\nu 0}, \\
 \mathcal{H}_{18}^{\mu\nu de} = 0, \\
 \mathcal{H}_{19}^{\mu\nu de} = \epsilon_{\rho\sigma}{}^{0\nu}A^{\rho d}A^{\sigma e}g^{\mu 0} + \epsilon_{\rho\sigma}{}^{0\mu}A^{\rho e}A^{\sigma d}g^{\nu 0}, \\
 \mathcal{H}_{20}^{\mu\nu de} = -2\epsilon_{\rho\sigma}{}^{\mu\nu}A^{\rho d}A^{\sigma e}, \\
 \mathcal{H}_{21}^{\mu\nu de} = 0, \\
 \mathcal{H}_{22}^{\mu\nu de} = 0, \\
 \mathcal{H}_{23}^{\mu\nu de} = 0, \\
 \mathcal{H}_{24}^{\mu\nu de} = 0, \\
 \mathcal{H}_{25}^{\mu\nu de} = \epsilon_\rho{}^{0\mu\nu}A^{\rho d}A^{0e} + \epsilon_\rho{}^{0\nu\mu}A^{\rho e}A^{0d}, \\
 \mathcal{H}_{26}^{\mu\nu de} = \epsilon_\rho{}^{0\mu\nu}A^{\rho e}A^{0d} + \epsilon_\rho{}^{0\nu\mu}A^{\rho d}A^{0e}, \\
 \mathcal{H}_{27}^{\mu\nu de} = 2A^{0d}A^{0e}g^{\mu\nu}, \\
 \mathcal{H}_{28}^{\mu\nu de} = -2A^{\nu d}A^{\mu e},
 \end{array} \right. \quad (29)$$



for those with two first-order derivatives and two additional vector fields.

For the terms with three first-order derivatives, we have

$$\begin{cases} \mathcal{H}_1^{\mu\nu de} = 3\epsilon^{de}{}_c (g^{0\mu} \partial^\nu A^{c0} - g^{0\nu} \partial^\mu A^{c0}), \\ \mathcal{H}_2^{\mu\nu de} = \epsilon^{de}{}_c (g^{0\mu} \partial^0 A^{c\nu} - g^{0\nu} \partial^0 A^{c\mu}) + \epsilon^{de}{}_c (\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu}), \\ \mathcal{H}_3^{\mu\nu de} = 0, \\ \mathcal{H}_4^{\mu\nu de} = \epsilon^{de}{}_c (\epsilon^{0\nu\rho\sigma} g^{0\mu} \partial_\rho A_\sigma^c - \epsilon^{0\mu\rho\sigma} g^{0\nu} \partial_\rho A_\sigma^c) + 2\epsilon^{de}{}_c \epsilon^{\rho\mu 0\nu} \partial_\rho A^{c0}, \\ \mathcal{H}_5^{\mu\nu de} = -2\epsilon^{de}{}_c \epsilon^{\mu\nu\rho\sigma} \partial_\rho A_\sigma^c + 4\epsilon^{de}{}_c \epsilon^{\rho\mu 0\nu} \partial^0 A_\rho^c. \end{cases} \quad (30)$$

With these partial Hessians, we now construct a basis of terms fulfilling the condition discussed above, i.e., such that  $\mathcal{H}^{0\mu de} = 0$  for all values of  $\mu, d$  and  $e$ ; see Sec. II D. To reach this goal, using notations already introduced in Ref. [68], we produce a Lagrangian by means of a linear combination of our test ones, namely,

$$\mathcal{L}_{\text{test}} = \sum_i x_i \mathcal{L}_i, \quad (31)$$

for a yet-unknown set of constant parameters  $x_i$ . The Hessian is then calculated for this Lagrangian, leading to algebraic equations for the  $x_i$  whose roots provide the required actions. It turns out to be easier to separately compute the cases  $\mu = 0$  and  $\mu = i$ , as well as  $d = e$  and  $d \neq e$ .

Let us begin with the case  $d = e$ . Test Lagrangians with two derivatives and no additional fields have only one Hessian component not identically vanishing, namely,

$$\mathcal{H}^{00dd} = 4(x_1 + x_2 + x_3), \quad (32)$$

while for two additional vector fields, there are four independent Hessian conditions, given by

$$\begin{aligned} \mathcal{H}^{00dd} &= 4(x_1 + x_3 + x_5)(A^b \cdot A_b) + 2(x_2 + x_4 + x_6)(A^d \cdot A^d) \\ &\quad - 2(x_9 + x_{10} + x_{14} + x_{16} + x_{17} + x_{27} + x_{28})(A^{0d} A^{0d}) \\ &\quad - 4(x_{11} + x_{13} + x_{15})(A^{0b} A_b^0), \end{aligned} \quad (33)$$

$$\begin{aligned} \mathcal{H}^{0idd} &= -(x_9 + x_{10} + x_{16} + x_{17} + 2x_{28})(A^{0d} A^{id}) \\ &\quad - 2(x_{11} + x_{15})(A^{0b} A_b^i) \\ &\quad - (x_{12} + x_{19} + 2x_{20})(\epsilon_{\rho\sigma}{}^{0i} A^{\rho d} A^{\sigma d}). \end{aligned} \quad (34)$$

On the other hand, the case  $d \neq e$  implies

$$\begin{aligned} \mathcal{H}^{00de} &= 2(x_2 + x_4 + x_6)(A^d \cdot A^e) \\ &\quad - 2(x_9 + x_{10} + x_{14} + x_{16} \\ &\quad + x_{17} + x_{27} + x_{28})(A^{0d} A^{0e}), \end{aligned} \quad (35)$$

$$\begin{aligned} \mathcal{H}^{0ide} &= -(x_9 + x_{17} + 2x_{28})(A^{0e} A^{id}) \\ &\quad - (x_{10} + x_{16})(A^{0d} A^{ie}) \\ &\quad - (-x_{12} + x_{19} + 2x_{20})(\epsilon^{0i}{}_{\rho\sigma} A^{\rho d} A^{\sigma e}). \end{aligned} \quad (36)$$

Making these four terms vanish can be done, without loss of generality (since all linear combinations of the resulting terms are all also acceptable):

$$\begin{aligned} x_3 &= -x_1 - x_5, \\ x_4 &= -x_2 - x_6, \\ x_{12} &= 0, \\ x_{13} &= 0, \\ x_{14} &= -x_{27} + x_{28}, \\ x_{15} &= -x_{11}, \\ x_{16} &= -x_{10}, \\ x_{17} &= -x_9 - 2x_{28}, \\ x_{19} &= -2x_{20}. \end{aligned} \quad (37)$$

With three derivatives, one finds that  $\mathcal{H}^{00dd}, \mathcal{H}^{00de}$  ( $d \neq e$ ) and  $\mathcal{H}^{0idd}$  identically vanish, whereas for  $d \neq e$ , we have

$$\mathcal{H}^{0ide} = \epsilon_c{}^{de} [(-3x_1 - x_2) \partial^i A^{0c} - (x_4 + 2x_5) \epsilon^{0i\rho\sigma} \partial_\rho A_\sigma^c], \quad (38)$$

thus leading to the conditions

$$\begin{aligned} x_2 &= -3x_1, \\ x_4 &= -2x_5. \end{aligned} \quad (39)$$

## B. Simplification of the Lagrangian

For one gradient and two vector fields, we can define the current

$$J^\mu = \epsilon_{abc} \epsilon^{\mu\nu\rho\sigma} A_\nu^a A_\rho^b A_\sigma^c, \quad (40)$$

showing that  $\mathcal{L}_2$  is a total derivative, namely,

$$\partial_\mu J^\mu = 3\mathcal{L}_2. \quad (41)$$

A similar technique applies for one derivative term and four additional vector fields: in this case, one forms the following two currents,

$$\begin{aligned} J_1^\mu &= \epsilon^\mu{}_{\nu\rho\sigma} A^{\nu a} A^{\rho b} A^{\sigma c} A^{ad} A_{ad} \epsilon_{abc}, \\ J_2^\alpha &= \epsilon_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A^{\rho c} A^{\sigma d} A_d^\alpha \epsilon_{abc}, \end{aligned} \quad (42)$$

yielding

$$\begin{aligned} \partial_\mu J_1^\mu &= 3(\mathcal{L}_3 - 2\mathcal{L}_5 + 2\mathcal{L}_6), \\ \partial_\alpha J_2^\alpha &= -\mathcal{L}_8. \end{aligned} \quad (43)$$

Finally, some terms involving two first-order derivatives can be described by

$$J^{\mu_1} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A^{\nu_1 a} \partial_{\mu_2} A_a^{\nu_2}, \quad (44)$$

$$J_\epsilon^\mu = \epsilon^{\mu\nu\rho\sigma} A_\nu^a (\partial_\rho A_{\sigma a}), \quad (45)$$

where we have used the definition  $\delta_{\nu_1\nu_2}^{\mu_1\mu_2} \equiv \delta_{\nu_1}^{\mu_1} \delta_{\nu_2}^{\mu_2} - \delta_{\nu_2}^{\mu_1} \delta_{\nu_1}^{\mu_2}$  stemming from Eq. (A2), leading to

$$\partial_{\mu_1} J^{\mu_1} = \mathcal{L}_1 - \mathcal{L}_3, \quad (46)$$

$$\partial_\mu J_\epsilon^\mu = \mathcal{L}_4. \quad (47)$$

Terms containing two derivatives and two fields are slightly more involved. We first make use of the identity [127,70]

$$A^{\mu\alpha} \tilde{B}_{\nu\alpha} + B^{\mu\alpha} \tilde{A}_{\nu\alpha} = \frac{1}{2} (B^{\alpha\beta} \tilde{A}_{\alpha\beta}) \delta_\nu^\mu, \quad (48)$$

valid for all antisymmetric tensors  $A$  and  $B$ . This provides the relations

$$(G^{\mu\alpha} \tilde{G}_{\nu\alpha}^b + G^{\mu ab} \tilde{G}_{\nu\alpha}^a) A_{\mu a} A_b^\nu = \frac{1}{2} (G^{\alpha\beta a} \tilde{G}_{\alpha\beta}^b) (A_a \cdot A_b) \quad (49)$$

and

$$G^{\mu\alpha a} \tilde{G}_{\nu\alpha a} A_\mu^b A_b^\nu = \frac{1}{4} (G^{\alpha\beta a} \tilde{G}_{\alpha\beta a}) (A^b \cdot A_b), \quad (50)$$

where  $G^{\mu\alpha a}$  is the Abelian form of the Faraday tensor as defined below in Eq. (59). From these, one then derives the following two identities relating the Lagrangians in Eq. (26):

$$\mathcal{L}_{25} + \mathcal{L}_{26} - \mathcal{L}_{22} - \mathcal{L}_{23} = \mathcal{L}_8 \quad (51)$$

and

$$2(\mathcal{L}_{24} - \mathcal{L}_{21}) = \mathcal{L}_7. \quad (52)$$

It is also possible to find total derivatives to reduce the number of independent terms. First, one can use the fact that  $\tilde{G}$  is divergence-free, introducing the currents

$$\begin{aligned} J_{G,1}^\mu &= \tilde{G}_a^{\mu\nu} A_\nu^a (A^b \cdot A_b), \\ J_{G,2}^\mu &= \tilde{G}_a^{\mu\nu} A_{\nu b} (A^a \cdot A^b), \end{aligned} \quad (53)$$

providing

$$\begin{aligned} \partial_\mu J_{G,1}^\mu &= \mathcal{L}_7 - 2\mathcal{L}_{22}, \\ \partial_\mu J_{G,2}^\mu &= \mathcal{L}_8 - \mathcal{L}_{21} - \mathcal{L}_{23}. \end{aligned} \quad (54)$$

One can subsequently use the antisymmetric forms written from  $\delta_{\nu_1\nu_2}^{\mu_1\mu_2}$ :

$$\begin{aligned} J_{\delta,1}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b A_a^{\nu_1} \partial_{\mu_2} A^{\nu_2 a}, \\ J_{\delta,2}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_\lambda^a A_\lambda^b A^{\nu_1 a} \partial_{\mu_2} A_b^{\nu_2}, \\ J_{\delta,3}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A^{\lambda a} A_a^{\nu_1} A_b^{\nu_2} \partial_{\mu_2} A_\lambda^b, \end{aligned} \quad (55)$$

resulting in

$$\begin{aligned} \partial_\mu J_{\delta,1}^\mu &= \mathcal{L}_1 - \mathcal{L}_5 + 2\mathcal{L}_{10} - 2\mathcal{L}_{16}, \\ \partial_\mu J_{\delta,2}^\mu &= \mathcal{L}_2 - \mathcal{L}_6 + \mathcal{L}_9 + \mathcal{L}_{11} - \mathcal{L}_{15} - \mathcal{L}_{17}, \\ \partial_\mu J_{\delta,3}^\mu &= \mathcal{L}_{14} + \mathcal{L}_9 + \mathcal{L}_{15} - \mathcal{L}_{27} - \mathcal{L}_{17} - \mathcal{L}_{11}. \end{aligned} \quad (56)$$

Finally, we can write

$$J_{\epsilon,1}^\mu = \epsilon^\mu{}_{\nu\rho\sigma} A^{\nu a} A^{\rho b} A_a^\alpha \partial^\sigma A_{\alpha b}, \quad (57)$$

implying

$$\partial_\mu J_{\epsilon,1}^\mu = \mathcal{L}_{18} + \mathcal{L}_{23} - \mathcal{L}_{21}. \quad (58)$$

All the above conditions are linearly independent. They allow us to write Lagrangians  $\mathcal{L}_9$ ,  $\mathcal{L}_{10} - \mathcal{L}_{16}$ ,  $\mathcal{L}_{11} - \mathcal{L}_{15}$ ,  $\mathcal{L}_{18}$ ,  $\mathcal{L}_{21}$ ,  $\mathcal{L}_{22}$ ,  $\mathcal{L}_{24}$ , and  $\mathcal{L}_{25}$  as functions of the other Lagrangians. Note, however, that one can always add these to other terms of the final basis for simplification purposes.

Lastly, the current  $J^\mu = \epsilon^\mu{}_{\nu\rho\sigma} \partial^\nu A^{\alpha a} \partial^\rho A_a^b A^{\sigma c} \epsilon_{abc}$  permits us to simplify one of the terms containing three first-order derivatives by making use of  $\partial_\mu J^\mu = \mathcal{L}_3$ .

### C. A new basis

One can now rewrite our basis of Lagrangians satisfying the Hessian condition, taking into account the extra relations stemming from the total derivatives and the identity of Ref. [127]. We group our terms to produce a new and more convenient basis, and for that purpose, we use the Abelian form of the Faraday tensor, namely,

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a, \quad (59)$$

as well as its Hodge dual  $\tilde{G}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma a}$ , also defined in the usual way. Using the Abelian form of the Faraday tensor to describe a non-Abelian vector field theory may seem a bit unusual, but it considerably simplifies our forthcoming considerations since this term naturally appears from the first-order derivatives of the vector field

and cancels in the scalar sector. We later move on to a formulation using the actual non-Abelian Faraday tensor, as is given by Eq. (2). We also make use of the symmetric counterpart of the Abelian Faraday tensor, namely,

$$S_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \quad (60)$$

For one first-order derivative of the vector field and two additional vector fields, we obtain

$$\tilde{\mathcal{L}}_1 = 2\mathcal{L}_1 = \epsilon_{abc}[G^{\mu\nu a} A_\mu^b A_\nu^c], \quad (61)$$

and with four additional vector fields, we obtain

$$\left\{ \begin{array}{l} \tilde{\mathcal{L}}_1 = 2\mathcal{L}_1 = \epsilon_{abc}[G^{\mu\nu d} A_\mu^a A_\nu^b](A^c \cdot A_d), \\ \tilde{\mathcal{L}}_2 = \mathcal{L}_2 + \mathcal{L}_3 = \epsilon_{abc}[S^{\mu\nu a} A_\mu^b A_\nu^d](A^c \cdot A_d), \\ \tilde{\mathcal{L}}_3 = \mathcal{L}_3 - \mathcal{L}_2 = \epsilon_{abc}[G^{\mu\nu a} A_\mu^b A_\nu^d](A^c \cdot A_d), \\ \tilde{\mathcal{L}}_4 = \mathcal{L}_4 = \epsilon_{abc}[\tilde{G}^{\mu\nu d} A_\mu^a A_\nu^b](A^c \cdot A_d), \\ \tilde{\mathcal{L}}_5 = \mathcal{L}_5 = \epsilon_{abc}[\tilde{G}^{\mu\nu a} A_\mu^d A_\nu^b](A^c \cdot A_d), \\ \tilde{\mathcal{L}}_6 = \mathcal{L}_6 - \mathcal{L}_7 = \epsilon_{abc}[\epsilon_{\mu\nu\rho\sigma} G^{\mu\alpha d} A_\mu^a A_\nu^{\rho\sigma} A^b A^c]. \end{array} \right. \quad (62)$$

Terms with two first-order derivatives and no additional fields can be written as

$$\mathcal{L}_1 = 2(\mathcal{L}_2 - \mathcal{L}_3) = G_a^{\mu\nu} G_{\mu\nu}^a, \quad (63)$$

and with two additional fields, they are given by

$$\left\{ \begin{array}{l} \tilde{\mathcal{L}}_1 = \mathcal{L}_1 - \mathcal{L}_5 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_a^{\nu_1}) (\partial_{\mu_2} A^{\nu_2 a}), \\ \tilde{\mathcal{L}}_2 = 2(\mathcal{L}_3 - \mathcal{L}_5) = G_a^{\mu\nu} G_{\mu\nu}^a (A^b \cdot A_b), \\ \tilde{\mathcal{L}}_3 = \mathcal{L}_2 - \mathcal{L}_6 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}), \\ \tilde{\mathcal{L}}_4 = 2(\mathcal{L}_4 - \mathcal{L}_6) = G_a^{\mu\nu} G_{\mu\nu b} (A^a \cdot A^b), \\ \tilde{\mathcal{L}}_5 = 2\mathcal{L}_7 = \tilde{G}_{\mu\nu}^a G_a^{\mu\nu} (A^b \cdot A_b), \\ \tilde{\mathcal{L}}_6 = 2\mathcal{L}_8 = \tilde{G}_{\mu\nu a} G_b^{\mu\nu} (A^a \cdot A^b), \\ \tilde{\mathcal{L}}_7 = \mathcal{L}_{18} + \mathcal{L}_{20} - 2\mathcal{L}_{19} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A^{b\nu} G_a^{\rho\alpha} G_{ab}^\sigma], \\ \tilde{\mathcal{L}}_8 = \mathcal{L}_{26} + \mathcal{L}_{23} = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{ab} S^{\alpha\sigma a}, \\ \tilde{\mathcal{L}}_9 = \mathcal{L}_{26} - \mathcal{L}_{23} = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{ab} G^{\alpha\sigma a}, \\ \tilde{\mathcal{L}}_{10} = \mathcal{L}_{14} - \mathcal{L}_{27} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{ab}), \\ \tilde{\mathcal{L}}_{11} = \mathcal{L}_{27} + \mathcal{L}_{28} - 2\mathcal{L}_{17} = A_\mu^a A_\nu^b G_{ab}^\mu G_a^{\nu\alpha}. \end{array} \right. \quad (64)$$

As anticipated, we obtain 11 independent terms, which correspond to 28 terms to begin with, with 8 constraints and 9 Hessian conditions.

Finally, the three-gradient case yields

$$\left\{ \begin{array}{l} \tilde{\mathcal{L}}_1 = 2(\mathcal{L}_1 - 3\mathcal{L}_2) = \epsilon_{abc} G_{\nu}^{\mu a} G_{\rho}^{\nu b} G_{\mu}^{\rho c}, \\ \tilde{\mathcal{L}}_2 = 2\mathcal{L}_4 - \mathcal{L}_3 - \mathcal{L}_5 = \epsilon_{abc} G^{\mu\alpha a} G_a^{\nu b} \tilde{G}_{\mu\nu}^c. \end{array} \right. \quad (65)$$

#### D. Scalar contribution

Let us now consider the scalar part of the previously developed Lagrangian, as explained in Sec. IID, making the substitution  $A_\mu^a \rightarrow \partial_\mu \pi^a$  and writing only those terms that do not identically vanish, using the results of Appendix A, where the useful Galileon Lagrangians are provided (Appendix A2), as well as the linear combinations leading to second-order equations (Appendix A4).

With one derivative and four vector fields, the only remaining term of the scalar sector out of the original three is

$$\tilde{\mathcal{L}}_2 = \epsilon_{abc} S^{\mu\nu a} A_\mu^b A_\nu^d (A^c \cdot A_d), \quad (66)$$

which does not yield second-order equations in the scalar limit.

Lagrangians involving two derivatives of the vector fields provide

$$\left\{ \begin{array}{l} \tilde{\mathcal{L}}_1 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_a^{\nu_1}) (\partial_{\mu_2} A^{\nu_2 a}), \\ \tilde{\mathcal{L}}_3 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}), \\ \tilde{\mathcal{L}}_{10} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{ab}), \end{array} \right. \quad (67)$$

leading to the corresponding scalar terms

$$\left\{ \begin{array}{l} \tilde{\mathcal{L}}_1|_\pi = \mathcal{L}_{4,I}^{\text{Gal},3}, \\ \tilde{\mathcal{L}}_3|_\pi = \mathcal{L}_{4,II}^{\text{Gal},3}, \\ \tilde{\mathcal{L}}_{10}|_\pi = \mathcal{L}_{4,II}^{\text{Gal},2} - \mathcal{L}_{4,III}^{\text{Gal},2}. \end{array} \right. \quad (68)$$

One can derive two linear combinations having second-order equations, namely,

$$\tilde{\mathcal{L}}_1|_\pi + 2\tilde{\mathcal{L}}_3|_\pi = \mathcal{L}_{4,I}^{\text{Gal},3} + 2\mathcal{L}_{4,II}^{\text{Gal},3} \quad (69)$$

[see Eq. (A33)] and

$$\begin{aligned} \tilde{\mathcal{L}}_{10}|_\pi + \tilde{\mathcal{L}}_3|_\pi &= \mathcal{L}_{4,II}^{\text{Gal},2} - \frac{1}{2}(2\mathcal{L}_{4,III}^{\text{Gal},2} + \mathcal{L}_{4,I}^{\text{Gal},3}) \\ &\quad + \frac{1}{2}(\mathcal{L}_{4,I}^{\text{Gal},3} + 2\mathcal{L}_{4,II}^{\text{Gal},3}), \end{aligned} \quad (70)$$

yielding second-order equations, as each of the three terms on the right-hand side of Eq. (70) does so, as shown in Appendix A [see Eqs. (A21), (A31) and (A33)].

### E. Final flat spacetime model

Let us regroup the results of the above sections to produce the final theory in flat spacetime with the Minkowskian metric. We first gather most of the new terms induced by the nonlinear contributions into an arbitrary function  $f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a)$ . Indeed, this is possible because they not only appear in the systematic procedure we have exposed, but they also satisfy all our conditions; this is equivalent to the general proof discussed in Ref. [70], where the typical term is built out of Levi-Civita tensors, necessarily inducing terms proportional to  $\epsilon^{00\dots}$  in the Hessian, and hence vanishing contributions.

Up to now, we have used the Abelian form of the Faraday tensor to express the relevant Lagrangians, although there can be situations in which working with the non-Abelian counterpart in Eq. (2) can be more

convenient, in particular, in view of the fact that this is the relevant tensor that appears naturally when one extends the theory to its gauged version. This is quite simple since the arbitrary function  $f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a)$  can be equivalently written as a new function  $\tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a)$  using Eq. (2). It is worth noting that such a change of variable implies no other terms than those already included in the original function.

Gathering the above considerations into a compact form, we obtain a first generic term, reminiscent of the Abelian case, namely,

$$\mathcal{L}_2 = f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a) = \tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a). \quad (71)$$

In addition to this term, all the remaining previously derived terms involving contractions with up to six Lorentz indices are

$$\begin{cases} \hat{\mathcal{L}}_1 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_a^{\nu_1}) (\partial_{\mu_2} A^{\nu_2 a}) + 2\delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}), \\ \hat{\mathcal{L}}_2 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}) + \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{\alpha b}), \\ \hat{\mathcal{L}}_3 = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} S^{\sigma\alpha a}, \end{cases} \quad (72)$$

the first two actually being equivalent in the pure scalar sector since they lead to the same equations of motion, i.e., those stemming from the Galileon Lagrangian containing four scalar fields in the 3 representation of SU(2). Note that there is no term containing only one gradient.

With this general basis, which we expand upon in the final discussion section, we can now turn to the covariantization required to apply this category of theories to cosmologically relevant situations.

## V. COVARIANTIZATION

### A. Procedure

Below we follow a procedure similar to that proposed for the Galileon case [18,19,117], the generalized Proca model [51,66,68,118], and the multi-Galileon situation [120,123,124]. The principle is simple: one first transforms all partial derivatives into covariant ones and then checks that only those terms leading to at most second-order equations of motion are kept.

The pure vector part now contains  $A$  and  $\nabla A$  terms, which translate into  $A$ ,  $\partial A$ ,  $g$  and  $\partial g$  terms. None of these terms could lead to any derivative of order higher than two in the equations of motion. On the other hand, the Faraday tensor terms do not yield metric derivatives since partial derivatives can be replaced by covariant ones by virtue of the anti-symmetry of these terms. We also leave these terms aside.

As for the scalar part, derivatives of order three or more could appear for the curvature. To fix this potential problem, we write the equations of motion in terms of covariant derivatives and commute them in order to

generate the curvature tensor, which contains only second-order derivatives of the metric: the problem is with the derivatives of the curvature terms. As these particular contributions stem from terms implying at least fourth-order derivatives of the scalar field, it is easy to identify them and to write down the required counterterms.

In practice, this does not show that the resulting equations of motion of the metric do not involve higher-order derivatives of the scalar field. We merely apply the results of Ref. [118], where it was shown that if the equations of motion for the scalar field are safe, then so are those for the metric. This result translates directly to our case.

For many of the terms discussed below, it turns out to be easier to write the Lagrangian as a function of the vector field rather than of its scalar part, even though we are ultimately interested in the latter. Indeed, the scalar Euler-Lagrange equation

$$0 = \frac{\partial \mathcal{L}}{\partial \pi_d} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \pi_d)} + \nabla_\nu \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \nabla_\nu \pi_d)} \quad (73)$$

can be written as

$$\begin{aligned} 0 &= -\nabla_\nu \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \pi_d)} + \nabla_\nu \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \nabla_\nu \pi_d)} \\ &= -\nabla_\nu \left( \frac{\partial \mathcal{L}}{\partial A_{\nu d}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu A_{\nu d})} \right) \end{aligned} \quad (74)$$

since the action is assumed to be local in  $A_\mu$  and therefore cannot contain terms involving nonderivative functions of the scalar field  $\pi$ .

In the following sections, we write those terms containing only the curvature and its derivative, or only its derivative, by the respective notation  $\mathcal{F}|_R$  or  $\mathcal{F}|_{\nabla R}$ , where  $\mathcal{F}$  is the term whose restriction is being considered. We concentrate on terms which are nonvanishing in the scalar sector only.

### B. Terms in $\mathcal{L}^{\text{Gal}}$

The Lagrangians we consider give, in the scalar sector,

$$\begin{cases} \hat{\mathcal{L}}_1|_\pi = \mathcal{L}_{4,\text{I}}^{\text{Gal},3} + 2\mathcal{L}_{4,\text{II}}^{\text{Gal},3}, \\ \hat{\mathcal{L}}_2|_\pi = \mathcal{L}_{4,\text{II}}^{\text{Gal},3} + \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2}, \end{cases} \quad (75)$$

where we use the Galileon Lagrangians of Appendix A. In the following, working in the vector sector, we substitute  $\partial_\mu \pi^a \rightarrow A_\mu^a$ . Equation (75) implies that only three independent counterterms are needed, i.e., those associated with  $\mathcal{L}_{4,\text{I}}^{\text{Gal},3}$ ,  $\mathcal{L}_{4,\text{II}}^{\text{Gal},3}$  and  $(\mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2})$ . We now proceed to find these counterterms.

First, we have

$$\begin{aligned} & \left\{ \nabla_\nu \nabla_\mu \left[ \frac{\partial \mathcal{L}_{4,\text{I}}^{\text{Gal},3}}{\partial (\nabla_\mu A_{\nu d})} \right] \right\} \Big|_R \\ &= -2A_b^\lambda A_\lambda^b R_{\mu\nu} \nabla^\nu A^{\mu d} - 2A_b^\lambda A_\lambda^b A^{\mu d} \nabla^\nu R_{\mu\nu}. \end{aligned} \quad (76)$$

Introducing

$$\mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3} = \frac{1}{4} A_b^\lambda A_\lambda^b A_\mu^a A_\mu^a R, \quad (77)$$

we find that

$$\left\{ \nabla_\nu \left[ \frac{\partial \mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3}}{\partial (A_{\nu d})} \right] \right\} \Big|_{\nabla R} = A_b^\lambda A_\lambda^b A^{\mu d} \nabla^\nu (g_{\mu\nu} R), \quad (78)$$

which finally implies the equation of motion (EOM)

$$\begin{aligned} & \text{EOM}_\pi(\mathcal{L}_{4,\text{I}}^{\text{Gal},3} + \mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3})|_{\nabla R} \\ &= -2A_b^\lambda A_\lambda^b A^{\mu\alpha} \nabla^\nu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0, \end{aligned} \quad (79)$$

vanishing by virtue of the properties of the Einstein tensor.

Similarly, for  $\mathcal{L}_{4,\text{II}}^{\text{Gal},3}$ , we have

$$\begin{aligned} & \left\{ \nabla_\nu \nabla_\mu \left[ \frac{\partial \mathcal{L}_{4,\text{II}}^{\text{Gal},3}}{\partial (\nabla_\mu A_{\nu d})} \right] \right\} \Big|_R \\ &= -2A_b^\lambda A_\lambda^d R_{\mu\nu} \nabla^\nu A^{\mu b} - 2A_b^\lambda A_\lambda^d A^{\mu b} \nabla^\nu R_{\mu\nu}. \end{aligned} \quad (80)$$

Introducing

$$\mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3} = \frac{1}{4} A_b^\lambda A_{\lambda a} A^{\mu b} A_\mu^a R, \quad (81)$$

which verifies

$$\left\{ \nabla_\nu \left[ \frac{\partial \mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3}}{\partial (A_{\nu d})} \right] \right\} \Big|_{\nabla R} = A_b^\lambda A_\lambda^d A^{\mu b} \nabla^\nu (g_{\mu\nu} R), \quad (82)$$

we obtain

$$\begin{aligned} & \text{EOM}_\pi(\mathcal{L}_{4,\text{II}}^{\text{Gal},3} + \mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3})|_{\nabla R} \\ &= -2A_b^\lambda A_\lambda^d A^{\mu b} \nabla^\nu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0. \end{aligned} \quad (83)$$

Finally, using the previous notation

$$\tilde{\mathcal{L}}_{10} = \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2}, \quad (84)$$

we have

$$\begin{aligned} & \left\{ \nabla_\nu \nabla_\mu \left[ \frac{\partial \tilde{\mathcal{L}}_{10}}{\partial (\nabla_\mu A_{\nu d})} \right] \right\} \Big|_R \\ &= -2A^{\mu d} A^{\lambda b} R_{\rho\lambda\mu}^\nu \nabla_\nu A_b^\rho - 2A^{\mu d} A^{\lambda b} A_b^\rho \nabla_\nu R_{\rho\lambda\mu}^\nu. \end{aligned} \quad (85)$$

We introduce the counterterm

$$\mathcal{L}_{10,\text{CT}} = -\frac{1}{2} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma R_{\mu\nu\rho\sigma}, \quad (86)$$

giving

$$\begin{aligned} & \nabla_\rho \left( \frac{\partial \mathcal{L}_{10,\text{CT}}}{\partial (A_{\rho d})} \right) \Big|_{\nabla R} = -2A^{\mu a} A_{\lambda a} A^{\rho d} \nabla^\nu R_{\rho\lambda\mu}^\nu \\ &= 2A^{\mu d} A^{\lambda b} A_b^\rho \nabla_\nu R_{\rho\lambda\mu}^\nu, \end{aligned} \quad (87)$$

which, as expected, results in

$$\text{EOM}_\pi(\tilde{\mathcal{L}}_{10} + \mathcal{L}_{10,\text{CT}})|_{\nabla R} = 0. \quad (88)$$

Then, to obtain the covariantized form of the action, it is sufficient to add the counterterms obtained in this part to the action given previously in flat spacetime. The result is summarized in Sec. VI.

### C. Coupling with curvature

Once the derivatives have been covariantized, one must also include possible direct coupling terms between the vector field and the curvature tensors, which we do below in a way entirely similar to that of Ref. [68]. First, we demand contractions with tensors whose divergences vanish on all indices (to ensure that integration by parts provides no higher order contributions in the equations of motion) [117,118]: this means the Einstein tensor as well as

$$\begin{aligned} L_{\mu\nu\rho\sigma} &= 2R_{\mu\nu\rho\sigma} + 2(R_{\mu\sigma}g_{\rho\nu} + R_{\rho\nu}g_{\mu\sigma} - R_{\mu\rho}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\rho}) \\ &+ R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu}), \end{aligned} \quad (89)$$

whose symmetries are those of the Riemann tensor, to which it is dual in the sense that it can be written as

$$L^{\alpha\beta\gamma\delta} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\rho\sigma}R_{\mu\nu\rho\sigma}. \quad (90)$$

Even limiting ourselves to the same number of fields as in the flat spacetime situation, many terms are *a priori* possible. To begin with, all contractions involving a single vector field are impossible. With two such fields, the reasoning is exactly equivalent to the Abelian case, which means the Lagrangians

$$\mathcal{L}_1^{\text{curv}} = G_{\mu\nu}A^{\mu a}A_a^\nu \quad (91)$$

and

$$\mathcal{L}_2^{\text{curv}} = L_{\mu\nu\rho\sigma}G^{\mu\nu a}G_a^{\rho\sigma} \quad (92)$$

are acceptable.

Terms in which at least one of the Abelian-like Faraday tensors is replaced by its Hodge dual can always be rewritten as a contraction between the Riemann tensor and two Abelian-like Faraday tensors, which cannot give second-order equations of motion [118]. One could envisage a contraction with a term like  $G^{\mu\rho a}G_a^{\nu\sigma}$ , but which is proportional to  $\mathcal{L}_2^{\text{curv}}$ : to show this, one needs to use the following identity,

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\rho\sigma\mu\nu} - \epsilon^{\alpha\rho\sigma\mu}\epsilon^{\beta\gamma\delta\nu} + \epsilon^{\alpha\gamma\delta\nu}\epsilon^{\beta\rho\sigma\mu} + \epsilon^{\alpha\beta\delta\nu}\epsilon^{\rho\gamma\sigma\mu} \\ - \epsilon^{\alpha\beta\gamma\nu}\epsilon^{\rho\delta\sigma\mu} = 0, \end{aligned} \quad (93)$$

and the first Bianchi identity.

With three fields, one can obtain a new nonvanishing term, in contrast to the Abelian case. This is mostly due to the fact that it is possible to have an antisymmetry in the exchange of two underived vector fields. We get

$$\mathcal{L}_3^{\text{curv}} = L_{\mu\nu\rho\sigma}\epsilon_{abc}G^{\mu\nu a}A^{\rho b}A^{\sigma c}, \quad (94)$$

which is shown to be proportional to  $L_{\mu\nu\rho\sigma}\epsilon_{abc}G^{\mu\rho a}A^{\nu b}A^{\sigma c}$ , by making use of the previous identity on the Levi-Civita tensor.

Four fields provide, again in contrast to the Abelian situation, the extra contribution

$$\mathcal{L}_4^{\text{curv}} = L_{\mu\nu\rho\sigma}A^{\mu a}A^{\nu b}A_a^\rho A_b^\sigma. \quad (95)$$

It is worth noticing at this point that it is possible to go from the expression of  $\mathcal{L}_2^{\text{curv}}$  and  $\mathcal{L}_3^{\text{curv}}$  using  $G_{\mu\nu}^a$  (the Abelian form of the Faraday tensor) to that using  $F_{\mu\nu}^a$  (the non-Abelian one),

both of which are equal in an Abelian theory: it is sufficient for this purpose to include the terms  $\mathcal{L}_3^{\text{curv}}$  and  $\mathcal{L}_4^{\text{curv}}$  only (they are generated by the transformation from  $G_{\mu\nu}^a$  to  $F_{\mu\nu}^a$ ).

## VI. FINAL MODEL, DISCUSSION

Let us summarize the results obtained for the generalized SU(2) Proca theory. First, we showed that any function of the vector field, Faraday tensor, and its Hodge dual (either in their Abelian or non-Abelian formulation) was possible, i.e.,

$$\mathcal{L}_2 = f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a) = \tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a). \quad (96)$$

Such a general  $\mathcal{L}_2$  term involving only gauge-invariant quantities for the derivatives is also present in the Abelian case; we will not discuss it any further since it appears similarly (and for the same reasons) in both the Abelian and non-Abelian theories.

Before presenting the other terms contained in the non-Abelian action, let us pursue the summary of what was found for its Abelian counterpart, as worked out in Refs. [66,68–70]; as usual, we denote  $\mathcal{L}_{n+2}$  the Lagrangians containing  $n \geq 1$  first-order derivatives of the vector field. First, the relation between the more general scalar and vector theories, i.e., the Galileon and generalized Proca models, provide, in this case, a deeper understanding through the use of the Stückelberg trick to go from one sector to another (i.e., switching between  $\partial_\mu\pi$  and  $A_\mu$ ). In the scalar Galileon theory, only one term exists in the Lagrangians  $\mathcal{L}_3$  to  $\mathcal{L}_5$ , each of which generates a contribution to the vector sector by the Stückelberg trick, i.e., those with a prefactor  $f_i(X)$  in the conclusion of Ref. [70]. An additional freedom stems from the fact that a given scalar Lagrangian can give different vector Lagrangians when permuting the second-order derivatives before introducing the vector field: although  $\partial_\mu\partial_\nu\pi = \partial_\nu\partial_\mu\pi$ , this symmetry is absent in the pure vector case since  $\partial_\mu A_\nu \neq \partial_\nu A_\mu$ . This property led to one additional contribution to the vector sector of each  $\mathcal{L}_4$  to  $\mathcal{L}_6$ . These contributions appear with the prefactor  $g_i(X)$  in Ref. [70]; they vanish in the pure scalar sector.

Coming back to the non-Abelian situation, and in addition to  $\mathcal{L}_2$ , we derived those relevant Lagrangians implying up to six contracted Lorentz indices and being nontrivial in flat spacetime. Contrary to the Abelian case, we found no such Lagrangian for  $n = 1$ . For  $n = 2$ , there are three possible terms; i.e.,  $\mathcal{L}_4$  contains

$$\left\{ \begin{aligned} \mathcal{L}_4^1 &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2}A_b^\lambda A_\lambda^b(\nabla_{\mu_1}A_a^{\nu_1})(\nabla_{\mu_2}A^{\nu_2 a}) + \frac{1}{4}A_b^\lambda A_\lambda^b A_a^\mu A_\mu^a R \\ &\quad + 2\delta_{\nu_1\nu_2}^{\mu_1\mu_2}A_a^\lambda A_{\lambda b}(\nabla_{\mu_1}A^{\nu_1 a})(\nabla_{\mu_2}A^{\nu_2 b}) + \frac{1}{2}A_b^\lambda A_{\lambda a}A^{\mu b}A_\mu^a R, \\ \mathcal{L}_4^2 &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2}A_a^\lambda A_{\lambda b}(\nabla_{\mu_1}A^{\nu_1 a})(\nabla_{\mu_2}A^{\nu_2 b}) + \frac{1}{4}A_b^\lambda A_{\lambda a}A^{\mu b}A_\mu^a R \\ &\quad + \delta_{\nu_1\nu_2}^{\mu_1\mu_2}A_{\mu_1}^a A_{\mu_2}^b(\nabla^{\nu_1}A_a^\alpha)(\nabla^{\nu_2}A_{\alpha b}) - \frac{1}{2}A^{\mu a}A^{\nu b}A_a^\rho A_b^\sigma R_{\mu\nu\rho\sigma}, \\ \mathcal{L}_4^3 &= \tilde{G}_{\mu\sigma}^b A_a^\mu A_{ab}^\sigma S^{\alpha\sigma a}, \end{aligned} \right. \quad (97)$$

the first two terms giving, once developed, the following forms:

$$\left\{ \begin{array}{l} \mathcal{L}_4^1 = (A_b \cdot A^b)[(\nabla \cdot A_a)(\nabla \cdot A^a) - (\nabla_\mu A_a^\nu)(\nabla^\mu A_\nu^a) + \frac{1}{4}A_a \cdot A^a R] \\ \quad + 2(A_a \cdot A_b)[(\nabla \cdot A^a)(\nabla \cdot A^b) - (\nabla_\mu A^{\nu a})(\nabla^\mu A_\nu^b) + \frac{1}{2}A^a \cdot A^b R], \\ \mathcal{L}_4^2 = (A_a \cdot A_b)[(\nabla \cdot A^a)(\nabla \cdot A^b) - (\nabla_\mu A^{\nu a})(\nabla^\mu A_\nu^b) + \frac{1}{4}A^a \cdot A^b R] \\ \quad + (A^{\mu a}A^{\nu b})[(\nabla_\mu A_a^\alpha)(\nabla_\nu A_{\alpha b}) - (\nabla_\nu A_a^\alpha)(\nabla_\mu A_{\alpha b}) - \frac{1}{2}A_b^\rho A^{\sigma b} R_{\mu\nu\rho\sigma}], \end{array} \right. \quad (98)$$

which are more easily compared with the equivalent results for the Abelian case. Finally, we also found four extra possibilities for the Lagrangians, implying a coupling with the curvature

$$\begin{aligned} \mathcal{L}_1^{\text{curv}} &= G_{\mu\nu} A^{\mu a} A_\nu^a, \\ \mathcal{L}_2^{\text{curv}} &= L_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_{\mu\nu}^a, \\ \mathcal{L}_3^{\text{curv}} &= L_{\mu\nu\rho\sigma} \epsilon_{abc} F^{\mu a} A^{\rho b} A^{\sigma c}, \\ \mathcal{L}_4^{\text{curv}} &= L_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma, \end{aligned} \quad (99)$$

thereby completing the full action at that order.

Let us first consider the actions whose equations of motion involve only second-order derivatives for the scalar (not first-order ones), which is equivalent to having only two vector fields together with the relevant gradients in the action. The multi-Galileon SU(2) model in the adjoint representation has been considered in [100], where it was shown that building a Lagrangian is only possible at the order of  $\mathcal{L}_4$  (not to mention the order  $\mathcal{L}_2$  already discussed above). The equivalent formulations of this Lagrangian are detailed in Appendix A. Following the previous considerations, no Lagrangian in the vector sector should appear at the order of  $\mathcal{L}_3$  since there is no such associated Lagrangian for the multi-Galileon at that order; we explicitly confirmed this expectation. In addition, two Lagrangians should appear at the order of  $\mathcal{L}_4$ , one associated with the multi-Galileon dynamics and one associated with the commutation of second-order derivatives of the scalar field. In fact, three Lagrangians have been found, two of them giving the multi-Galileon dynamics in the scalar sector. We then interpret these two previous terms as contributions which are equivalent in the scalar case but not in the vector case. The fact that there are two non-vanishing Lagrangians in the scalar sector is also due to a commutation of the second-order derivatives of the scalar fields but in a current term, which implies that it is not possible to describe this commutation with a Lagrangian vanishing in the pure scalar sector. This additional term is specific to the non-Abelian case: the term in  $\delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_{\mu_1}^a A_{\mu_2}^b (\nabla^{\nu_1} A_a^\alpha) (\nabla^{\nu_2} A_{\alpha b})$  vanishes in the Abelian case, while  $\mathcal{L}_4^1$  and  $\mathcal{L}_4^2$  both reduce to  $\mathcal{L}_4^{\text{Abelian}} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A^\lambda A_\lambda (\nabla_{\mu_1} A^{\nu_1}) (\nabla_{\mu_2} A^{\nu_2})$ .

To go further, let us first consider terms implying more derivatives, i.e., having  $n \geq 3$ . At the order of  $\mathcal{L}_5$ , and since

there is no possible dynamics for the SU(2) adjoint multi-Galileon, we expect that no term having a nonvanishing pure scalar contribution is possible. This suggests that the only possible term is

$$\mathcal{L}_5 = \epsilon_{abc} (A^a \cdot A^d) \tilde{G}_d^{\alpha\mu} \tilde{G}_{\mu}^{\beta b} S_{\alpha\beta}^c, \quad (100)$$

with the other SU(2) index contractions giving a vanishing result. At the order of  $\mathcal{L}_6$ , the only possibility seems to be the independent possible contractions of SU(2) indices on  $\mathcal{L}_6^{\text{Abelian}} = (A \cdot A) \tilde{G}^{\alpha\beta} \tilde{G}^{\mu\nu} S_{\alpha\mu} S_{\beta\nu}$ , since there is no possibility of having a term that does not vanish in the pure scalar sector. However, one should verify that there is no other term vanishing in the pure scalar sector, not included in  $\mathcal{L}_2$ , and whose dynamics is not described by the previous ones. This kind of terms would be specific to a non-Abelian theory, as is the second term of  $\mathcal{L}_4^2$ , and they would vanish for a vector field in a trivial group representation.

Concerning the Lagrangians with more than two vector fields, together with the relevant gradients, one has to pay attention to the fact that fully factorizing an  $f(A_\mu^a)$  as in the Abelian case is not guaranteed to lead to a valid procedure, although factorizing such an arbitrary function in front of any valid contribution also leads to another valid contribution. In addition, one could think that if there is no valid Lagrangian with only a few nongradient vector fields at a given derivative order, it is fairly probable that there is also no such valid Lagrangian at all at this order. For instance, we showed explicitly that terms at the order of  $\mathcal{L}_3$  are not possible with up to four vector fields, and this questions the possibility of having such a term even with a higher number of vector fields. An interesting point is that if a Lagrangian is allowed which does not vanish in the pure scalar sector, it corresponds to a possible term in the multi-Galileon action, which shows that both theories are closely related.

To conclude, this discussion showed that even if the full action of the model has not been obtained yet, discussing the low order terms permits us to identify and understand the whole Lagrangian structure. The above discussion is not specific to the SU(2) case and therefore can be extended to other group representations. For a theory with a vector field transforming under any representation of any group, a systematic study of all possible terms in the action should be performed in parallel with the corresponding multi-Galileon theory.

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## APPENDIX A: SU(2) GALILEON LAGRANGIAN EQUIVALENT FORMULATIONS

### 1. Introduction

The purpose of this appendix is to write explicitly all the Lagrangians describing the multi-Galileon dynamics in the three-dimensional representation of SU(2), focusing on the Lagrangians containing only four Galileon fields, i.e., those which are useful in this article. A Lagrangian describing this dynamics is given in Ref. [123], namely,

$$\mathcal{L}_m^\pi = \alpha^{i_1 \dots i_m} \delta_{[\nu_2 \dots \nu_m]}^{\mu_2 \dots \mu_m} \pi_{i_1} \partial_{\mu_2} \partial^{\nu_2} \pi_{i_2} \dots \partial_{\mu_m} \partial^{\nu_m} \pi_{i_m}, \quad (\text{A1})$$

with  $m$  running from 1 to 5, and with the notation

$$\begin{aligned} & \frac{1}{(D-n)!} \epsilon^{i_1 \dots i_n \sigma_1 \dots \sigma_{D-n}} \epsilon_{j_1 \dots j_n \sigma_1 \dots \sigma_{D-n}} \\ & = n! \delta_{[i_1 \dots i_n]}^{j_1 \dots j_n} = \delta_{i_1 \dots i_n}^{j_1 \dots j_n} = \delta_{i_1}^{j_1} \dots \delta_{i_n}^{j_n} \pm \dots, \end{aligned} \quad (\text{A2})$$

for  $n$  running from 1 to 4 (in a four-dimensional spacetime). Other equivalent formulations are possible, which is the purpose of this appendix.

This investigation is necessary for two reasons. First, the formulation given in Eq. (A1) cannot be obtained from a vector Lagrangian using the switch  $A_\mu^a \rightarrow \partial_\mu \pi^a$  since a scalar field without derivatives is present. Second, if different Lagrangians are equivalent in the scalar sector, they could give Lagrangians that are not equivalent in the vector sector. We thus expect that different Lagrangians valid in the vector sector become different but equivalent formulations of the multi-Galileon dynamics when considering the pure scalar part of the action.

For this purpose, we use the results of Ref. [16], which describe equivalent formulations of the Galileon theory in the Abelian case, introducing a Lagrangian similar to that in Eq. (A1), together with the following Lagrangians:

$$\mathcal{L}_m^{\text{Gal},1} = \delta_{[\nu_1 \dots \nu_{m-1}]}^{\mu_1 \dots \mu_{m-1}} \partial_{\mu_1} \pi \partial^{\nu_1} \pi \partial_{\mu_2} \partial^{\nu_2} \pi \dots \partial_{\mu_{m-1}} \partial^{\nu_{m-1}} \pi, \quad (\text{A3})$$

$$\mathcal{L}_m^{\text{Gal},2} = \delta_{[\nu_1 \dots \nu_{m-2}]}^{\mu_1 \dots \mu_{m-2}} \partial_{\mu_1} \pi \partial_\lambda \pi \partial^{\nu_1} \partial^\lambda \pi \dots \partial_{\mu_{m-2}} \partial^{\nu_{m-2}} \pi, \quad (\text{A4})$$

$$\mathcal{L}_m^{\text{Gal},3} = \delta_{[\nu_1 \dots \nu_{m-2}]}^{\mu_1 \dots \mu_{m-2}} \partial_\lambda \pi \partial^\lambda \pi \partial_{\mu_1} \partial^{\nu_1} \pi \dots \partial_{\mu_{m-2}} \partial^{\nu_{m-2}} \pi, \quad (\text{A5})$$

for  $m \geq 2$ , the case  $m = 1$  giving  $\mathcal{L} = \pi$ . These Lagrangians all give second-order equations of motion.

### 2. Lagrangians

We first write all possible Lagrangians appearing when we add the group indices to the previous Lagrangians, restricting ourselves to the case  $m = 4$ . They are more numerous than in the multi-Galileon case since we have an additional freedom when choosing the group index contractions.

The only possible Lagrangian associated with the formulation of Ref. [123] is

$$\mathcal{L}_4^{\text{PSZ}} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial_{\mu_1} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b. \quad (\text{A6})$$

The Lagrangians appearing in Ref. [16], given in Eqs. (A3)–(A5), can be endowed with SU(2) indices in several ways, namely, two possibilities for  $\mathcal{L}_4^{\text{Gal},1}$ :

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b \quad (\text{A7})$$

and

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^a \partial_{\mu_3} \partial^{\nu_3} \pi^b; \quad (\text{A8})$$

three possibilities for  $\mathcal{L}_4^{\text{Gal},2}$ :

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi^a \partial^\lambda \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A9})$$

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi_b \partial^\lambda \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A10})$$

and

$$\mathcal{L}_{4,\text{III}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi_b \partial^\lambda \partial^{\nu_1} \pi^b \partial_{\mu_2} \partial^{\nu_2} \pi^a; \quad (\text{A11})$$

and finally two possibilities for  $\mathcal{L}_4^{\text{Gal},3}$ :

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},3} = \partial_\lambda \pi_a \partial^\lambda \pi^a \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b \quad (\text{A12})$$

and

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},3} = \partial_\lambda \pi_a \partial^\lambda \pi_b \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b. \quad (\text{A13})$$

Looking for the Lagrangians implying second-order equations of motion, one can quickly verify that  $\mathcal{L}_4^{\text{PSZ}}$ ,  $\mathcal{L}_{4,\text{I}}^{\text{Gal},1}$  and  $\mathcal{L}_{4,\text{II}}^{\text{Gal},1}$  have this property due to the symmetry properties of  $\delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3}$ . However, the other Lagrangians do not give *a priori* second-order equations of motion.<sup>2</sup> We then investigate, in the following, the relations among the different Lagrangians.

<sup>2</sup>The automatic cancellation between third-order derivatives discussed in Ref. [16] is not valid anymore since this cancellation can be spoiled by the group indices.



### 3. Relations among the Lagrangians

#### a. Between PSZ and Gal,1

We first relate  $\mathcal{L}_4^{\text{PSZ}}$  and the Lagrangians  $\mathcal{L}_4^{\text{Gal},1}$  by means of conserved currents. Indeed,

$$J_{0,I}^{\mu_1} = J_{4,I}^{\text{PSZ-Gal},\mu_1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b \quad (\text{A14})$$

gives

$$\partial_{\mu_1} J_{0,I}^{\mu_1} = \partial_{\mu_1} J_{4,I}^{\text{PSZ-Gal},\mu_1} = \mathcal{L}_4^{\text{PSZ}} + \mathcal{L}_{4,I}^{\text{Gal},1}, \quad (\text{A15})$$

and

$$J_{0,II}^{\mu_1} = J_{4,II}^{\text{PSZ-Gal},\mu_1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^a \partial_{\mu_3} \partial^{\nu_3} \pi^b \quad (\text{A16})$$

gives

$$\partial_{\mu_1} J_{0,II}^{\mu_1} = \partial_{\mu_1} J_{4,II}^{\text{PSZ-Gal},\mu_1} = \mathcal{L}_4^{\text{PSZ}} + \mathcal{L}_{4,II}^{\text{Gal},1}. \quad (\text{A17})$$

It is also possible to make a direct correspondence between  $\mathcal{L}_{4,I}^{\text{Gal},1}$  and  $\mathcal{L}_{4,II}^{\text{Gal},1}$  with the current

$$J_{0,I \rightarrow II}^{\mu_2} = J_{4,I \rightarrow II}^{\text{Gal},1,\mu_2} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi^a \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A18})$$

yielding

$$\partial_{\mu_2} J_{0,I \rightarrow II}^{\mu_2} = \partial_{\mu_2} J_{4,I \rightarrow II}^{\text{Gal},1,\mu_2} = \mathcal{L}_{4,I}^{\text{Gal},1} - \mathcal{L}_{4,II}^{\text{Gal},1}. \quad (\text{A19})$$

#### b. Between Gal,2 and Gal,3

Introducing

$$J_1^{\mu_1} = J_{4,I}^{\text{Gal},2-3,\mu_1} = \partial_\lambda \pi_a \partial^\lambda \pi^a \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A20})$$

we get

$$\partial_{\mu_1} J_1^{\mu_1} = \partial_{\mu_1} J_{4,I}^{\text{Gal},2-3,\mu_1} = 2\mathcal{L}_{4,III}^{\text{Gal},2} + \mathcal{L}_{4,I}^{\text{Gal},3}. \quad (\text{A21})$$

In a similar way, from

$$J_2^{\mu_1} = J_{4,II}^{\text{Gal},2-3,\mu_1} = \partial_\lambda \pi_a \partial^\lambda \pi_b \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A22})$$

we obtain

$$\partial_{\mu_1} J_2^{\mu_1} = \partial_{\mu_1} J_{4,II}^{\text{Gal},2-3,\mu_1} = \mathcal{L}_{4,I}^{\text{Gal},2} + \mathcal{L}_{4,II}^{\text{Gal},2} + \mathcal{L}_{4,II}^{\text{Gal},3}. \quad (\text{A23})$$

#### c. Between Gal,1, Gal,2 and Gal,3 through Kronecker properties

We use the following identity given in Ref. [16]:

$$\delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} = \delta_{\nu_1}^{\mu_1} \delta_{\nu_2 \dots \nu_n}^{\mu_2 \dots \mu_n} + \sum_{i=2}^n (-1)^{i-1} \delta_{\nu_i}^{\mu_i} \delta_{\nu_1 \nu_2 \dots \nu_{i-1} \nu_{i+1} \dots \nu_n}^{\mu_1 \mu_2 \dots \mu_n}, \quad (\text{A24})$$

which gives, for  $n = 3$ ,

$$\delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} = \delta_{\nu_1}^{\mu_1} \delta_{\nu_2 \nu_3}^{\mu_2 \mu_3} - \delta_{\nu_2}^{\mu_2} \delta_{\nu_1 \nu_3}^{\mu_1 \mu_3} + \delta_{\nu_3}^{\mu_3} \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2}. \quad (\text{A25})$$

It is then possible to obtain two additional relations among the different Lagrangians. Indeed, applying this identity to  $\mathcal{L}_{4,I}^{\text{Gal},1}$  and  $\mathcal{L}_{4,II}^{\text{Gal},1}$ , we get

$$\mathcal{L}_{4,I}^{\text{Gal},1} = -2\mathcal{L}_{4,I}^{\text{Gal},2} + \mathcal{L}_{4,I}^{\text{Gal},3} \quad (\text{A26})$$

and

$$\mathcal{L}_{4,II}^{\text{Gal},1} = -\mathcal{L}_{4,II}^{\text{Gal},2} - \mathcal{L}_{4,III}^{\text{Gal},2} + \mathcal{L}_{4,II}^{\text{Gal},3}. \quad (\text{A27})$$

#### 4. Lagrangians with second-order equations of motion

Using the results of the previous subsections, we can summarize the Lagrangians that give second-order equations of motion:

$$\mathcal{L}_4^{\text{PSZ}}, \quad (\text{A28})$$

$$\mathcal{L}_{4,I}^{\text{Gal},1} = -2\mathcal{L}_{4,I}^{\text{Gal},2} + \mathcal{L}_{4,I}^{\text{Gal},3} = -\mathcal{L}_4^{\text{PSZ}} - \partial_\mu J_{0,I}^\mu, \quad (\text{A29})$$

$$\mathcal{L}_{4,II}^{\text{Gal},1} = -\mathcal{L}_{4,II}^{\text{Gal},2} - \mathcal{L}_{4,III}^{\text{Gal},2} + \mathcal{L}_{4,II}^{\text{Gal},3} = -\mathcal{L}_4^{\text{PSZ}} - \partial_\mu J_{0,II}^\mu, \quad (\text{A30})$$

$$\mathcal{L}_{4,II}^{\text{Gal},2} = \frac{1}{4} \mathcal{L}_{4,I}^{\text{Gal},1} - \frac{1}{2} \mathcal{L}_{4,II}^{\text{Gal},1} - \frac{1}{4} \partial_\mu J_1^\mu + \frac{1}{2} \partial_\mu J_2^\mu, \quad (\text{A31})$$

$$\mathcal{L}_{4,I}^{\text{Gal},2} + \mathcal{L}_{4,III}^{\text{Gal},2} = -\frac{1}{2} \mathcal{L}_{4,I}^{\text{Gal},1} + \frac{1}{2} \partial_\mu J_1^\mu, \quad (\text{A32})$$

and

$$\mathcal{L}_{4,I}^{\text{Gal},3} + 2\mathcal{L}_{4,II}^{\text{Gal},3} = \frac{1}{2} \mathcal{L}_{4,I}^{\text{Gal},1} + \mathcal{L}_{4,II}^{\text{Gal},1} + \frac{1}{2} \partial_\mu J_1^\mu + \partial_\mu J_2^\mu. \quad (\text{A33})$$

- [1] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, *J. High Energy Phys.* **10** (2012) 140.
- [2] F. L. Bezrukov and M. Shaposhnikov, *Phys. Lett. B* **659**, 703 (2008).
- [3] A. Mazumdar and J. Rocher, *Phys. Rep.* **497**, 85 (2011).
- [4] D. H. Lyth and A. Riotto, *Phys. Rep.* **314**, 1 (1999).
- [5] M. P. Hertzberg and F. Wilczek, [arXiv:1407.6010](https://arxiv.org/abs/1407.6010).
- [6] S. Ferrara, R. Kallosh, A. Linde, and M. Porrati, *Phys. Rev. D* **88**, 085038 (2013).
- [7] K. A. Olive, *Phys. Rep.* **190**, 307 (1990).
- [8] D. Baumann and L. McAllister, *Inflation and String Theory* (Cambridge University Press, Cambridge, England, 2015).
- [9] A. Ashtekar and D. Sloan, *Phys. Lett. B* **694**, 108 (2010).
- [10] A. Barrau, *Proc. Sci.*, ICHEP2010 (2010) 461.
- [11] M. Ostrogradski, *Mem. Ac. St. Petersburg* **VI**, 385 (1850).
- [12] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [13] R. P. Woodard, *Lect. Notes Phys.* **720**, 403 (2007).
- [14] R. P. Woodard, *Scholarpedia* **10**, 32243 (2015).
- [15] A. Nicolis, R. Rattazzi, and E. Trincherini, *Phys. Rev. D* **79**, 064036 (2009).
- [16] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, *Phys. Rev. D* **84**, 064039 (2011).
- [17] C. Deffayet and D. A. Steer, *Classical Quantum Gravity* **30**, 214006 (2013).
- [18] C. Deffayet, G. Esposito-Farese, and A. Vikman, *Phys. Rev. D* **79**, 084003 (2009).
- [19] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **80**, 064015 (2009).
- [20] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Prog. Theor. Phys.* **126**, 511 (2011).
- [21] T. Kobayashi, N. Tanahashi, and M. Yamaguchi, *Phys. Rev. D* **88**, 083504 (2013).
- [22] P. Creminelli, A. Nicolis, and E. Trincherini, *J. Cosmol. Astropart. Phys.* **11** (2010) 021.
- [23] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Phys. Rev. Lett.* **105**, 231302 (2010).
- [24] S. Mizuno and K. Koyama, *Phys. Rev. D* **82**, 103518 (2010).
- [25] C. Burrage, C. de Rham, D. Seery, and A. J. Tolley, *J. Cosmol. Astropart. Phys.* **01** (2011) 014.
- [26] P. Creminelli, G. D'Amico, M. Musso, J. Norena, and E. Trincherini, *J. Cosmol. Astropart. Phys.* **02** (2011) 006.
- [27] K. Kamada, T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Phys. Rev. D* **83**, 083515 (2011).
- [28] M. Libanov, S. Mironov, and V. Rubakov, *J. Cosmol. Astropart. Phys.* **08** (2016) 037.
- [29] S. Banerjee and E. N. Saridakis, [arXiv:1604.06932](https://arxiv.org/abs/1604.06932).
- [30] S. Hirano, T. Kobayashi, and S. Yokoyama, [arXiv:1604.00141](https://arxiv.org/abs/1604.00141).
- [31] R. Brandenberger and P. Peter, [arXiv:1603.05834](https://arxiv.org/abs/1603.05834).
- [32] S. Nishi and T. Kobayashi, *J. Cosmol. Astropart. Phys.* **04** (2016) 018.
- [33] N. Chow and J. Khoury, *Phys. Rev. D* **80**, 024037 (2009).
- [34] F. P. Silva and K. Koyama, *Phys. Rev. D* **80**, 121301 (2009).
- [35] T. Kobayashi, *Phys. Rev. D* **81**, 103533 (2010).
- [36] R. Gannouji and M. Sami, *Phys. Rev. D* **82**, 024011 (2010).
- [37] S. Tsujikawa, *Lect. Notes Phys.* **800**, 99 (2010).
- [38] A. De Felice and S. Tsujikawa, *Phys. Rev. Lett.* **105**, 111301 (2010).
- [39] A. Ali, R. Gannouji, and M. Sami, *Phys. Rev. D* **82**, 103015 (2010).
- [40] A. Padilla, P. M. Saffin, and S.-Y. Zhou, *J. High Energy Phys.* **01** (2011) 099.
- [41] A. De Felice and S. Tsujikawa, *Phys. Rev. D* **84**, 124029 (2011).
- [42] D. F. Mota, M. Sandstad, and T. Zlosnik, *J. High Energy Phys.* **12** (2010) 051.
- [43] S. Nesseris, A. De Felice, and S. Tsujikawa, *Phys. Rev. D* **82**, 124054 (2010).
- [44] G. Gabadadze and S. Yu, [arXiv:1608.01060](https://arxiv.org/abs/1608.01060).
- [45] J. Neveu, V. Ruhlmann-Kleider, P. Astier, M. Besancon, J. Guy, A. Möller, and E. Babichev, [arXiv:1605.02627](https://arxiv.org/abs/1605.02627).
- [46] V. Salvatelli, F. Piazza, and C. Marinoni, *J. Cosmol. Astropart. Phys.* **09** (2016) 027.
- [47] M. Shahalam, S. K. J. Pacif, and R. Myrzakulov, *Eur. Phys. J. C* **76**, 410 (2016).
- [48] M. Minamitsuji, *Gen. Relativ. Gravit.* **48**, 26 (2016).
- [49] E. N. Saridakis and M. Tsoukalas, *Phys. Rev. D* **93**, 124032 (2016).
- [50] M. Biswas and U. Debnath, *Commun. Theor. Phys.* **65**, 121 (2016).
- [51] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *Phys. Rev. Lett.* **114**, 211101 (2015).
- [52] D. Langlois and K. Noui, *J. Cosmol. Astropart. Phys.* **02** (2016) 034.
- [53] D. Langlois and K. Noui, *J. Cosmol. Astropart. Phys.* **07** (2016) 016.
- [54] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, *J. Cosmol. Astropart. Phys.* **07** (2016) 033.
- [55] M. Crisostomi, K. Koyama, and G. Tasinato, *J. Cosmol. Astropart. Phys.* **04** (2016) 044.
- [56] T. Harko, F. S. N. Lobo, E. N. Saridakis, and M. Tsoukalas, [arXiv:1609.01503](https://arxiv.org/abs/1609.01503).
- [57] E. Babichev, C. Charmousis, and A. Lehébel, *Classical Quantum Gravity* **33**, 154002 (2016).
- [58] J. Sakstein, H. Wilcox, D. Bacon, K. Koyama, and R. C. Nichol, *J. Cosmol. Astropart. Phys.* **07** (2016) 019.
- [59] T. Kobayashi, *Phys. Rev. D* **94**, 043511 (2016).
- [60] M. Lagos, T. Baker, P. G. Ferreira, and J. Noller, *J. Cosmol. Astropart. Phys.* **08** (2016) 007.
- [61] N. Frusciante, G. Papadomanolakis, and A. Silvestri, *J. Cosmol. Astropart. Phys.* **07** (2016) 018.
- [62] T. Qiu, *Phys. Rev. D* **93**, 123515 (2016).
- [63] Y. Akita and T. Kobayashi, *Phys. Rev. D* **93**, 043519 (2016).
- [64] G. W. Horndeski, *J. Math. Phys. (N.Y.)* **17**, 1980 (1976).
- [65] C. Deffayet, A. E. Gümrükçüoğlu, S. Mukohyama, and Y. Wang, *J. High Energy Phys.* **04** (2014) 082.
- [66] L. Heisenberg, *J. Cosmol. Astropart. Phys.* **05** (2014) 015.
- [67] G. Tasinato, *J. High Energy Phys.* **04** (2014) 067.
- [68] E. Allys, P. Peter, and Y. Rodríguez, *J. Cosmol. Astropart. Phys.* **02** (2016) 004.
- [69] J. Beltrán Jimenez and L. Heisenberg, *Phys. Lett. B* **757**, 405 (2016).

- [70] E. Allys, J. P. Beltrán Almeida, P. Peter, and Y. Rodríguez, *J. Cosmol. Astropart. Phys.* **09** (2016) 026.
- [71] L. Heisenberg, R. Kase, and S. Tsujikawa, *Phys. Lett. B* **760**, 617 (2016).
- [72] R. Kimura, A. Naruko, and D. Yoshida, arXiv:1608.07066.
- [73] G. Tasinato, *Classical Quantum Gravity* **31**, 225004 (2014).
- [74] M. Hull, K. Koyama, and G. Tasinato, *J. High Energy Phys.* **03** (2015) 154.
- [75] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y.-l. Zhang, and G.-B. Zhao, *Phys. Rev. D* **93**, 104016 (2016).
- [76] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y.-l. Zhang, *J. Cosmol. Astropart. Phys.* **06** (2016) 048.
- [77] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y.-l. Zhang, *Phys. Rev. D* **94**, 044024 (2016).
- [78] L. Heisenberg, R. Kase, and S. Tsujikawa, arXiv:1607.03175.
- [79] M.-a. Watanabe, S. Kanno, and J. Soda, *Phys. Rev. Lett.* **102**, 191302 (2009).
- [80] K. Dimopoulos, *Phys. Rev. D* **74**, 083502 (2006).
- [81] K. Dimopoulos, M. Karciuskas, and J. M. Wagstaff, *Phys. Rev. D* **81**, 023522 (2010).
- [82] A. Golovnev, V. Mukhanov, and V. Vanchurin, *J. Cosmol. Astropart. Phys.* **06** (2008) 009.
- [83] C. Armendariz-Picon, *J. Cosmol. Astropart. Phys.* **07** (2004) 007.
- [84] Y. Rodríguez, L. G. Gomez, and C. M. Nieto, in *Proceedings 2nd Argentinian-Brazilian Meeting on Gravitation, Relativistic Astrophysics and Cosmology (GRACO II): Buenos Aires, Argentina, 2014*, edited by G. S. Vila, F. L. Vieyro, and J. C. Fabris (2015), pp. 250.
- [85] A. Maleknejad and M. M. Sheikh-Jabbari, *Phys. Lett. B* **723**, 224 (2013).
- [86] A. Maleknejad and M. M. Sheikh-Jabbari, *Phys. Rev. D* **84**, 043515 (2011).
- [87] P. Adshead and M. Wyman, *Phys. Rev. Lett.* **108**, 261302 (2012).
- [88] A. Maleknejad, M. M. Sheikh-Jabbari, and J. Soda, *J. Cosmol. Astropart. Phys.* **01** (2012) 016.
- [89] R. Namba, E. Dimastrogiovanni, and M. Peloso, *J. Cosmol. Astropart. Phys.* **11** (2013) 045.
- [90] P. Adshead, E. Martinec, and M. Wyman, *J. High Energy Phys.* **09** (2013) 087.
- [91] D. G. Boulware, *Ann. Phys. (N.Y.)* **56**, 140 (1970).
- [92] K.-i. Shizuya, *Nucl. Phys.* **B94**, 260 (1975).
- [93] C. Grosse-Knetter, *Phys. Rev. D* **48**, 2854 (1993).
- [94] N. Banerjee, R. Banerjee, and S. Ghosh, *Ann. Phys. (N.Y.)* **241**, 237 (1995).
- [95] J.-C. Su, *Nuovo Cim. B* **117**, 203 (2002).
- [96] P. Senjanovic, *Ann. Phys. (N.Y.)* **100**, 227 (1976); **209**, 248 (E) (1991).
- [97] R. Banerjee and J. Barcelos-Neto, *Nucl. Phys.* **B499**, 453 (1997).
- [98] R. Slansky, *Phys. Rep.* **79**, 1 (1981).
- [99] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists* (Cambridge University Press, Cambridge, England, 2003).
- [100] A. Padilla, P. M. Saffin, and S.-Y. Zhou, *Phys. Rev. D* **83**, 045009 (2011).
- [101] M. L. Metha, J. M. Normand, and V. Gupta, *Commun. Math. Phys.* **90**, 69 (1983).
- [102] K.-i. Maeda and K. Yamamoto, *Phys. Rev. D* **87**, 023528 (2013).
- [103] P. Adshead and M. Wyman, *Phys. Rev. D* **86**, 043530 (2012).
- [104] C. M. Nieto and Y. Rodríguez, *Mod. Phys. Lett. A* **31**, 1640005 (2016).
- [105] E. Davydov and D. Galtsov, *Phys. Lett. B* **753**, 622 (2016).
- [106] S. Alexander, D. Jyoti, A. Kosowsky, and A. Marciano, *J. Cosmol. Astropart. Phys.* **05** (2015) 005.
- [107] M. Sharif and R. Saleem, *Astropart. Phys.* **62**, 100 (2015).
- [108] F. Darabi and A. Parsiya, *Mod. Phys. Lett. A* **29**, 1450161 (2014).
- [109] A. Maleknejad and E. Erfani, *J. Cosmol. Astropart. Phys.* **03** (2014) 016.
- [110] K.-i. Maeda and K. Yamamoto, *J. Cosmol. Astropart. Phys.* **12** (2013) 018.
- [111] M. R. Setare and V. Kamali, *Gen. Relativ. Gravit.* **46**, 1642 (2014).
- [112] A. Maleknejad, M. M. Sheikh-Jabbari, and J. Soda, *Phys. Rep.* **528**, 161 (2013).
- [113] J. A. R. Cembranos, A. L. Maroto, and S. J. N. Jareño, *Phys. Rev. D* **87**, 043523 (2013).
- [114] E. Dimastrogiovanni and M. Peloso, *Phys. Rev. D* **87**, 103501 (2013).
- [115] A. Ghalee, *Phys. Lett. B* **717**, 307 (2012).
- [116] W. Li, arXiv:1508.03247.
- [117] C. de Rham and L. Heisenberg, *Phys. Rev. D* **84**, 043503 (2011).
- [118] J. Beltrán Jiménez, R. Durrer, L. Heisenberg, and M. Thorsrud, *J. Cosmol. Astropart. Phys.* **10** (2013) 064.
- [119] M. Hull, K. Koyama, and G. Tasinato, *Phys. Rev. D* **93**, 064012 (2016).
- [120] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **82**, 061501 (2010).
- [121] A. Padilla, P. M. Saffin, and S.-Y. Zhou, *J. High Energy Phys.* **12** (2010) 031.
- [122] K. Hinterbichler, M. Trodden, and D. Wesley, *Phys. Rev. D* **82**, 124018 (2010).
- [123] A. Padilla and V. Sivanesan, *J. High Energy Phys.* **04** (2013) 032.
- [124] V. Sivanesan, *Phys. Rev. D* **90**, 104006 (2014).
- [125] P. Ramond, *Group Theory: A Physicist's Survey* (Cambridge University Press, Cambridge, England, 2010).
- [126] R. Feger and T. W. Kephart, *Comput. Phys. Commun.* **192**, 166 (2015).
- [127] P. Fleury, J. P. B. Almeida, C. Pitrou, and J.-P. Uzan, *J. Cosmol. Astropart. Phys.* **11** (2014) 043.