Analytical solutions in rotating linear dilaton black holes: Resonant frequencies, quantization, greybody factor, and Hawking radiation

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Charged massive scalar field perturbations are studied in the gravitational, electromagnetic, dilaton, and axion fields of rotating linear dilaton black holes. In this geometry, we separate the covariant Klein-Gordon equation into radial and angular parts and obtain the exact solutions of both the equations in terms of the confluent Heun functions. Using the radial solution, we study the problems of resonant frequencies, entropy/area quantization, and greybody factor. We also analyze the behavior of the wave solutions near the event horizon of the rotating linear dilaton black hole and derive its Hawking temperature via the Damour-Ruffini-Sannan method.

I. INTRODUCTION

Analytical solutions to a wave equation in a black hole background (in particular, for a stationary black hole) are of remarkable importance in theoretical and mathematical physics. Obtaining an exact solution to the wave equation is widely applicable in black hole physics. For example, one can study the quasinormal modes [1–4], analyze the entropy and area quantization [5], compute the greybody factor and absorption rate of a black hole [6–12], and study black hole perturbation [13]. The behavior of scalar fields in the backgrounds of black holes is studied to understand the physics of spin-0 particles. Thus, it is important to seek analytical solutions to the Klein-Gordon equation (KGE) and analyze physical phenomena such as the emission of scalar particles from black holes. Therefore, exact solutions of the KGE in various black hole geometries have been studied, e.g., [14–27] and references therein.

The subject of quantization of black holes was introduced in the early 1970s by Bekenstein [28,29]. He conjectured that a black hole has an equidistant area (A_{BH}) spectrum: $A_{BH\_n} = 8\pi n\hbar$, ($n = 0, 1, 2, …$) [30–32]. To this end, he used the Ehrenfest's principle by considering the black hole area as an adiabatic invariant quantity. Since those seminal works of Bekenstein, numerous theoretical models have been proposed for quantizing the black holes; for a topical review see [33]. Among them, the method of Maggiore [34] has gained much attention in the literature. According to Maggiore, a black hole can be considered as a damped harmonic oscillator with a frequency (ω), which is identical to the complex resonant frequencies or quasinormal modes. For the high damping oscillations, the imaginary part of the resonant frequencies always becomes dominant over the real part. Based on this fact Maggiore used the transition frequency $\Delta\omega \approx \text{Im}\,\omega_{n-1} - \text{Im}\,\omega_n$ in the adiabatic invariance formula of Kunstatter [35] and proved that the area spectrum is exactly equal to Bekenstein's original result [32].

Hawking [36] proved that black holes can thermally create and emit quantum particles, known as Hawking radiation, until they exhaust their energy and evaporate completely. According to this theory, black holes are therefore neither completely black nor able to last forever. In fact, the Hawking radiation is a phenomenon in which both the general theory of relativity and quantum theory simultaneously play an active role. Especially, it can be seen as the onset of the theory of quantum gravity. In the last four decades, various methods have been proposed to compute the Hawking radiation, e.g., [37–48]. The numerous publications on this subject to date clearly show that the Hawking radiation remains central in theoretical physics. Moreover, in experimental physics, Steinhauer [49] has almost succeeded in creating a laboratory-scale imitation of a black hole that emits Hawking radiation, that is, the particles that escape black holes because of quantum mechanical effects. One of the most valuable contributions to the original derivation of Hawking was made by Damour, Ruffini [38], and Sannan [50] (DRS). They demonstrated that to obtain the decay or emission rate, it is necessary to build a damped part in the outgoing wave function. Therefore, one should apply a simple analytic continuation to the outgoing wave function (available in the exterior region) to obtain its internal region structure. Such an analytical extension produces a damping factor and yields the scattering probability (i.e., emission rate) of the scalar wave at the event horizon. Furthermore, the DRS method only requires the existence of a future horizon and is independent from the dynamical details of the horizon formation process. For applications of the DRS method, the reader may refer to [26,27,51–57]. This method is also closely related to the near-horizon conformal property of the black hole geometry [58,59]. Although many studies of the DRS method have focused on the asymptotically flat black holes, the applications of the DRS method on the nonasymptotically flat black holes have remained very limited [59–61]. Our present study aims to decrease this paucity in the literature.

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In this study, we mainly focus on the analytical solutions of the KGE for a charged massive scalar field in rotating linear dilaton black holes (RLDBHs) [62]. These black holes represent nonasymptotically flat (like the Friedmann-Lemaître-Robertson-Walker spacetime [63], which is the universe model that most theorists currently use) solutions to the Einstein-Maxwel-dilaton-axion (EMDA) gravity theory. The experiments [64–67] on dilaton and axion fields that naturally exist in the RLDBH geometry may vindicate the dark matter in the near future. At the present time, the studies for the RLDBH [24,68–73] in the literature are relatively fewer than the studies that exist for static linear dilaton black holes [73–80]. In particular, for the problems of absorption cross section and decay rate for the massless and chargeless bosons emitted by a RLDBH, the reader is referred to [71]. We show that the obtained charged massive scalar wave function solutions are expressed in terms of the confluent Heun functions [81–84]. The solutions cover the region between the outer horizon and spatial infinity. Inspired by the very recent study of Vieira and Bezerra [57], we derive the resonant frequencies of the RLDBH by using the analytical solution of the radial equation and in the sequel derive the equally spaced entropy/area spectra of this black hole. We also study the greybody problem of the RLDBH spacetime. But the limited linear transformation formulas of the confluent Heun functions lead us to consider the case of chargeless and massless scalar fields. By obtaining the greybody factor, we reveal which waves are eligible to travel from the horizon to the asymptotic region in the RLDBH geometry. We then investigate the Hawking emission of the chargeless and massless spin-0 particles by computing the emission rate within the framework of the DRS method.

The paper is divided into the following sections. In Sec. II, we introduce the metric of the RLDBH spacetime and demonstrate its thermodynamic features. Section III is devoted to the KGE for charged massive scalar fields in the RLDBH geometry. Moreover, we separate the KGE into angular and radial parts. In Sec. IV, the analytical solutions of the angular and radial equations are represented in terms of the confluent Heun functions. In Sec. V, we present the applications of the wave solution. In this regard, the problems of resonant frequencies, entropy/area quantization, greybody factor, and Hawking radiation are elaborately studied. Finally, we summarize our discussions in the conclusion section. (We use geometrized units where $G = c = 1$, so that energy and time have units of length. The appendix lists the prominent symbols that are used throughout the paper.)

II. ROTATING LINEAR DILATON BLACK HOLE SPACETIME

The EMDA theory is described by the following action [85],

$$S = \frac{1}{16\pi} \int d^4x \sqrt{|g|} \left(\Re - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \varkappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right.$$
$$\left. - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \chi \partial^\mu \chi \right), \tag{1}$$

where ϕ and χ represent the dilaton and axion fields, respectively. The Maxwell field is governed by $F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ in which $\mathcal{A}$ is an Abelian vector field (i.e., electromagnetic vector potential), $\tilde{F}^{\mu\nu}$ is the dual of $F_{\mu\nu}$, and $\Re$ denotes the Ricci scalar. The line element that is the solution to Eq. (1) corresponds to the rotating linear dilaton metric with mass term, M, rotation parameter, a, and background electric charge, Q. In Boyer-Lindquist coordinates, the RLDBH spacetime [62] is given by

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + \xi \left[d\theta^2 + \sin^2\theta \left(d\varphi - \frac{a}{\xi} dt \right)^2 \right], \tag{2}$$

where

$$f = \Delta \xi^{-1}, \tag{3}$$

$$\xi = r r_0, \tag{4}$$

and the horizon surface equation ($f = 0$) is obtained from the condition

$$\Delta = (r - r_+)(r - r_-), \tag{5}$$

in which r_+ and r_- represent the event and inner (Cauchy) horizons, respectively. Those radii are given by

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{6}$$

where M is twice the quasilocal mass ($M = 2M_{QL}$) [86]. The charge parameter $r_0 = \sqrt{2}Q$ and the rotation parameter a are related with the angular momentum (J) as $2J = a r_0$ [62]. Once the rotation ceases, the stationary metric (2) clearly becomes static [46]. Furthermore, the dilaton and axion fields are given by

$$e^{-2\phi} = \frac{\xi}{r^2 + a^2 \cos^2\theta}, \tag{7}$$

$$\chi = -\frac{r_0 a \cos\theta}{r^2 + a^2 \cos^2\theta}. \tag{8}$$

The electromagnetic vector potential is

$$A = A_t dt + \mathcal{A}_\varphi d\varphi = \frac{1}{\sqrt{2}} (e^{2\phi} dt + a \sin^2\theta d\varphi), \tag{9}$$

and the Maxwell 2-form is derived as follows:

$$F = \frac{1}{\sqrt{2}} \left[\frac{r^2 - a^2 \cos^2\theta}{r\xi} dr \wedge dt + a \sin 2\theta d\theta \wedge \left(d\varphi - \frac{a}{\xi} dt \right) \right]. \tag{10}$$

A. Thermodynamics

In this section, we discuss the thermodynamic features of the RLDBHs. First, we consider a particle near the horizon with the following 4-velocity,

$$\mathbf{u} = u^t \left(\partial_t + \frac{a}{\xi} \partial_\varphi \right), \tag{11}$$

which satisfies the normalization condition

$$1 = u^\mu u_\mu. \tag{12}$$

Hence, one finds (near the horizon)

$$u^t = \frac{1}{\sqrt{g_{tt}}}. \tag{13}$$

Because all the metric components are only functions of r and θ, particle acceleration can be obtained from

$$a^\mu = \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = -g^{\mu\alpha} \partial_\alpha \ln u^t. \tag{14}$$

The definition of the surface gravity (κ) is given by [87]

$$\kappa = \lim_{r \to r_+} \frac{\sqrt{a^\mu a_\mu}}{u^t}. \tag{15}$$

After a straightforward calculation, Eq. (15) becomes

$$\kappa = \frac{1}{2} \frac{df}{dr} \bigg|_{r=r_+} = \frac{r_+ - r_-}{2r_+ r_0}. \tag{16}$$

Thus, we obtain the Hawking temperature as

$$T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar(r_+ - r_-)}{4\pi r_+ r_0}. \tag{17}$$

The angular velocity (Ω_H) and the black hole area are given by

$$\Omega_H = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \bigg|_{r=r_+} = \frac{a}{r_+ r_0}, \tag{18}$$

$$A_{\mathrm{BH}} = \int_0^{2\pi} d\varphi \int_0^\pi \sqrt{-g} d\theta = 4\pi r_+ r_0. \tag{19}$$

It is worth noting that in the evaluation of integral (19), $\sqrt{-g}$ is considered for the metric tensor of the RLDBH horizon (setting $dr = dt = 0$),

$$g_{\mu\nu} = \begin{pmatrix} r_+ r_0 & 0 \\ 0 & r_+ r_0 \sin^2\theta \end{pmatrix}. \tag{20}$$

Hence, the entropy of the black hole, S_{BH}, is given by

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4\hbar} = \frac{\pi r_+ r_0}{\hbar}. \tag{21}$$

The quantities described by Eqs. (17), (18), and (21) satisfy the first law of thermodynamics,

$$dM_{QL} = T_H dS_{\mathrm{BH}} + \Omega_H dJ. \tag{22}$$

III. SEPARATION OF THE KGE IN RLDBH GEOMETRY

In this section, we consider the wave equation of the charged massive scalar particles propagating in the geometry of RLDBH.

The KGE for a charged massive scalar particle is given by (e.g., [88])

$$\frac{1}{\sqrt{-g}} (\partial_\alpha - iq\mathcal{A}_\alpha)(\sqrt{-g} g^{\alpha\nu}(\partial_\nu - iq\mathcal{A}_\nu)\Psi) - \mu_s^2 \Psi = 0, \tag{23}$$

where μ_s and q represent the mass and charge of the scalar field Ψ, respectively. Owing to the axial symmetry and time independence of the spacetime, the scalar field can be written as

$$\Psi = \Psi(\mathbf{r}, t) = R(r)S(\theta)e^{im\varphi}e^{-i\omega t}, \tag{24}$$

where ω is the energy (frequency) corresponding to the flux of particles at spatial infinity and m denotes the azimuthal quantum number. Thus, Eq. (23) takes the following form in the RLDBH spacetime,

$$\frac{1}{2\xi f}[q^2(r^2 + a^2)^2 - \sqrt{2}qam(a^2 + r^2) + 2m^2 a^2]$$
$$+ \frac{\omega}{f}[\omega\xi + \sqrt{2}q(r^2 + a^2) - 2am] + \sqrt{2}qma$$
$$- \xi\mu_s^2 + \frac{1}{R(r)} \frac{d}{dr}\left[\xi f \frac{d}{dr} R(r)\right] - \frac{m^2}{\sin^2\theta} - \frac{1}{2}q^2 a^2 \sin^2\theta$$
$$+ \frac{1}{\sin\theta S(\theta)} \frac{d}{d\theta}\left[\cos\theta \frac{d}{d\theta} S(\theta)\right] = 0, \tag{25}$$

and by using an eigenvalue λ one can separate Eq. (25) into an angular equation,

$$\frac{d^2}{d\theta^2} S(\theta) + \cot\theta \left(\frac{d}{d\theta} S(\theta)\right)$$
$$- \left(\frac{m^2}{\sin^2\theta} + \frac{1}{2}q^2 a^2 \sin^2\theta - \lambda\right) S(\theta) = 0, \tag{26}$$

and a radial equation,

$$\frac{d}{dr}\left[\xi f\left(\frac{d}{dr} R(r)\right)\right] + \left\{\frac{1}{2\xi f} - \sqrt{2}qam(a^2 + r^2) + 2m^2 a^2\right.$$
$$+ \frac{\omega}{f}[\xi\omega + \sqrt{2}q(r^2 + a^2) - 2am]$$
$$+ \left. \sqrt{2}qma - \xi\mu_s^2 \right\} R(r) = 0. \tag{27}$$

As demonstrated in the following sections, the above separations enable us to find the solutions of the angular and radial equations in terms of the confluent Heun

functions. In particular, the solution of Eq. (27) can help us compute the standard Hawking radiation of the RLDBH.

IV. ANALYTICAL SOLUTIONS OF THE ANGULAR AND RADIAL EQUATIONS

In this section, we discuss the exact solutions of the angular and radial parts of the KGE.

A. Angular equation

By changing the independent variable θ to a new variable y,

$$\theta = \cos^{-1}(1 - 2y), \tag{28}$$

Eq. (26) transforms into

$$\frac{d^2}{dy^2}S(y) + \frac{2y-1}{y(y-1)}\frac{d}{dy}S(y)$$
$$-\frac{1}{4}\left[\left(\frac{m}{y(y-1)}\right)^2 + \frac{4\lambda}{y(y-1)} + 8q^2a^2\right]S(y) = 0. \tag{29}$$

We also introduce a new function $H(y)$ via

$$S(y) = e^{\tau y}\left(\frac{y}{y-1}\right)^m H(y), \tag{30}$$

where

$$\tau = 2aq^*, \tag{31}$$

in which $q^* = \frac{q}{\sqrt{2}}$. Function $H(y)$ satisfies the following equation,

$$\frac{d^2}{dy^2}H(y) + \left[2\tau + \frac{2}{y-1} + \frac{m-1}{y(y-1)}\right]\frac{d}{dy}H(y)$$
$$+ \frac{(2y-1+m)\tau - \lambda}{y(y-1)}H(y) = 0, \tag{32}$$

which can be rewritten as the confluent Heun equation [84],

$$\frac{d^2}{dy^2}H(y) + \left(\tilde{\alpha} + \frac{\tilde{\beta}+1}{y} + \frac{\tilde{\gamma}+1}{y-1}\right)\frac{d}{dy}H(y)$$
$$+ \left(\frac{\tilde{\mu}}{y} + \frac{\tilde{\nu}}{y-1}\right)H(y) = 0. \tag{33}$$

The parameters $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are given by

$$\tilde{\alpha} = 2\tau, \qquad \tilde{\beta} = -\tilde{\gamma} = -m. \tag{34}$$

By setting

$$\tilde{\eta} = \frac{m^2}{2} - \lambda, \qquad \tilde{\delta} = 0, \tag{35}$$

the other two parameters $\tilde{\mu}$ and $\tilde{\nu}$ in Eq. (32) become

$$\tilde{\mu} = \frac{1}{2}(\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma} + \tilde{\alpha}\tilde{\beta} - \tilde{\beta}\tilde{\gamma}) - \tilde{\eta} = \tau(1-m) + \lambda, \tag{36}$$

$$\tilde{\nu} = \frac{1}{2}(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\alpha}\tilde{\gamma} + \tilde{\beta}\tilde{\gamma}) + \tilde{\delta} + \tilde{\eta} = \tau(1+m) - \lambda. \tag{37}$$

The solution of Eq. (32) is given by [89]

$$H(y) = C_1 \text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; y)$$
$$+ C_2 y^{-\tilde{\beta}}\text{HeunC}(\tilde{\alpha}, -\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; y), \tag{38}$$

where C_1 and C_2 are the integral constants. Thus, the general exact solution of the angular part (28) of the KGE for a charged massive scalar field in the RLDBH geometry and over the entire range $0 \leq y < \infty$ is

$$S(y) = e^{\tau y}\left(\frac{y}{y-1}\right)^m [C_1 \text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; y)$$
$$+ C_2 y^{-\tilde{\beta}}\text{HeunC}(\tilde{\alpha}, -\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; y)]. \tag{39}$$

B. Radial equation

We follow the procedure described in Sec. IV A to show that the radial equation (27) can also be transformed into the confluent Heun equation (33). Thus, we first set

$$z = \frac{r - r_+}{r_- - r_+}, \tag{40}$$

and using this new coordinate, Eq. (27) transforms into

$$\frac{d^2}{dz^2}R(z) + \frac{(1-2z)\frac{d}{dz}R(z)}{z(1-z)} + \left\{\frac{q^{*2}(r_+ - r_-)^2 z^2}{(1-z)^2} - \frac{(2q^*\omega r_0 + 4q^{*2}r_+ - r_0\mu_s^2)(r_+ - r_-)z}{(1-z)^2}\right.$$
$$+ \frac{6q^* r_0 r_+ \omega + r_0^2\omega^2 + \mu_s^2(r_- - 2r_+)r_0 + 2(3r_+^2 + a^2)q^{*2} - \lambda}{(1-z)^2} - \frac{2}{(r_+ - r_-)(1-z)^2 z}$$
$$\times [2q^*(3r_+^2\omega r_0 - mar_+ + a(a\omega r_0 - mr_-)) + 4q^{*2}r_+^3 - r_0\mu_s^2 r_+^2 + (\mu_s^2 r_0 r_- + 4a^2 q^{*2} - \lambda + 2r_0^2\omega^2)r_+$$
$$+ \lambda r_- - 2r_0\omega am] + \frac{(r_+^2 + a^2)^2}{(r_+ - r_-)^2 z^2(1-z)^2}\left(q^* - \frac{(ma - \omega r_0 r_+)}{r_+^2 + a^2}\right)^2\right\}R(z) = 0. \tag{41}$$

Moreover, when we apply a particular s-homotopic transformation [27] to the dependent variable $R(z) \rightarrow U(z)$, where

$$R(z) = e^{\beta_1 z}z^{\beta_2}(1-z)^{\beta_3}U(z), \tag{42}$$

and coefficients β_1, β_2, and β_3 are given by

$$\beta_1 = iq^*(r_+ - r_-), \tag{43}$$

$$\beta_2 = \frac{i[\omega r_+ r_0 - ma + q^*(r_+^2 + a^2)]}{r_+ - r_-}, \tag{44}$$

$$\beta_3 = \frac{i[\omega r_- r_0 - ma + q^*(r_-^2 + a^2)]}{r_+ - r_-}. \tag{45}$$

Function $U(z)$ satisfies the confluent Heun equation (33) with the following parameters:

$$\tilde{\alpha} = 2\beta_1, \qquad \tilde{\beta} = 2\beta_2, \qquad \tilde{\gamma} = 2\beta_3, \tag{46}$$

$$\tilde{\delta} = (r_+ - r_-)[r_0\mu_s^2 - 2q^{*2}(r_+ + r_-) - 2q^*\omega r_0], \tag{47}$$

$$\begin{aligned}
\tilde{\eta} = {}& \frac{1}{(r_+ - r_-)^2}[2q^{*2}(r_+^4 - 2r_+^3 r_- - a^4 - 2a^2 r_+ r_-) \\
& + 2q^*(ma(r_+^2 + r_-^2 + 2a^2) \\
& - \omega r_0(r_- a^2 + 3r_- r_+^2 - r_+^3 + a^2 r_+)) \\
& - 2(ma - \omega r_0 r_+)(ma - \omega r_0 r_-)] - \lambda - r_0\mu_s^2 r_+.
\end{aligned} \tag{48}$$

We recall that

$$\tilde{\mu} = \frac{1}{2}(\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma} + \tilde{\alpha}\tilde{\beta} - \tilde{\beta}\tilde{\gamma}) - \tilde{\eta}, \tag{49}$$

$$\tilde{\nu} = \frac{1}{2}(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\alpha}\tilde{\gamma} + \tilde{\beta}\tilde{\gamma}) + \tilde{\delta} + \tilde{\eta}. \tag{50}$$

Thus, using solution (38) of the confluent Heun equation, the general solution of Eq. (40) in the exterior region of the event horizon ($0 \le z < \infty$) is given by

$$\begin{aligned}
R(z) = {}& e^{\beta_1 z}(1 - z)^{\beta_3}[C_1 z^{\beta_2}\text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; z) \\
& + C_2 z^{-\beta_2}\text{HeunC}(\tilde{\alpha}, -\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; z)],
\end{aligned} \tag{51}$$

where C_1 and C_2 are constants.

Around the regular singular point $u = 0$, the confluent Heun function [82–84] behaves as

$$\begin{aligned}
& \text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; u) \\
& = 1 + u\frac{(\tilde{\beta}(\tilde{\gamma} - \tilde{\alpha} + 1) + 2\tilde{\eta} - \tilde{\alpha} + \tilde{\gamma})}{2(1 + \tilde{\beta})} + \frac{u^2}{8(1 + \tilde{\beta})(2 + \tilde{\beta})} \\
& \times (\tilde{\beta}^2(\tilde{\alpha} - \tilde{\gamma})^2 - 4\tilde{\eta}\tilde{\beta}(\tilde{\alpha} - \tilde{\gamma}) + 2\tilde{\alpha}\tilde{\beta}(2\tilde{\alpha} - \tilde{\beta} - 3\tilde{\gamma}) \\
& + 4\tilde{\beta}\tilde{\gamma}(\tilde{\beta} + \tilde{\gamma}) + 3(\tilde{\alpha}^2 + \tilde{\beta}^2) + 4\tilde{\eta}(\tilde{\eta} - 2(\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma} - 1)) \\
& - 4\tilde{\alpha}(\tilde{\beta} + \tilde{\gamma}) + 4\tilde{\beta}\tilde{\delta} + \tilde{\gamma}(10\tilde{\beta} + 3\tilde{\gamma}^2) + 4(\tilde{\beta} + \tilde{\delta} + \tilde{\gamma})) + \cdots,
\end{aligned} \tag{52}$$

which is an auxiliary mathematical expression in the analysis of the Hawking radiation.

V. APPLICATIONS OF THE WAVE SOLUTION

In this section, we first study the resonant frequencies of the charged and massive scalar waves propagating in the RLDBH geometry by using the solution (51). Next, the obtained resonant frequencies are applied in the rotational adiabatic invariant quantity [69,90,91] to derive the quantum entropy/area spectra of the RLDBH. This problem is nothing but the extension of [69] in which the massless and chargeless scalar waves were considered. Then, we obtain the reflection coefficient and greybody factor for particular scalar waves. Finally, we apply the DRS method to compute the Hawking radiation of the RLDBH.

A. Resonant frequencies and spectroscopy of RLDBH

In this subsection, we follow the recently developed technique of [57] for computing the resonant frequencies for charged massive scalar waves propagating in the RLDBH background. By using those frequencies, we show how one can derive the equally spaced entropy/area spectra of the RLDBH.

Resonant frequencies are the proper modes at which a black hole freely oscillates when excited by a perturbation. In fact, the resonant frequencies are called quasinormal modes, in contrast to the normal modes of Newtonian gravity, because they are damped by the emission of gravitational waves; as a consequence, the corresponding eigenfrequencies are complex [92]. The imaginary component of the frequency tells us how quickly the oscillation will die away. So if one sets a black hole to vibrating, it radiates the energy away in gravitational waves. Both rotating and static black hole solutions have not just one resonant frequency but a whole series of resonant frequencies (see for example [93] and references therein). The resonant frequencies are associated with the radial solution (51) for the certain boundary conditions. The solution should be finite on the horizon and well behaved at asymptotic infinity. The latter remark requires that $R(z)$ must have a polynomial form, which is possible with the $\tilde{\delta}_n$-condition [57,84],

$$1 + \frac{\tilde{\beta} + \tilde{\gamma}}{2} + \frac{\tilde{\delta}}{\tilde{\alpha}} = -n, \qquad n = 0, 1, 2, \ldots. \tag{53}$$

By using the Frobenius method and putting the power series expansion into the confluent Heun's differential equation (33), the three-term recursive relation of coefficients starts to appear. To obtain the Heun polynomials, one should impose the condition of $\mathbf{C}_{n+2} = 0$, where $\mathbf{C}_n$ is one of the elements of the three-term recurrence relation [84]. In fact, $\mathbf{C}_{n+2} = 0$ is equivalent to the $\tilde{\delta}_n$-condition (for the technical details, we refer the interested reader to [94] and the references therein). In particular, Fiziev [95] showed that the confluent Heun's polynomials obtained from the $\tilde{\delta}_n$-condition (53) admit the most general class of solutions to the Teukolsky master equation, which is correspondent

with the Teukolsky-Starobinsky identities (TSIs) [6,7,96]. Please be reminded that TSIs are the key element of the theory of perturbations to a gravitational field (including the subject of quasinormal modes [97]) of rotating relativistic objects. Detailed studies on the TSIs can be seen in the seminal works of Chandrasekhar [13,98].

From Eq. (53), we get

$$1 + i\frac{[4(r_+^2 + a^2)q^{*2} + 4q^*(\omega r_0 r_+ - ma) - \mu_s^2 r_0 (r_+ - r_-)]}{2q^*(r_+ - r_-)}$$

$$= -n. \tag{54}$$

Similar to the very recent work of Ref. [57], one can find an analytic expression for the resonant frequencies ω_n from Eq. (54) as follows,

$$\omega_n = m\Omega_H + q\Phi_e + 2\kappa r_0 \frac{\mu_s^2}{q^*} + i(n+1)\kappa, \tag{55}$$

where

$$\Phi_e = -\frac{r_+^2 + a^2}{\sqrt{2}r_+ r_0},$$

$$= -\mathcal{A}_t|_{r=r_+, \theta=(0,\pi)}, \tag{56}$$

which is the electric potential of the RLDBH measured at the north/south poles of the 2-sphere with radius r_+ [99]. From Eq. (55), one can get the transition frequency between two highly damped $(n \to \infty)$ neighboring states as follows:

$$\Delta\omega \approx \text{Im}\omega_{n-1} - \text{Im}\omega_n, \qquad (n \to \infty)$$

$$= \kappa = \frac{2\pi T_H}{\hbar}. \tag{57}$$

For a black hole system with total energy E, the natural adiabatic invariant quantity I_{adb} is given by [69,90,91]

$$I_{adb} = \int \frac{dE}{\Delta\omega} \equiv \int \frac{T_H dS_{\text{BH}}}{\Delta\omega}. \tag{58}$$

For large quantum numbers $(n \to \infty)$, the Bohr-Sommerfeld quantization condition [100] applies and I_{adb} acts as a quantized quantity $(I_{adb} \simeq n\hbar)$ [35]. Inserting the transition frequency (57) into Eq. (58), one finds

$$I_{adb} = \frac{\hbar S_{\text{BH}}}{2\pi} = n\hbar. \tag{59}$$

From above, we read the entropy spectrum as

$$S_{\text{BH}\_n} = 2\pi n. \tag{60}$$

Since $S_{\text{BH}} = \frac{A_{\text{BH}}}{4\hbar}$, the area spectrum is then obtained as

$$A_{\text{BH}\_n} = 8\pi n\hbar, \tag{61}$$

and the minimum change in the area becomes

$$\Delta A_{\text{BH}}^{\min} = 8\pi\hbar. \tag{62}$$

As can be seen from Eqs. (60) and (61), both entropy and area spectra are equally spaced and independent of the black hole parameters. Besides, Eq. (62) shows that the RLDBH horizon is made by patches of equal area $8\pi\hbar$. Namely, the results obtained are fully in agreement with the Bekenstein conjecture [32] and with Ref. [69].

B. Greybody factor and Hawking radiation

The greybody factor accounts for the fact that waves need to travel from the horizon to spatial infinity in the curved geometry [99]. Analytical greybody factor computations require us to know the behavior of the general radial solution (51) near spatial infinity $r \to \infty$ or $z \to \infty$. To this end, there is a need for a transformation similar to the following:

$$\text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; z) \to \text{gamma functions}$$

$$\times \text{HeunC}\left(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}; \frac{1}{z}\right). \tag{63}$$

The parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$ and the gamma functions introduced in the above equation should be related with $\tilde{\alpha}$, $\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}$ according to the transformation rules of the special functions [101]. The key point here is the normalization condition [82]: $\text{HeunC}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}; \frac{1}{z} = 0) = 1$ while $z \to \infty$. Thus, the asymptotic solution of the transformed [via Eq. (63)] radial equation (51) would describe the pure asymptotic ingoing and outgoing waves [like the wave solution illustrated in Eq. (75)], which allow us to evaluate original flux coming from infinity and compare it to the flux at the black hole horizon. Then, the calculation of the greybody factor would be possible for the RLDBH. But unfortunately such a transformation (63) currently does not exist in the literature. In fact, unlike many classical hypergeometric transformations [102], the confluent Heun functions have very limited transformations [103]. Nevertheless, we are not completely helpless. The following transformation [89] enables us to transform the confluent Heun functions to the hypergeometric functions.

$$\text{HeunC}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}; z) = (1-z)^{-\Xi} F\left(\Xi, \Xi - \tilde{\gamma}; 1 + \tilde{\beta}; \frac{z}{z-1}\right),$$

$$(\tilde{\alpha} = 0, \tilde{\beta} + 1 \neq 0, \tilde{\delta} = 0, z \neq 1), \tag{64}$$

where

$$\Xi = \frac{1 + \tilde{\beta} + \tilde{\gamma} + \sqrt{\tilde{\beta}^2 + \tilde{\gamma}^2 + 1 - 4\tilde{\eta}}}{2}. \tag{65}$$

However, the conditions of $\tilde{\alpha} = 0$ and $\tilde{\delta} = 0$ for nonextremal $(r_+ \neq r_-)$ RLDBH are simultaneously satisfied if and only if the chargeless $(q = 0 \to \beta_1 = 0)$ and massless $(\mu_s = 0)$ scalar fields are taken into account [see Eqs. (43),

(46), and (47)]. In this case, the general radial solution (51) reduces to the following form,

$$R(z) = C_1 z^{\beta_2}(1-z)^{\beta_3-\Xi} F\left(\Xi, \Xi-\tilde{\gamma}; 1+\tilde{\beta}; \frac{z}{z-1}\right)$$
$$+ C_2 z^{-\beta_2}(1-z)^{\beta_3-\hat{\Xi}} F\left(\hat{\Xi}, \hat{\Xi}-\tilde{\gamma}; 1-\tilde{\beta}; \frac{z}{z-1}\right),$$

(66)

where

$$\hat{\Xi} = \Xi(\tilde{\beta} \to -\tilde{\beta}).$$ (67)

If one changes the independent variable z to a new variable u via the following transformation,

$$u = \frac{z}{z-1} = \frac{r-r_+}{r-r_-}, \to z = \frac{u}{u-1},$$ (68)

Eq. (66) recasts in

$$R(u) = C_1 u^{\hat{\alpha}}(1-u)^{\hat{\beta}} F(\hat{a}, \hat{b}, \hat{c}; u)$$
$$+ C_2 u^{-\hat{\alpha}}(1-u)^{\hat{\beta}} F(\hat{a}-\hat{c}+1, \hat{b}-\hat{c}+1, 2-\hat{c}; u),$$

(69)

where

$$\hat{\alpha} = -i\frac{\omega r_0 r_+ - ma}{r_+ - r_-},$$
$$\hat{\beta} = \frac{1}{2} - \sqrt{\lambda^2 + \frac{1}{4} - \omega^2 r_0^2},$$ (70)

and

$$\hat{a} = \hat{\beta} - i\omega r_0,$$
$$\hat{b} = \hat{\beta} - i\frac{\omega r_0(r_+ + r_-) - 2ma}{r_+ - r_-},$$
$$\hat{c} = 1 - \frac{2i(\omega r_0 r_+ - ma)}{r_+ - r_-}.$$ (71)

Meanwhile, in the absence of charge and mass the angular equation (29) admits a normalizable solution when it is expressed in terms of the spheroidal harmonics [102] with eigenvalues $\lambda = -l(l+1)$ in which l denotes the orbital quantum number. In fact, radial solution (69) was thoroughly studied by Li [24] several years ago. From now on, we review the computations of [24] to obtain the greybody factor of the RLDBH.

To study the absorption features of black holes, one should consider two boundary conditions: pure ingoing modes near the horizon and both ingoing and outgoing modes at spatial infinity. Following [24], one can infer from the ingoing boundary condition of the horizon (no outgoing wave survives at the horizon) that the coefficient C_2 in Eq. (69) must vanish. Thus, considering one of the hypergeometric transformations [see Eq. (15.3.6) of [102]] we can obtain the asymptotic behavior of the radial solution as

$$R(r) = C_1\left[\left(\frac{r}{r_+ - r_-}\right)^{-\hat{\beta}} \frac{\Gamma(\hat{c}-\hat{a}-\hat{b})\Gamma(\hat{c})}{\Gamma(\hat{c}-\hat{b})\Gamma(\hat{c}-\hat{a})}\right.$$
$$\left.+ \left(\frac{r}{r_+ - r_-}\right)^{\hat{\beta}-1} \frac{\Gamma(\hat{a}+\hat{b}-\hat{c})\Gamma(\hat{c})}{\Gamma(\hat{a})\Gamma(\hat{b})}\right].$$ (72)

On the other hand, the radial equation (27) with $\mu_s = q = 0$ near spatial infinity of the RLDBH is reduced to the following simple second order differential equation:

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [\omega^2 r_0^2 - l(l+1)]R = 0.$$ (73)

The solution of the above equation is given by

$$R(r) = D_1 r^{-\hat{\beta}} + D_2 r^{\hat{\beta}-1}.$$ (74)

When $\omega r_0 > l + 1/2$ (high-energy mode), Eq. (74) can also be expressed as a complex solution,

$$R(r) = \frac{1}{\sqrt{r}}(D_1 e^{i\sigma \ln r} + D_2 e^{-i\sigma \ln r}),$$ (75)

where

$$\sigma = \sqrt{\omega^2 r_0^2 - (l+1/2)^2}.$$ (76)

It is obvious from Eq. (75) that the first term represents the outgoing wave while the second term stands for the ingoing wave. Comparing the asymptotic solutions (72) and (74), one can get the following relationships between the coefficients:

$$D_1 = C_1(r_+ - r_-)^{\hat{\beta}} \frac{\Gamma(\hat{c}-\hat{a}-\hat{b})\Gamma(\hat{c})}{\Gamma(\hat{c}-\hat{b})\Gamma(\hat{c}-\hat{a})},$$
$$D_2 = C_1(r_+ - r_-)^{1-\hat{\beta}} \frac{\Gamma(\hat{a}+\hat{b}-\hat{c})\Gamma(\hat{c})}{\Gamma(\hat{a})\Gamma(\hat{b})}.$$ (77)

The conserved flux is given by

$$\mathcal{F} = -\frac{i}{2}\sqrt{-g}g^{rr}(R^*\partial_r R - R\partial_r R^*).$$ (78)

After substituting the asymptotic solution (75) of the high-energy modes into Eq. (78), we get the asymptotic flux as follows:

$$\mathcal{F}^{asy} = \frac{\sigma}{2}(|D_1|^2 - |D_2|^2).$$ (79)

Thus, one can remark that $\frac{1}{2}\sigma|D_1|^2$ and $\frac{1}{2}\sigma|D_2|^2$ terms denote the outgoing and ingoing fluxes, respectively. Hence, we can compute the reflection coefficient for the high-energy modes as

$$\mathbb{R} = \frac{|D_1|^2}{|D_2|^2} = \left|\frac{\Gamma[\frac{1}{2}-i(\sigma+\Bbbk)]\Gamma[\frac{1}{2}-i(\Bbbk+\omega r_0)]}{\Gamma[\frac{1}{2}+i(\sigma-\Bbbk)]\Gamma[\frac{1}{2}+i(\Bbbk-\omega r_0)]}\right|^2,$$
$$= \frac{\cosh\pi(\sigma-\Bbbk)\cosh\pi(\sigma-\omega r_0)}{\cosh\pi(\sigma+\Bbbk)\cosh\pi(\sigma+\omega r_0)},$$ (80)

where

$$\mathbb{k} = \frac{(r_+ + r_-)\omega r_0 - 2ma}{2\kappa r_+ r_0}. \tag{81}$$

It is worth noting that cosh forms of the gamma functions seen in Eq. (80) come from the Euler's reflection formulas of the gamma function [102]. The greybody factor or the absorption probability is given by

$$\gamma_{GB} = 1 - \mathbb{R}. \tag{82}$$

One can check that when we increase the frequency of the waves from the starting value $\frac{l+1/2}{r_0}$, the greybody factor γ_{GB} rapidly goes to 1: $\lim_{\omega(>\frac{l+1/2}{r_0})\to\infty}\gamma_{GB} \to 1$ (see also Fig. 1 depicted in [24]). The particle flux [104], in general, obeys $\mathcal{F}_p(\omega) \propto \frac{\gamma_{GB}}{e^{\omega/T}-1}$, where T denotes the temperature. The interpretation of this expression is that Hawking radiation is produced with a thermal spectrum at the horizon and then the spacetime curvature between the horizon and infinity can scatter some of the radiation back down the black hole. Thus, the observer at spatial infinity could detect a nonthermal radiation. On the other hand, when $\gamma_{GB} \to 1$, the spectrum of Hawking radiation observed by an asymptotic observer is pure thermal (blackbody radiation). Namely, the highly energetic thermal waves can overpass the gravitational barrier located at the outside of the RLDBH and travel from the horizon to the asymptotic region. As a final remark, it should be kept in mind that the identification of the ingoing and outgoing fluxes at spatial infinity is governed by the parameter $\hat{\beta}$. According to Eq. (70), $\hat{\beta}$ is real for the low-energy modes $\omega < \frac{r_0}{2}$ and complex for the high-energy modes $\omega > \frac{l+1/2}{r_0}$. We have shown above that high-energy modes render possible the calculation of the greybody factor. However, in the case of the low-energy modes ($\omega < \frac{r_0}{2}$) the identification of the fluxes becomes a very hard task. This is in fact due to the nonasymptotically flat structure of the RLDBH. This issue was also discussed in detail in [24].

Now, we want to employ the DRS method to investigate the Hawking radiation of the RLDBH. Since we now know that the chargeless and massless scalar (thermal) waves with $\omega > \frac{l+1/2}{r_0}$ and $\gamma_{GB} \to 1$ can smoothly reach the observer at spatial infinity, the radial solution (51) with $q = \mu_s = 0$ can be used for the application of the DRS method. From Eq. (52), we can see that Eq. (51) near the exterior event horizon ($r \to r_+$, $z \to 0$) behaves as

$$R(z) \sim C_1 z^{\beta_2^0} + C_2 z^{-\beta_2^0}, \tag{83}$$

where $\beta_2^0 = \beta_2|_{q=\mu_s=0}$. Thus, the near-horizon wave solution can be approximated to

$$\Psi \sim e^{-i\omega t} z^{\pm\beta_2^0}. \tag{84}$$

Taking cognizance of Eqs. (16), (18), and (44), the parameter β_2^0 reads

$$\beta_2^0 = \frac{i}{2\kappa}[\omega - m\Omega_H]. \tag{85}$$

Moreover, if we set

$$\varpi = \omega - m\Omega_H, \tag{86}$$

Eq. (85) can be rewritten as

$$\beta_2^0 = i\frac{\varpi}{2\kappa}. \tag{87}$$

Performing Taylor series expansion, we find the structure of the metric function f around the event horizon to be

$$\begin{aligned} f_{EH} &\simeq \frac{df}{dr}\bigg|_{r=r_+}(r - r_+) + O(r - r_+)^2, \\ &\simeq 2\kappa(r_+ - r_-)x, \end{aligned} \tag{88}$$

where $x = -z = \frac{r-r_+}{r_+-r_-}$. Thus, we can express the tortoise coordinate (r^*) [13] near the horizon as

$$r^* \simeq \int \frac{dr}{f_{EH}} = \int \frac{dx}{2\kappa x} = \frac{1}{2\kappa}\ln x \approx \frac{1}{2\kappa}\ln(r - r_+), \tag{89}$$

which corresponds to

$$r - r_+ \simeq e^{2\kappa r_*}. \tag{90}$$

Therefore, the ingoing and outgoing wave solutions on the black hole event horizon surface become

$$\Psi_{in} = e^{-i\omega t}e^{-i\varpi r_*}, \tag{91}$$

$$\Psi_{out}(r > r_+) = e^{-i\omega t}e^{i\varpi r_*}, \tag{92}$$

respectively. To reveal the features of the waves near the event horizon, we first define

$$\tilde{r} = \frac{\varpi}{\omega}r_*, \tag{93}$$

and then introduce the Eddington-Finkelstein coordinate,

$$v = t + \tilde{r}. \tag{94}$$

Hence, the ingoing and outgoing wave solutions become

$$\Psi_{in} = e^{-i\omega(t+\tilde{r})} = e^{-i\omega v}, \tag{95}$$

$$\begin{aligned} \Psi_{out}(r > r_+) &= e^{-i\omega(t-\tilde{r})} = e^{-i\omega v}e^{2i\varpi r_*}, \\ &= e^{-i\omega v}(r - r_+)^{\frac{i\varpi}{\kappa}}. \end{aligned} \tag{96}$$

One can easily observe that the $\Psi_{out}(r > r_+)$ solution is not analytical at the event horizon and the analytic continuation produces a damping factor, which makes it possible to attain an expression for the decay rate γ_{dy} [105]. To clarify the latter remark, we use the DRS method by rotating $-\pi$ through the lower-half complex r plane,

$$(r - r_+) \to |r - r_+|e^{-i\pi} = (r_+ - r)e^{-i\pi}. \tag{97}$$

Thus, the outgoing wave solution in the internal region $(r < r_+)$ yields

$$\Psi_{\text{out}}(r < r_+) = e^{-i\omega v}(r_+ - r)^{i\frac{\varpi}{\kappa}} e^{\pi\frac{\varpi}{\kappa}}. \qquad (98)$$

Therefore, Eqs. (96) and (98) represent the analytically continuous outgoing wave propagating around the RLDBH's event horizon. Thus, the emission rate or the relative scattering probability [27] of the scalar wave at the surface of the event horizon surface becomes

$$\gamma_{dy} = \left| \frac{\Psi_{\text{out}}(r > r_+)}{\Psi_{\text{out}}(r < r_+)} \right|^2 = e^{-2\pi\frac{\varpi}{\kappa}}. \qquad (99)$$

Following the other DRS applications like [51–57], the Hawking radiation spectrum of the scalar particles emitted from a black hole is obtained via the normalization condition as follows,

$$|N(\varpi)|^2 = \frac{\gamma_{dy}}{1 - \gamma_{dy}} = \frac{1}{e^{2\pi\frac{\varpi}{\kappa}} - 1}, \qquad (100)$$

where $N(\varpi)$ is the normalization constant. Equation (100) suggests that the emission of scalar particles has a thermal character analogous to the well-known blackbody spectrum with temperature $T_H = \frac{\kappa}{2\pi}$, which fully agrees with Eq. (17).

It is also worth noting that since the rotating dilaton black hole contains an ergosphere outside the horizon, it might have a super-radiant instability. In asymptotically flat spacetimes, such as Kerr black holes, the super-radiance shows itself as an (quantum) emission of certain modes ejaculating the angular momentum of the black hole to spatial infinity. On the other hand, when the considered spacetime has a nonasymptotically flat geometry, the asymptotic behavior of the modes should be carefully analyzed. As it was shown in [62,71], the super-radiant modes of the RLDBH do not propagate to spatial infinity. Instead, they are confined in a region outside the event horizon of the RLDBH. So we have an exponential growth for the super-radiant modes. However, this gives rise to the classical instability of the RLDBH solution. Besides, $|N(\varpi)|^2$ would diverge. Therefore, in addition to the condition of $\omega > \frac{l+1/2}{r_0}$, the energies/frequencies of the chargeless and massless scalar fields performing the Hawking emission should also be higher than the threshold [106] frequency $(m\Omega_H)$ of the super-radiance: $\omega > m\Omega_H$.

VI. CONCLUSION

We have presented complete analytical solutions to the covariant KGE for a charged massive scalar field in the RLDBH spacetime. Both the angular and radial exact solutions are demonstrated in terms of the confluent Heun functions [81–84], and they cover the whole range of the observable space $0 \le z < \infty$.

In particular, the radial solution has enabled us to analyze the resonant frequencies of the RLDBH with the help of the $\tilde{\delta}_n$-condition [57,84,95]. We have used the resonant frequencies to study the spectroscopy of the RLDBH. Both entropy and area spectra are found to be equally spaced and independent of the RLDBH parameters. Therefore, our results support the conjecture of Bekenstein [32]. We then studied the greybody factor problem of the RLDBH in order to explore which waves can propagate from the horizon to the spatial infinity. While doing this computation, the lack of the inverse transformation of the confluent Heun functions that is essential to find the exact asymptotic form of the radial solution forced us to consider the case of chargeless and massless scalar fields, which was previously considered in [24,69–71]. Using a particular transformation between the confluent Heun and hypergeometric functions, we have expressed the radial solution in terms of the hypergeometric functions that possess a broad spectrum of linear transformation features [102]. We then obtained the ingoing and outgoing fluxes at spatial infinity for the high-energy modes $(\omega > \frac{l+1/2}{r_0})$ that admit the identification of the fluxes, asymptotically. On the other hand, the low-energy modes $(\omega < \frac{r_0}{2})$ did not let us make the distinction between the ingoing and outgoing fluxes at spatial infinity; we simply ignored that case. Afterwards, we performed an analytical computation of the greybody factor. From the limit of $\lim_{\omega(>\frac{l+1/2}{r_0})\to\infty} \gamma_{GB} \to 1$, we have deduced that the high-energetic thermal waves can pass over the gravitational barriers and move from the horizon to the observer located at the asymptotic region. For those high-energy modes, we have considered the DRS method, which is a powerful mathematical tool of the analytic continuation for identifying the Hawking temperature of the considered black hole. To this end, we have used the series expansion of the confluent Heun function [see Eq. (52)] and obtained the outgoing wave solution in the exterior and interior regions of the event horizon. Finally, the Hawking radiation spectrum of the scalar particles emitted from the RLDBH was derived, and Hawking temperature (17) of the RLDBH was successfully obtained.

The results of the study are promising and motivate further work in this direction. In particular, the results can be extended to other particles with nonzero spin and to the black holes of higher dimensions.

APPENDIX: LIST OF SYMBOLS

For reference, the following is a list of symbols that are used often throughout the text.

M, M_{QL}	Mass parameter of the RLDBH spacetime and quasilocal mass ($M_{QL} = M/2$).
$\mathcal{A}$; ϕ, χ,	Electromagnetic vector potential; Dilaton and axion fields.
Q, r_0	Background electric charge and charge parameter ($r_0 = \sqrt{2}Q$).
J, a, Ω_H	Angular momentum, rotation parameter, $a = 2J/r_0 \in [0, M]$, and angular velocity ($\Omega_H = \frac{a}{r_+ r_0}$).
r_+, r_-	Outer horizon and inner horizon.
κ, T_H	Surface gravity and Hawking temperature ($T_H = \frac{\hbar \kappa}{2\pi}$).
A_{BH}, S_{BH}	Black hole area and entropy.
Ψ; μ_s, q	Scalar field (wave function); Mass parameter and charge parameter of the scalar field.
ω	Frequency. The time dependence of any field is $\sim e^{-i\omega t}$.
n	Overtone numbers of the eigenfrequencies. It starts from a fundamental mode with $n = 0$.
ω_n, $\Delta\omega$	Resonant (quasinormal mode) frequency and transition frequency: $\Delta\omega \approx \mathrm{Im}\,\omega_{n-1} - \mathrm{Im}\,\omega_n$.
m, λ	Azimuthal number with respect to the axis of rotation and eigenvalue.
Φ_e, I_{adb}	Electric potential of the RLDBH and adiabatic invariant quantity.
$\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{\eta}$	Parameters of the confluent Heun's function (HeunC).
y, z	Independent variables of the confluent Heun's function: $y = \frac{1-\cos(\theta)}{2}$ and $z = \frac{r-r_+}{r_- - r_+}$.
$\tilde{\mu}$, $\tilde{\nu}$	$\tilde{\mu} = \frac{1}{2}(\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma} + \tilde{\alpha}\tilde{\beta} - \tilde{\beta}\tilde{\gamma}) - \tilde{\eta}$ and $\tilde{\nu} = \frac{1}{2}(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\alpha}\tilde{\gamma} + \tilde{\beta}\tilde{\gamma}) + \tilde{\delta} + \tilde{\eta}$.
$\hat{a}$, $\hat{b}$, $\hat{c}$	Parameters of the hypergeometric function.
u	Independent variables of the hypergeometric function: $u = \frac{z}{z-1} = \frac{r-r_+}{r-r_-}$.
$\mathbb{R}$, γ_{GB}	Reflection coefficient and greybody factor ($\gamma_{GB} = 1 - \mathbb{R}$).
r_*, ϖ	Tortoise coordinate and wave frequency detected by the observer rotating with the horizon: $\varpi = \omega - m\Omega_H$.
γ_{dy}, $N(\varpi)$	Emission rate and normalization constant.
l	Integer angular number, related to the eigenvalue $\lambda = -l(l+1)$.
Ψ_{in}, Ψ_{out}	Ingoing and outgoing wave solutions around the event horizon.
$\mathcal{F}$, $\mathcal{F}^{asy}$, $\mathcal{F}_p$	Conserved flux, asymptotic flux, and particle flux.
C_1, C_2, D_1, D_2	Integral constants of the wave solutions.
q^*, τ, β_2^0, $\tilde{r}$	$q^* = \frac{q}{\sqrt{2}}$, $\tau = 2aq^*$, $\beta_2^0 = i\frac{\varpi}{2\kappa}$, and $\tilde{r} = \frac{\varpi}{\omega} r_*$.
Ξ, $\hat{\Xi}$	$\Xi = \frac{1 + \tilde{\beta} + \tilde{\gamma} + \sqrt{\tilde{\beta}^2 + \tilde{\gamma}^2 + 1 - 4\tilde{\eta}}}{2}$ and $\hat{\Xi} = \Xi(\tilde{\beta} \to -\tilde{\beta})$.
σ, $\mathbb{k}$	$\sigma = \sqrt{\omega^2 r_0^2 - (l + 1/2)^2}$ and $\mathbb{k} = \frac{(r_+ + r_-)\varpi r_0 - 2ma}{2\kappa r_+ r_0}$.
$\tilde{\delta}_n$-condition	$1 + \frac{\tilde{\beta} + \tilde{\gamma}}{2} + \frac{\tilde{\delta}}{\tilde{\alpha}} = -n$.

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