PHYSICAL REVIEW D **94,** 084038 (2016)

Existence of a constant mean curvature foliation in the extended Schwarzschild spacetime

Kuo-Wei Lee*

Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

(Received 28 July 2016; published 24 October 2016)

We construct a T-axisymmetric, spacelike, spherically symmetric, constant mean curvature (CMC) hypersurface foliation in the Kruskal extension with properties such that the mean curvature varies in each slice and ranges from minus infinity to plus infinity. This family of hypersurfaces extends the CMC foliation discussions by Malec and Ó Murchadha in 2009 [Phys. Rev. D **80**, 024017 (2009)].

DOI: 10.1103/PhysRevD.94.084038

I. INTRODUCTION

Spacelike constant mean curvature (CMC) hypersurfaces in spacetimes are very important objects in general relativity. They are broadly used in the analysis of Einstein constraint equations [1,2] and in the gauge condition in the Cauchy problem of the Einstein equations [3,4]. In addition, CMC foliation properties have been identified as the absolute time function in cosmological spacetimes [5].

How to define a canonical absolute time function is an important issue in the field of relativistic cosmology. At first, the cosmological time function was considered, which is defined by the supremum of the lengths of all past-directed timelike curves starting at some point. The idea of the cosmological time function is natural, but its drawback is poor regularity.

In 1971, York [5] suggested the CMC time function, which is a real-value function $f(x)$ defined on a spacetime such that every level set $\{f(x) = H\}$ is a Cauchy hypersurface with constant mean curvature H. In cosmological spacetimes, by the maximum principle, if the CMC time function exists, then it is unique. Furthermore, the CMC time function has better regularity than the cosmological time function. These properties indicate another viewpoint of the absolute time function. By definition, if the CMC time function exists, the spacetime is foliated by Cauchy hypersurfaces with constant mean curvature, and the mean curvature of these Cauchy hypersurfaces increases with time. This phenomenon leads us to concern ourselves with the CMC foliation problem in spacetimes.

Many CMC foliation results are proven for cosmological spacetimes (spatially compact spacetimes) with constant sectional curvature in Ref. [6] and references therein. However, CMC foliation properties are not well understood for spatially noncompact spacetimes such as the Schwarzschild spacetime (specifically, the Kruskal extension), which is the simplest model of a universe containing a star. In [7], Malec and Ó Murchadha constructed a family of T-axisymmetric, spacelike, spherically symmetric, constant mean curvature (TSS-CMC) hypersurfaces in the Kruskal extension, where each slice has the same mean curvature, and they conjectured this family foliates the Kruskal extension. In [8], the author used the shooting method and Lorentzian geometric analysis to prove the existence and uniqueness of the Dirichlet problem for a spacelike, spherically symmetric, constant mean curvature (SS-CMC) equation with symmetric boundary data in the Kruskal extension. As an application, the author completely proved Malec and Ó Murchadha's TSS-CMC foliation conjecture.

In [9], Malec and Ó Murchadha discussed different TSS-CMC foliation properties, and they posited whether there is a TSS-CMC foliation with varied constant mean curvature in each slice. One result is that if the relation between the mean curvature H and the TSS-CMC hypersurface parameter c is proportional, that is, $c = -8M^3H$, then there is a family of TSS-CMC hypersurfaces such that H ranges from minus infinity to plus infinity, but where all hypersurfaces intersect at the bifurcation sphere (the origin in the Kruskal extension).

In this paper, we will construct another family of TSS-CMC hypersurfaces with varied constant mean curvature in each slice. If H and c have a nonlinear relation, then there exists a TSS-CMC hypersurface foliation in the Kruskal extension. The main theorem is stated as follows.

Theorem 1: There exists a family of hypersurfaces $\{\Sigma_{H(c),c}\}$, $c \in \mathbb{R}$ in the Kruskal extension satisfying the following properties:

- (i) Every $\Sigma_{H(c),c}$ is a T-axisymmetric, spacelike, spherically symmetric, constant mean curvature hypersurface.
- (ii) Any two hypersurfaces in $\{\Sigma_{H(c),c}\}$ are disjoint.
- (iii) Every point (T', X') in the Kruskal extension belongs to $\Sigma_{H(c'),c'}$ for some $c' \in \mathbb{R}$.
- (iv) When $\{\Sigma_{H(c),c}\}$ foliates the Kruskal extension from the bottom to the top, the corresponding constant mean curvature H ranges from $-\infty$ to ∞ and the parameter c ranges from ∞ to $-\infty$.
- (v) $\{\Sigma_{H(c),c}\}$ is invariant under the reflection with respect to the X axis.

*d93221007@gmail.com

It is remarkable that, by similar argument, we can construct many different TSS-CMC hypersurface foliations with varied H and with X-axis symmetry, such that the TSS-CMC foliation in the Kruskal extension is not unique. Furthermore, we can also get TSS-CMC hypersurface foliations with varied H but without X-axis symmetry. By Lorentzian isometry, there are SS-CMC hypersurface foliations with varied H but without T-axis symmetry.

The organization of this paper is as follows. In Sec. II, we first give a brief introduction to the Schwarzschild spacetime and Kruskal extension, and then we summarize results of the TSS-CMC hypersurfaces in the Kruskal extension in order to construct a TSS-CMC foliation. The main theorem is stated and proved in Sec. III. Some discussions about TSS-CMC foliation properties are in Sec. IV.

II. PRELIMINARY

A. The Kruskal extension

In this paper, we mainly focus on the Kruskal extension, which is the maximal analytic extended Schwarzschild spacetime. The Schwarzschild spacetime is a four-dimensional time-oriented Lorentzian manifold equipped with the metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{(1-\frac{2M}{r})}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2,$$

where $M > 0$ is a constant. The metric is not defined at $r = 2M$, but in fact is a coordinate singularity. That is, after coordinates change, the metric is smooth at $r = 2M$,

$$ds^2 = \frac{16M^2 e^{-\frac{r}{2M}}}{r}(-dT^2 + dX^2) + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{1}$$

where

$$\begin{cases} (r-2M)e^{\frac{r}{2M}} = X^2 - T^2 \\ \frac{t}{2M} = \ln\left|\frac{X+T}{X-T}\right|. \end{cases} \tag{2}$$

The Kruskal extension is the union of two Schwarzschild spacetimes equipped with the extended metric (1). Figure 1 points out the correspondences between the Kruskal extension (left panel, T-X plane) and Schwarzschild spacetimes (right panel, t-r plane). We refer the reader to Wald [10] or Ref. [11] for more discussions on the Kruskal extension.

Remark that each point in the Kruskal T-X plane or the Schwarzschild t-r plane is topologically a sphere $\mathbb{S}^2$, which is parameterized by θ and ϕ. In this article, we are interested in the spherically symmetric hypersurfaces. It implies that every such hypersurface is a curve in both the T-X plane and the t-r plane. For convenience in our notation, we will ignore parameters θ and ϕ in this paper.

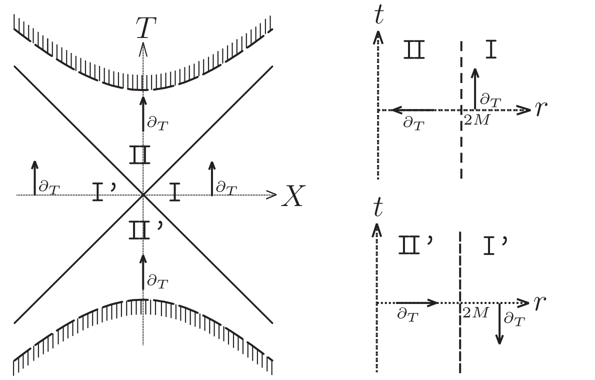

FIG. 1. The Kruskal extension and the Schwarzschild spacetimes.

We take ∂_T as a future-directed timelike vector field in the Kruskal extension, which is also shown in Fig. 1. Once ∂_T is chosen, for a spacelike hypersurface Σ, we will choose $\vec{n}$ as the future-directed unit normal vector of Σ in the Kruskal extension; the mean curvature H of Σ is defined by $H = \frac{1}{3}g^{ij}\langle\nabla_{e_i}\vec{n}, e_j\rangle$, where $\{e_i\}_{i=1}^3$ is a basis on Σ.

B. *T*-axisymmetric, spacelike, spherically symmetric, constant mean curvature hypersurfaces in the Kruskal extension

Let Σ: $(T = F(X), X)$ be a SS-CMC hypersurface in the Kruskal extension. In [11], we computed the SS-CMC equation

$$F''(X) + e^{-\frac{r}{2M}}\left(\frac{6M}{r^2} - \frac{1}{r}\right)(-F(X) + F'(X)X)(1-(F'(X))^2) + \frac{12HMe^{-\frac{r}{4M}}}{\sqrt{r}}(1-(F'(X))^2)^{\frac{3}{2}} = 0, \tag{3}$$

where the spacelike condition is $1-(F'(X))^2 > 0$, and $r = r(T,X) = r(F(X),X)$ satisfies Eq. (2), namely, $(r-2M)e^{\frac{r}{2M}} = X^2 - T^2 = X^2 - (F(X))^2$.

Because Eq. (3) contains r, which is a nonlinear relation between $T = F(X)$ and X, it is challenging to get results—such as the existence, uniqueness, and behavior of the solution—from this equation. Instead of dealing with Eq. (3), in Refs. [8,11,12] we solved and analyzed the SS-CMC equation in each Schwarzschild spacetime region. Suppose that Σ: $(t = f(r), r)$ is a SS-CMC hypersurface in the Schwarzschild spacetime; then, $f(r)$ satisfies the following SS-CMC equation:

$$f'' + \left(\left(\frac{1}{h} - (f')^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right)f' \pm 3H\left(\frac{1}{h} - (f')^2 h\right)^{\frac{3}{2}} = 0, \tag{4}$$

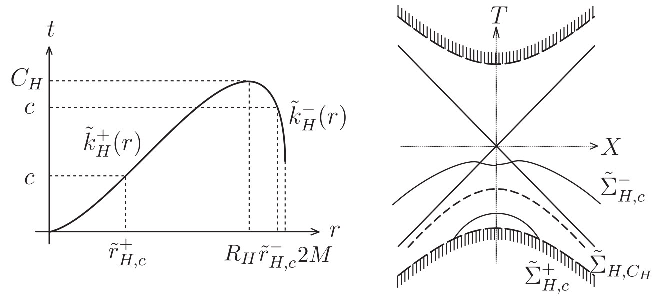

FIG. 2. Each point on the graph of $\tilde{k}_H(r)$ determines a TSS-CMC hypersurface and its T intercept in the Kruskal extension.

where $h(r) = 1 - \frac{2M}{r}$, and the spacelike condition is $\frac{1}{h} - (f')^2 h > 0$. Remark that the choice of $\pm$ signs in (4) depends on different regions and different pieces of SS-CMC hypersurfaces. Because Eq. (4) is a second-order ordinary differential equation, the solution is solved explicitly, and we can completely characterize SS-CMC hypersurfaces in the Kruskal extension through relations (2).

Here we summarize results in Refs. [11,12] regarding the construction of the TSS-CMC hypersurfaces. These results will be used for further discussions in this article. In Ref. [11], the solution of Eq. (4) in the Schwarzschild interior, which maps to the Kruskal extension $\mathtt{II'}$, is

$$f(r;H,c,\bar{c}) = \begin{cases} \int_{r_0}^{r} \frac{l(x;H,c)}{-h(x)\sqrt{l^2(x;H,c)-1}}\,\mathrm{d}x + \bar{c}, & \text{if } f'(r) > 0 \\ \int_{r_0}^{r} \frac{l(x;H,c)}{h(x)\sqrt{l^2(x;H,c)-1}}\,\mathrm{d}x + \bar{c}, & \text{if } f'(r) < 0, \end{cases}$$

where r_0 is a point in the domain of $f(r)$, $l(r;H,c) = \frac{1}{\sqrt{-h(r)}}(Hr + \frac{c}{r^2})$, and c, $\bar{c}$ are two constants of integration. Here we require $l(r;H,c) > 1$ so that the function $f(r)$ is meaningful; it is equivalent to $c > -Hr^3 + r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}}$, so it is natural to define the function

$$\tilde{k}_H(r) = -Hr^3 + r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}}$$

to analyze the domain of the solution $f(r)$.

Now we look at the case where $H \leq 0$. Given H, as Fig. 2 shows, the function $\tilde{k}_H(r)$ has a maximum value C_H at $r = R_H$, where R_H is determined by $-3HR_H^{\frac{5}{2}}(2M - R_H)^{\frac{1}{2}} = 2R_H - 3M$ (see Proposition 11 in [11]). Denote the increasing part and decreasing part of the function $\tilde{k}_H(r)$ by $\tilde{k}_H^+(r)$ and $\tilde{k}_H^-(r)$, respectively. For $c \in (0, C_H)$, the solution of $\tilde{k}_H^+(r) = c$ is denoted by $r = \tilde{r}^+_{H,c}$; then, $(0, \tilde{r}^+_{H,c}]$ is the domain of the SS-CMC solution $f(r)$. Remark that $r = \tilde{r}^+_{H,c}$ belongs to the domain of $f(r)$ because the behavior $f'(r) \sim O((\tilde{r}^+_{H,c} - r)^{-\frac{1}{2}})$ implies that $f(\tilde{r}^+_{H,c})$ is a finite value. Consider the SS-CMC hypersurface that is the union of two graphs of $t = f(r)$, where one satisfies $f'(r) > 0$ and the other satisfies $f'(r) < 0$, and two graphs are smoothly joined at the point $(t, r) = (0, \tilde{r}^+_{H,c})$. This SS-CMC hypersurface is symmetric about $t = 0$. Because $t = 0$ in the Schwarzschild interior is the T-axis in the Kruskal extension $\mathtt{II'}$, this hypersurface maps to a TSS-CMC hypersurface $\tilde{\Sigma}^+_{H,c}$ in the Kruskal extension $\mathtt{II'}$, and $\tilde{\Sigma}^+_{H,c}$ intersects the T axis at $T = -\sqrt{2M - \tilde{r}^+_{H,c}}\,e^{\frac{\tilde{r}^+_{H,c}}{4M}}$, see Fig. 2.

For $c \in (-8M^3 H, C_H)$, the solution of $\tilde{k}_H^-(r) = c$ is denoted by $r = \tilde{r}^-_{H,c}$; then, $[\tilde{r}^-_{H,c}, \infty)$ is the domain of the SS-CMC solution $f(r)$. Remark that $f(r)$ is defined as $r = 2M$ in the sense of Kruskal extension, and $\tilde{r}^-_{H,c}$ belongs to the domain because of $f'(r) \sim O((r - \tilde{r}^-_{H,c})^{-\frac{1}{2}})$. The union of graphs of functions $t = f(r)$, which are smoothly joined at the point $(t, r) = (0, \tilde{r}^-_{H,c})$, maps to a TSS-CMC hypersurface $\tilde{\Sigma}^-_{H,c}$ in the Kruskal extensions $\mathtt{I}$, $\mathtt{II}$, and $\mathtt{I'}$. Furthermore, $\tilde{\Sigma}^-_{H,c}$ intersects the T axis at $T = -\sqrt{2M - \tilde{r}^-_{H,c}}\,e^{\frac{\tilde{r}^-_{H,c}}{4M}}$.

The dotted curve between $\tilde{\Sigma}^+_{H,c}$ and $\tilde{\Sigma}^-_{H,c}$ in Fig. 2 is the TSS-CMC hypersurface $\tilde{\Sigma}_{H,C_H}$, which corresponds to the point with maximum value of $\tilde{k}_H(r)$. The hypersurface $\tilde{\Sigma}_{H,C_H}$ is a hyperbola $X^2 - T^2 = (R_H - 2M)e^{\frac{R_H}{2M}}$ with $T < 0$ in the Kruskal extension $\mathtt{II'}$, and it is a cylindrical hypersurface $r = R_H$ in the Schwarzschild interior.

From the above discussion, we establish a one-to-one correspondence from each point on the graph of $\tilde{k}_H(r)$ to a TSS-CMC hypersurface $\tilde{\Sigma}_{H,c}$.

C. The construction of TSS-CMC foliation with varied H

In order to construct a TSS-CMC hypersurface foliation with varied H in each slice, we will view H as a variable and thus consider the two-variable function

$$\tilde{k}(H, r) = -Hr^3 + r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}},$$

where $r \in [0, 2M]$ and $H \leq 0$. Here we only consider $H \leq 0$ because we will use the symmetry property to get the $H \geq 0$ part. First, we introduce the following proposition.

Proposition 1: There exists a function $y(r)$ defined on $(0, 2M]$ satisfying the following properties:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}r} \neq \frac{3y}{r} + \frac{r^{\frac{1}{2}}(-3M+r)}{(2M-r)^{\frac{1}{2}}} \\ \frac{\mathrm{d}y}{\mathrm{d}r} < 0 \text{ for all } r \in (0, 2M) \\ y(2M) = 0 \quad \text{and} \quad \lim_{r\to 0^+} y(r) = \infty. \end{cases} \tag{5}$$

Before proving Proposition 1, we briefly explain why we need the function $y(r)$; more discussions about $y(r)$ and TSS-CMC foliation are in Sec. IV. The first property of (5)

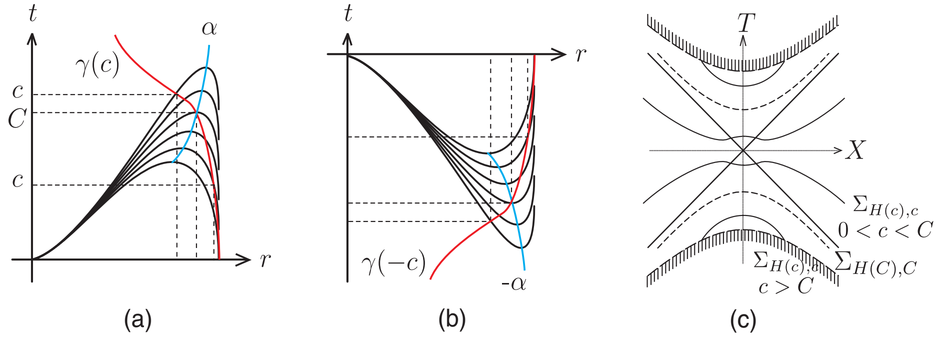

FIG. 3. The curve $\gamma(c)$ is the graph of a decreasing function and each point on $\gamma(c)$ corresponds to a TSS-CMC hypersurface $\Sigma_{H(c),c}$ in the Kruskal extension. (a) For $H \le 0$, the curve $\gamma(c)$ is a graph of a function $y(r)$ which satisfies all properties in Proposition 1. (b) For $H \ge 0$, the curve $\gamma(-c)$ is a graph of a function $y(r)$ which satisfies all properties in Proposition 1. (c) TSS-CMC hypersurfaces are shown in the Kruskal extension.

means that the graph of $y(r)$ intersects the graph of $\tilde{k}(H, r)$ with fixed H once, and this will be used to show the constructed TSS-CMC hypersurfaces are disjoint and have different mean curvatures. The property $\frac{\mathrm{d}y}{\mathrm{d}r} < 0$ implies that the mean curvature is monotonic along the TSS-CMC foliation. The property $y(2M) = 0$ shows that the hypersurface passing through the bifurcation point (the origin in the Kruskal extension) is maximal. The property $\lim_{r\to 0^+} y(r) = \infty$ shows the mean curvature will range to all real numbers.

Proof of Proposition 1.—First, we compute

$$\frac{\partial \tilde{k}}{\partial r}(H, r) = -3Hr^2 + \frac{r^{\frac{1}{2}}(3M - 2r)}{(2M - r)^{\frac{1}{2}}} = \frac{3y}{r} + \frac{r^{\frac{1}{2}}(-3M + r)}{(2M - r)^{\frac{1}{2}}}.$$

Here we replace H with y and r by the relation $y = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$ in the last equality. To find the function $y(r)$, it suffices to find a function $q(r) > 0$ such that

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}r} - \frac{3y}{r} = \frac{r^{\frac{1}{2}}(-3M+r)}{(2M-r)^{\frac{1}{2}}} - q(r) \\ y(2M) = 0. \end{cases} \tag{6}$$

When multiplying the integrating factor $\mathrm{e}^{\int -\frac{3}{r}\mathrm{d}r} = r^{-3}$ on both sides of the differential equation (6), it becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^{-3}y(r)) = \frac{-3M + r}{r^{\frac{5}{2}}(2M - r)^{\frac{1}{2}}} - \frac{q(r)}{r^3}.$$

After integration, the function $y(r)$ is solved,

$$y(r) = r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}} + r^3 \int_r^{2M} \frac{q(x)}{x^3}\mathrm{d}x.$$

Next, we calculate

$$y'(r) = \frac{r^{\frac{1}{2}}(3M - 2r)}{(2M - r)^{\frac{1}{2}}} + 3r^2 \int_r^{2M} \frac{q(x)}{x^3}\mathrm{d}x - q(r).$$

Consider the function $q(r)$, which is of the form $q(r) = Cr^{-p}$, where C and p are positive numbers to be determined. Then,

$$y'(r) = \frac{r^{\frac{1}{2}}(3M - 2r)}{(2M - r)^{\frac{1}{2}}} - C\left(\frac{3}{(p + 2)(2M)^{p+2}} + \frac{(p - 1)}{(p + 2)r^p}\right).$$

Because the function $g(r) = \frac{r^{\frac{1}{2}}(3M-2r)}{(2M-r)^{\frac{1}{2}}}$ has a global maximum value $g(r_*) = \sqrt{6\sqrt{3} - 9}M$ at $r_* = \frac{(3-\sqrt{3})M}{2}$, we can choose any value $p > 1$ and then choose the constant C large enough such that $y'(r) < 0$ for all $r \in (0, 2M)$.

Finally, we check the limit behavior,

$$\lim_{r\to 0^+} y(r) = \lim_{r\to 0^+}\left(r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}} + r^3 \int_r^{2M} \frac{C}{x^{p+3}}\mathrm{d}x\right)$$
$$= C(p + 2)\lim_{r\to 0^+}\left(\frac{1}{r^{p-1}} - \frac{r^3}{(2M)^{p+2}}\right) = \infty.$$

■

In the following paragraphs, we will use the notation $\tilde{k}_H(r)$ if we consider the function $\tilde{k}(H, r)$ with fixed H. From Proposition 1, we find a strictly decreasing function $y(r)$ such that the equation $y(r) = \tilde{k}_H(r)$ has a unique solution for every $H \le 0$. Figure 3(a) illustrates the curve γ, which is the graph of $y(r)$, and we set the curve $\gamma(c)$ with parameter c by $c = y(r)$. Because $y(r)$ is strictly decreasing, we have $r = y^{-1}(c)$, and the mean curvature can be expressed as $H(c)$ by the relation $c = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$. Thus, there is an one-to-one correspondence from each point on $\gamma(c)$ to a TSS-CMC hypersurface $\Sigma_{H(c),c}$, where $\Sigma_{H(c),c}$ intersects the T axis at $T = -\sqrt{2M - y^{-1}(c)}\mathrm{e}^{\frac{y^{-1}(c)}{4M}}$, as Fig. 3(c) shows.

In Fig. 3(a), we trace another curve α. The curve α consists of all points (t, r) satisfying $t = \max_{r\in[0,2M]}\tilde{k}_H(r)$ for every $H \le 0$. It is easy to see that the curve α is a graph of an increasing function such that the curves γ and α intersect once; we denote the intersection point by (C, R).

The point (C, R) corresponds to the hyperbola $X^2 - T^2 = (R - 2M)e^{\frac{R}{2M}}$ with $T < 0$, which is the dotted curve in Fig. 3(c).

When $c = 0$, we get $r = 2M$ and $H = 0$. The TSS-CMC hypersurface $\Sigma_{H(0),0}$ is a maximal hypersurface passing through $(T, X) = (0, 0)$ such that $\Sigma_{H(0),0}$ is $T \equiv 0$, or X axis. So far, we have constructed the TSS-CMC foliation in the region $T \leq 0$.

Next, we consider another two-variable function

$$k(H, r) = -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}},$$

where $r \in [0, 2M]$ and $H \geq 0$. The function $k(H, r)$ comes from the inequality $l(r; H, c) = \frac{1}{\sqrt{-h(r)}}(-Hr - \frac{c}{r^2}) > 1$, and it will determine the domain of a SS-CMC solution in the Schwarzschild interior that maps to the Kruskal extension $\mathtt{II}$. In fact, the functions $k(H, r)$ for $H \geq 0$ and $\tilde{k}(H, r)$ for $H \leq 0$ are symmetric about the r axis. Thus, for the construction of TSS-CMC hypersurfaces in the $T \geq 0$ part of the Kruskal extension, in Fig. 3(b), we choose the curve $\gamma(-c)$ by the reflection of the curve $\gamma(c)$ with respect to the r axis. Each point on $\gamma(-c)$ will correspond on a one-to-one manner to a TSS-CMC hypersurface $\Sigma_{H(-c),-c}$ in the Kruskal extension, and $\Sigma_{H(-c),-c}$ intersects the T axis at $T = \sqrt{2M - (-y)^{-1}(-c)}e^{\frac{(-y)^{-1}(-c)}{4M}}$, as Fig. 3(c) shows. Furthermore, hypersurfaces $\Sigma_{H(-c),-c}$ and $\Sigma_{H(c),c}$ are symmetric about the X axis.

From the above discussion, we collect TSS-CMC hypersurfaces from $\gamma(c)$ with $c \geq 0$ and from $\gamma(-c)$ with $c \geq 0$, and we use the notation $\{\Sigma_{H(c),c}\}, c \in \mathbb{R}$. Remark that when the parameter c ranges from ∞ to $-\infty$, the mean curvature H ranges from $-\infty$ to ∞. In next section, we will show that $\{\Sigma_{H(c),c}\}, c \in \mathbb{R}$, forms a TSS-CMC foliation in the Kruskal extension.

III. THE EXISTENCE OF TSS-CMC FOLIATION

Now we are ready to prove the family $\{\Sigma_{H(c),c}\}, c \in \mathbb{R}$, which we constructed in Sec. II C, foliates the Kruskal extension.

Theorem 2: There exists a family of hypersurfaces $\{\Sigma_{H(c),c}\}$, $c \in \mathbb{R}$, in the Kruskal extension satisfying the following properties:

(i) Every $\Sigma_{H(c),c}$ is a T-axisymmetric, spacelike, spherically symmetric, constant mean curvature hypersurface.

(ii) Any two hypersurfaces in $\{\Sigma_{H(c),c}\}$ are disjoint.

(iii) Every point (T', X') in the Kruskal extension belongs to $\Sigma_{H(c'),c'}$ for some $c' \in \mathbb{R}$.

(iv) When $\{\Sigma_{H(c),c}\}$ foliates the Kruskal extension from the bottom to the top, the corresponding constant mean curvature H ranges from $-\infty$ to ∞ and the parameter c ranges from ∞ to $-\infty$.

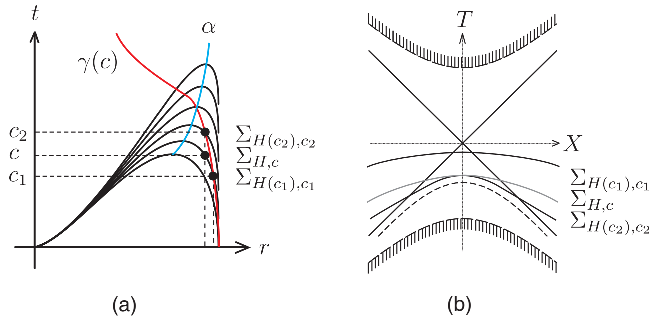

FIG. 4. $\Sigma_{H(c_1),c_1}$ and $\Sigma_{H(c_2),c_2}$ are disjoint by comparing with $\Sigma_{H,c}$.

(v) $\{\Sigma_{H(c),c}\}$ is invariant under the reflection with respect to the X axis.

From the construction in Sec. II C, because $\{\Sigma_{H(c),c}\}$ consists of TSS-CMC hypersurfaces, property (i) is true. In addition, the parameter c ranges for all $\mathbb{R}$, which implies that H ranges for all $\mathbb{R}$ as well, so property (iv) holds. Because we collect $\{\Sigma_{H(c),c}\}$ for $c \geq 0$ first and then use the X-axis reflection to get $\{\Sigma_{H(c),c}\}$ for $c \leq 0$, we state the symmetric property in (v). Next, we will prove properties (ii) and (iii), and it suffices to prove the $T \leq 0$ part because of the symmetry property (v).

Proof of property (ii).—Here we provide a proof of property (ii) for the case when TSS-CMC hypersurfaces lie between $T = 0$ and the hyperbola $X^2 - T^2 = (R - 2M)e^{\frac{R}{2M}}$ with $T < 0$; the property is similarly proved if the TSS-CMC hypersurfaces lie below the hyperbola. Given any two TSS-CMC hypersurfaces $\Sigma_{H(c_1),c_1}$ and $\Sigma_{H(c_2),c_2}$ with $c_1 < c_2$, we compare these two hypersurfaces with $\Sigma_{H,c}$, which is a TSS-CMC hypersurface with mean curvature $H = H(c_1)$ and which has the same T intercept as $\Sigma_{H(c_2),c_2}$, see Fig. 4(a). Because $H = H(c_1)$, two points in the t-r plane corresponding to $\Sigma_{H,c}$ and $\Sigma_{H(c_1),c_1}$ lie on the same function $\tilde{k}_{H(c_1)}(r)$. Because $\Sigma_{H,c}$ and $\Sigma_{H(c_2),c_2}$ have the same T intercept, two points in the t-r plane corresponding to $\Sigma_{H,c}$ and $\Sigma_{H(c_2),c_2}$ have the same r value.

Because $c_1 < c_2$, we have $H(c_2) < H(c_1) = H$, which implies T values of $\Sigma_{H(c_2),c_2}\colon (T_{H(c_2),c_2}(X), X)$ and $\Sigma_{H,c}\colon (T_{H,c}(X), X)$ in the Kruskal extension satisfy

$$T_{H(c_2),c_2}(X) < T_{H,c}(X) \quad \text{for all } X \neq 0. \tag{7}$$

The inequality (7) holds because of $\frac{\partial f'}{\partial H} > 0$ for all $X \geq 0$ in the Schwarzschild spacetime. Furthermore, in Ref. [8], we proved that for every fixed $H \in \mathbb{R}$, the curve formed by the union of the graphs of $\tilde{k}_H(r)$ and $k_H(r)$ corresponds to a TSS-CMC hypersurface family $\{\Sigma_H\}$, and $\{\Sigma_H\}$ foliates the Kruskal extension. Because both $\Sigma_{H,c}\colon (T_{H,c}(X), X)$ and $\Sigma_{H(c_1),c_1}\colon (T_{H(c_1),c_1}(X), X)$ belong to $\{\Sigma_H\}$, and their T intercepts satisfy $T_{H,c}(0) < T_{H(c_1),c_1}(0)$, we have $T_{H,c}(X) < T_{H(c_1),c_1}(X)$. Therefore, $T_{H(c_2),c_2}(X) < T_{H(c_1),c_1}(X)$ for all

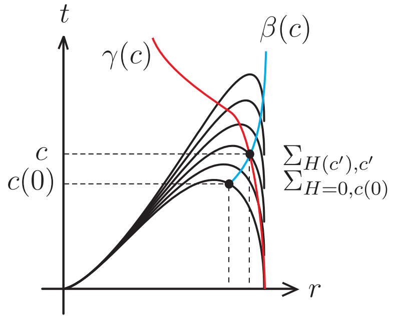

FIG. 5. Proof of the family $\{\Sigma_{H(c),c}\}$ covering the Kruskal extension.

X, and, hence, $\Sigma_{H(c_1),c_1}$ and $\Sigma_{H(c_2),c_2}$ are disjoint; see Fig. 4(b). ■

Proof of property (iii).—Recall that in Theorems 3 and 7 of [8], we proved the existence and uniqueness of the Dirichlet problem for the TSS-CMC equation with symmetric boundary data. In other words, for any fixed $H \in \mathbb{R}$ and given (T', X') in the Kruskal extension, there exists a unique TSS-CMC hypersurface, denoted by $\Sigma_{H,c(H)}$, passing through (T', X') and $(T', -X')$.

Here we prove the case when (T', X') lies between $T = 0$ and the hyperbola $X^2 - T^2 = (R - 2M)e^{\frac{R}{2M}}$ with $T < 0$; it is similarly proved if (T', X') lies in other regions, see Fig. 5. When $H = 0$, there exists a unique value $c(0)$ such that $(T', X') \in \Sigma_{H=0,c(0)}$. For every $H \leq 0$, we can find a point on the graph of $\tilde{k}_H^-(r)$ corresponding to a TSS-CMC hypersurface $\Sigma_{H,c(H)}$ passing through (T', X'). Set $\beta(c)$ be all such points. We know that $\beta(c)$ is a continuous curve because Theorem 6 in [8] shows these solutions are continuously varied with the mean curvature. When $H \to -\infty$, we have $\tilde{k}_H^-(r) \to \infty$, so the t value of $\beta(c)$ tends to infinity and r value of $\beta(c)$ tends to $2M$. By the intermediate value theorem, two curves $\gamma(c)$ and $\beta(c)$ must intersect at some point c', and, hence, $(T', X') \in \Sigma_{H(c'),c'}$. ■

IV. DISCUSSION

We consider a family of TSS-CMC hypersurfaces in the Kruskal extension. The mean curvature is constant on each slice but changes from slice to slice. We first construct this TSS-CMC hypersurface family with one more symmetry, which we call the X-axis symmetry. That is, after getting TSS-CMC hypersurfaces in the region $T \leq 0$, we use the reflection with respect to the X axis to derive TSS-CMC hypersurfaces in the region $T \geq 0$. Based on the result of TSS-CMC hypersurface foliation with fixed mean curvature in [8], we prove these TSS-CMC hypersurfaces foliate the Kruskal extension.

Two functions $\tilde{k}(H, r)$ and $k(H, r)$ play important roles in this foliation argument. Each point on the graphs of $\tilde{k}(H, r)$ and $k(H, r)$ will correspond in a one-to-one manner to a TSS-CMC hypersurface $\Sigma_{H,c}$, where H, r, c satisfy relations $c = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$ or $c = -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$, respectively. The argument in Sec. II C indicates that if we find a curve $\gamma(c)$ that intersects every $\tilde{k}_H(r)$ and $k_H(r)$ exactly once, and $\gamma(c)$ is the union of two monotonic functions, then we can prove the TSS-CMC foliation property.

In Proposition 1, Eq. (5), there are three conditions the curve $\gamma(c)$ should be satisfied. The first condition $\frac{dy}{dr} \neq \frac{3y}{r} + \frac{r^{\frac{1}{2}}(-3M+r)}{(2M-r)^{\frac{1}{2}}}$ implies that the curve $\gamma(c)$ intersects every $\tilde{k}_H(r)$ exactly once. The second condition $\frac{dy}{dr} < 0$ for all $r \in (0, 2M)$ indicates that c is decreasing and H is increasing along TSS-CMC hypersurfaces. The condition $\lim_{r\to 0^+} y(r) = \infty$ coupled with symmetry $\lim_{r\to 0^+} -y(r) = -\infty$ states that mean curvatures of TSS-CMC hypersurfaces range from $-\infty$ to ∞. The condition $y(2M) = 0$ will impose that the TSS-CMC hypersurface passing through the bifurcation sphere $(T, X) = (0, 0)$ be the maximal hypersurface, which is $T \equiv 0$. In this case, we can use the X-axis symmetry to get the whole TSS-CMC family $\{\Sigma_{H,c}\}$.

Our first remark is that the existence of $\gamma(c)$ is not unique. This is because $\frac{dy}{dr} \neq \frac{3y}{r} + \frac{r^{\frac{1}{2}}(-3M+r)}{(2M-r)^{\frac{1}{2}}}$ is an open condition. Therefore, we can find many different TSS-CMC foliations.

Our second remark is that the condition $y(2M) = 0$ is more flexible. In fact, we can consider a more general setting to solve the function $y(r)$ in Proposition 1 by satisfying (5) but replacing the condition $y(2M) = 0$ with $y(2M) = A$, where $A \in \mathbb{R}$. There still exists a function in the general setting, which we denote by $y_A(r)$. Let the curve $\gamma_A(c)$ be the graph of $y_A(r)$ with parameter $c = y_A(r)$. Thus, we have derived TSS-CMC hypersurfaces below $\Sigma_{H(A),A}$. How do we get TSS-CMC hypersurfaces above $\Sigma_{H(A),A}$? Recall that these TSS-CMC hypersurfaces are determined by the function $k(H, r)$. Notice that $k(H, r) = -\tilde{k}(-H, r)$, so we consider the curve $\gamma_{-A}(c) + 2A$, which is the curve by moving $\gamma_{-A}(c)$ along the t direction by $2A$. Then, $\gamma_A(c)$ and $\gamma_{-A}(c) + 2A$ are joined at $r = 2M$ and $\gamma_A(c) \cup (\gamma_{-A}(c) + 2A)$ intersects every $\tilde{k}_H(r)$ and $k_H(r)$ exactly once. Hence, the corresponding TSS-CMC hypersurface family $\{\Sigma_{H(c),c}\}$, $c \in \mathbb{R}$ foliates the Kruskal extension, but it is not X axisymmetric.

Next, every TSS-CMC foliation can be changed to be a SS-CMC foliation, without the T-axisymmetric property, by the Lorentzian isometry. All of the foliations have the property that mean curvatures range from $-\infty$ to ∞. This phenomenon is more close to the definition of the CMC time function. However, only TSS-CMC hypersurfaces across the Kruskal regions I and I′ are Cauchy hypersurfaces.

Finally, our TSS-CMC foliation construction extends the results and discussions in Malec and Ó Murchadha's paper

[9]. They considered TSS-CMC foliations where the mean curvature H and the TSS-CMC hypersurface parameter c are proportional, that is, $c = -8M^3H$; then, there is a family of TSS-CMC hypersurfaces such that H ranges from minus infinity to plus infinity, but all hypersurfaces intersect at the origin in the Kruskal extension. In order to break the phenomenon of intersection, we consider the TSS-CMC hypersurface family with a nonlinear relation between c and H, and where both variables range for all real numbers. This consideration will fulfill the TSS-CMC foliation with varied mean curvature in each slice in the Kruskal extension.

In Ref. [9], they computed the formula of the lapse function of the CMC foliation [Eq. (8) in [9]] from the Einstein equation. Remark that the relations between notations c, H in this paper and C, K in paper [9] are $H = \frac{K}{3}$ and $c = -C$. If we set a new parameter t such that $c = c(t)$ and $H = H(t)$ and require the lapse function $N(r,t) \to 1$ when $r \to \infty$, then $\dot{c} = c'(t)$ and $\dot{H} = H'(t)$ will satisfy $\dot{c}X + \dot{H}Y = 1$, which is Eq. (34) in [9], where

$$X = X(c,H,r) = -\int_r^\infty \frac{6H^2x^2 + 12c^2}{k(x)(2M + 2H^2x^3 - 2Hc - \frac{4c^2}{x^3})^2}\mathrm{d}x,$$

$$Y = Y(c,H,r) = -\int_r^\infty \frac{x^2(6M - 6Hc - \frac{24c^2}{x^3})}{k(x)(2M + 2H^2x^3 - 2Hc - \frac{4c^2}{x^3})^2}\mathrm{d}x,$$

$$k^2(r) = 1 - \frac{2M}{r} + \left(Hr + \frac{c}{r^2}\right)^2.$$

In Theorem 2, once we get a TSS-CMC foliation $\{\Sigma_{H(c),c}\}$ in the Kruskal extension, we can reparameterize the family by $\{\Sigma_{H(t),c(t)}\} = \{\Sigma_{H(c(t)),c(t)}\}$ such that the corresponding lapse function $N(r,t) \to 1$ as $r \to \infty$ in parameter t; that is, $\dot{c}X + \dot{H}Y = 1$. This is because $H = H(c)$, $\dot{H} = \frac{\mathrm{d}H}{\mathrm{d}c}\dot{c}$ and $\frac{\mathrm{d}H}{\mathrm{d}c}$ is given; then, $\dot{c}X + \dot{H}Y = 1$ implies $\dot{c}$ will satisfy $\dot{c} = \frac{1}{X + \frac{\mathrm{d}H}{\mathrm{d}c}Y}$, which is solvable due to the existence of the first-order differential equation.

ACKNOWLEDGMENTS

The author would like to thank Yng-Ing Lee, Mao-Pei Tsui, Mu-Tao Wang, and referees for their interest in and discussions about the topic of this paper. The author is supported by MOST, Grant No. 103-2115-M-002-013-MY3.

[1] Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation, Vol. 1* (Plenum, New York, 1980), pp. 99–172.

[2] A. Lichnerowicz, L'intégration des équations de la gravitation relativiste et le problème des n corps, J. Math. Pures Appl. **23**, 37 (1944).

[3] L. Andersson and V. Moncrief, Elliptic-hyperbolic systems and the Einstein equations, Ann. Inst. Henri Poincaré **4**, 1 (2003).

[4] Y. Choquet-Bruhat and T. Ruggeri, Hyperbolicity of the $3+1$ system of Einstein equations, C. R. Seances Acad. Sci. Ser. 1 **294**, 425 (1982).

[5] J. W. York, Jr., Gravitational Degrees of Freedom and the Initial-Value Problem, Phys. Rev. Lett. **26**, 1656 (1971).

[6] L. Andersson, T. Barbot, F. Beguin, and A. Zeghib, Cosmological time versus CMC time in spacetimes of constant curvature, Asian J. Math. **16**, 37 (2012).

[7] E. Malec and N. Ó Murchadha, Constant mean curvature slices in the extended Schwarzschild solution and the collapse of the lapse, Phys. Rev. D **68**, 124019 (2003).

[8] K.-W. Lee, Dirichlet problem for the constant mean curvature equation and CMC foliation in the extended Schwarzschild spacetime, Classical Quantum Gravity **33**, 185001 (2016).

[9] E. Malec and N. Ó Murchadha, General spherically symmetric constant mean curvature foliations of the Schwarzschild solution, Phys. Rev. D **80**, 024017 (2009).

[10] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[11] K.-W. Lee and Y.-I. Lee, Spacelike spherically symmetric CMC hypersurfaces in Schwarzschild spacetimes (I): Construction, arXiv:1111.2679v2.

[12] K.-W. Lee and Y.-I. Lee, Spacelike spherically symmetric CMC foliation in the extended Schwarzschild spacetime, Ann. Inst. Henri Poincaré, **17**, 1477 (2016).