5D Lovelock gravity: New exact solutions with torsion

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Five-dimensional Lovelock gravity is investigated in the first order formalism. A new class of exact solutions is constructed: the Bañados, Teitelboim, Zanelli black rings with and without torsion. We show that our solution with torsion exists in a different sector of the Lovelock gravity, as compared to the Lovelock Chern-Simons sector or the one investigated by Canfora *et al*. The conserved charges of the solutions are found using Nester's formula, and the results are confirmed by the canonical method. We show that the theory linearized around the background with torsion possesses two additional degrees of freedom with respect to general relativity.

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I. INTRODUCTION

The general theory of relativity introduced a revolution in our understanding of space-time and gravity, the influence of which on modern physics can hardly be emphasized enough—almost all present investigations in high-energy physics are, in certain way, related to it. On one hand, the general theory of relativity has been very successful in explaining experimental results, but on the other, it produced a lot of problems for physicists to solve. The first of them is the problem of singularities, appearing quite often in gravitational solutions; there are theorems which show that singularities must appear under certain physically reasonable assumptions [1]. This situation inspired research in the direction of alternative theories of gravity, with an idea of finding a singularity free theory that reproduces experimental results equally as well as general relativity.

The second problem is quantization of tje general theory of relativity. The inability to quantize general relativity in a standard way, like Yang-Mills theories, motivated physicists to search for alternatives, on one side for a different quantization procedure (loop quantum gravity) and on the other for modifications of the original theory (extra dimensions, supersymmetry, string theory, alternative theories of gravity) [2–5]. In this paper, we shall focus on an alternative theory of gravity with one extra dimension—Lovelock gravity in five dimensions (5D).

Lovelock gravity is one of many generalizations of general relativity, physically appealing because of its similarity to the former. It possesses equations of motion which are the second order differential equations; it is ghost free; etc. But beyond this, most of its basic properties are not well known, and as the old saying says, "The devil is in the details." First, not many solutions are known, and those constructed usually are torsionless or belong to some special point in the parameter space [6–10]. Second, symmetries and local degrees of freedom of the theory are not known for the generic choice of parameters but only for the special case of Lovelock Chern-Simons gravity [11].

In this paper, we shall introduce new solutions with(out) torsion within Lovelock gravity in 5D by using the first order formulation. The most interesting of them are the Bañados, Teitelboim, Zanelli (BTZ) black rings with(out) torsion, the properties of which can be analyzed by using the canonical formalism. The canonical analysis is a powerful tool for studying gauge theories, but it is not limited solely to them. It gives a well-defined procedure for determining symmetries of a theory, construction of the symmetry generators, and for counting the number of local degrees of freedom. Applying the canonical analysis to a theory is extremely rewarding because of the already mentioned results it gives. Note, in particular, that the most reliable approach to conserved charges in gravity is based on the canonical analysis [12,13]. The main aspect of this approach consists in demanding the canonical generators to have well-defined functional derivatives. For a given asymptotic behavior of the fields, this condition usually requires the form of the generators to be improved by adding suitable surface terms.

The paper is organized as follows. Section II contains a short review of the Poincaré gauge theory of gravity and Lovelock gravity. Section III is devoted to the new solutions of 5D Lovelock gravity—the BTZ black rings with(out) torsion. The conserved charges for these solutions are computed by using Nester formula [14]. In Sec. IV, we construct the canonical generator of gauge transformations, local translations, and Lorentz rotations and compute the canonical conserved charges for the solutions constructed in Sec. III, confirming the results obtained in Sec. III. In Sec. V, we investigate the canonical structure of the theory linearized around the solution with torsion and conclude that in this sector the theory exhibits additional degrees of freedom.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, and the Greek

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indices refer to the coordinate frame; the first letters of both alphabets $(a, b, c, \ldots; \alpha, \beta, \gamma, \ldots)$ run over $1, 2, \ldots D-1$, and the middle alphabet letters $(i, j, k, \ldots; \mu, \nu, \lambda, \ldots)$ run over $0, 1, 2, \ldots D-1$; the signature of space-time is $\eta = (+, -, \ldots, -)$; and the totally antisymmetric tensor $\varepsilon^{i_1 i_2 \ldots i_D}$ and the related tensor density $\varepsilon^{\mu_1 \mu_2 \ldots \mu_D}$ are both normalized so that $\varepsilon^{01\ldots D-1} = 1$. The symbol $\wedge$ of the exterior (wedge) product between forms is omitted for simplicity.

II. LOVELOCK GRAVITY

A. PGT in brief

The basic gravitational variables in poincaré gauge theory (PGT) are the vielbein e^i and the Lorentz connection $\omega^{ij} = -\omega^{ji}$ (1-forms). The field strengths corresponding to the gauge potentials e^i and ω^{ij} are the torsion T^i and the curvature R^{ij} (2-forms): $T^i = de^i + \omega^i{}_m \wedge e^m$ $R^{ij} = d\omega^{ij} + \omega^i{}_m \wedge \omega^{mj}$. Gauge symmetries of the theory are local translations and local Lorentz rotations, parametrized by ξ^μ and ε^{ij}.

In local coordinates x^μ, we can expand the vielbein and the connection 1-forms as $e^i = e^i{}_\mu dx^\mu$, $\omega^i = \omega^i{}_\mu dx^\mu$. Gauge transformation laws have the form

$$\begin{aligned} \delta_0 e^i{}_\mu &= \varepsilon^{ij} e_{j\mu} - (\partial_\mu \xi^\rho) e^i{}_\rho - \xi^\rho \partial_\rho e^i{}_\mu =: \delta_{\mathrm{PGT}} e^i{}_\mu, \\ \delta_0 \omega^{ij}{}_\mu &= \nabla_\mu \varepsilon^{ij} - (\partial_\mu \xi^\rho) \omega^{ij}{}_\rho - \xi^\rho \partial_\rho \omega^{ij}{}_\mu =: \delta_{\mathrm{PGT}} \omega^{ij}{}_\mu, \end{aligned} \tag{2.1}$$

and the field strengths are given as

$$\begin{aligned} T^i &= \nabla e^i \equiv de^i + \omega^{ij} \wedge e_j = \frac{1}{2} T^i{}_{\mu\nu} dx^\mu \wedge dx^\nu, \\ R^{ij} &= d\omega^{ij} + \omega^{ik} \wedge \omega_k{}^j = \frac{1}{2} R^{ij}{}_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{2.2}$$

where $\nabla = dx^\mu \nabla_\mu$ is the covariant derivative.

To clarify the geometric meaning of the above structure, we introduce the metric tensor as a specific, bilinear combination of the vielbeins,

$$\begin{aligned} g &= \eta_{ij} e^i \otimes e^j = g_{\mu\nu} dx^\mu \otimes dx^\nu, \\ g_{\mu\nu} &= \eta_{ij} e^i{}_\mu e^j{}_\nu, \quad \eta_{ij} = (+, -, -, -, -). \end{aligned}$$

Although the metric and connection are in general independent dynamical/geometric variables, the antisymmetry of ω^{ij} in PGT is equivalent to the so-called *metricity condition*, $\nabla g = 0$. The geometry of which the connection is restricted by the metricity condition (metric-compatible connection) is called *Riemann-Cartan geometry*. Thus, PGT has the geometric structure of Riemann-Cartan space.

The connection ω^{ij} determines the parallel transport in the local Lorentz basis. Being a true geometric operation, parallel transport is independent of the basis. This property is incorporated into PGT via the so-called *vielbein postulate*, the vanishing of the total covariant derivative of $e^i{}_\mu$,

$$D_\mu(\omega + \Gamma) e^i{}_\nu \coloneqq \partial_\mu e^i{}_\nu + \omega^{ij}{}_\mu e_{j\nu} - \Gamma^\rho{}_{\nu\mu} e^i{}_\rho = 0,$$

where $\Gamma^\rho{}_{\nu\mu}$ is the affine connection and the torsion is defined by $T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}$. The previous relation implies the identity

$$\omega_{ijk} = \Delta_{ijk} + K_{ijk}, \tag{2.3}$$

where Δ is Riemannian (Levi-Civitá) connection and $K_{ijk} = -\frac{1}{2}(T_{ijk} - T_{kij} + T_{jki})$ is the contortion. Latin indices are changed into Greek and vice versa by means of vielbeins (and its inverse). Namely, $X^i = e^i{}_\mu X_\mu$ and $X^\mu = e_i{}^\mu X^i$. For details, see Ref. [13].

B. Lovelock action and equations of motion

Lovelock gravity can also be considered in the framework of PGT. Dimensionally continued Euler density L_p in D dimensions is defined as

$$L_p = \varepsilon_{i_1 i_2 \ldots i_D} R^{i_1 i_2} \ldots R^{i_{2p-1} i_{2p}} e^{i_{2p+1}} \ldots e^{i_D}, \tag{2.4}$$

where p is the number of curvature tensors in Euler density. In the previous relation, we omitted the wedge product for simplicity. The general form of the Lovelock gravity Lagrangian [15] in 5D is a linear combination of all dimensionally continued Euler densities in five dimensions,

$$I = \frac{\alpha_0}{5} I_0 + \frac{\alpha_1}{3} I_1 + \alpha_2 I_2, \tag{2.5a}$$

where

$$\begin{aligned} I_0 &= \int \varepsilon_{ijkln} e^i e^j e^k e^l e^n, \\ I_1 &= \int \varepsilon_{ijkln} R^{ij} e^k e^l e^n, \\ I_2 &= \int \varepsilon_{ijkln} R^{ij} R^{kl} e^n. \end{aligned} \tag{2.5b}$$

C. Field equations

Variation of the action with respect to vielbein e^i and connection ω^{ij} yields the gravitational field equations:

$$\varepsilon_{ijkln}(\alpha_0 e^j e^k e^l e^n + \alpha_1 R^{jk} e^l e^n + \alpha_2 R^{jk} R^{ln}) = 0, \tag{2.6a}$$

$$\varepsilon_{ijkln}(\alpha_1 e^k e^l + 2\alpha_2 R^{kl}) T^n = 0. \tag{2.6b}$$

Let us note that in the generic case the field equations (2.6) imply that torsion can be nonvanishing.

For later convenience, let us present the tensor form of the field equations,

$$\varepsilon^{\mu\nu\rho\sigma\tau}_{ijkln}\left(\alpha_0 e^j{}_\nu e^k{}_\rho e^l{}_\sigma e^n{}_\tau + \frac{1}{2}\alpha_1 R^{jk}{}_{\nu\rho} e^l{}_\sigma e^n{}_\tau + \frac{1}{4}\alpha_2 R^{jk}{}_{\nu\rho} R^{ln}{}_{\sigma\tau}\right) = 0, \tag{2.7a}$$

$$\varepsilon^{\mu\nu\rho\sigma\tau}_{ijkln}(\alpha_1 e^k{}_\nu e^l{}_\rho + \alpha_2 R^{kl}{}_{\nu\rho}) T^n{}_{\sigma\tau} = 0, \tag{2.7b}$$

where $\varepsilon^{\mu\nu\rho\sigma\tau}_{ijkln} \coloneqq \varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon_{ijkln}$.

D. Consequences of field equations

If we take covariant derivative of (2.6a), make use of the Bianchi identities, and multiply (2.6b) with e^j, we get the following system:

$$\varepsilon_{ijkln}(2\alpha_0 e^j e^k e^l + \alpha_1 R^{jk} e^l) T^n = 0,$$
$$\varepsilon_{ijkln}(\alpha_1 e^j e^k e^l + 2\alpha_2 R^{jk} e^l) T^n = 0.$$

In the case $4\alpha_0\alpha_2 - \alpha_1^2 \neq 0$, the previous set of equations reduces to the following conditions,

$$v_i \coloneqq T^j{}_{ji} = 0, \tag{2.8a}$$

$$R^{jk}{}_{ir} T^r{}_{jk} - 2\mathrm{Ric}^j{}_k T^k{}_{ij} = 0, \tag{2.8b}$$

where $\mathrm{Ric}^j{}_k \coloneqq R^{jl}{}_{kl}$ is the Ricci tensor.

Therefore, in the generic case, torsion is *traceless*, and the second irreducible component of torsion ${}^{(2)}T_i$ vanishes. For details on irreducible decomposition of torsion and curvature in PGT, see Ref. [16]. Let us note that the condition $4\alpha_0\alpha_2 - \alpha_1^2 \neq 0$ is violated in the case of Lovelock Chern-Simons gravity.

E. Maximally symmetric solution

The field equation admits the existence of the maximally symmetric Riemannian solution (maximally symmetric Riemannian background) defined by

$$\bar{R}^{ij} = -\Lambda e^i e^j, \qquad \bar{T}^i = 0, \tag{2.9}$$

where Λ is the effective cosmological constant iff

$$\alpha_0 - \alpha_1\Lambda + \alpha_2\Lambda^2 = 0. \tag{2.10}$$

This equation can be solved for Λ:

$$\Lambda_\pm = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}}{2\alpha_2}. \tag{2.11}$$

The solution is unique for $\alpha_1^2 - 4\alpha_0\alpha_2 = 0$, which is the case in Lovelock Chern-Simons gravity.

Let us note that in terms of $\Lambda_\pm$ equations of motion (2.6) take an elegant form:

$$\varepsilon_{ijkln}(R^{jk} + \Lambda_+ e^j e^k)(R^{ln} + \Lambda_- e^l e^n) = 0, \tag{2.12a}$$

$$\varepsilon_{ijkln}\left(R^{kl} + \frac{\Lambda_+ + \Lambda_-}{2} e^k e^l\right) T^n = 0. \tag{2.12b}$$

In obtaining these equations, we assumed that $\alpha_2 \neq 0$, and this condition will be used in the rest of the paper, because for $\alpha_2 = 0$ the theory reduces to general relativity.

III. NEW CLASS OF SOLUTIONS

The search for a new class of solutions is inspired by Canfora *et al.* [17], who found a solution of the type $\mathrm{AdS}_2 \times S_3$ when the coupling constants satisfy the relation

$$\alpha_1^2 = 12\alpha_0\alpha_2, \tag{3.1}$$

which is different from the one satisfied in Lovelock Chern-Simons gravity. We shall now construct another class of solutions of the "complementary" type $\Sigma_3 \times \Gamma_2$, where Σ_3 and Γ_2 are three- and two-dimensional manifolds, determined by solving the equations of motion. We start from the following anzatz for curvature,

$$R^{ab} = A e^a e^b,$$
$$R^{3a} = R^{4a} = 0,$$
$$R^{34} = B e^3 e^4, \tag{3.2}$$

and torsion,

$$T^a = p\varepsilon^{abc} e_b e_c,$$
$$T^3 = T^4 = 0. \tag{3.3}$$

In the anzatz, we used the notation $a, b, c, \ldots \in \{0, 1, 2\}$ and $\varepsilon^{abc} \coloneqq \varepsilon^{abc34}$, and A, B, and p are some functions restricted by the equations of motion. Note that torsion is totally antisymmetric, and thus only the third irreducible component ${}^{(3)}T^i$ is nonvanishing; see Ref. [16]. Let us now check whether the anzatz solves the equations of motion (2.12). From (2.12b), we obtain

$$\left(B + \frac{\Lambda_- + \Lambda_+}{2}\right) p = 0.$$

Thus, one can have a vanishing torsion for $p = 0$ or a nonvanishing torsion for

$$B = -\frac{\Lambda_- + \Lambda_+}{2}. \tag{3.4}$$

From (2.12a), we obtain

$$A(\Lambda_- + \Lambda_+) + 2\Lambda_-\Lambda_+ = 0, \quad (3.5)$$

$$4\Lambda_-\Lambda_+ + (A+B)(\Lambda_- + \Lambda_+) + 2AB = 0. \quad (3.6)$$

If $\Lambda_- + \Lambda_+ = 0$, which is equivalent to $\alpha_1 = 0$, Eq. (3.5) implies $\alpha_0 = 0$, whereas A remains undetermined; otherwise, for $\alpha_1 \neq 0$, we have

$$A = -\frac{2\Lambda_-\Lambda_+}{\Lambda_- + \Lambda +}. \quad (3.7)$$

Let us first analyze the case with nonvanishing torsion and $\alpha_1 \neq 0$, when A and B are both determined. By combining Eqs. (3.4), (3.5), and (3.6) and using Vieta's formulas, $\Lambda_- + \Lambda_+ = \frac{\alpha_1}{\alpha_2}$ and $\Lambda_-\Lambda_+ = \frac{\alpha_0}{\alpha_2}$, we obtain that the solution exists in the sector:

$$\alpha_1^2 = 8\alpha_0\alpha_2. \quad (3.8)$$

This sector is different from the one in Ref. [17], and the above solution is the first one in this sector. Using Eqs. (3.4), (3.10), and (3.8), we obtain

$$A = \frac{B}{2}. \quad (3.9)$$

Now, we turn to the solution with vanishing torsion and $\alpha_1 \neq 0$. In this case, A is determined, and B is arbitrary, which can be used to insure the validity of (3.6), which takes the form

$$2\frac{\alpha_0}{\alpha_2} + B\left(\frac{\alpha_1}{\alpha_2} - 4\frac{\alpha_0}{\alpha_1}\right) = 0. \quad (3.10)$$

We see that if $\alpha_1^2 - 4\alpha_0\alpha_2 = 0$, which is the Lovelock Chern-Simons gravity, for the validity of (3.10), one must have $\alpha_0 = 0$. These two conditions imply $\alpha_1 = 0$, which is in contradiction with our assumption; hence, the solution does not exist in the Lovelock Chern-Simons case. If $\alpha_1^2 - 4\alpha_0\alpha_2 \neq 0$ and $\alpha_1 \neq 0$ (recall that we are not interested in general relativity, so $\alpha_2 \neq 0$ also), we can choose any value of parameters obeying this conditions and get a solution. So, this class of solutions exists generically i.e. for almost any choice of parameters.

For clarity of the exposure, we devote next few sections to the most interesting solutions which belong to the class derived in this section.

A. BTZ black ring with torsion

For this case, the curvature takes the following form,

$$R^{ab} = qe^a e^b,$$
$$R^{3a} = R^{4a} = 0,$$
$$R^{34} = -\frac{1}{r_0^2} e^3 e^4, \quad (3.11)$$

while the torsion is given by

$$T^a = p\varepsilon^{abc} e_b e_c,$$
$$T^3 = T^4 = 0. \quad (3.12)$$

The Bianchi identity implies that p is *constant*, and the Riemannian curvature reads

$$\tilde{R}^{ab} = \left(q + \frac{p^2}{4}\right) e^a e^b,$$
$$\tilde{R}^{3a} = \tilde{R}^{4a} = 0,$$
$$\tilde{R}^{34} = -\frac{1}{r_0^2} e^3 e^4. \quad (3.13)$$

Therefore, we can introduce the AdS_3 radius ℓ as

$$\frac{1}{\ell^2} := q + \frac{p^2}{4}.$$

Identity (3.9) implies the following relation:

$$\frac{1}{\ell^2} = -\frac{1}{2r_0^2} + \frac{p^2}{4}. \quad (3.14)$$

In the AdS_3 sector, the anzatz for curvature and torsion is solved by the AdS_3 solution with torsion as well as by the BTZ black hole [18] with torsion. In the latter, physically more appealing case, the 5D vielbein reads

$$e^0 = Ndt, \quad e^1 = N^{-1}dr, \quad e^2 = r(d\varphi + N_\varphi dt),$$
$$e^3 = r_0 d\theta, \quad e^4 = r_0 \sin\theta d\chi, \quad (3.15a)$$

where

$$N^2 = -2m + \frac{r^2}{\ell^2} + \frac{j^2}{r^2}, \qquad N_\varphi = \frac{j}{r^2},$$

where m and j are (dimensionless) parameters. The Cartan connection is given by

$$\omega^{ab} = \tilde{\omega}^{ab} - \varepsilon^{abc}\frac{p}{2}e_c,$$
$$\tilde{\omega}^{01} = -\frac{r}{\ell^2}dt - \frac{j}{r}d\varphi,$$
$$\tilde{\omega}^{12} = Nd\varphi,$$
$$\tilde{\omega}^{20} = N^{-1}\frac{j}{r^2}dr,$$
$$\omega^{34} = \tilde{\omega}^{34} = -\cos\theta d\chi, \quad (3.15b)$$

where $\tilde{\omega}^{ij}$ is the Riemannian connection. Let us note that the coordinate ranges are

$$-\infty < t < +\infty, \quad 0 \le r < +\infty, \quad 0 \le \varphi \le 2\pi,$$
$$0 \le \theta \le \pi, \quad 0 \le \chi \le 2\pi.$$

1. Killing vectors

The maximal number of Killing vectors of the solution with field strengths (3.11), (3.12), and (3.13) is $9 = 6 + 3$, since the AdS_3 solution with(out) torsion has six Killing vectors; see Ref. [19]. The solution (3.15) has *five* Killing vectors, since the BTZ solution possesses two Killing vectors. They are given by

$$\xi^{(1)} = \ell \frac{\partial}{\partial t}, \qquad \xi^{(2)} = \frac{\partial}{\partial \varphi}, \qquad \xi^{(3)} = \frac{\partial}{\partial \chi},$$
$$\xi^{(4)} = \sin\chi \frac{\partial}{\partial\theta} + \cot\theta\cos\chi\frac{\partial}{\partial\chi},$$
$$\xi^{(5)} = \cos\chi \frac{\partial}{\partial\theta} - \cot\theta\sin\chi\frac{\partial}{\partial\chi}. \tag{3.16}$$

B. Riemannian BTZ ring

For this case, the curvature (Riemannian) takes the following form,

$$R^{ab} = \frac{1}{\ell^2} e^a e^b,$$
$$R^{3a} = R^{4a} = 0,$$
$$R^{34} = -\frac{1}{r_0^2} e^3 e^4, \tag{3.17}$$

while the torsion equals zero, $T^i = 0$.

Let us note that since torsion is zero there are no further constraints on B, so we can chose $B = -\frac{1}{r_0^2}$. In terms of the action constants, we get

$$\frac{1}{\ell^2} = -\frac{2\alpha_0}{\alpha_1}, \qquad \frac{1}{r_0^2} = \frac{2\alpha_0\alpha_1}{\alpha_1^2 - 4\alpha_0\alpha_2}. \tag{3.18}$$

The solution exists provided that $\alpha_0\alpha_1 < 0$ and $\alpha_1^2 - 4\alpha_0\alpha_2 < 0$. Let us note this solution does not solve equations of motion in Lovelock Chern-Simons gravity.

The vielbein fields and connection take the same form as in (3.15) with $p = 0$, while Killing vectors are identical and given by (3.16).

C. Conserved charges

In order to compute conserved charges, we shall make use of Nester formula. Let us denote the difference between any variable X and its reference value $\bar{X}$ by $\Delta X = X - \bar{X}$. In 5D, the boundary term B is a 3-form. With a suitable set of boundary conditions for the fields, the proper boundary term reads [14]

$$B = (\xi\rfloor b^i)\Delta\tau_i + \Delta b^i(\xi\rfloor\bar{\tau}_i) + \frac{1}{2}(\xi\rfloor\omega^i{}_j)\Delta\rho_i{}^j + \frac{1}{2}\Delta\omega^i{}_j(\xi\rfloor\bar{\rho}_i{}^j), \tag{3.19}$$

where ξ is an asymptotically Killing vector, while τ_i and ρ_{ij} are covariant momenta corresponding to torsion and curvature, respectively. The covariant momenta for the Lovelock action (2.5) are given by

$$\tau_i \coloneqq \frac{\partial L}{\partial T^i} = 0, \tag{3.20}$$

$$\rho_{ij} \coloneqq \frac{\partial L}{\partial R^{ij}} = 2\varepsilon_{ijkln}\left(\frac{\alpha_1}{3} e^k e^l + 2\alpha_2 R^{kl}\right) e^n. \tag{3.21}$$

Consequently, we obtain

$$\rho_{ab} = 4\varepsilon_{abc}\left(\alpha_1 - \frac{2\alpha_2}{r_0^2}\right) e^c e^3 e^4,$$
$$\rho_{a3} = 2\varepsilon_{abc}(\alpha_1 + 2\alpha_2 q) e^b e^c e^4 = \alpha_1\varepsilon_{abc} e^b e^c e^4,$$
$$\rho_{a4} = 2\varepsilon_{abc}(\alpha_1 + 2\alpha_2 q) e^b e^c e^3 = \alpha_1\varepsilon_{abc} e^b e^c e^3,$$
$$\rho_{34} = 2\varepsilon_{abc}\left(\frac{\alpha_1}{3} + 2\alpha_2 q\right) e^a e^b e^c = -\frac{\alpha_1}{3} e^a e^b e^c. \tag{3.22}$$

In our calculations of the boundary integrals, we use the coordinates $x^\mu = (t, r, \varphi, \theta, \chi)$. The background configuration is the one defined by zero values of parameters $m = 0$ and $j = 0$ of the solution (3.15). For the solutions with Killing vectors ∂_t and ∂_φ, the conserved charges are the energy and angular momentum, respectively,

$$E = \int_{\partial\Sigma} B(\partial_t) = \int_{\partial\Sigma} e^i{}_t\Delta\tau_i + \Delta e^i\bar{\tau}_{it} + \frac{1}{2}\omega^{ij}{}_t\Delta\rho_{ij} + \frac{1}{2}\Delta\omega^{ij}\bar{\rho}_{ijt}, \tag{3.23a}$$

$$J = \int_{\partial\Sigma} B(\partial_\varphi) = \int_{\partial\Sigma} e^i{}_\varphi\Delta\tau_i + \Delta e^i\bar{\tau}_{i\varphi} + \frac{1}{2}\omega^{ij}{}_\varphi\Delta\rho_i + \frac{1}{2}\Delta\omega^{ij}\bar{\rho}_{ij\varphi}, \tag{3.23b}$$

where $\partial\Sigma$ is a boundary $S^1 \times S^2$, located at infinity, described by coordinates φ, θ, χ.

Thus, conserved charges for the black ring with torsion and the Riemannian black ring are given by

$$E = 8\pi^2 r_0^2\left(\alpha_1 - \frac{2\alpha_2}{r_0^2}\right) m, \qquad J = 8\pi^2 r_0^2\left(\alpha_1 - \frac{2\alpha_2}{r_0^2}\right) j. \tag{3.24}$$

Let us note that the solution with torsion exists in the sector $\alpha_1^2 = 8\alpha_0\alpha_2$, where both conserved charges vanish.

IV. CANONICAL GAUGE GENERATOR

As an important step in our examination of the asymptotic structure of space-time, we are going to construct the canonical gauge generator, which is our basic tool for studying asymptotic symmetries and conserved charges of 5D Lovelock gravity.

A. Hamiltonian and constraints

The best way to understand the dynamical content of gauge symmetries is to explore the canonical generator, which acts on the basic dynamical variables via the Poisson bracket (PB) operation. To begin the canonical analysis, we rewrite the action (2.5) as

$$I = \int d^5x\mathcal{L}$$

$$\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma\tau}_{ijkln}\int_{\mathcal{M}} d^5x\left(\frac{\alpha_0}{5}e^i{}_\mu e^j{}_\nu e^k{}_\rho e^l{}_\sigma + \frac{\alpha_1}{6}R^{ij}{}_{\mu\nu}e^k{}_\rho e^l{}_\sigma + \frac{\alpha_2}{4}R^{ij}{}_{\mu\nu}R^{kl}{}_{\rho\sigma}\right)e^n{}_\tau. \tag{4.1}$$

1. Primary constraints and canonical Hamiltonian

The basic Lagrangian variables $(e^i{}_\mu, \omega^{ij}{}_\mu)$ and the corresponding canonical momenta $(\pi_i{}^\mu, \pi_{ij}{}^\mu)$ are related to each other through the set of primary constraints:

$$\begin{aligned}
&\phi_i{}^0 \coloneqq \pi_i{}^0 \approx 0, \qquad \phi_{ij}{}^0 \coloneqq \pi_{ij}{}^0 \approx 0,\\
&\phi_i{}^\alpha \coloneqq \pi_i{}^\alpha \approx 0,\\
&\phi_{ij}{}^\alpha \coloneqq \pi_{ij}{}^\alpha - 2\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}\left(\frac{\alpha_1}{3}e^k{}_\beta e^l{}_\gamma + \alpha_2 R^{kl}{}_{\beta\gamma}\right)e^n{}_\delta \approx 0.
\end{aligned} \tag{4.2}$$

The algebra of primary constraints is displayed in the Appendix.

The canonical Hamiltonian is defined by

$$\mathcal{H}_c = \pi_i{}^\mu \dot{e}^i{}_\mu + \frac{1}{2}\pi_{ij}{}^\mu\dot{\omega}^{ij}{}_\mu - \mathcal{L}.$$

Since the Lagrangian is linear in velocities, the canonical Hamiltonian in the formula given above reduces to $\mathcal{H}_c = -\mathcal{L}(\dot{e}^i{}_\mu = 0, \dot{\omega}^{ij}{}_\mu = 0)$. It is linear in unphysical variables:

$$\begin{aligned}
\mathcal{H}_c &= e^i{}_0\mathcal{H}_i + \frac{1}{2}\omega^{ij}{}_0\mathcal{H}_{ij} + \partial_\alpha D^\alpha,\\
\mathcal{H}_i &= -\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}\left(\alpha_0 e^j{}_\alpha e^k{}_\beta e^l{}_\gamma e^n{}_\delta + \frac{1}{2}\alpha_1 R^{jk}{}_{\alpha\beta}e^l{}_\gamma e^n{}_\delta + \frac{1}{4}\alpha_2 R^{jk}{}_{\alpha\beta}R^{ln}{}_{\gamma\delta}\right),\\
\mathcal{H}_{ij} &= -\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}(\alpha_1 e^k{}_\alpha e^l{}_\beta + \alpha_2 R^{kl}{}_{\alpha\beta})T^n{}_{\gamma\delta},\\
D^\alpha &= \varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}\omega^{ij}{}_0(\alpha_1 e^k{}_\beta e^l{}_\gamma + \alpha_2 R^{kl}{}_{\beta\gamma})e^n{}_\delta.
\end{aligned} \tag{4.3}$$

2. Secondary constraints

Going over to the total Hamiltonian,

$$\mathcal{H}_T = \mathcal{H}_c + u^i{}_\mu\phi_i{}^\mu + \frac{1}{2}u^{ij}{}_\mu\phi_{ij}{}^\mu, \tag{4.4}$$

we find that the consistency conditions of the primary constraints $\pi_i{}^0$ and π^0_{ij} yield the secondary constraints:

$$\mathcal{H}_i \approx 0, \qquad \mathcal{H}_{ij} \approx 0. \tag{4.5}$$

Let us note that these constraints reduce to the $\mu = 0$ components of the Lagrangian field equations (2.7).

The consistency of the remaining primary constraints $\phi_i{}^\alpha$ and $\phi_{ij}{}^\alpha$ leads to the relations for multipliers $u^i{}_\beta$ and $u^{ij}{}_\beta$,

$$\begin{aligned}
&\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}[\underline{R}^{jk}{}_{0\beta}(\alpha_1 e^l{}_\gamma e^n{}_\delta + \alpha_2 R^{ln}{}_{\gamma\delta}) + (\alpha_1 R^{jk}{}_{\beta\gamma} + 4\alpha_0 e^j{}_\beta e^k{}_\gamma)e^l{}_0 e^n{}_\delta] = 0,\\
&\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}[\underline{T}^k{}_{0\beta}(\alpha_1 e^l{}_\gamma e^n{}_\delta + \alpha_2 R^{ln}{}_{\gamma\delta}) + \alpha_2\underline{R}^{kl}{}_{0\beta}T^n{}_{\gamma\delta} + \alpha_1 e^k{}_0 e^l{}_\beta T^n{}_{\gamma\delta}] = 0,
\end{aligned} \tag{4.6}$$

where $\underline{T}^i{}_{0\alpha} = T^i{}_{0\alpha}(\dot{e}^i{}_\alpha \to u^i{}_\alpha)$ and $\underline{R}^{ij}{}_{0\alpha} = R^{ij}{}_{0\alpha}(\dot{\omega}^{ij}{}_\alpha \to u^{ij}{}_\alpha)$. Using the Hamiltonian equations of motion $\dot{e}^i{}_\alpha = u^i{}_\beta$ and $\dot{\omega}^{ij}{}_\alpha = u^{ij}{}_\alpha$, these relations reduce to the $\mu = \alpha$ components of the Lagrangian field equations (2.7).

3. Further consistency procedure

Some of the relations (4.6) can be solved in terms of the multipliers $u^i{}_\alpha$ and $u^{ij}{}_\alpha$, while the others may lead to ternary constraints, the consistency of which has to be examined as well. However, this procedure is extremely sensitive to the particular sector of the theory as we shall illustrate in the next section (for the pure Lovelock theory, see Ref. [20]). The final form of the total Hamiltonian is given by

$$\begin{aligned}
\mathcal{H}_T &= \bar{\mathcal{H}}_T + u^i{}_0\pi_i{}^0 + \frac{1}{2}u^{ij}{}_0\pi_{ij}{}^0 + (u\cdot\phi),\\
\bar{\mathcal{H}}_T &= e^i{}_0\bar{\mathcal{H}}_i + \frac{1}{2}\omega^{ij}{}_0\bar{\mathcal{H}}_{ij} + \partial_\alpha\bar{D}^\alpha,\\
\bar{\mathcal{H}}_i &= \mathcal{H}_i + (\bar{u}\cdot\phi),\\
\bar{\mathcal{H}}_{ij} &= \mathcal{H}_{ij} + (\bar{u}\cdot\phi),\\
\bar{D}^\alpha &= D^\alpha + (\bar{u}\cdot\phi),
\end{aligned} \tag{4.7}$$

where by $(u\cdot\phi)$ we denoted terms stemming form the *undetermined* multipliers and belonging to the set $(u^i{}_\beta, u^{ij}{}_\beta)$, and by $(\bar{u}\cdot\phi)$ we denoted terms stemming form the *determined* multipliers belonging to the same set.

B. Canonical generator and charges

The sure symmetries of the theory are local translations and local Lorentz rotations. The general form of the canonical generator of the local Poincaré transformations constructed by the Castellani procedure [21] is given by

$$
\begin{aligned}
G &= -G_1 - G_2,\\
G_1 &= \dot{\xi}^\rho\left(e^i{}_\rho\pi_i{}^0 + \frac{1}{2}\omega^{ij}{}_\rho\pi^0_{ij}\right)\\
&\quad + \xi^\rho\left(e^i{}_\rho\bar{\mathcal{H}}_i + \frac{1}{2}\omega^{ij}{}_\rho\bar{\mathcal{H}}_{ij} + C_{\rm PFC}\right),\\
G_2 &= \frac{1}{2}\dot{\varepsilon}^{ij}\pi_{ij}{}^0 + \frac{1}{2}\varepsilon^{ij}(\bar{\mathcal{H}}_{ij} + C_{\rm PFC}),
\end{aligned}
$$

where $C_{\rm PFC}$ are terms proportional to sure primary first class constraints $(\pi_i{}^0, \pi_{ij}{}^0)$.

The canonical generator acts on dynamical variables via the PB operation, and hence, it should have well-defined functional derivatives. In order to ensure this property, we have to improve the form of G by adding a suitable surface term Γ, such that $\tilde{G} = G + \Gamma$ is a well-defined canonical generator. In this process, the asymptotic conditions play a crucial role; see for instance Refs. [22,23]. Though we did not construct the exact form of the canonical generator, it still allows us to compute canonical charges for the solutions found in Sec. III. Namely, if we adopt the general principle that the quantities that vanish on shell have an arbitrary fast asymptotic decrease, we obtain that the on-shell variation of the generator takes the following form,

$$
\delta G(\xi^t = \ell, \xi^\varphi = 1) \approx \delta\Gamma = -\ell\delta E_c - \delta J_c, \tag{4.8}
$$

where

$$
E_c = 8\pi^2 r_0^2\left(\alpha_1 - \frac{2\alpha_2}{r_0^2}\right)m, \qquad J_c = 8\pi^2 r_0^2\left(\alpha_1 - \frac{2\alpha_2}{r_0^2}\right)j \tag{4.9}
$$

are the *canonical* conserved charges, which are identical to the expressions (3.24), obtained from the Nester formula.

V. LINEARIZED THEORY

The canonical structure of the full nonlinear theory crucially depends on the relations (4.6), as we already mentioned in the previous section. In order to get a deeper insight into the structure of the Lovelock gravity in the sector $\alpha_1^2 = 8\alpha_0\alpha_2$, we shall consider the theory linearized around the BTZ black ring with torsion (3.15). The linearization is based on the expansion of the basic dynamical variables $(e^i{}_\mu, \omega^{ij}{}_\mu)$ and the related conjugate momenta $(\pi_i{}^\mu, \pi_{ij}{}^\mu)$ denoted shortly by Q_A,

$$
Q_A = \bar{Q}_A + \tilde{Q}_A, \tag{5.1}
$$

where $\bar{Q}_A$ refers to the background [solution (3.15) with $m = j = 0$ and $p \neq 0$], while $\tilde{Q}_A$ denotes small excitations.

From the linearized form of the 60 relations (4.6), we conclude that out of $60 = 5\times 4 + 10\times 4$ multipliers $(\tilde{u}^i{}_\alpha, \tilde{u}^{ij}{}_\alpha)$ 46 are determined, while among 14 remaining relations, there are 12 new constraints (since two pairs of them are identical), the explicit form of which is given by

$$
\alpha_1\tilde{R}^{24}{}_{r\chi} + \alpha_1\sin\theta\tilde{R}^{23}{}_{r\theta} + 4\alpha_0 r_0\sin\theta\tilde{e}^2{}_r \approx 0, \tag{5.2a}
$$

$$
\alpha_1\tilde{R}^{14}{}_{\varphi\chi} + \alpha_1\sin\theta\tilde{R}^{13}{}_{\varphi\theta} + 4\alpha_0 r_0\sin\theta\tilde{e}^2{}_r \approx 0, \tag{5.2b}
$$

$$
\begin{aligned}
&\frac{r^2}{\ell}(\alpha_1\tilde{R}^{14}{}_{r\chi} + \alpha_1\sin\theta\tilde{R}^{13}{}_{r\theta} + 2\alpha_0 r_0\sin\theta\tilde{e}^1{}_r)\\
&\quad -\alpha_1\tilde{R}^{24}{}_{\varphi\chi} - \alpha_1\sin\theta\tilde{R}^{23}{}_{\varphi\theta} - 2\alpha_0 r_0\sin\theta\tilde{e}^2{}_\varphi \approx 0
\end{aligned} \tag{5.2c}
$$

and

$$
\tilde{T}^4{}_{r\chi} + \sin\theta\tilde{T}^3{}_{r\theta} \approx 0, \tag{5.3a}
$$

$$
p(\alpha_1 r_0(\tilde{e}^4{}_\chi + \sin\theta\tilde{e}^3{}_\theta) + 2\alpha_2\tilde{R}^{34}{}_{\theta\chi}) \approx 0, \tag{5.3b}
$$

$$
\tilde{T}^4{}_{\varphi\chi} + \sin\theta\tilde{T}^3{}_{\varphi\theta} \approx 0, \tag{5.3c}
$$

$$
\alpha_1\frac{r}{\ell}r_0\sin\theta\tilde{T}^2{}_{r\theta} - 2p(\alpha_1 r_0\sin\theta\tilde{e}^0{}_\theta + 2\alpha_2\tilde{R}^{04}{}_{\theta\chi}) \approx 0, \tag{5.3d}
$$

$$
\alpha_1\frac{r}{\ell}r_0\tilde{T}^2{}_{r\chi} - 2p(\alpha_1 r_0\sin\theta\tilde{e}^0{}_\chi - 2\alpha_2\tilde{R}^{03}{}_{\theta\chi}) \approx 0, \tag{5.3e}
$$

$$
\alpha_1 r_0\tilde{T}^1{}_{\varphi\chi} + 2pr(\alpha_1 r_0\tilde{e}^0{}_\chi - 2\alpha_2\tilde{R}^{03}{}_{\theta\chi}) \approx 0, \tag{5.3f}
$$

$$
\tilde{R}^{03}{}_{r\chi} \approx 0, \tag{5.3g}
$$

$$
\tilde{R}^{02}{}_{\theta\chi} \approx 0, \tag{5.3h}
$$

$$
\tilde{R}^{01}{}_{\theta\chi} \approx 0. \tag{5.3i}
$$

Let us denote 12 constraints (5.2a) and (5.3a) by $\tilde{\psi}_A$. The consistency conditions of $\tilde{\psi}_A$ leads to the determination of 12 additional multipliers, thus finishing the consistency procedure. Thus, out of 60 multipliers $(\tilde{u}^i{}_\alpha, \tilde{u}^{ij}{}_\alpha)$, 58 are determined, while 2 remain undetermined. By using the PB algebra from the Appendix, we find

$$
\begin{aligned}
&\{\tilde{\phi}_{12}{}^r, \tilde{\phi}_i{}^\alpha\} \approx 0, \qquad \{\tilde{\phi}_{12}{}^r, \tilde{\phi}_{ij}{}^\alpha\} \approx 0,\\
&\{\tilde{\phi}_{12}{}^r, \tilde{\psi}_A\} \approx 0,\\
&\{\tilde{\phi}_{12}{}^\varphi, \tilde{\phi}_i{}^\alpha\} \approx 0, \qquad \{\tilde{\phi}_{12}{}^\varphi, \tilde{\phi}_{ij}{}^\alpha\} \approx 0,\\
&\{\tilde{\phi}_{12}{}^\varphi, \tilde{\psi}_A\} \approx 0.
\end{aligned}
$$

TABLE I. Classification of constraints.

	First class	Second class
Primary	$\tilde{\phi}_i{}^0$, $\tilde{\phi}_{ij}{}^0$, $\tilde{\phi}_{12}{}^r$, $\tilde{\phi}_{12}{}^\varphi$	$\tilde{\phi}_i{}^\alpha$, $\tilde{\phi}_{ij}{}^\alpha$ $ij \neq 12 \wedge \alpha \neq r, \varphi$
Secondary	$\tilde{\bar{\mathcal{H}}}_i$, $\tilde{\bar{\mathcal{H}}}_{ij}$	$\tilde{\psi}_A$

The undetermined multipliers correspond to the constraints $\tilde{\phi}_{12}{}^r$ and $\tilde{\phi}_{12}{}^\varphi$ which are first class (FC). The final classification of constraints is given in Table I. In total, there are $N_1 = 32$ FC constraints and $N_2 = 70$ second class (SC) constraints. The number of propagating degrees of freedom in phase space is

$$N^* = 2N - 2N_1 - N_2 = 150 - 64 - 70 = 14.$$

In the configuration space, there are seven degrees of freedom: five of them correspond to general relativity in $D = 5$, and two are additional degrees of freedom. The presence of two primary FC constraints $\tilde{\phi}_{12}{}^r$, $\tilde{\phi}_{12}{}^\varphi$ implies that there is an additional gauge symmetry in the theory, as a consequence of the fact that variables $\tilde{\omega}^{12}{}_r$ and $\tilde{\omega}^{12}{}_\varphi$ do not appear in the linearized equations of motion.

VI. CONCLUSION

In this paper, we found a new class of solutions of Lovelock gravity in 5D, in the first order formalism. The most interesting solutions are the BTZ black rings with(out) torsion. It is shown that the solution with torsion exists provided that the parameters of the theory satisfy the relation $\alpha_1^2 = 8\alpha_0\alpha_2$. This sector of the parameter space is different from the one of Lovelock Chern-Simons gravity, as well as from the sector investigated by Canfora *et al.* [17]. Restricting our attention to the basic properties of the solutions, we calculated the values of conserved charges by using Nester's formula and the canonical method. The canonical structure of the theory linearized around the background with torsion shows that there are two additional degrees of freedom, compared to general relativity.

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APPENDIX: ALGEBRA OF CONSTRAINTS

The structure of the PB algebra of constraints is an important ingredient in the analysis of the Hamiltonian consistency conditions. Starting from the fundamental PB $\{e^i{}_\mu, \pi_j{}^\nu\} = \delta^i_j\delta^\nu_\mu\delta(\mathbf{x}-\mathbf{x}')$ and $\{\omega^{ij}{}_\mu, \pi_{kl}{}^\nu\} = 2\delta^{[i}_k\delta^{j]}_l\delta^\nu_\mu\delta(\mathbf{x}-\mathbf{x}')$, we find PB between primary constraints:

$$\{\phi_i{}^\alpha, \phi_{jk}{}^\beta\} = -2\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}(\alpha_1 e^l{}_\gamma e^n{}_\delta + \alpha_2 R^{ln}{}_{\gamma\delta})\delta,$$
$$\{\phi_{ij}{}^\alpha, \phi^\beta_{kl}\} = -8\alpha_2\varepsilon^{0\alpha\beta\gamma\delta}_{ijkln}T^n{}_{\gamma\delta}\delta. \quad \text{(A1)}$$

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