

Solar System constraints on renormalization group extended general relativity: The PPN and Laplace-Runge-Lenz analyses with the external potential effect

Davi C. Rodrigues,^{1,*} Sebastião Mauro,^{2,3,†} and Álefe O. F. de Almeida^{1,‡}

¹*Departamento de Física, CCE, Universidade Federal do Espírito Santo, Avenida Fernando Ferrari 514, Vitória, Espírito Santo 29075-910, Brazil*

²*Departamento de Física, ICE, Universidade Federal de Juiz de Fora, Juiz de Fora, Minas Gerais 36036-330, Brazil*

³*Instituto Federal de Educação, Ciências e Tecnologia do Sul de Minas—IFSuldeMinas, 37410-000 Três Corações, Minas Gerais, Brazil*

(Received 18 September 2016; published 24 October 2016)

General relativity extensions based on renormalization group effects are motivated by a known physical principle and constitute a class of extended gravity theories that have some unexplored unique aspects. In this work we develop in detail the Newtonian and post-Newtonian limits of a realization called renormalization group extended general relativity (RGGR). Special attention is given to the external potential effect, which constitutes a type of screening mechanism typical of RGGR. In the Solar System, RGGR depends on a single dimensionless parameter $\bar{\nu}_\odot$, and this parameter is such that for $\bar{\nu}_\odot = 0$ one fully recovers GR in the Solar System. Previously this parameter was constrained to be $|\bar{\nu}_\odot| \lesssim 10^{-21}$, without considering the external potential effect. Here we show that under a certain approximation RGGR can be cast in a form compatible with the parametrized post-Newtonian (PPN) formalism, and we use both the PPN formalism and the Laplace-Runge-Lenz technique to put new bounds on $\bar{\nu}_\odot$, either considering or not the external potential effect. With the external potential effect the new bound reads $|\bar{\nu}_\odot| \lesssim 10^{-16}$. We discuss the possible consequences of this bound on the dark matter abundance in galaxies.

DOI: [10.1103/PhysRevD.94.084036](https://doi.org/10.1103/PhysRevD.94.084036)

I. INTRODUCTION

There are diverse motivations for extending gravity beyond general relativity (e.g., Refs. [1–3]), including quantum gravity, the avoidance of singularities, understanding inflation, theoretical and observational improvements on dark matter, alleviating coincidence issues related to dark energy, and others. Nonetheless, these extensions must be capable of explaining the success of general relativity (GR) in the Solar System. This apparently simple test of gravity is, for many cases, a hard test.

Among the possible extensions of GR that have consequences at astrophysical or cosmological scales, we consider here extensions that are based on the renormalization group (RG) framework applied to gravity [4–21]. In this context, either from quantum field theory (QFT) in curved spacetime [7,22], or in certain quantum-gravity theories, like the asymptotic safety program [17,23,24], the GR constants G and Λ necessarily run in the ultraviolet limit. Moreover, in the low-energy limit, the β functions of G and Λ need not be zero, since the Appelquist-Carazzone decoupling [25] does not hold in this context (contrary to the coupling constants associated to the

high-derivative terms that appear in QFT in curved spacetime [26]). Therefore, large-scale variations of these “constants” can be a sign of these RG effects, which may as well provide leads to the underlying quantum-gravity theory [15]. Here we will focus on the realization that was named renormalization group extended general relativity (RGGR) [27,28].

Besides being motivated from known physical effects, these gravity extensions based on the RG represent a new route to GR extensions on their own. They share similarities with scalar-tensor gravity in the sense that they depend on a metric and additional quantities that transform as scalars, like G , Λ or the RG scale μ . These similarities at large scales were particularly explored in Refs. [29–31].¹ As detailed in Ref. [28], which follows and extends the approaches of Refs. [29,33], they may also include features that are either unnatural or clearly outside the usual scalar-tensor approaches: i) natural and simple potentials from the RG perspective are rather complex from the scalar-tensor perspective, ii) potentials within the RG perspective need not be universal, as they

¹There are others approaches that can be promptly spotted as not similar to scalar-tensor theories, in particular those whose RG effects are not implemented at the action level (e.g., Ref. [21]), or those that use additional dynamical tensors (e.g., Ref. [32]), but these are not in the focus of this work.

*davi.rodrigues@cosmo-ufes.org

†sebastiao.filho@ifsulde Minas.edu.br

‡alefe@cosmo-ufes.org

can be derived for each system and can be different from system to system, and iii) the identification of the physical meaning of the RG scale μ is an important step from the RG perspective, and this identification leads to the imposition of a constraint between μ and the matter fields. In conclusion, although there are similarities, there are important differences. Also, these differences can lead to new forms of screening mechanisms [34–37], and this is one of the main points of this work, namely to show that RGGR has a type of screening mechanism that depends on the external Newtonian potential of a system (in general it depends on the scalar $U^\alpha U^\beta h_{\alpha\beta}$, which will be explained latter).

There are some procedures for testing gravity beyond Newtonian gravity, and the most general and cited one is the parametrized post-Newtonian (PPN) formalism (e.g., Refs. [38–49]). Here we consider both a version of the PPN formalism and the Laplace-Runge-Lenz (LRL) vector technique (e.g., Refs. [50–52]).

Solar System tests of RGGR were evaluated in Refs. [53,54], where bounds on the dimensionless parameter $\bar{\nu}_\odot \equiv \alpha_\odot \nu$ were found to be respectively $|\bar{\nu}_\odot| \lesssim 10^{-17}$ and $|\bar{\nu}_\odot| \lesssim 10^{-21}$. In Ref. [27] it was argued that if the effective constant $\bar{\nu}$ approximately runs linearly with the system mass, then the galaxy results could be explained and also the first bound [53] would be satisfied. If RGGR has significant impact on dark matter then $\bar{\nu}$ should be about $\sim 10^{-9}$ – 10^{-7} in galaxies [55,56]. Since the baryonic mass of the studied galaxies ranged from $10^8 M_\odot$ to $10^{11} M_\odot$, the first bound is compatible with the linear behavior, while the second one is not (this was also discussed in Ref. [54]). The precise mechanism for the running of $\bar{\nu}$ was unclear.

The previous works on Solar System constraints [53,54] have not considered the external potential effect, which is presented in detail here, in Sec. III B. This effect is inherent to the RGGR approach and works as a (partial) screening mechanism.

In this work, we also clarify the correspondence and the validity of using the RGGR noncovariant approach (introduced in Ref. [27]) as an approximation to the covariant version [28]. Appendices A and B are devoted to this issue.

The starting point of this work is the covariant RGGR formulation proposed in Ref. [28]. Section II is devoted to a brief review of RGGR. Section III presents a proper post-Newtonian framework, introduces the external potential effect and presents in detail a point-particle solution. Section IV applies the previous results to the Solar System within the PPN formalism and the LRL technique. This technique can be applied either with or without the external potential effect, hence allowing for an evaluation of the effect relevance. Considering the external potential effect, both techniques can be applied and are compatible. Finally, in Sec. V we present our conclusions and discussions.

II. RENORMALIZATION GROUP EXTENDED GENERAL RELATIVITY: A BRIEF REVIEW

A. The action

In Ref. [28] we proposed the following action for describing the large-scale RG effects in gravity, which we will in general label as RGGR,

$$S = \int \left[\frac{R - 2\Lambda\{\mu\}}{16\pi G(\mu)} + \lambda(\mu - f(g, \gamma, \Psi)) \right] \sqrt{-g} d^4x + S_m, \quad (1)$$

where $S = S[g, \gamma, \mu, \lambda, \Psi]$, $S_m = S_m[g, \Psi]$, Ψ stands for any matter fields of any nature, and μ is the RG scale, whose relation to all the other fields is stated in the action in a constraint-like way, as imposed by the Lagrange multiplier λ . The field $\gamma_{\alpha\beta}$, which only appears inside f and without derivatives, is a tensor that works as a reference metric. Reference or background metrics commonly appear in QFT in curved spacetime and in some quantum gravity approaches. As shown in Ref. [28], $\gamma_{\alpha\beta}$ is important for guaranteeing energy-momentum conservation, and for presenting certain noncovariant scale settings in covariant form. About the scalars G and Λ , they are respectively the gravitational coupling and the cosmological “constant,” and they both depend on the RG scale μ . Namely, G is a standard function of μ , which is fixed at the action level. This means that the form of this dependence is independent on the other fields and their boundary conditions (i.e., if, for instance, $G = \mu^2$ for cosmology, then $G = \mu^2$ for the Solar System, for all the galaxies, for vacuum and for any other system). On the other hand, the relation between Λ and μ is not assumed to be universal, it is system dependent. It is not fixed at the action level, but it can and must be derived from the field equations. This is why we introduced in Ref. [28] different notations for these dependences; we write $G(\mu)$ and $\Lambda\{\mu\}$.

From the RG perspective, the difference between $G(\mu)$ and $\Lambda\{\mu\}$ is that for the first one we assume the existence of a universal β function, that is, a β function that is independent of any other properties of the system, while for $\Lambda\{\mu\}$ the corresponding β function is system dependent. Since β functions in general depend on the presence of other fields (and gravity interacts with everything), the use of “ $\{\mu\}$ ” should not come as a surprise. Since there are diverse works that suggest a simple and specific form for $G(\mu)$, for the latter only we use the usual fixed dependence “ (μ) .” Whenever it is necessary to specify a function $G(\mu)$ we use the following simple expression that has been derived from different approaches (e.g., Refs. [6,7,29,33,57]):

$$G^{-1}(\mu) = 1 + 2\nu \ln \mu, \quad (2)$$

where ν is a small dimensionless constant, and GR is recovered for $\nu = 0$.

B. A class of covariant scale settings and $T^{\alpha\beta}$ conservation

The action (1) in general spoils the energy-momentum tensor conservation, but there is a particular class of scale settings (i.e., f functions) that preserves energy-momentum conservation, namely [28],

$$\mu = f(U^\alpha U^\beta h_{\alpha\beta}), \quad (3)$$

where $h_{\alpha\beta} \equiv g_{\alpha\beta} - \gamma_{\alpha\beta}$. The tensor U^α denotes the four-velocity field. The coupling of the RGGR action to a general perfect fluid is presented in Ref. [28], which uses the fluid action description of Ref. [58].

On the energy-momentum conservation, the action (1) does not lead in general to $\nabla_\alpha T^{\alpha\beta} = 0$, where $T^{\alpha\beta}$ is the matter energy-momentum tensor. The reason is that it is not possible to write $S = S_g[g, \Phi_G] + S_m[g, \Psi]$, where Φ_G denotes fields of any nature that do not appear in S_m , while Ψ denotes fields that do not appear in S_g (for a review, see Appendix E of Ref. [59]). Since λ is the term that prevents the action from splitting into gravitational and matter parts, in general $\nabla_\alpha T^{\alpha\beta} \propto \lambda$. Hence, if λ is set to zero at the level of the field equations, it is possible to guarantee that $\nabla_\alpha T^{\alpha\beta} = 0$. This is precisely achieved by using the scale setting proposed in Eq. (3). Indeed, since $\gamma_{\alpha\beta}$ only appears in the action inside f , the variation of S with respect to $\gamma_{\alpha\beta}$ leads to

$$\lambda f' U^\alpha U^\beta = 0. \quad (4)$$

Hence either $\lambda = 0$ or $f' = 0$. The last option simply implies that μ is a constant, leading to standard GR. A GR extension can be found if the other solution holds, namely if $\lambda = 0$ at the level of the field equations, which implies that $\nabla_\alpha T^{\alpha\beta} = 0$.

For the scale setting (3), the variation of the action (1) with respect to the metric reads

$$\mathcal{G}_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \quad (5)$$

where

$$\mathcal{G}_{\alpha\beta} \equiv G_{\alpha\beta} + g_{\alpha\beta} G \square G^{-1} - G \nabla_\alpha \nabla_\beta G^{-1}, \quad (6)$$

$\square \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$, and ∇_α is the usual covariant derivative. From the energy-momentum tensor conservation, one derives that [28,30,55]

$$\nabla_\alpha \left(\frac{\Lambda}{G} \right) = \frac{1}{2} R \nabla_\alpha G^{-1}. \quad (7)$$

The equation above can also be derived from the action variation with respect to μ . Further details on the RGGR action can be found in Ref. [28].

C. The noncovariant scale setting

An issue that any RG approach to gravity must answer is the physical meaning of the scale μ , that is the relation of μ with other physical quantities (which in the end is the same as specifying the f function).

In the context of stationary, slow-velocity and weak-field systems, some of us have introduced in previous works the scale setting [27]

$$\mu = \left(\frac{\Phi}{\Phi_0} \right)^\alpha, \quad (8)$$

where Φ_0 and α are constants that describe the system, and Φ is the Newtonian potential, defined by

$$\nabla^2 \Phi = 4\pi G_0 \rho, \quad \text{with } \Phi(r \rightarrow \infty) = 0. \quad (9)$$

In the above, ρ is the matter density, G_0 is the gravitational constant at some spacetime point, and μ is the RG scale written in dimensionless form. In the following, a system of units such that $G_0 = 1$ is always used. The constant Φ_0 is actually irrelevant in any perturbative expansion up to the first order of ν .

The noncovariant scale setting above can be seen as an approximation to the covariant one, as detailed in Appendix A.

The scale setting (8) has achieved interesting phenomenological consequences for galaxy systems. In particular we considered its implications for dark matter [27,55,56,60–63], and it was found that this approach can have a significant impact on the necessary amount of dark matter in galaxies. Indeed, the internal dynamics of galaxies alone shows that good results are achievable in galaxies even without dark matter.

As previously stated, α is not a universal constant (contrary to ν): it depends on the system. Considering galaxy rotation curves (e.g., Refs. [27,56]), it is used as a constant inside a galaxy, but it changes from galaxy to galaxy.² Since all the dynamical tests do not depend only on α , but on the combination $\alpha\nu$, effectively one can replace the two constants α and ν by a single system-dependent constant $\bar{\nu} \equiv \nu\alpha$. In Ref. [53] it was found that, for the Solar System internal dynamics, $|\bar{\nu}_\odot| \lesssim 10^{-17}$. More recently, Ref. [54] used more precise data for the Solar System and arrived at the condition $|\bar{\nu}_\odot| \lesssim 10^{-21}$. Nevertheless, neither of these references considered the external potential effect, which is detailed in Sec. III B.

²This behavior can be qualitatively described by the covariant formulation, as detailed in the Appendices. Nonetheless there is no known covariant expression that can quantitatively explain these variations.

III. POST-NEWTONIAN FRAMEWORK, THE EXTERNAL POTENTIAL EFFECT AND THE POINT-PARTICLE SOLUTION

This section is devoted to three items not covered in previous publications: i) introducing a perturbative scheme that will allow for the application of the PPN formalism, ii) presenting the external potential effect, which is a kind of screening mechanism that is part of RGGR, and iii) presenting a detailed evaluation of the point-particle solution from the field equations such that it can be used in the PPN and LRL analyses in the next section.

Although all the results present in this section are derived from the noncovariant scale setting, the same results can also be derived from the covariant one, as shown in Appendices A and B.

A. GR and RG perturbations

Consider the following perturbative scheme about the metric $g_{\alpha\beta}^{(0)}$:

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + g_{\alpha\beta}^{(1,0)} + g_{\alpha\beta}^{(0,1)} + \dots, \quad (10)$$

$$T_{\alpha\beta} = T_{\alpha\beta}^{(0)} + T_{\alpha\beta}^{(1,0)} + T_{\alpha\beta}^{(0,1)} + \dots, \quad (11)$$

$$G(\mu) \equiv 1 + \delta G(\mu) = 1 + G^{(1)}(\mu) + \dots, \quad (12)$$

$$\Lambda(\mu) \equiv \Lambda_0 + \delta\Lambda(\mu) = \Lambda_0 + \Lambda^{(1)}(\mu) + \dots \quad (13)$$

The metric $g_{\alpha\beta}^{(0)}$ satisfies the Einstein equation with the energy-momentum tensor $T_{\alpha\beta}^{(0)}$, the gravitational constant G_0 (which is set to be 1) and the cosmological constant Λ_0 .

The terms of the type $X_{\alpha\beta}^{(n,0)}$ refer to some perturbative expansion within GR; for instance, n may refer to the order of a post-Newtonian expansion. The terms of the type $X_{\alpha\beta}^{(n,m)}$ are the RG correction of m th order to the GR perturbation of order n .

The background is here picked to be Minkowski, that is,

$$g_{\alpha\beta}^{(0)} = \eta_{\alpha\beta}, \quad \Lambda_0 = 0, \quad T_{\alpha\beta}^{(0)} = 0. \quad (14)$$

Within this case, it was shown in detail in Ref. [28] that, up to first order in both of the perturbations, if $\tilde{g}_{\alpha\beta}$ is a solution of the Einstein equation given by $\tilde{G}_{\alpha}^{\beta} = 8\pi\tilde{T}_{\alpha}^{\beta}$ then the metric solution for the field equation (5) can be found from the conformal transformation

$$g_{\alpha\beta} = G\tilde{g}_{\alpha\beta} + O(2,2). \quad (15)$$

The symbol $O(m, n)$ designates any terms of the m th or higher order in the GR perturbation, and of n th or higher order in the RG perturbation. For the particular case of $O(2, 2)$, when it is present it is implied that the mixed terms in the perturbations are not explicitly written (since the terms of the type $X^{(1,1)}$ are necessarily equal or smaller than either $X^{(2,0)}$ or $X^{(0,2)}$). The use of $O(\infty, m)$, which will appear later, implies that the expression is exact if $\nu = 0$; it is an exact GR expression with RG corrections up to the order $m - 1$.

In order to illustrate the notation and review an important result that can also be found in Refs. [27,28], let $\tilde{g}_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}$ and $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$; therefore, using Eq. (15),

$$\begin{aligned} h_{\alpha\beta} &= g_{\alpha\beta} - \eta_{\alpha\beta} \\ &= (\eta_{\alpha\beta} + \tilde{h}_{\alpha\beta})G - \eta_{\alpha\beta} + O(2,2). \end{aligned} \quad (16)$$

Hence, in particular,

$$\begin{aligned} h_{00} &= -2\Phi - G^{(1)} + O(2,2) \\ &= -2\Phi + 2\nu \ln \mu + O(2,2). \end{aligned} \quad (17)$$

Since h_{00} is twice the effective potential (i.e., the potential whose gradient yields the acceleration), the above equation expresses the relation between the RGGR effective potential and the Newtonian potential (up to first order in both of the perturbations).

B. The external potential effect

Here we consider the dynamical effect of an external Newtonian potential. It is shown that the larger the absolute value of the external potential, the smaller the non-Newtonian effects of the considered system. Hence, an external potential acts as a screening mechanism for RGGR, in the sense that the environment reduces the RGGR non-Newtonian (and non-GR) contribution. To be clear, this effect is not an *ad hoc* feature; it is already part of the theory.

The spherically symmetric case. With the scale setting (8) or (A1), which sets a relation between μ and Φ , gravity is in general sensitive to the Newtonian potential value, such that the dynamics of a system may change due to a constant shift of the Newtonian potential.

In particular, for a static system with spherical symmetry, it is possible to define an effective additional mass of RGGR (δM_{RGGR}) which can be expressed as [55]

$$\begin{aligned} \delta M_{\text{RGGR}}(r) &\equiv (\Phi'_{\text{RGGR}} - \Phi')r^2 \\ &= \bar{\nu} \frac{r}{1 + \frac{4\pi r}{M(r)} \int_r^\infty \rho(a) da}, \end{aligned} \quad (18)$$

where Φ_{RGGR} is the effective potential of RGGR (i.e., its gradient yields the acceleration of a test particle). If the mass distribution is simply that of a particle of mass M , the term with the integral is zero for $r > 0$, and the effective additional mass increases linearly with r (and hence the additional force decreases with r). Contrary to Newtonian gravity or pure general relativity, this result is sensitive to the existence of a spherical mass distribution at large radius. If there is such a mass distribution, the integral $\int_r^\infty \rho(a)ada$ will be greater than zero, and hence δM_{RGGR} will be suppressed.

In conclusion, considering a static spherical mass distribution, the larger the amount of mass outside a given region, the smaller the non-Newtonian effects in that region. Equation (18) reexpresses the external potential effect as an external mass density effect, with the hypothesis that stationary spherical symmetry holds.

Average potentials and subsystems. Let S' be a subsystem of a system S , much smaller than S , such that at S' the average Newtonian potential of S (denoted by Φ_s) is a constant. The use of the term ‘‘average’’ is to be explicit that structures whose characteristic sizes are much smaller than the size of S are not individually considered. Similarly, the (average) Newtonian potential of a galaxy does not consider individual stars, and the (average) Newtonian potential inside of a star does not consider individual particles. Hence the Newtonian potentials of S and S' can be written as,

$$\Phi_s = \phi_s + \phi_e, \quad (19)$$

$$\begin{aligned} \Phi_{s'} &= \phi_{s'} + \Phi_s|_{s'} \\ &= \phi_{s'} + \phi_s|_{s'} + \phi_e \\ &= \phi_{s'} + \phi_{e'} + \phi_e. \end{aligned} \quad (20)$$

In the above, ϕ_s stands for the Newtonian potential generated by the system S , and ϕ_e refers to the total external contribution. The universe is not static or stationary at large scales, and hence ϕ_e is *a priori* an unknown effective constant.

Since S' is a small subsystem of S , the entire external contribution is $\Phi_s|_{s'}$, which is the total (average) Newtonian potential of the system S at the position of the system S' . The external potential contribution to the system S' can be described by two terms. One of them is the same ϕ_e constant that appears in Φ_s , and the other is $\phi_{e'} = \phi_s|_{s'}$.

Consequences for G in the system and subsystem. The expression for G associated to the system S , with explicit reference to the constants G_0 and Φ_0 , reads [using Eqs. (2) and (8)]

$$G_s^{-1}(\phi_s) = G_0^{-1} \left(1 + 2\bar{\nu}_s \ln \frac{\phi_s + \phi_e}{\Phi_0} \right). \quad (21)$$

It is also possible to express the same function G_s^{-1} with respect to another reference potential. In particular, using ϕ_e as the reference potential, one finds,

$$G_s^{-1}(\phi_s) = G_e^{-1} \left[1 + 2\bar{\nu}_e \ln \left(1 + \frac{\phi_s}{\phi_e} \right) \right], \quad (22)$$

with G_e and $\bar{\nu}_e$ such that

$$G_e^{-1}\bar{\nu}_e = G_0^{-1}\bar{\nu}_s, \quad (23)$$

$$G_e^{-1} = G_0^{-1} \left(1 + 2\bar{\nu}_s \ln \frac{\phi_e}{\Phi_0} \right). \quad (24)$$

The above equations are found by demanding that the two expressions for $G_s^{-1}(\phi_s)$ above are compatible among themselves and that the relation between G_0 , G_e , Φ_0 and ϕ_e must be a constant, that is, that it cannot depend on ϕ_s .

Using the relations just presented for changing the reference potential, the $G_{s'}$ function corresponding to subsystem S' can be written as

$$\begin{aligned} G_{s'}^{-1}(\phi_{s'}) &= G_{0'}^{-1} \left(1 + 2\bar{\nu}_{s'} \ln \frac{\phi_{s'} + \phi_{e'} + \phi_e}{\Phi_{0'}} \right) \\ &= G_{e'}^{-1} \left[1 + 2\bar{\nu}_{e'} \ln \left(1 + \frac{\phi_{s'} + \phi_e}{\phi_{e'}} \right) \right]. \end{aligned} \quad (25)$$

If the unknown constant ϕ_e satisfies $|\phi_e| \ll |\phi_s|$ in a given region (which implies that $|\phi_e| \ll |\phi_{e'}|$), then in this region one can write

$$G_s^{-1}(\phi_s) \approx G_e^{-1} \left(1 + 2\bar{\nu}_e \ln \frac{\phi_s}{\phi_e} \right), \quad (26)$$

$$\begin{aligned} G_{s'}^{-1}(\phi_{s'}) &\approx G_{e'}^{-1} \left[1 + 2\bar{\nu}_{e'} \ln \left(1 + \frac{\phi_{s'}}{\phi_{e'}} \right) \right] \\ &\approx G_{e'}^{-1} \left[1 + 2\bar{\nu}_{e'} \left(\frac{\phi_{s'}}{\phi_{e'}} - \frac{1}{2} \frac{\phi_{s'}^2}{\phi_{e'}^2} \right) \right]. \end{aligned} \quad (27)$$

In the above we used that $\ln(1+X) = \ln X + O(1/X)$, for $|X| \gg 1$, and $\ln(1+x) = x - x^2/2 + O(x^3)$, for $|x| \ll 1$. The last expansion assumes $\phi_{s'}/\phi_{e'} < 1$, which is a condition realized in the Solar System context, as it will be shown.

The expansion (27) will prove useful for the PPN application. Henceforth, for simplicity, we will always consider $|\phi_e| \ll |\phi_s|$. Also, this condition is necessary in order to find compatibility with the galaxy results of Ref. [27].

The Solar System as a subsystem of the Galaxy. For this application, the system S will be the Milky Way, designated by the subscript ‘‘MW’’, and the subsystem S' will be the Solar System, which is designated by ‘‘ \odot .’’ From Refs. [64,65], one arrives at the following estimates for

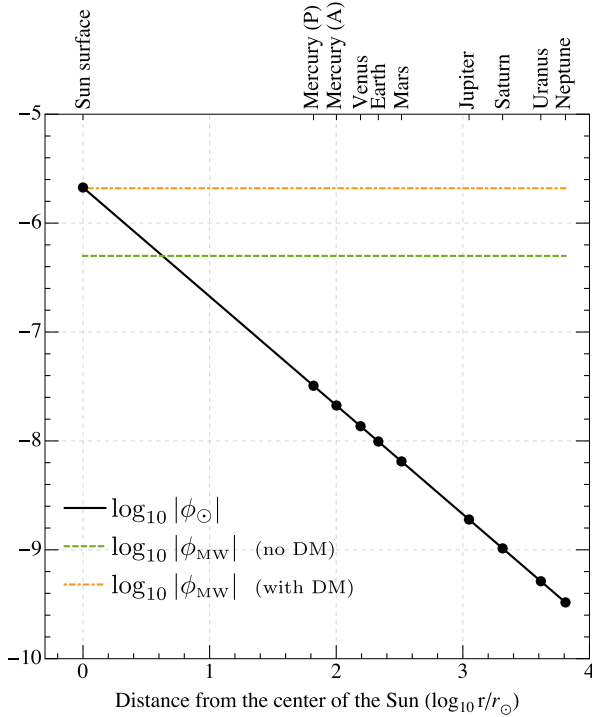


FIG. 1. The Newtonian potential generated by the Sun ϕ_{\odot} across the Solar System, and the value of the Newtonian potentials generated by the Galaxy at the Solar System ($\phi_{\text{MW}}|_{\odot}$). The letters “P” and “A” after Mercury refer to its perihelion and aphelion.

the value of $\phi_{\text{MW}}|_{\odot}$, that is, the Newtonian potential generated by the Milky Way evaluated at the Solar System position: -5×10^{-7} , if only the baryonic matter is considered, or -2.1×10^{-6} considering both the baryonic matter and a standard dark matter halo (both in unities of c^2 , which is set to be one). Since the RG effects may, at least in part, mimic dark matter-like effects, the true value of $\phi_{\text{MW}}|_{\odot}$ should lie in between these two cases.

Figure 1 shows the Newtonian potential generated by the Sun (ϕ_{\odot}) across the Solar System and the values of $\phi_{\text{MW}}|_{\odot}$ either with or without dark matter. The Solar System data was derived from Ref. [66].

The Solar System is a subsystem of the Milky Way, and hence its external potential $\phi_{e'}$ is ϕ_{MW} . For all the planets one finds $\phi_{\odot}/\phi_{e'} \lesssim 10^{-2}$, which shows that the expansion (27) can be used in this context. To be more precise, the largest value of $\phi_{\odot}/\phi_{e'}$ comes from Mercury at its perihelion, and it reads 6.4×10^{-2} for the case without dark matter, and 1.5×10^{-2} for the case with dark matter.

C. Point-particle solution of RGGR: Detailed derivation from the field equations

The purpose of this subsection is to present a derivation of the point-particle solution directly from the field equations (5). This is not the first time that point-particle

solutions have been considered in the RGGR context; see for instance Ref. [27] and Eq. (17), where it was used as a conformal transformation method. However, beyond being useful for verifying the result without the use of conformal transformations (whose employment in gravity theories has led to diverse debates on its meaning), the latter method only holds for the first-order perturbation in both the GR and the RG expansions, while from the direct use of the field equations it is possible to find analytical results valid up to the order $O(\infty, 2)$, that is, up to arbitrary order in the GR parameters, and apart from second-order corrections on the RG parameter $\bar{\nu}$.

Considering the proper spacetime symmetry, and without loss of generality in this context, let

$$ds^2 = g_{00}(r)dt^2 + g_{11}(r)dr^2 + r^2 d\Omega^2, \quad (28)$$

with $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$.

With the above line element, it is straightforward to show the following identities:

$$\begin{aligned} \mathcal{G}_{\mu}^{\nu} &= 0 \quad \forall \mu \neq \nu, \\ \mathcal{G}_2^2 &= \mathcal{G}_3^3. \end{aligned} \quad (29)$$

It is useful to use two field equations, from Eq. (5), rearranged such that Λ does not appear explicitly, that is,

$$\mathcal{G}_0^0 = \mathcal{G}_1^1, \quad (30)$$

$$\mathcal{G}_1^1 = \mathcal{G}_2^2. \quad (31)$$

The remaining nontrivial field equation simply presents the solution for Λ , and it can be written as $\Lambda = -\mathcal{G}_0^0$. This approach was also used in Refs. [55,67]. From the first one, the relation between g_{11} and g_{00} up to first order in $\bar{\nu}$ is derived to be

$$g_{11} = -\frac{K}{g_{00}} \left[1 + 2\bar{\nu} \left(r \frac{\mu'}{\mu} - \ln \frac{\mu}{\mu_1} \right) \right] + O(\infty, 2), \quad (32)$$

where a prime means a derivative with respect to r and μ_1 are integration constants. Within GR, the constant K can be trivially eliminated by a time redefinition ($t \rightarrow t/\sqrt{K}$). Within RGGR, both the constants K and μ_1 can be absorbed through a time redefinition. To prove this, one only needs to note that

$$\begin{aligned} &K \left(1 + 2\bar{\nu} \left(r \frac{\mu'}{\mu} - \ln \frac{\mu}{\mu_1} \right) \right) \\ &\approx K(1 + 2\bar{\nu} \ln \mu_1) \left(1 + 2\bar{\nu} \left(r \frac{\mu'}{\mu} - \ln \mu \right) \right), \end{aligned} \quad (33)$$

up to first order in $\bar{\nu}$. And hence, after the time redefinition $t \rightarrow t/\sqrt{K(1 + 2\bar{\nu} \ln \mu_1)}$, the μ_1 constant is eliminated. Equivalently, it can be fixed to be 1.

From Eqs. (31) and (32), the g_{00} solution is derived to be

$$g_{00} = -1 + \frac{C_1}{r} + C_2 r^2 + 2\bar{\nu} \left(1 - \frac{3C_1}{2r}\right) \ln \mu + O(\infty, 2). \quad (34)$$

The constants C_1 and C_2 within general relativity ($\bar{\nu} = 0$) are associated respectively to a mass at $r = 0$ and to a cosmological constant. As expected, the derived solution is an extension of the Schwarzschild–de Sitter solution. The above result, in the context of a point particle, extends the solution (17).

IV. THE PPN AND LRL ANALYSES FOR THE SOLAR SYSTEM

A. PPN with the external potential effect

Reviews of the PPN formalism can be found in diverse references (e.g., Refs. [45,48,49,68,69]). The full PPN approach, as used in Ref. [45], depends on ten parameters, but scalar-tensor theories have only two parameters whose values can be different from the corresponding GR values, which are commonly denoted by γ and β . As analyzed in detail in Ref. [28], RGGR can be seen as a peculiar type of scalar-tensor theory with a certain constraint and with a potential that is system dependent. Since in a given system the potential is fixed, in the Solar System RGGR is expected to have only two nontrivial PPN parameters. To find their values, we consider the line element (28) and the following metric expansion [45,68]:

$$g_{00} \approx -1 + \frac{2M}{r} - 2(\beta - \gamma) \left(\frac{M}{r}\right)^2, \quad (35)$$

$$g_{11} \approx 1 + 2\gamma \frac{M}{r}. \quad (36)$$

In the above, γ and β are constants, and for GR both of them are equal to 1. A full detailed analysis of RGGR directly from its covariant expression and without the restriction to the Solar System is beyond the scope of this work, and is currently a work in progress. This simpler PPN application, sometimes also referred to as the Eddington expansion, is nonetheless both useful for the Solar System application and to introduce procedures that will be useful for the full PPN development in this context.

Within the PPN framework, it is not uncommon to associate an order of smallness such that $v^n \sim (M/r)^{n/2} \sim O(n)$, where v is the test particle velocity. Nevertheless, for clarity in the present context, we use the convention in which n is associated with the power on the metric perturbation, and hence we use $(M/r)^n \sim O(n)$.

Comparing Eqs. (32) and (34) with Eqs. (35) and (36), one sees that a minimum condition for applying this parametrization is $|C_2| r^2 \approx 0$ for the range of r considered

and $C_1/r \sim O(1)$. With these considerations, Eqs. (32) and (34) can be written as

$$g_{00} = -1 + \frac{C_1}{r} + 2\bar{\nu} \left(1 - \frac{3C_1}{2r}\right) \ln \mu + O(\infty, 2), \quad (37)$$

$$g_{11} = 1 + \frac{C_1}{r} + 2\bar{\nu} \left(r \frac{\mu'}{\mu} + C_1 \left(\frac{\mu'}{\mu} - \frac{\ln \mu}{2r}\right)\right) + O(2, 2). \quad (38)$$

In the above, since C_2 is no longer considered, $O(n, m)$ refers to terms of n th order or higher in C_1 , and terms of m th order or higher in $\bar{\nu}$. All the terms that depend on $C_1 \bar{\nu}$ are considered.

The “ln” terms can be expanded as in Eq. (27). It is convenient to write explicitly the dependence on r ; hence let

$$\frac{\phi_{\odot}}{\phi_e} \equiv \frac{k}{r}. \quad (39)$$

With these considerations,

$$g_{00} = -1 + \frac{C_1}{r} + \frac{k}{r} \bar{\nu} \left(-2 - \frac{k}{r} - \frac{2k^2}{3r^2} + 3 \frac{C_1}{r} + \frac{3kC_1}{2r^2} + \frac{k^2 C_1}{r^3}\right) + O(\infty, 2, 4), \quad (40)$$

$$g_{11} = 1 + \frac{C_1}{r} + 2 \frac{k}{r} \bar{\nu} \left(1 + \frac{3C_1}{2r}\right) + O(2, 2, 2), \quad (41)$$

where $O(\infty, 2, 4)$ refer to terms of arbitrary order in C_1 , of second or higher order in $\bar{\nu}$ and of fourth or higher order in k . The meaning of $O(2, 2, 2)$ follows analogously. There are no terms with C_1^2 in g_{00} since there is no such terms up to first order in $\bar{\nu}$ [see Eq. (34)].

To apply the formalism, it is necessary to relate the expansions used above; otherwise it is impossible to know whether, say, C_1^2/r^2 can be neglected while $k^2 \bar{\nu}/r^2$ is considered. For the case of the planet Mercury, the terms of order $O(1)$ are of the same order as $\phi_{\odot}(r_{\text{Mer}}) = -2.7 \times 10^{-8}$ (the Newtonian potential generated by the Sun at Mercury’s orbit). This should correspond, apart from higher-order corrections, to $-M/r$. In other words, this is the assumption that, at the Newtonian level in the Solar System, RGGR must agree with GR and Newtonian theory; otherwise there is no hope to be compatible with the Solar System data. Moreover, $\bar{\nu}$ needs to be sufficiently small. Considering galactic internal dynamics, the upper bound found for $\bar{\nu}$ was $|\bar{\nu}| \lesssim 10^{-7}$ [27], which will be used as a starting point, but soon a stronger bound will be shown.

The value of k/r_{Mer} can be computed from its definition (39), and it corresponds to $O(0.17)$ for the case without

dark matter and $O(0.25)$ for the case with standard dark matter.

With the above analysis, the relation between the three expansions is clarified, and it is possible to sort the expansion terms. Hence, Eqs. (40) and (41) can be expressed as

$$g_{00} \approx -1 + \frac{C_1}{r} - \frac{k}{r} \bar{\nu} \left(2 + \frac{k}{r}\right), \quad (42)$$

$$g_{11} \approx 1 + \frac{C_1}{r} + 2 \frac{k}{r} \bar{\nu}. \quad (43)$$

The relation between C_1 and M is fixed by comparing Eqs. (35) and (42), and it yields,

$$C_1 = 2M + 2\bar{\nu}k. \quad (44)$$

From the coefficient of r^{-1} in g_{11} , γ is found to be

$$\gamma = 1 + \frac{2\bar{\nu}k}{M} = 1 - \frac{2\bar{\nu}}{\phi_{e'}}. \quad (45)$$

According to Ref. [49], γ is constrained from Solar System experiments and observations to satisfy $|\gamma - 1| \lesssim 10^{-5}$. Since $|\phi_{e'}| \sim 10^{-6}$, we derive from the above that $|\bar{\nu}| \lesssim 10^{-11}$.

From the coefficient of r^{-2} in Eq. (42), the value of γ and Eq. (35), β can be derived as

$$\beta = 1 + \frac{\bar{\nu}k^2}{2M^2} = 1 + \frac{\bar{\nu}}{2\phi_{e'}^2}. \quad (46)$$

Since $|\beta - 1| \lesssim 10^{-4}$ [49], the above implies that

$$|\bar{\nu}_\odot| \lesssim 10^{-16}. \quad (47)$$

We stress that the external potential effect is essential for the PPN application in its standard form [45], and that this bound above considers it. In the next subsection, by using the LRL approach, this bound with the external potential effect is confirmed, and it will be compared to the case without it.

B. LRL vector dynamics with and without the external potential effect

This section is devoted to estimating the upper bound on $\bar{\nu}$ in the Solar System from the LRL vector dynamics using the perihelion precession data from Ref. [70], which is more recent and precise than the data set used in Ref. [53].

Reviews on the LRL vector can be found in Refs. [50–52]. The notation and approach used here follow closely those of Ref. [53].

One of the most important results of general relativity is predicting a correction to the precession of the orbit of the

TABLE I. Estimated corrections, in milliarseconds per century, to the standard Newtonian-Einsteinian secular precessions of the perihelion determined with the INPOP10a and the EPM2011 ephemerides [71,72]. The relevant data for the purposes of this work are the uncertainties in this table.

Planet	EPM2011	INPOP10a
Mercury	-2.0 ± 3.0	0.4 ± 0.6
Venus	2.6 ± 1.6	0.2 ± 1.5
Earth	0.19 ± 0.19	-0.20 ± 0.90
Mars	-0.020 ± 0.037	-0.040 ± 0.150
Jupiter	58.7 ± 28.3	-41.0 ± 42.0
Saturn	-0.32 ± 0.47	0.15 ± 0.65

planets. New determinations of the corrections to the usual Newtonian-Einsteinian secular precession of perihelion of the planets constitute a relevant data set to constrain modified gravitation models in the Solar System; see Table I.

From Eq. (17), the RGGR gravitational potential, apart from the terms $O(\nu^2)$ and $O(\nu\Phi)$, reads

$$\Phi_{\text{RGGR}} = \Phi - \nu \ln \mu. \quad (48)$$

The above potential is essentially the one that appears in Eq. (42) and was used for PPN, but without using the k expansion. Although the PPN approach started from a more precise framework, with computations valid to arbitrary order in C_1 , in the end there was no significant change in the RGGR potential in comparison with the one derived from the conformal transformation. It should be stressed, however, that the LRL technique only tests the planetary orbits, while the PPN approach used here tests both the orbits (from β) and the light deflection due to the Sun (from γ). Therefore, since the major RGGR constraint found from the PPN formalism came from the β observational constraints, the LRL analyses should yield essentially the same bound on $\bar{\nu}_\odot$. This will be confirmed in this section.

From Eq. (48), for a point particle of mass m the force is given by

$$\mathbf{F}_{\text{RGGR}} = \mathbf{F}_{\text{N}} + m\nu \nabla \ln \mu, \quad (49)$$

where \mathbf{F}_{N} is the Newtonian gravitational force for a point particle. Using the effective μ from Eq. (8), and considering the external potential effect,

$$\mathbf{F}_{\text{RGGR}} = \mathbf{F}_{\text{N}} - \frac{m\bar{\nu}r_0}{r(r+r_0)} \hat{\mathbf{r}}, \quad (50)$$

where $r_0 = -M_\odot/\phi_{e'}$ and $\hat{\mathbf{r}}$ is the standard unit radial vector with origin at the Sun. The above is the gravitational force acting on a point particle of mass m in the weak-field regime.

TABLE II. Orbital parameters of the planets. Here, τ is the orbital period, ϵ is the eccentricity and a is the major semiaxis of the orbit [66].

Planet	τ (years)	ϵ	a (10^{10} m)
Mercury	0.241	0.2056	5.791
Venus	0.615	0.0067	10.82
Earth	1	0.0167	14.96
Mars	1.881	0.0935	22.792
Jupiter	11.862	0.0489	77.857
Saturn	29.457	0.0565	143.353

The LRL vector associated to the Sun with mass M_\odot and a planet of mass m is given by

$$\mathbf{A} = \mathbf{p} \times \boldsymbol{\ell} - m^2 M_\odot \hat{\mathbf{r}}, \quad (51)$$

where \mathbf{p} is the linear momentum of the particle of mass m , and $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$ is the angular momentum. Some important properties of this vector are that $\mathbf{A} \cdot \boldsymbol{\ell} = 0$ and that, for an unperturbed Newtonian gravity, $d\mathbf{A}/dt = \mathbf{0}$ and \mathbf{A} is collinear to the major axis. In the case of any perturbation, in general \mathbf{A} will not be a constant of motion and will slowly precess. Also, the magnitude of the LRL vector yields a relation to the eccentricity ϵ ,

$$|\mathbf{A}| = m^2 M_\odot \epsilon. \quad (52)$$

The average precession of the orbit is derived from the computation of $\langle d\mathbf{A}/dt \rangle$ and reads [53]

$$\Omega = -\frac{\langle F_p \cos \varphi \rangle}{M_\odot m^2 \epsilon} \boldsymbol{\ell}, \quad (53)$$

where φ is the angle between \mathbf{A} and the major semiaxis, the symbol $\langle \rangle$ means an average in the sense

$$\langle X \rangle \equiv \frac{m}{\ell \tau} \int_0^{2\pi} r^2(\varphi) X(r(\varphi), \varphi) d\varphi, \quad (54)$$

and τ is the period of the unperturbed motion.

TABLE III. Upper bound on $|\bar{\nu}_\odot|$ either with or without the external potential effect (EPE), using $\phi_e \sim 10^{-6}$. The data for Ω came from the uncertainties in Table I. Among the two samples in that table, the smallest uncertainties were selected for each planet.

Planet	$ \bar{\nu}_\odot $ (with EPE)	$ \bar{\nu}_\odot $ (without EPE)
Mercury	$\lesssim 10^{-16}$	$\lesssim 10^{-19}$
Venus	$\lesssim 10^{-15}$	$\lesssim 10^{-19}$
Earth	$\lesssim 10^{-16}$	$\lesssim 10^{-20}$
Mars	$\lesssim 10^{-16}$	$\lesssim 10^{-20}$
Jupiter	$\lesssim 10^{-12}$	$\lesssim 10^{-17}$
Saturn	$\lesssim 10^{-13}$	$\lesssim 10^{-19}$

From Eq. (53), the orbit precession velocity associated to the force (50) reads,

$$\Omega = \frac{2\pi \bar{\nu} a (1 - \epsilon^2)}{M_\odot \tau \epsilon^2} \left[\frac{1}{\sqrt{1 - \left[\frac{r_0 \epsilon}{r_0 + a(1 - \epsilon^2)} \right]^2}} - 1 \right], \quad (55)$$

where a is the major semiaxis of the ellipse. The above generalizes the Ω expression for RGGR of Ref. [53], by considering the presence of an external potential. Namely, the expression of Ref. [53] is found in the limit $r_0 \rightarrow \infty$.

Using Eq. (55) with the values of the perihelion precession of the planets, as given by Table I, together with the values of a , τ and ϵ from Table II, one can find an upper bound on $\bar{\nu}$ for each of the planetary orbits. The $\bar{\nu}$ bounds with and without the external potential can be seen respectively in Table III.

V. CONCLUSIONS

Currently there are research lines that look for GR extensions based on the RG flow of G and Λ , and consider their possible effects on large-scale (infrared) physics. These extend GR by using principles that are well established in other contexts. The approaches that are natural within this RG framework are either impossible or unnatural to achieve by other means, and thus it introduces new paths for extending and evaluating gravity. Also, finding nontrivial flows of G and Λ on large scales may provide clues about quantum gravity [15,17].

In this work we focused on a particular realization named RGGR [27,28]. It is based on and extends the approaches of Refs. [29,33]. This extension depends on an effective dimensionless quantity $\bar{\nu}$ that measures the strength of the RG in a given system, and it is such that in the limit $\bar{\nu} \rightarrow 0$ the theory becomes pure GR. Two previous works have found bounds on $\bar{\nu}$ in the Solar System ($\bar{\nu}_\odot$) [53,54], but they did not consider the external potential effect, which we presented here for the first time. This effect is part of RGGR within either the noncovariant or the covariant formulations, and it acts as a new kind of screening mechanism.

Considering this external potential effect, we found that $|\bar{\nu}_\odot| \lesssim 10^{-16}$ either from the PPN formalism or the LRL vector technique. The external potential effect could alleviate the bounds associated to the Solar System planetary orbits from 3 to 6 orders of magnitude (see Table III). However, this effect alone cannot fully explain the difference between the effective $\bar{\nu}$ in the Solar System from that in a galaxy, in case RGGR does have a significant impact on galactic dark matter.

The external potential effect acts as a screening mechanism for RGGR in the sense that, the larger the external potential, the smaller the non-GR corrections. In other words, the environment can in principle hide the RG effects. Quantitatively, the external potential effect is not

sufficient to completely hide the RG effects in the Solar System, if the RG effects are relevant for galactic dynamics; but, in the end, it is a significant dynamical effect that should always be considered.

The change of $\bar{\nu}$ from system to system may follow a linear correlation to the system mass, as argued in Refs. [27,56], and also be compatible with the bounds derived here for the Solar System. The Appendices develop further the effective changes of $\bar{\nu}$ from system to system, but the precise mechanism that may allow for a variation of $\bar{\nu}$ of about 8 to 10 orders of magnitude from the Solar System to a galaxy, if there is one, is still unclear.

Independent of the possible connection to dark matter, here we have evaluated the RGGR Solar System bounds, introduced in detail the external potential effect for the first time, developed an approach for applying the standard PPN formalism to RGGR (this approach requires the use of the external potential effect), and used the LRL technique to evaluate the magnitude of the external potential effect. This latter effect may open new possibilities for screening mechanisms, not necessarily related to the RGGR approach.

ACKNOWLEDGMENTS

We thank Tomi Koivisto, David Mota and Ilya Shapiro for commenting on previous versions of this work, and José de Freitas Pacheco for a discussion on the PPN parameters. S. M. and A. O. F. A. thank CAPES (Brazil) for support. D. C. R. thanks CNPq (Brazil) and FAPES (Brazil) for partial financial support.

APPENDIX A: A SPECIFIC COVARIANT SCALE SETTING

Consider the following simple realization of the covariant setting (3):

$$\mu = f(U^\alpha U^\beta h_{\alpha\beta}) = A + BU^\alpha U^\beta h_{\alpha\beta}, \quad (\text{A1})$$

where A and B are constants. This simple covariant scale setting was introduced in Ref. [28], and it will be shown in detail in this and the next appendix that, under certain reasonable limits, it is a covariant extension of the scale setting (8).

Adopting a comoving coordinate system ($U^i = 0$), the scalar $U^\alpha U^\beta h_{\alpha\beta}$ can be expressed as, with $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$,

$$\begin{aligned} U^\alpha U^\beta h_{\alpha\beta} &= U^0 U^0 h_{00} \\ &= -\frac{h_{00}}{g_{00}} \\ &= -\left(1 + \frac{1}{G\bar{g}_{00}}\right) \\ &= -1 + \frac{G^{-1}}{1 - \bar{h}_{00}}. \end{aligned} \quad (\text{A2})$$

In the above, we used the fact that $\bar{g}_{00} \equiv G^{-1}g_{00}$. The above expression fixes a relation between μ and \bar{h}_{00} . For a Minkowski background, following Sec. III A, the relation between \bar{h}_{00} and the Newtonian potential Φ , reads

$$\bar{h}_{00} = -2\Phi + O(2, 2), \quad (\text{A3})$$

where we used the fact that the metric that solves the Einstein equation $\tilde{G}_{\alpha\beta} = 8\pi\tilde{T}_{\alpha\beta}$ is $\tilde{g}_{\alpha\beta}$, whose time-time component satisfies $\tilde{g}_{00} = -1 - 2\Phi + O(2)$, and that $\tilde{g}_{00} = G^{-1}g_{00} + O(2, 2) = \bar{g}_{00} + O(2, 2)$.

The function $\mu(\bar{h}_{00})$, or $\mu(\Phi)$, will not be an analytical function in general. Indeed, considering the $G(\mu)$ expression as given in Eq. (2), the equation (A2) is a transcendental one for μ .

Far away from any mass, $h_{\alpha\beta}$ should become zero (i.e., the metric $g_{\alpha\beta}$ should coincide with the background), and hence in this limit $\mu = A$, which in turn implies that $G^{-1} = 1 + 2\nu \ln A$. Using unities such that $G|_{h_{\alpha\beta}=0} = G_0 = 1$, one finds

$$A = 1. \quad (\text{A4})$$

To avoid any singularity in G for any $\mu \in [1, \infty)$, ν needs to be positive, and this is always assumed henceforth.

Combining the previous equations,

$$\begin{aligned} \mu &= 1 + B\left(-1 + \frac{G^{-1}}{1 - \bar{h}_{00}}\right) \\ &= 1 + B\left(-1 + \frac{1 + 2\nu \ln \mu}{1 + 2\Phi} + O(2, 2)\right). \end{aligned} \quad (\text{A5})$$

This is a transcendental equation for μ , but it can be solved for Φ ,

$$\Phi = \frac{1}{2}\left(\frac{1 + 2\nu \ln(1 + \delta\mu)}{1 + \frac{\delta\mu}{B}} - 1\right) + O(\nu^2). \quad (\text{A6})$$

In the above, we introduced $\delta\mu \equiv \mu - 1 > 0$. As expected, from the above one finds $\lim_{\delta\mu \rightarrow 0} \Phi = 0$.

Up to this point, B is simply any real number, but from the previous results, and two considerations, its value can be found. For sufficiently small $\delta\mu$, Φ reads

$$\Phi|_{\substack{\delta\mu \ll |B| \\ \delta\mu \ll 1}} \approx \frac{1}{2}\left(2\nu\delta\mu - \frac{\delta\mu}{B}\right) = \frac{1}{2}\delta\mu(2\nu - B^{-1}). \quad (\text{A7})$$

The first consideration is that the inequality $\Phi \leq 0$ must be satisfied, and hence, since $\nu > 0$,

$$0 < B \leq \frac{1}{2\nu}. \quad (\text{A8})$$

The second consideration is that when $\delta\mu \rightarrow 0$ or equivalently when $h_{\alpha\beta} \rightarrow 0$, Φ should smoothly go to zero, implying that

$$\lim_{\delta\mu \rightarrow 0} \partial_{\delta\mu} \Phi = 0. \quad (\text{A9})$$

Therefore,

$$B = \frac{1}{2\nu}. \quad (\text{A10})$$

With the above, Eq. (A6) can now be simply written as

$$\Phi = \nu \ln(1 + \delta\mu) - \nu\delta\mu + O(\nu^2). \quad (\text{A11})$$

To clarify the meaning of the above equation, it is stating a correlation between $\delta\mu$ and Φ , and this correlation, naturally, only exists if $\nu \neq 0$. This correlation is the one that comes from the covariant scale setting, and should be compared with the noncovariant one (8). The above equation cannot be solved analytically to express either $\delta\mu$ or μ as a function of Φ , and hence Eq. (8) should be seen as a local analytical approximation for the function $\mu(\Phi)$.

Figure 2 shows a parametric plot of the evolution of δG as a function of Φ for different values of ν . The highest value of ν used in that figure corresponds to the value used in galaxies without dark matter (e.g., Refs. [55,56]), and the smallest one is close to the Solar System bound derived in Ref. [54]. It can be seen that changes of many orders of magnitude in Φ translate into a much smaller variation in δG . The range of Φ includes values corresponding to the surface of a neutron star ($\sim 10^{-1}$), and down to 10^{-10} , which is about the Newtonian potential generated by the baryonic

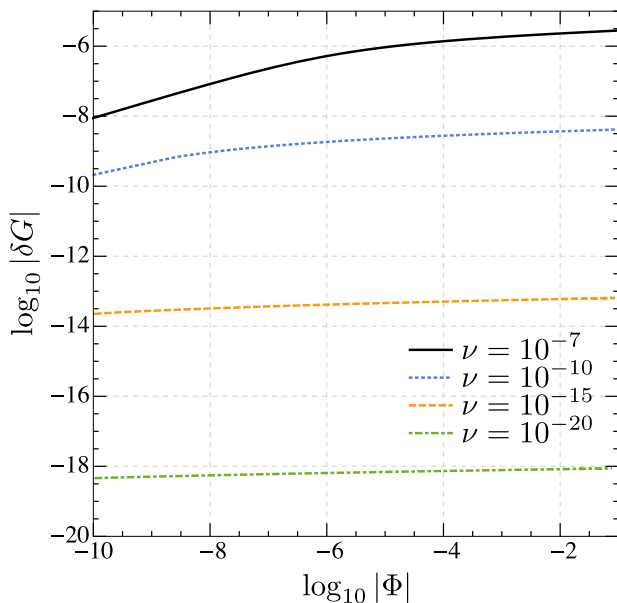


FIG. 2. The relation between $|\delta G| = |G - 1| \approx 2\nu \ln \mu$ [from Eq. (2)] and $|\Phi|$ [from Eq. (A6)] for four different values of ν .

matter of dwarf galaxies at their farthest observed rotation curve radius.

APPENDIX B: THE NONCOVARIANT SCALE SETTING AS AN APPROXIMATION FOR THE COVARIANT ONE

Equation (8) implies that

$$\alpha = \frac{1}{\mu \partial_{\mu} \ln(-\Phi)}, \quad (\text{B1})$$

where ∂_{μ} is the derivative with respect to μ . In Eq. (8), α appears as a constant, but from the perspective of the covariant scale setting, α should in general be a function of μ , as given by the above equation. If, for a given system, α is close to a constant, then for that system the noncovariant scale setting may work as a good approximation. The relation between α and Φ can be seen in Fig. 3, which indeed shows that α changes slowly even if Φ changes by some orders of magnitude.

By using the noncovariant approach, one is using an approximation to derive the PPN parameters of the covariant approach. This approximation must be sufficiently precise. One way to evaluate this error is to consider, from a given value of Φ , the relative error between the two μ 's inferred from Eqs. (8) and (A11). Since it is possible to analytically express $\Phi(\mu)$, it is more convenient to adopt the inverse route, that is, from a given μ , to find the relative error between the potentials inferred by Eqs. (8) and (A11), which we call here Φ and Φ_A respectively. If the maximum relative error between Φ and Φ_A , along Mercury's orbit, is

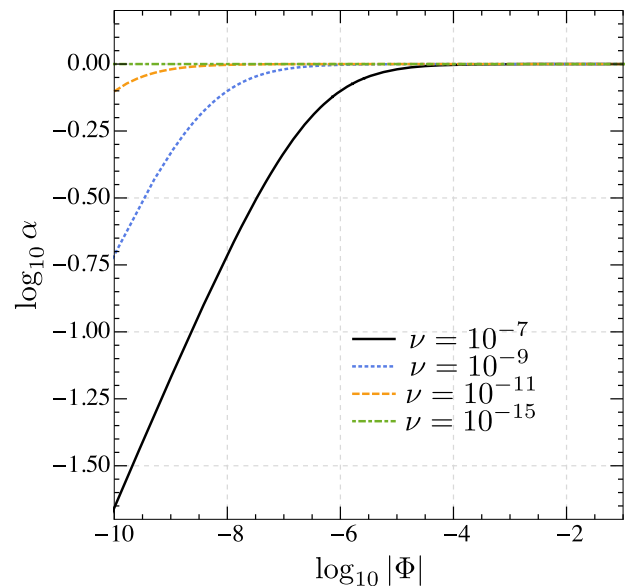


FIG. 3. The relation between α and Φ for different values of ν . It shows that the noncovariant approximation, where α is a constant, can be a good approximation for many systems, since large changes of Φ lead to much smaller changes in α .

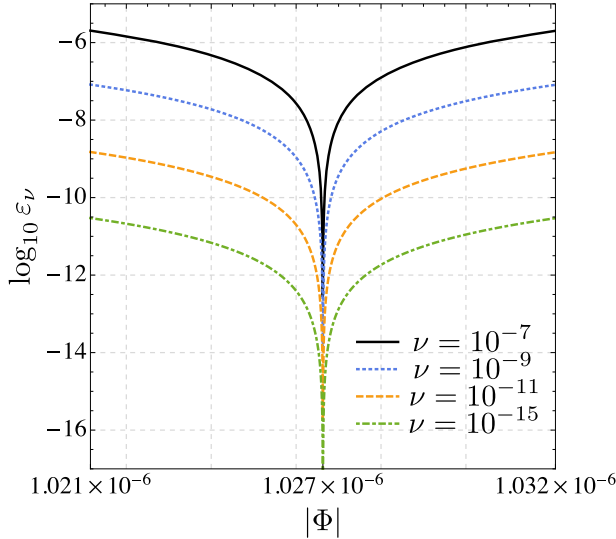


FIG. 4. The relative errors (B3) introduced by the use of the approximation (8), in the context of Mercury’s orbit with the external potential of the Galaxy. The range of Φ in this plot spans the variation of the Newtonian potential through the planet orbit added by 10^{-6} . Dividing or multiplying the latter value by 5, does not change the value of $\log_{10} \epsilon$ significantly. The plot indicates that the noncovariant scale setting works as a good approximation for the covariant one, in the context of Mercury’s orbit.

ϵ_ν^{\max} , then the PPN parameter γ , and consequently β , acquires an additional uncertainty of $\pm \gamma \epsilon_\nu^{\max}$ when inferred from the other approach. In particular, if $\epsilon_\nu^{\max} \sim 10$, it is no longer possible to use one approach (the noncovariant scale setting) to state precisely the order of magnitude of either γ or β of the other approach (the covariant scale setting case).

In order to show that both of the approaches lead to compatible bounds for Mercury’s orbit, we compare the Newtonian potential from Eq. (A11), $\Phi(\mu)$, to an approximated potential given by Eq. (8), namely

$$\Phi_A(\mu, \mu_0) \equiv \mu^{1/\alpha(\mu_0)} \Phi_0(\mu_0), \quad (\text{B2})$$

where $\alpha(\mu)$ is given by Eq. (B1), and $\Phi_0(\mu_0)$ is such that $\Phi_A(\mu_0, \mu_0) = \Phi(\mu_0)$. In the plot of Fig. 3, the above approximation corresponds to a straight line approximation at μ_0 to the $\alpha(\mu)$ curve. To quantify the approximation, we use the relative error that is given by

$$\epsilon_\nu(\mu, \mu_0) \equiv \left| 1 - \frac{\Phi(\mu)}{\Phi_A(\mu, \mu_0)} \right|. \quad (\text{B3})$$

Without considering the external potential effect, the range of Φ values of relevance is from -3.2×10^{-8} to -2.1×10^{-8} . The contribution of the Milky Way to the local potential depends on whether dark matter is being considered or not, but for both cases it is about $\phi_{\text{MW}}(r_\odot) \sim 10^{-6}$ at the Solar System position. This means that the range of variation of the Newtonian potential along the orbit of Mercury is $[-(K_{\text{MW}} + 0.032) \times 10^{-6}, -(K_{\text{MW}} + 0.021) \times 10^{-6}]$, where K_{MW} is a number roughly equal to unity whose precise value depends on the amount of dark matter in the Milky Way.

Figure 4 shows that the relative error introduced by the approximation (8) is small enough to allow for a PPN evaluation for the planet Mercury for all the relevant values of ν . For the case without the external potential effect, one can draw a similar plot as that of Fig. 4, with higher values of the relative errors, but no higher than 10^{-2} .

Since the main focus here is on order-of-magnitude evaluations of the Solar System bounds, the above shows that there exists a value for α such that Eq. (8) can work as a satisfactory approximation to the covariant scale setting (A1), considering the post-Newtonian analysis of Mercury’s orbit.

-
- [1] V. Faraoni and S. Capozziello, *Beyond Einstein Gravity* (Springer, New York, 2011).
 - [2] S. Capozziello and M. De Laurentis, *Phys. Rep.* **509**, 167 (2011).
 - [3] T. Clifton, P.G. Ferreira, A. Padilla, and C. Skordis, *Phys. Rep.* **513**, 1 (2012).
 - [4] J. Julve and M. Tonin, *Nuovo Cimento Soc. Ital. Fis.* **46B**, 137 (1978).
 - [5] A. Salam and J. Strathdee, *Phys. Rev. D* **18**, 4480 (1978).
 - [6] E. Fradkin and A. A. Tseytlin, *Nucl. Phys.* **B201**, 469 (1982).
 - [7] B. Nelson and P. Panangaden, *Phys. Rev. D* **25**, 1019 (1982).
 - [8] J. T. Goldman, J. Perez-Mercader, F. Cooper, and M. M. Nieto, *Phys. Lett. B* **281**, 219 (1992).
 - [9] I. L. Shapiro and J. Solà, *Phys. Lett. B* **475**, 236 (2000).
 - [10] A. Bonanno and M. Reuter, *Phys. Rev. D* **62**, 043008 (2000).
 - [11] A. Bonanno and M. Reuter, *Phys. Lett. B* **527**, 9 (2002).
 - [12] M. Reuter and F. Saueressig, *Phys. Rev. D* **65**, 065016 (2002).
 - [13] A. Bonanno and M. Reuter, *Phys. Rev. D* **65**, 043508 (2002).
 - [14] E. Bentivegna, A. Bonanno, and M. Reuter, *J. Cosmol. Astropart. Phys.* **01** (2004) 001.
 - [15] M. Reuter and H. Weyer, *J. Cosmol. Astropart. Phys.* **12** (2004) 001.

- [16] A. Bonanno, G. Esposito, and C. Rubano, *Classical Quantum Gravity* **21**, 5005 (2004).
- [17] M. Niedermaier and M. Reuter, *Living Rev. Relativ.* **9**, 5 (2006).
- [18] I. L. Shapiro and J. Solà, *Phys. Lett. B* **682**, 105 (2009).
- [19] S. Weinberg, *Phys. Rev. D* **81**, 083535 (2010).
- [20] A. Bonanno, *Phys. Rev. D* **85**, 081503 (2012).
- [21] J. Solà, *J. Phys. Conf. Ser.* **453**, 012015 (2013).
- [22] I. L. Shapiro, *Classical Quantum Gravity* **25**, 103001 (2008).
- [23] R. Percacci, in *Approaches to Quantum Gravity*, edited by D. Oriti (Cambridge University Press, Cambridge, England, 2007), p. 111.
- [24] M. Reuter and F. Saueressig, *New J. Phys.* **14**, 055022 (2012).
- [25] T. Appelquist and J. Carazzone, *Phys. Rev. D* **11**, 2856 (1975).
- [26] E. V. Gorbar and I. L. Shapiro, *J. High Energy Phys.* **02** (2003) 021.
- [27] D. C. Rodrigues, P. S. Letelier, and I. L. Shapiro, *J. Cosmol. Astropart. Phys.* **04** (2010) 020.
- [28] D. C. Rodrigues, B. Chauvineau, and O. F. Piattella, *J. Cosmol. Astropart. Phys.* **09** (2015) 009.
- [29] M. Reuter and H. Weyer, *Phys. Rev. D* **69**, 104022 (2004).
- [30] B. Koch and I. Ramirez, *Classical Quantum Gravity* **28**, 055008 (2011).
- [31] M. Hindmarsh and I. D. Saltas, *Phys. Rev. D* **86**, 064029 (2012).
- [32] E. Manrique, M. Reuter, and F. Saueressig, *Ann. Phys. (Amsterdam)* **326**, 463 (2011).
- [33] I. L. Shapiro, J. Solà, and H. Stefancic, *J. Cosmol. Astropart. Phys.* **01** (2005) 012.
- [34] A. Vainshtein, *Phys. Lett.* **39B**, 393 (1972).
- [35] J. Khoury and A. Weltman, *Phys. Rev. Lett.* **93**, 171104 (2004).
- [36] K. Hinterbichler and J. Khoury, *Phys. Rev. Lett.* **104**, 231301 (2010).
- [37] T. S. Koivisto, D. F. Mota, and M. Zumalacarregui, *Phys. Rev. Lett.* **109**, 241102 (2012).
- [38] K. Nordtvedt, Jr., *Astrophys. J.* **161**, 1059 (1970).
- [39] C. M. Will and K. Nordtvedt, Jr., *Astrophys. J.* **177**, 757 (1972).
- [40] L. Blanchet, T. Damour, and G. Schaefer, *Mon. Not. R. Astron. Soc.* **242**, 289 (1990).
- [41] T. Damour, M. Soffel, and C.-m. Xu, *Phys. Rev. D* **43**, 3273 (1991).
- [42] T. Damour, M. Soffel, and C.-m. Xu, *Phys. Rev. D* **45**, 1017 (1992).
- [43] T. Damour, M. Soffel, and C.-m. Xu, *Phys. Rev. D* **47**, 3124 (1993).
- [44] T. Damour, M. Soffel, and C.-m. Xu, *Phys. Rev. D* **49**, 618 (1994).
- [45] C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).
- [46] S. A. Klioner and M. H. Soffel, *Phys. Rev. D* **62**, 024019 (2000).
- [47] L. Blanchet, T. Damour, and G. Esposito-Farese, *Phys. Rev. D* **69**, 124007 (2004).
- [48] L. Blanchet, *Living Rev. Relativ.* **17**, 2 (2014).
- [49] C. M. Will, *Living Rev. Relativ.* **17**, 4 (2014).
- [50] H. Goldstein, C. P. P. Jr., and J. L. Saffo, *Classical Mechanics*, 3rd ed. (Pearson, New York, 2001).
- [51] J. Sivardière, *Am. J. Phys.* **52**, 909 (1984).
- [52] U. Ben-Ya'acov, *J. Math. Phys.* **51**, 122902 (2010).
- [53] C. Farina, W. Kort-Kamp, S. Mauro, and I. L. Shapiro, *Phys. Rev. D* **83**, 124037 (2011).
- [54] S.-S. Zhao and Y. Xie, *Phys. Rev. D* **92**, 064033 (2015).
- [55] D. C. Rodrigues, *J. Cosmol. Astropart. Phys.* **09** (2012) 031.
- [56] D. C. Rodrigues, P. L. de Oliveira, J. C. Fabris, and G. Gentile, *Mon. Not. R. Astron. Soc.* **445**, 3823 (2014).
- [57] F. Bauer, *Classical Quantum Gravity* **22**, 3533 (2005).
- [58] J. R. Ray, *J. Math. Phys.* **13**, 1451 (1972).
- [59] R. M. Wald, *General Relativity* (University of Chicago, Chicago, 1984).
- [60] D. C. Rodrigues, P. S. Letelier, and I. L. Shapiro, in *Proceedings of the International Conference on Two Cosmological Models* (Plaza y Valdés, Madrid, 2012), p. 151.
- [61] J. C. Fabris, P. L. de Oliveira, D. C. Rodrigues, A. M. Velasquez-Toribio, and I. L. Shapiro, *Int. J. Mod. Phys. A* **27**, 1260006 (2012).
- [62] D. C. Rodrigues, P. L. de Oliveira, J. C. Fabris, and I. L. Shapiro, *AIP Conf. Proc.* **1471**, 98 (2011).
- [63] P. de Oliveira, J. de Freitas Pacheco, and G. Reinisch, *Gen. Relativ. Gravit.* **47**, 12 (2015).
- [64] F. Iocco, M. Pato, and G. Bertone, *Nat. Phys.* **11**, 245 (2015).
- [65] M. Pato and F. Iocco, *Astrophys. J.* **803**, L3 (2015).
- [66] NASA space science data coordinated archive. <http://nssdc.gsfc.nasa.gov/planetary/> (2016).
- [67] D. C. Rodrigues, B. Koch, O. F. Piattella, and I. L. Shapiro, *AIP Conf. Proc.* **1647**, 57 (2015).
- [68] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, England, 2003).
- [69] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).
- [70] L. Iorio, *Int. J. Mod. Phys. D* **24**, 1530015 (2015).
- [71] A. Fienga, J. Laskar, P. Kuchynka, H. Manche, G. Desvignes, M. Gastineau, I. Cognard, and G. Theureau, *Celest. Mech. Dyn. Astron.* **111**, 363 (2011).
- [72] E. V. Pitjeva and N. P. Pitjev, *Mon. Not. R. Astron. Soc.* **432**, 3431 (2013).