

Gravitational bending angle of light for finite distance and the Gauss-Bonnet theorem

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We discuss a possible extension of calculations of the bending angle of light in a static, spherically symmetric and asymptotically flat spacetime to a nonasymptotically flat case. We examine a relation between the bending angle of light and the Gauss-Bonnet theorem by using the optical metric. A correspondence between the deflection angle of light and the surface integral of the Gaussian curvature may allow us to take account of the finite distance from a lens object to a light source and a receiver. Using this relation, we propose a method for calculating the bending angle of light for such cases. Finally, this method is applied to two examples of the nonasymptotically flat spacetimes to suggest finite-distance corrections: the Kottler (Schwarzschild–de Sitter) solution to the Einstein equation and an exact solution in Weyl conformal gravity.

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I. INTRODUCTION

The gravitational bending of light by mass led to the first experimental confirmations of the theory of general relativity. In modern astronomy and cosmology, the gravitational lensing is widely used as one of the important tools for probing extrasolar planets, dark matter and dark energy.

The light bending is also of theoretical importance, especially for studying a null structure of a spacetime. A rigorous form of the bending angle plays an important role in understanding properly a strong gravitational field [1–8]. For example, strong gravitational lensing in a Schwarzschild black hole was considered by Frittelli *et al.* [1], by Virbhadra and Ellis [2] and more comprehensively by Virbhadra [3]; Virbhadra *et al.* [4] studied distinctive lensing features of naked singularities. Virbhadra and Ellis [5] and Virbhadra and Keeton [6] later described the strong gravitational lensing by naked singularities; DeAndrea and Alexander [9] discussed the lensing by naked singularities to test the cosmic censorship hypothesis; Eiroa *et al.* [7] treated Reissner-Nordström black hole lensing; Perlick [8] discussed the lensing by a Barriola-Vilenkin monopole and also that by an Ellis wormhole. Kitamura *et al.* proposed a lens model whose gravitational potential declines as $1/r^n$ [10] in order to study the gravitational lensing by exotic matter (or energy) [11–14] that might follow a nonstandard equation of state. See Tsukamoto *et al.* [15] for its possible connection to the Tangherlini solution to the higher-dimensional Einstein equation.

Some recent papers give the expressions for the deflection of light for the Kottler (often called Schwarzschild–de Sitter) spacetime [16–23] and for the spherical, static and vacuum exact solution in Weyl conformal gravity [24–28]. However, their results are not in agreement with each other and hence they are controversial. The apparent inconsistency among the previous works might be caused, because

the spacetimes are not asymptotically flat and their methods are no longer appropriate for treating such a nonasymptotically flat spacetime. In the nonasymptotically flat spacetime, we can never assume that the source of light is located at infinite distance from a gravitational lens object. The main purpose of this paper is to discuss a possible extension of calculations of the bending angle of light in a static, spherically symmetric and asymptotically flat spacetime, particularly in order to find finite-distance corrections.

For this purpose, we shall examine a relation between the bending angle of light and the Gauss-Bonnet theorem in differential geometry. In this sense, the present paper may discuss a possible extension of Gibbons and Werner [29]. They considered two different domains: one (say, \mathcal{D}) bounded by two light rays, to exhibit the connection between topology and multiple images, and the other (say, \mathcal{D}') bounded by one light ray and a nongeodesic circular arc, to compute the asymptotic deflection angle. They suggested that the asymptotic deflection angle of light can be written as the surface integral of the Gaussian curvature over the domain \mathcal{D}' . They did integrate only for the asymptotic case, for which they assumed the observer and source are in the asymptotically Euclidean region. Namely, the angles at the location of the observer and source are defined only in Euclidean space [29].

Throughout this paper, we use the unit of $G = c = 1$. In the following, the observer may be called the receiver in order to avoid a confusion between r_O and r_0 by using r_R .

II. LIGHT PROPAGATION, OPTICAL METRIC AND GAUSS-BONNET THEOREM

A. Static and spherically symmetric spacetime

We consider a static and spherically symmetric (SSS) spacetime. The SSS spacetime can be described as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2, \end{aligned} \quad (1)$$

where the origin of the spatial coordinates is chosen as the location of a lens object, μ and ν run from 0 to 3, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. By introducing two functions as $A(r) \equiv -g_{tt}$ and $B(r) \equiv g_{rr}$, Eq. (1) is rewritten as

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 d\Omega^2. \quad (2)$$

B. Optical metric

Light rays satisfy the null condition as $ds^2 = 0$, which is rearranged via Eq. (2) as

$$\begin{aligned} dt^2 &= \gamma_{ij} dx^i dx^j \\ &= \frac{B(r)}{A(r)} dr^2 + \frac{r^2}{A(r)} d\Omega^2, \end{aligned} \quad (3)$$

where i and j denote 1, 2 and 3, and γ_{ij} is often called the optical metric [30]. The optical metric defines a three-dimensional Riemannian space (denoted as M^{opt}), in which the light ray is expressed as a spatial curve.

For the spherically symmetric spacetime, without the loss of generality, we can choose the photon orbital plane as the equatorial plane ($\theta = \pi/2$). The two-dimensional coordinates on the equatorial plane are denoted as x^I ($I = 1, 2$), where I may mean r, ϕ particularly in the polar coordinates. The nonvanishing components of the optical metric are

$$\gamma_{rr} = \frac{B(r)}{A(r)}, \quad (4)$$

$$\gamma_{\phi\phi} = \frac{r^2}{A(r)}. \quad (5)$$

Let us suppose the tangent vector field along the light ray. The unit tangential vector along the light ray in M^{opt} can be defined as

$$K^I \equiv \frac{dx^I}{dt}. \quad (6)$$

This is a spatial vector. Note that K^I is defined in terms of γ_{IJ} but not g_{IJ} , because we consider light rays.

C. Impact parameter

In the SSS spacetime, there are two constants of motion for a massless particle such as a photon. They are the specific energy and the specific angular momentum as

$$E = A(r) \frac{dt}{d\lambda}, \quad (7)$$

$$L = r^2 \frac{d\phi}{d\lambda}, \quad (8)$$

where λ denotes the affine parameter along the light ray. As usual, we define the impact parameter of the light ray as

$$\begin{aligned} b &\equiv \frac{L}{E} \\ &= \frac{r^2}{A(r)} \frac{d\phi}{dt}. \end{aligned} \quad (9)$$

In terms of the impact parameter and the metric components at the position of the massless particle, the components of K^I can be expressed as

$$(K^r, K^\phi) = \frac{bA(r)}{r^2} \left(\frac{dr}{d\phi}, 1 \right). \quad (10)$$

Here, the unity of the vector K^I leads to the orbit equation as

$$\left(\frac{dr}{d\phi} \right)^2 + \frac{r^2}{B(r)} = \frac{r^4}{b^2 A(r) B(r)}. \quad (11)$$

This can be also derived directly from $ds^2 = 0$.

Is it safe for us to call what is defined as b the impact parameter of light? Let us briefly mention this. If there were no lens objects, then the spacetime would be Minkowskian, namely $A(r) = 1$ and $B(r) = 1$ in the polar coordinates, and b would thus be equal to the closest distance according to Eq. (11). Therefore, b can be safely called the impact parameter of the orbit.

D. Angles

We can define the dyad as

$$e_{\text{rad}}^I = \left(\frac{1}{\sqrt{\gamma_{rr}}}, 0 \right), \quad (12)$$

$$e_{\text{ang}}^I = \left(0, \frac{1}{\sqrt{\gamma_{\phi\phi}}} \right), \quad (13)$$

which correspond to the unit vector along the radial direction from the center of the lens object and that along the angular direction, respectively.

Let Ψ denote the angle of the light ray measured from the radial direction. It can be defined by

$$\cos \Psi \equiv \gamma_{IJ} e_{\text{rad}}^I K^J, \quad (14)$$

where we used that e_{rad}^I and K^J are unit vectors. This expression is rewritten more explicitly as

$$\begin{aligned} \cos \Psi &= \gamma_{rr} e_{\text{rad}}^r K^r \\ &= \frac{\sqrt{\gamma_{rr}} b A(r)}{r^2} \frac{dr}{d\phi}. \end{aligned} \quad (15)$$

This leads to

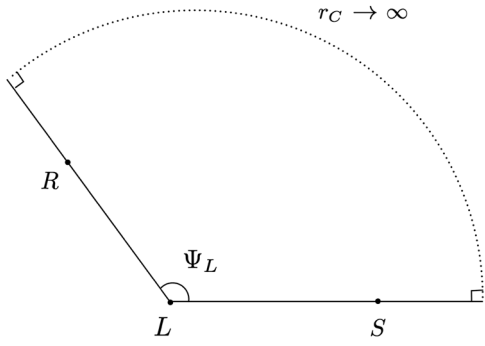


FIG. 3. Embedded triangle ${}^{\infty}\nabla_L^{\infty}$. It consists of a circular arc segment C_r of coordinate radius r_C centered at the lens with taking $r_C \rightarrow \infty$, and two radial geodesics through either the receiver or the source. One can determine Ψ_L at the point L from this figure by using the Gauss-Bonnet theorem. See Eq. (23).

Therefore, we obtain

$$\alpha_{\Psi} = \iint_{{}^R\nabla_L^S} K dS. \quad (22)$$

Equation (22) shows clearly that α_{Ψ} is invariant in differential geometry. The definition by Eq. (18) is thus justified [34]. The Gaussian curvature can be related with the Riemannian tensor. See e.g. Werner for this relation [35,36].

However, it seems impossible to define Ψ_L for a case of a black hole, because L is the singularity. On the other hand, ϕ_{RS} seems preferred for practical calculations in order to avoid such a problem associated with Ψ_L , because ϕ_{RS} can be defined outside the horizon for a black hole case.

We begin by considering another embedded triangle, which consists of a circular arc segment C_r of coordinate radius r_C centered at the lens which intersects the radial geodesic through the receiver or the source. See Fig. 3, in which we assume the asymptotically flat spacetime and a sufficiently large r_C , for which the embedded triangle is denoted by ${}^{\infty}\nabla_L^{\infty}$. Then, $\kappa_g \rightarrow 1/r_C$ and $d\ell \rightarrow r_C d\phi$ as $r_C \rightarrow \infty$ (see e.g. [29]). Hence, we obtain $\int_{C_r} \kappa_g d\ell \rightarrow \phi_{RS}$. Applying this result to the Gauss-Bonnet theorem for ${}^{\infty}\nabla_L^{\infty}$ leads to

$$\Psi_L = \phi_{RS} + \iint_{{}^{\infty}\nabla_L^{\infty}} K dS. \quad (23)$$

By substituting Eq. (23) into Ψ_L in α_{Ψ} of Eq. (22), we obtain

$$\begin{aligned} \alpha &= \Psi_R - \Psi_S + \phi_{RS} \\ &= - \iint_{{}^R\nabla_L^S} K dS, \end{aligned} \quad (24)$$

where we use Eq. (17) and ${}^{\infty}\square_S^{\infty}$ denotes an oriented area of ${}^{\infty}\nabla_L^{\infty}$ subtracted by ${}^R\nabla_L^S$. Equation (24) shows that α is

invariant in differential geometry. Moreover, it follows that $\alpha = 0$ in Euclidean space.

Both α and α_{Ψ} are geometrically invariant. The integration domain for α_{Ψ} includes the lens position. Therefore, α_{Ψ} might not be suitable for a black hole case.

On the other hand, it is likely that α can avoid such a problem. Furthermore, in the next section, we shall see that (1) α by Eq. (17) recovers the known formula of the bending angle for the asymptotic receiver and source and (2) it can be done in practice to calculate α without encountering an infinitely large term for a nonasymptotically flat model, though the justification of α by Eq. (24) is currently limited within an asymptotically flat case.

III. METHOD OF CALCULATING THE BENDING ANGLE OF LIGHT

There are two ways of calculating α , because Eq. (17) always agrees with Eq. (24). One method is to use Eq. (17). For this method, all we have to do is to calculate the three angles of Ψ_R , Ψ_S and ϕ_{RS} . The other method is to use Eq. (24), where we first calculate the Gaussian curvature K by using the optical metric and next we integrate K over the quadrilateral ${}^{\infty}\square_S^{\infty}$. Note that the integration domain ${}^{\infty}\square_S^{\infty}$, especially an expression of the geodesic curve from S to R , is unknown *a priori* and hence it must be looked for, though the calculation must be straightforward but tedious. Let us suppose an asymptotic receiver and source of light in the Schwarzschild spacetime for instance. Even in this case, it is a quite elaborate task to calculate the surface integral to recover the known formula $\alpha = 4M/b$. See Gibbons and Werner [29]. As a result, it is likely that the first method is much easier than the second one.

A. Asymptotically flat case

Let us consider the case of the asymptotic flatness. Then, we can assume $A(r) \rightarrow 1$ and $B(r) \rightarrow 1$ as $r \rightarrow \infty$. As usual, we assume also that the source and receiver are located at null infinity. Namely, we assume $r_R \rightarrow \infty$ and $r_S \rightarrow \infty$. Then, let us examine whether Eq. (17) can recover the textbook formula for the deflection angle of light. For this purpose, we assume $\Psi_R = 0$ and $\Psi_S = \pi$, because we keep b constant with $r_R \rightarrow \infty$ and $r_S \rightarrow \infty$. Hence, we obtain

$$\alpha = \phi_{RS} - \pi. \quad (25)$$

All we have to do is to compute ϕ_{RS} .

The orbit equation for the light ray in the SSS spacetime is in a general form as

$$\left(\frac{du}{d\phi}\right)^2 = F(u), \quad (26)$$

where u denotes the inverse of r . Please see Eq. (11) for more detail.

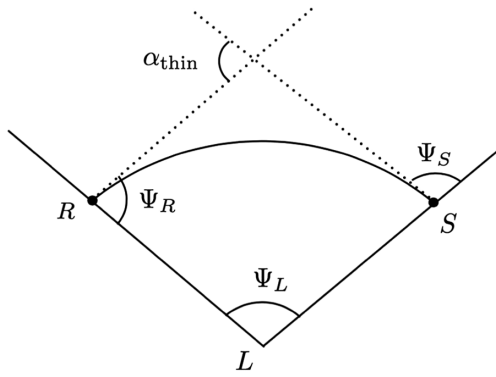


FIG. 4. Thin lens approximation. In this approximation, the light ray deflects only at the lens plane. Namely, we assume that the spacetime is flat except for the location of the thin lens, so that Ψ_L in this figure can be identified with ϕ_{RS} . Let α_{thin} denote the deflection angle of light in the thin lens approximation. The dotted straight lines are tangential to the light ray at the receiver or at the source. For the quadrilateral in Euclidean space, $\alpha_{\text{thin}} = \Psi_R - \Psi_S + \phi_{RS}$, because the sum of the inner angles is 2π .

Integrating Eq. (26) leads to the angle ϕ_{RS} as

$$\phi_{RS} = 2 \int_0^{u_0} \frac{du}{\sqrt{F(u)}}, \quad (27)$$

where u_0 is the inverse of the closest approach (often denoted as r_0). Therefore, substituting this into Eq. (17) gives

$$\alpha = 2 \int_0^{u_0} \frac{du}{\sqrt{F(u)}} - \pi. \quad (28)$$

This is exactly the deflection angle of light in the literature. Therefore, α may be interpreted as the deflection angle of light. See also Fig. 4 for the thin lens approximation. One can see that α is likely to correspond to α_{thin} , where α_{thin} denotes the deflection angle of light in the thin lens approximation.

B. Finite-distance cases

In practice, the thin lens approximation works well for most cases in astronomy so far. This approximation is almost the same as an assumption that a light source and a receiver are nearly at null infinity in the asymptotically flat case. To be more specific, the present paper assumes that the distance from the source to the receiver is finite because every observed star and galaxy is located at a finite distance from us (e.g., at finite redshift in cosmology) and the distance is much larger than the size of the lens. Hence, we keep r_R and r_S finite. Then, let u_R and u_S denote the inverse of r_R and r_S , respectively. Equation (17) becomes

$$\alpha = \int_{u_R}^{u_0} \frac{du}{\sqrt{F(u)}} + \int_{u_S}^{u_0} \frac{du}{\sqrt{F(u)}} + \Psi_R - \Psi_S. \quad (29)$$

Equation (28) is thus corrected.

Equations (17), (24) and (29) are equivalent to each other. They are different from the deflection angle that is often used (or argued) in the recent papers [37].

For the Schwarzschild spacetime, the line element becomes

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (30)$$

Then, $F(u)$ is

$$F(u) = \frac{1}{b^2} - u^2 + r_g u^3. \quad (31)$$

By using Eq. (16), $\Psi_R - \Psi_S$ in the Schwarzschild spacetime is expanded in a power series in r_g as

$$\begin{aligned} \Psi_R^{\text{Sch}} - \Psi_S^{\text{Sch}} &\equiv [\arcsin(bu_R) + \arcsin(bu_S) - \pi] \\ &\quad - \frac{1}{2} b r_g \left(\frac{u_R^2}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2 u_S^2}} \right) \\ &\quad + O(b r_g^2 u_S^3, b r_g^2 u_R^3). \end{aligned} \quad (32)$$

It follows that $\Psi_R - \Psi_S$ for the Schwarzschild case approaches π as $u_S \rightarrow 0$ and $u_R \rightarrow 0$.

IV. NONASYMPTOTICALLY FLAT CASES

Finally, we consider a nonasymptotically flat spacetime such as the Kottler solution to the Einstein equation and an exact solution in the Weyl conformal gravity. For such cases, we cannot assume the source at the past null infinity ($r_S \rightarrow \infty$) or the receiver at the future null infinity ($r_R \rightarrow \infty$), because $A(r)$ diverges or does not exist as $r \rightarrow \infty$. Hence, we should keep the source and the receiver at a finite distance from the lens object. It is Eq. (29) that we can use for such a case. As mentioned already, Eq. (16) is more convenient for calculating Ψ_R and Ψ_S than Eq. (14), since Eq. (16) needs a local quantity but not any derivatives. For the two cases, the explicit expressions are as follows.

A. Kottler case

For the Kottler spacetime [16], the line element is

$$ds^2 = -\left(1 - \frac{r_g}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r} - \frac{\Lambda}{3} r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (33)$$

where Λ denotes the cosmological constant.

By using Eq. (16), $\Psi_R - \Psi_S$ is expanded in terms of r_g and Λ as

$$\begin{aligned} \Psi_R - \Psi_S &= \Psi_R^{\text{Sch}} - \Psi_S^{\text{Sch}} - \frac{b\Lambda}{6u_R\sqrt{1-b^2u_R^2}} - \frac{b\Lambda}{6u_S\sqrt{1-b^2u_S^2}} \\ &+ \frac{bu_R(-1+2b^2u_R^2)}{8(1-b^2u_R^2)^{3/2}} \left(r_g^2u_R^2 + \frac{2r_g\Lambda}{3u_R} + \frac{\Lambda^2}{9u_R^4} \right) \\ &+ \frac{bu_S(-1+2b^2u_S^2)}{8(1-b^2u_S^2)^{3/2}} \left(r_g^2u_S^2 + \frac{2r_g\Lambda}{3u_S} + \frac{\Lambda^2}{9u_S^4} \right) \\ &+ O(r_g^3, r_g^2\Lambda, r_g\Lambda^2, \Lambda^3), \end{aligned} \quad (34)$$

where $\Psi_R^{\text{Sch}} - \Psi_S^{\text{Sch}}$ is a part existing in Schwarzschild spacetime. Note that the above expansion of $\Psi_R - \Psi_S$ is divergent as $u_S \rightarrow 0$ and $u_R \rightarrow 0$. This is because the spacetime is not asymptotically flat and hence it does not allow the limit of $u_S \rightarrow 0$ and $u_R \rightarrow 0$. Hence, the power series form by Eq. (34) must be used within a certain finite radius of convergence.

For the Kottler case, $F(u)$ becomes

$$F(u) = \frac{1}{b^2} - u^2 + r_g u^3 + \frac{\Lambda}{3}. \quad (35)$$

Hence, we obtain

$$\begin{aligned} \phi_{RS} &= \pi - \arcsin(bu_R) - \arcsin(bu_S) \\ &+ \frac{r_g}{b} \left[\frac{1}{\sqrt{1-b^2u_R^2}} \left(1 - \frac{1}{2}b^2u_R^2 \right) \right. \\ &+ \left. \frac{1}{\sqrt{1-b^2u_S^2}} \left(1 - \frac{1}{2}b^2u_S^2 \right) \right] \\ &+ \frac{\Lambda b^3}{6} \left[\frac{u_R}{\sqrt{1-b^2u_R^2}} + \frac{u_S}{\sqrt{1-b^2u_S^2}} \right] \\ &+ \frac{r_g\Lambda b}{12} \left[\frac{2-3b^2u_R^2}{(1-b^2u_R^2)^{3/2}} + \frac{2-3b^2u_S^2}{(1-b^2u_S^2)^{3/2}} \right] + O(r_g^2, \Lambda^2). \end{aligned} \quad (36)$$

By using Eqs. (34) and (36), we obtain the correct deflection angle of light as

$$\begin{aligned} \alpha &= \frac{r_g}{b} \left[\sqrt{1-b^2u_R^2} + \sqrt{1-b^2u_S^2} \right] \\ &- \frac{\Lambda b}{6} \left[\frac{\sqrt{1-b^2u_R^2}}{u_R} + \frac{\sqrt{1-b^2u_S^2}}{u_S} \right] \\ &+ \frac{r_g\Lambda b}{12} \left[\frac{1}{\sqrt{1-b^2u_R^2}} + \frac{1}{\sqrt{1-b^2u_S^2}} \right] + O(r_g^2, \Lambda^2). \end{aligned} \quad (37)$$

Some terms in this expression may apparently diverge in the limit as both $bu_R \rightarrow 0$ and $bu_S \rightarrow 0$. Note that this limit

has no relevance with astronomical observations in the Kottler spacetime. Therefore, the apparent divergence does not matter.

Aghili *et al.* have recently discussed the numerical effects of a slowly varying Hubble parameter on the gravitational lensing [38]. It is left as a future work to examine an application of the present approach to such a cosmological model with a slowly varying Hubble parameter.

B. Weyl conformal gravity case

Weyl conformal gravity introduces three independent parameters (often denoted as β , γ and k) into the spherical solution, for which Birkhoff's theorem was proven in conformal gravity [25]. The line element with the three parameters is [24]

$$\begin{aligned} ds^2 &= -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ A(r) &= 1 - 3m\gamma - \frac{2m}{r} + \gamma r - kr^2, \end{aligned} \quad (38)$$

where we defined $m \equiv \beta(2-3\beta\gamma)/2$. The term with the coefficient k makes the same contribution as the cosmological constant in the Kottler spacetime that has been studied above. Henceforth, we omit the r^2 term for brevity.

By using Eq. (16), $\Psi_R - \Psi_S$ is expanded in a power series in β and γ as

$$\begin{aligned} \Psi_R - \Psi_S &\equiv \Psi_R^{\text{Sch}} - \Psi_S^{\text{Sch}} \\ &+ \frac{b\gamma}{2} \left(\frac{u_R}{\sqrt{1-b^2u_R^2}} + \frac{u_S}{\sqrt{1-b^2u_S^2}} \right) \\ &- \frac{m\gamma}{2} \left[\frac{bu_R(2-b^2u_R^2)}{(1-b^2u_R^2)^{3/2}} + \frac{bu_S(2-b^2u_S^2)}{(1-b^2u_S^2)^{3/2}} \right] \\ &+ O(m^2, \gamma^2). \end{aligned} \quad (39)$$

Note that this series expansion of $\Psi_R - \Psi_S$ is divergent as $u_S \rightarrow 0$ and $u_R \rightarrow 0$. This is because the nonasymptotic flatness of the spacetime does not allow the limit of $u_S \rightarrow 0$ and $u_R \rightarrow 0$. Hence, we must use Eq. (39) within its certain radius of convergence.

For the conformal gravity case with $k=0$, $F(u)$ becomes

$$F(u) = \frac{1}{b^2} - u^2 + 2mu^3 + \Gamma u^2 - \gamma u. \quad (40)$$

Then, ϕ_{RS} is obtained as

$$\begin{aligned}
 \phi_{RS} = & [\pi - \arcsin(bu_R) - \arcsin(bu_S)] \\
 & + \frac{m}{b} \left(\frac{2 - b^2u_R^2}{\sqrt{1 - b^2u_R^2}} + \frac{2 - b^2u_S^2}{\sqrt{1 - b^2u_S^2}} \right) \\
 & - \frac{\gamma}{2} \left(\frac{b}{\sqrt{1 - b^2u_R^2}} + \frac{b}{\sqrt{1 - b^2u_S^2}} \right) \\
 & + \frac{m\gamma}{2} \left[\frac{b^3u_R^3}{(1 - b^2u_R^2)^{3/2}} + \frac{b^3u_S^3}{(1 - b^2u_S^2)^{3/2}} \right] \\
 & + O(m^2, \gamma^2). \tag{41}
 \end{aligned}$$

In total, we obtain α for the Weyl conformal gravity case as

$$\begin{aligned}
 \alpha = & \frac{2m}{b} \left(\sqrt{1 - b^2u_R^2} + \sqrt{1 - b^2u_S^2} \right) \\
 & - m\gamma \left(\frac{bu_R}{\sqrt{1 - b^2u_R^2}} + \frac{bu_S}{\sqrt{1 - b^2u_S^2}} \right) + O(m^2, \gamma^2). \tag{42}
 \end{aligned}$$

The terms linear in γ cancel out in the expression for the deflection angle of light. Hence, this might correct the results in previous papers [26–28] that reported nonzero contributions from γ .

C. Far source and receiver

Finally, let us consider an asymptotic case as $bu_S \ll 1$ and $bu_R \ll 1$, which means that both the source and the receiver are very far from the lens object. Note that $bu_S \rightarrow 0$ and $bu_R \rightarrow 0$ might cause the divergent terms in the deflection angle. Hence, we focus on the dominant part of each term in a series expansion without taking the limit. Let us write down approximate expressions for the deflection of light.

(1) Kottler case:

It follows that the expression for ϕ_{RS} in the far approximation coincides with the seventh and eighth terms of Eq. (5) in [20], the third and fifth terms of Eq. (15) in [21], and the second term of Eq. (14) in [23]. However, they [20,21,23] did not consider $\Psi_R - \Psi_S$. Equation (37) becomes

$$\alpha \sim \frac{2r_g}{b} - \frac{1}{6} \Lambda b \left(\frac{1}{u_R} + \frac{1}{u_S} \right) + \frac{1}{6} r_g \Lambda b. \tag{43}$$

This might give a correction to the previous results [20,21,23]. For instance, Sereno [20] considered only ϕ_{RS} . See [37] for a more subtle case associated with Rindler and Ishak's approach.

(2) Weyl conformal gravity case:

For the Weyl conformal gravity, the deflection angle of light in the far approximation becomes

$$\alpha \sim \frac{4m}{b} + O(m^2, \gamma^2). \tag{44}$$

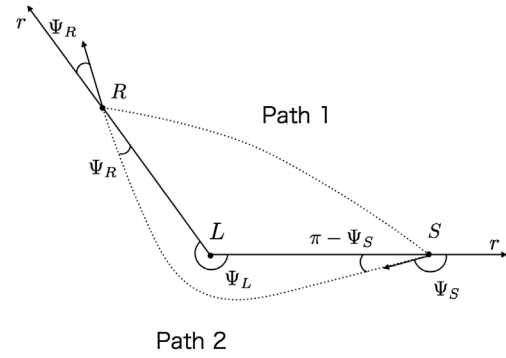


FIG. 5. Two light paths. In the present paper, we focus on the path 1 in this figure, because it corresponds to the brightest lensed image and it plays a crucial role in astronomy. There is another possible path (path 2 in this figure). The two light paths are denoted by dotted lines.

Note that $m\gamma$ parts from $\Psi_R - \Psi_S$ and from ψ_{RS} cancel out. See Eqs. (39) and (41).

Before closing this section, we briefly mention another light path (path 2 in Fig. 5). The bending angle for this path is computable if we take account of the orientation of path 2 (see Fig. 5).

V. CONCLUSION

In this paper, we studied a connection between the bending angle of light and the Gauss-Bonnet theorem by using the optical metric in the SSS spacetimes. A correspondence of the deflection angle of light to the surface integral of Gaussian curvature may allow us to take account of the finite distance of a light source and a receiver from a lens object.

The proposed approach of calculating the deflection angle of light by Eq. (17) was applied to two examples of the nonasymptotically flat spacetimes: the Kottler solution to the Einstein equation and an exact solution in Weyl conformal gravity. For both cases, we suggested finite-distance corrections to the deflection angle of light without encountering an infinitely large term, as a conjecture, because the justification of α by Eq. (24) cannot be applied to such a nonasymptotically flat case as it is. It would be interesting to examine whether or not the justification of Eq. (17) is extended to a nonasymptotically flat case. If it is not, we may find new corrections to the deflection angle of light. It would be interesting to study along this direction.

Moreover, let us suppose that the light ray passes near a relativistic compact object. For this case, the deflection angle of light may exceed 2π to produce the relativistic images. For such a large deflection case, the orbit has the winding number W that may be unity or more. Equations (17) and (22) still work, because the light ray lives on a single plane in M^{opt} [39]. Further study along the direction of the relativistic strong lensing by using the present approach is left for future work.

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- [30] To be more precise, γ_{ij} might be denoted as γ_{ij}^{opt} , since the notation as γ_{ij} is often used for the induced metric on a hypersurface in the 3 + 1 formulation in the literature. However, we use γ_{ij} for denoting the optical metric for its simplicity in this paper, because the induced metric does not appear in this paper and hence readers would not be confused.
- [31] Let us suppose two scalars. One scalar $\xi(A)$ is defined at a point A . The other scalar $\eta(B)$ is at a very different point B . The sum of $\xi(A)$ and $\eta(B)$ and the difference between $\xi(A)$ and $\eta(B)$ have no invariant meaning in general.
- [32] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Englewood Cliffs, NJ, 1976), pp. 268–269.
- [33] Fermat’s principle in a spacetime is the principle that the light path makes the arrival time stationary. It is expressed as $\delta \int dt = 0$ along the light ray [8]. In our case, hence, Eq. (3) implies that the orbit of light is a geodesic in M^{opt} with the optical metric γ_{ij} .
- [34] It is likely that mathematicians prefer to define α by Eq. (22). However, it is quite elaborate in practice to perform the areal integral of Eq. (22). In order to do it, we have to find out the integration domain first, especially the domain boundary connecting the receiver and the source. This boundary is the light ray that is unknown *a priori* for most cases and hence the expression of the boundary can be obtained only after the orbit equation is solved.
- [35] M. C. Werner, *Gen. Relativ. Gravit.* **44**, 3047 (2012).
- [36] $K = R_{r\phi r\phi} / \det(\gamma_{IJ})$ in our notation. See e.g. Eq. (16) in Werner [35]. K can diverge at $r = 0$ for black hole cases. For such cases, it is likely that Eq. (22) requires more mathematical treatments, because the singularity at $r = 0$ is hidden by a horizon. Further study along this mathematical direction is left for future work.
- [37] In discussing the effect of the cosmological constant on the bending of light, Rindler and Ishak have proposed a

different definition of the deflection angle of light as $2\Psi_S$, where the source (and also the receiver) is assumed to be at rest at the very special point that is defined as an intersection between the light ray and the $\phi = 0$ curve. Note that their calculations and results, even if they were correct, can be applied only for the particular positions of the receiver and source ($\phi_S = 0$ and $\phi_R = \pi$) and hence the applicability of their argument seems very narrow. Moreover, their deflection angle (denoted as α_{RI}) can be written as $\alpha_{\text{RI}} = \Psi_R - \Psi_S + \pi$ in the notation of the present paper. Apparently, α_{RI} agrees with Eq. (17) only for a particular case $\phi_{RS} = \pi$ that is chosen by hand, such that the geometrical configuration of the lens, receiver and source can be the same as that in Rindler and Ishak. There is a significant difference, in spite of the apparent coincidence between the two definitions. Note that they use a simple spatial metric g_{ij} to define a covariant angle Ψ_{RI} that is different from the angle Ψ in the present paper. The definition of Ψ_R and Ψ_S in the present paper is based on the optical metric γ_{ij} that respects describing the null geodesics projected onto a spatial sector M^{opt} . Practically speaking, γ_{ij} is different from g_{ij} by a factor $A(r)$ in a SSS spacetime. See also Eqs. (4) and (5). Rindler and Ishak's approach has been criticized by numerous authors (see [19,20,22,23]). Following the pioneering work by Lake [17], for instance, Arakida and Kasai [23] claimed that, even if the cosmological constant is taken into account, the deflection angle of light can be rearranged as $\alpha_{\text{AK}} = 2M/B + 15\pi M^2/8B^2 + O(M^3/B^3)$ to agree with that in Schwarzschild spacetime, if the impact parameter b is replaced by a new impact parameter B in order to absorb the influence of the cosmological constant. Arakida and Kasai continued to argue that the orbit equation in the presence of the cosmological constant becomes the same as that for the

Schwarzschild case, after the impact parameter is suitably shifted. Therefore, they concluded that the deflection angles for both cases agreed with each other. However, this final step in their argument is misleading. The receiver and source can be located at the null infinity for the Schwarzschild case. On the other hand, we can never assume $r_R \rightarrow \infty$ and $r_S \rightarrow \infty$ for the Kottler case, because the receiver and source go across de Sitter horizon in this limit. In other words, they should have integrated the orbit equation from a finite-distance source to a finite-distance receiver. If they had done it, they might have obtained the deflection angle with certain corrections due to the finiteness of distance. Note that their paper did not even define the deflection angle of light for finite distance or take care of the optical metric. The observer at rest in a static slice of the spacetime may be in motion in a different slice. This is the case, for instance in the Kottler solution, where an observer at rest at the static chart can be in motion in the cosmological coordinates of the same spacetime. See e.g. Park [19] and Simpson *et al.* [22] for detailed discussions on the corrections to the deflection angle of light due to different time slices.

- [38] M. E. Aghili, B. Bolen, and L. Bombelli, [arXiv:1408.0786](https://arxiv.org/abs/1408.0786).
- [39] It is likely that the nonvanishing contributions to α at the linear order of Γ and γ in the previous works are because they did not use the optical metric to define the angles at the receiver and the source and they employed a different definition of α by following [18]. In particular, γ does not contribute at the leading order. It might be interesting to interpret this thing by a projective equivalence [40] between the solution in the Weyl conformal gravity and another spacetime.
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