Dark energy and dark matter from an additional adiabatic fluid

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The dark sector is described by an *additional* barotropic fluid which evolves adiabatically during the Universe's history and whose adiabatic exponent γ is derived from the standard definitions of specific heats. Although in general γ is a function of the redshift, the Hubble parameter and its derivatives, we find that our assumptions lead necessarily to solutions with $\gamma =$ constant in a Friedmann-Lemaître-Robertson-Walker universe. The adiabatic fluid acts effectively as the sum of two distinct components, one evolving like nonrelativistic matter and the other depending on the value of the adiabatic index. This makes the model particularly interesting as a way of simultaneously explaining the nature of both dark energy and dark matter, at least at the level of the background cosmology. The ΛCDM model is included in this family of theories when $\gamma = 0$. We fit our model to supernovae Ia, $H(z)$ and baryonic acoustic oscillation data, discussing the model selection criteria. The implications for the early Universe and the growth of small perturbations in this model are also discussed.

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I. INTRODUCTION

The present time cosmic expansion may be described in terms of a late-time fluid which dominates over the other contributions to the cosmic matter budget [1]. The simplest assumption is based on the hypothesis that such a fluid is *perfect* [2] and enters *by hand* the Einstein equations as the source for speeding up the Universe today [3]. This component (referred to as "dark energy") is required to have a negative equation of state in order to guarantee that the Universe undergoes an accelerated phase at late times [4], and the search for its nature is the focus of much current research in cosmology. The *minimal model* for dark energy is the one where the cosmological constant Λ [5] dominates over the other species including pressureless matter [6]. Although appealing and now very well established, the cosmological constant suffers from several shortcomings and consequently the ΛCDM model cannot be considered the complete explanation for the universe dynamics [8].[1]

It is for these reasons that models for an evolving dark energy contribution have attracted considerable attention over the past two decades [9]. There exist several explanations for *evolving dark energy*, which range from modifying Einstein's gravity, including additional degrees of freedom arising from quantum backgrounds, to proposing different energy momentum tensors for this dark sector.[2] In every case, all evolving dark energy contributions should be compatible with the laws of thermodynamics and be described by perfect fluids, at least at the level of the background cosmology [11]. The problem of describing properties of equilibrium thermodynamics in terms of a nonequilibrium dark energy fluid is one of the challenges of modern day cosmology [12].

Within the framework of a homogeneous and isotropic universe, this problem can be avoided by assuming that at any given epoch, the fluid evolution is at least described by a quasistatic process. More recently, it has been shown that it is possible to formulate the thermodynamic quantities of interest for a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology [13]. In particular, it has been argued that the role of specific heats in cosmology can be confronted with observational data [14]. In doing so, an investigation of the simplest assumptions on specific heats leading to an evolving dark energy contribution has been recently presented [15].

In this paper we show that by investigating how heat capacities evolve at arbitrary redshifts it may be possible to

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[1]All conceivable approaches to dark energy are practically indistinguishable at the level of the background, leading to a degeneracy problem and only model-independent measures of the evolution of the equation of state would indicate whether the ΛCDM model really is the favored cosmological framework [7].

[2]For a representative but incomplete list, see for example [10] and references therein.

construct a cosmological model with an evolving dark energy term, which is a natural extension to the standard ΛCDM model.

In what follows, a few basic requirements for the heat capacities are assumed:

(1) they evolve in time,
(2) they are related to the internal energy and enthalpy of the Universe as required by standard thermodynamics,
(3) they have been evaluated for all perfect fluids making up the energy budget of the Universe,
(4) the process of thermal exchange is purely adiabatic, so that the volume scales as the third power of the scale factor.

Moreover, as a consequence of the above prescriptions, the pressureless matter contribution turns out to be an emergent phenomenon and the inferred dark energy contribution is weakly interacting, behaving like a gaseous fluid source for Einstein's equations.

In particular, we assume that the adiabatic index γ may take particular values, excluding regions in which it cannot span. To this end, we try to give either a thermodynamic explanation for dark energy or to formulate a cosmological model from basic principles which take into account the laws of thermodynamics. We explore both the cases of varying and constant adiabatic indices and find cosmological models which differ slightly from the concordance paradigm. In this way we provide a new approach in which dark energy *emerges* as a consequence of the Universe's thermodynamics. We propose tight bands of available values for the adiabatic index and describe how to determine the difference between our thermodynamic dark energy contribution from a pure cosmological constant, even at the level of background cosmology.

We investigate either the late-time or early-time universe and we show that our model is compatible with the basic requirements of the standard paradigm. We noticed that our approach becomes a pure dark fluid contribution as the adiabatic index runs to vanish. Finally, we compare our approach with data, by means of supernovae Ia (SNIa), $H(z)$ and baryonic acoustic oscillation (BAO) data sets. Our numerical results are compatible with the standard model, showing that our paradigm works fairly well in describing the Universe's expansion history at different stages of the Universe's evolution. Slight departures are accounted for in the shift of linear perturbations, whose corresponding plots are inside the 10% of discrepancies with respect to the ΛCDM model.

The paper is structured as follows. In Sec. II, we consider the properties of heat capacities in the context of a homogeneous and isotropic universe. We describe how to build up physical definitions for them and how to obtain the corresponding adiabatic indices, emphasizing how to understand their physical meaning. In Sec. III, we describe the cosmological consequences of inducing dark energy by examining either the case of constant or variable indices. We discuss the case of a purely gaseous dark energy contribution and how to obtain the liming case of a dark fluid as $\gamma \to 0$. In Sec. VI, we investigate some consequences of our approach in the high redshift regime, while in Sec. IV, we consider two fitting procedures involving supernova and BAO data in order to obtain observational constraints for our model. Finally, in Sec. VII, we present our conclusions and give some perspectives for future work.

II. HEAT CAPACITIES IN OBSERVABLE COSMOLOGY

In this section, we apply standard thermodynamics to the case of a homogeneous and isotropic universe in order to obtain expressions for the specific heats. The thermodynamic laws can be used either in a classical or quantum regime, assuming that the Universe does not allow for the exchange of heat with the environment. It follows that the simplest choice for modeling the Universe takes into account that its evolution is purely adiabatic. While matter creation may occur within this framework [16], we do not assume this possibility in our approach.

In the case of a pure FLRW line element

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(\sin^2\theta d\phi^2 + d\theta^2)\right], \quad (1)$$

the dynamics of the Universe obeys the Friedmann equations,

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Omega_k H_0^2}{a^2}, \quad (2a)$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \quad (2b)$$

where dots represent derivatives with respect to the cosmic time t.

From Eq. (2b), it is easy to show that the present day dynamics of the Universe is sourced by a perfect fluid whose equation of state is negative, to guarantee that $\ddot{a} > 0$.

In general, the gravitational field is determined by the same source at all stages of its evolution, so that in order to guarantee that at both early and late times the source is the same perfect fluid, one is forced to assume that it behaves like a thermodynamic system [17], whose evolution is described in terms of the redshift. Hence, we expect the laws of thermodynamics to hold and to be mathematically consistent with the FLRW universe [18].

However, the problem with the standard requirements imposed by thermodynamics in a FLRW universe is that it is difficult to construct a self-consistent definition of temperature, because eventual departures from the background radiation temperature must be associated with cosmological fluids [19]. For example, it has been recently proposed that the specific heats of the Universe, given by

$$C_V = \frac{\partial U}{\partial T}, \tag{3a}$$

$$C_P = \frac{\partial h}{\partial T}, \tag{3b}$$

are compatible with FLRW cosmology, but T needs to be fixed to a precise value inside viable intervals. In general, a possible dark energy temperature may evolve as the Universe expands, so that T could be considered as a function of the redshift rather than a constant. For these reasons, a direct comparison of Eqs. (3a) and (3b) may be affected by theoretical shortcomings on its validity.

In principle, both the internal energy and enthalpy are functions of the volume, pressure and temperature. In this case, one naturally obtains

$$C_V = \frac{1}{T'}\left[h' - \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial P}{\partial S}\right)_V V' - VP'\right], \tag{4a}$$

$$C_P = C_V + \frac{1}{T'}\left[\left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial P}{\partial S}\right)_V V' + VP' - \left(\frac{\partial h}{\partial P}\right)_T P'\right]. \tag{4b}$$

However, since all state variables evolve in terms of the redshift, it seems natural to assume the simplest hypothesis in which both energy and enthalpy are functions of T only. To do so, it is possible to assume $U = \rho V$ and $h = (\rho + P)V$, where V is the volume of the Universe. In this way, one splits the functional dependence of U, h in terms of V, assuming that $\rho = \rho(T)$ and $P = P(T)$. The standard definition of the volume is $V = V_0 a^3$ [20], which represents the simplest assumption reflecting both early and late times of the Universe's evolution. Even though its introduction seems natural, other approaches suggest alternative forms of the volume, for example an apparent horizon volume definition $V \propto r^3 = V_0 H^{-3}$ would reproduce a causal region where the entropy becomes $\propto H^{-2}$ [21]. Moreover, employing the weak energy conditions

$$T_{\mu\nu} k^\mu k^\nu \geq 0, \tag{5}$$

$$\rho \geq 0, \tag{6}$$

$$\rho + P \geq 0, \tag{7}$$

one finds that both U and h must be positive definite.

Under the simplest hypothesis, in which dark energy weakly interacts, state functions depend on the temperature only and $V \propto a^3$; one simply obtains

$$C_V = \frac{3V_0}{8\pi G T'(1+z)^4}[2(1+z)HH' - 3H^2 + \Omega_k(1+z)^2], \tag{8a}$$

$$C_P = C_V + \frac{V_0}{T'(1+z)^3}\left(P' - \frac{3P}{1+z}\right), \tag{8b}$$

with no restrictions on their evolutions for different epochs of the evolution of the Universe. Here primes denote derivatives with respect to redshift z. The above forms of the heat capacities are not easy to compare to cosmic predictions, due to the complexity of their dependence on H and its derivatives. One intriguing way to investigate their physical consequences is to frame C_V and C_P in terms of observable quantities. In order to achieve this, let us consider expanding the scale factor $a(t)$ [22],

$$a(t) \sim 1 + H_0\Delta t - \frac{q_0}{2}H_0^2\Delta t^2 + \frac{j_0}{6}H_0^3\Delta t^3 + \cdots, \tag{9}$$

where

$$H = \frac{1}{a}\frac{da}{dt}, \tag{10a}$$

$$q = -\frac{1}{aH^2}\frac{d^2a}{dt^2}, \tag{10b}$$

$$j = \frac{1}{aH^3}\frac{d^3a}{dt^3}, \tag{10c}$$

and since [23]

$$H = H_0\left(1 + \frac{H'}{H_0}\bigg|_{z=0} z + \frac{H''}{2H_0}\bigg|_{z=0} z^2 + \cdots\right), \tag{11}$$

by comparing Eqs. (11) with Eq. (11), we get [24]

$$H' \equiv \frac{1+q}{1+z}H, \qquad H'' \equiv \frac{j-q^2}{(1+z)^2}H, \tag{12}$$

and by virtue of Eqs. (9), we obtain

$$C_P = \frac{2V_0}{T'}\frac{(j-1)H^2 + \Omega_k H_0^2(1+z)^2}{(1+z)^4}, \tag{13a}$$

$$C_V = \frac{3V_0}{T'}\frac{(2q-1)H^2 + \Omega_k H_0^2(1+z)^2}{(1+z)^4}. \tag{13b}$$

The above expressions give us direct information on the behaviour of dark energy in cases where one is able to describe the temperature as a function of redshift z. On the other hand, only the adiabatic index

$$\gamma \equiv \frac{C_P}{C_V} \tag{14}$$

is necessary to describe the cosmological evolution of our model. Using (13b) and (13b), the adiabatic index becomes

$$\gamma = \frac{2[(j-1)H^2 + \Omega_k H_0^2(1+z)^2]}{3[(2q-1)H^2 + \Omega_k H_0^2(1+z)^2]}. \tag{15}$$

Notice that there exists a solution for which $C_P = 0$ and $C_V = 0$. For $\Omega_k = 0$, this occurs when $q \to 1/2$ and $j \to 1$. In general, this could happen in standard cosmology at a redshift $z \gg 1$, under the hypothesis of de Sitter contribution to dark energy.

Another interesting fact is how γ behaves at the transition redshift $z_{\rm tr}$ [25], i.e., when dark energy starts to dominate over matter. Assuming for simplicity that $\Omega_k = 0$, the adiabatic index becomes

$$\gamma_{\rm tr} = -\frac{8\pi G(1+z_{\rm tr})P'}{3H^2}\bigg|_{z=z_{\rm tr}} = (1+z_{\rm tr})\frac{d\ln P}{dz}\bigg|_{z=z_{\rm tr}}, \tag{16}$$

where we have made use of $P = 8\pi G H^2(2q-1)$ which is a direct consequence of Eq. (2b), and $q(z_{\rm tr}) = 0$. From the above considerations, it is easy to see that the expression (16) is valid for *any* cosmological model. This heuristically shows that the adiabatic index is intimately related to the variation of the pressure. As a consequence, via the dynamics of the Friedmann equations, this determines how dark energy evolves and how possible departures from the standard concordance model may arise. This issue is addressed in the next sections, in which we investigate the consequences of Eq. (15) in cosmology.

III. THERMODYNAMICS OF ADIABATIC DARK ENERGY

Standard thermodynamics states that the combination of the first and second principles leads to

$$TdS = d(\rho V) + PdV = d[(\rho+P)V] - VdP, \tag{17}$$

and since $\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$, one gets $dP = (\rho + P)dT/T$. It is therefore easy to demonstrate that

$$dS = d\left[\frac{(\rho+P)V}{T}\right], \tag{18}$$

where any arbitrary constant is assumed to be 0 for simplicity. The basic requirements of thermodynamics suggest that $S \equiv V(\rho+P)/T$. Taking the combination of the first and second Friedmann equations, one gets

$$\dot{\rho} + 3H(P+\rho) = 0, \tag{19}$$

which can be recast as $d(\rho V) + PdV = 0$.

The conservation law, by virtue of Eq. (18), becomes

$$d\left[\frac{(\rho+P)V}{T}\right] = 0, \tag{20}$$

leading to the fact that $S = \text{const}$. This is equivalent to a thermodynamic system in which there is no heat exchange, i.e., is adiabatic.

We postulate a model in which the Universe is filled with an adiabatic fluid characterized by an adiabatic index γ, which, in principle, is a function of redshift as evident from (15), and an *external* nonrelativistic matter component $\Omega_m^{(\rm ext)}$, which can be chosen arbitrarily in a way that will become clearer later.

The complete system, rewritten in terms of the redshift, is

$$\frac{H^2}{H_0^2} = \frac{8\pi G\rho}{3H_0^2} + \Omega_m^{(\rm ext)}(1+z)^3 + \Omega_k(1+z)^2, \tag{21a}$$

$$(1+z)HH' - H^2 = 4\pi G\left[P + \frac{\rho}{3} + \Omega_m^{(\rm ext)}H_0^2(1+z)^2\right], \tag{21b}$$

$$P = P_0 V^{-\gamma} = P_0(1+z)^{3\gamma}, \tag{21c}$$

$$\gamma = \frac{C_P}{C_V} = \frac{(\rho' + P')V + (\rho+P)V'}{\rho' V + \rho V'}. \tag{21d}$$

From (21a) and (21b) we can simplify (21d) to obtain

$$\gamma = \frac{(\rho' + P')V + (\rho+P)V'}{\rho' V + \rho V'} = -\frac{P'V}{PV'} = \frac{(1+z)P'}{3P}. \tag{22}$$

On the other hand, taking (21c) gives

$$\frac{(1+z)P'}{3P} = \gamma + (1+z)\gamma' \ln(1+z). \tag{23}$$

Clearly, this is compatible with (22) only for $\gamma' = 0$, that is, $\gamma = \text{constant}$. In this case, it is easy to see that ρ is given by

$$\rho(z) = \left(\rho_0 + \frac{P_0}{1-\gamma}\right)(1+z)^3 - \frac{P_0}{1-\gamma}(1+z)^{3\gamma}. \tag{24}$$

First of all, we notice that there appears a term scaling as $(1+z)^3$, which corresponds to nonrelativistic matter; this term can, in principle, be identified with cosmological dark matter, but not necessarily (see below). Moreover, there appears a second term which instead scales as $(1+z)^{3\gamma}$. Choosing $\gamma = 0$ one recovers the dark fluid [26], whereby the corresponding term assumes the role of a pure Λ term.

To construct our model, we have used general thermodynamic notions to describe a universe filled with an adiabatic fluid and a standard nonrelativistic matter component. Given the standard definitions of C_P, C_V, γ, Eq. (21c) essentially defines our models. We have made no assumptions about the cosmological evolution of the adiabatic fluid beside (21c) and adiabaticity, and find that there is both an emergent dark energy and a dark matter component. In other words, postulating an additional adiabatic fluid with rather standard thermodynamical properties leads to the appearance of two separate components, one evolving as nonrelativistic matter (dark matter) and the other as a (dynamical) dark energy component with $\rho \sim (1+z)^{3\gamma}$.

We stress that this is not merely a different formulation of the FLRW cosmology, but a new model entirely.

Solutions with γ evolving with redshift are, in principle, possible, although in order for this to be possible one has to relax at least one condition between (21c) and (21d).

For instance, one could try defining the polytropic behavior using $P \sim \rho^\gamma$ instead of $P \sim V^{-\gamma}$, which is strictly related to the approach of Chaplygin gas models [27].

From (21c) and (24) we find that the equation of state $w \equiv P/\rho$ of the adiabatic fluid evolves as

$$w(z) \equiv \frac{P(z)}{\rho(z)} = -\frac{(1-\gamma)w_0(1+z)^{3\gamma}}{w_0(1+z)^{3\gamma} - (1-\gamma+w_0)(1+z)^3}. \tag{25}$$

Defining

$$\Omega_m \equiv \frac{8\pi G}{3H_0^2}\left(\rho_0 + \frac{P_0}{1-\gamma}\right) + \Omega_m^{(\mathrm{ext})}, \tag{26}$$

we can write the Hubble parameter in the simple form

$$\frac{H(z)^2}{H_0^2} = \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + (1-\Omega_m-\Omega_k)(1+z)^{3\gamma}. \tag{27}$$

The analysis presented in Sec. IV is performed using Ω_m, Ω_k and γ as the independent parameters. The parameter Ω_m describes the total matter content of the Universe, be it external (baryons and/or dark matter) or due to the evolution of the polytropic fluid under study. As one can see from (26), fixing Ω_m and γ still leaves freedom in choosing the value of $\Omega_m^{(\mathrm{ext})}$ and $w_0 = P_0/\rho_0$, so essentially one can insert the desired amount of external nonrelativistic matter by hand. This is a very interesting result because by choosing any value of $\Omega_m^{(\mathrm{ext})}$, the "right" amount of dustlike fluid can automatically be accounted for. The most relevant possibilities are

(i) $\Omega_m^{(\mathrm{ext})} = \Omega_m$, that is,

$$w(z) = w_0 = \gamma - 1. \tag{28}$$

We are basically tuning our fluid so that its dust component vanishes. In this picture, the new fluid only contributes to dark energy, and baryons and CDM are assumed as external and independent components. Moreover, the dark energy component has constant equation of state $w = \gamma - 1$. Notably, in this case our model is equivalent to ωCDM.

(ii) $\Omega_m^{(\mathrm{ext})} = \Omega_b$: in this case, the Universe is filled with just baryons and the new fluid, which is mimicking both dark energy and cold dark matter. For fixed γ, Ω_m and Ω_b, we must choose P_0 so that

$$\frac{8\pi G}{3H_0^2}\left(\rho_0 + \frac{P_0}{1-\gamma}\right) = \Omega_m - \Omega_b \equiv \Omega_{\mathrm{CDM}}, \tag{29}$$

which corresponds to

$$P_0 = (\gamma - 1)\left(\rho_0 - \frac{3H_0^2}{8\pi G}\Omega_{\mathrm{CDM}}\right). \tag{30}$$

Let us stress that these two possibilities, and indeed any other combination of P_0 and $\Omega_m^{(\mathrm{ext})}$ resulting in the same Ω_m, do not need separate analyses. As shown in (27), Ω_m and γ are the only parameters associated to the fluid relevant for cosmological fits.

IV. COSMOLOGICAL CONSTRAINTS

We test our model using a Metropolis-Hastings Monte Carlo code using $L = \exp(-\chi^2_{\mathrm{tot}}/2)$ as the likelihood function, with

$$\chi^2_{\mathrm{tot}} = \chi^2_{\mathrm{SN}} + \chi^2_H + \chi^2_{\mathrm{BAO}},$$

equipped with a Gelman-Rubin convergence diagnostic. We use several cosmological data sets: SNIa data from the Union2.1 compilation [28], $H(z)$ data (as quoted in [29]), and BAO [30–32].

A. Differential age and $H(z)$ data

Independent reconstructions of Hubble measurements constitute a novel approach to track the evolution of the Universe without invoking a model *a priori*. In particular, employing massive early-type galaxies as cosmic chronometers, it would be possible to match astronomical and cosmological measurements to evaluate the differential age, i.e., the ratio $\frac{dt}{dz}$. Since, differentiating the scale factor definition with respect to the redshift, one obtains

$$\frac{dz}{dt} = -(1+z)H(z), \tag{31}$$

it is possible, knowing the redshift at which the measure has been performed, to evaluate H at different stages of the Universe's evolution.

We compare $H(z)$ data with the exact solution (27). The χ^2 is

$$\chi^2_H = \sum_i \frac{[H^{(\mathrm{model})}(z_i) - H^{(\mathrm{exp})}(z_i)]^2}{\sigma_i^2}. \tag{32}$$

B. Supernovae Ia data

We use SNIa data from the Union2.1 catalogue, containing 580 data points. Type-Ia supernovae observations have been extensively employed during the last few decades for cosmological model parameter fitting. Supernovae Ia are widely thought to be standard candles, i.e., objects whose luminosity curves are intimately related to distances.[3] The Union2.1 catalogue is built up to extend previous versions, with the advantage that the whole systematics is mostly reduced. So one can assume that systematic errors do not influence numerical outcomes. Moreover, errors on the redshift measurements are assumed to be negligibly small.

The observable quantity associated to SNIa is the distance modulus, namely the difference between the apparent magnitude m and absolute magnitude M of each object,

$$\mu(z) \equiv m(z) - M = 5\log_{10}\frac{d_L(z)}{10\ \mathrm{pc}} \tag{33}$$

where

$$d_L(z) = \frac{(1+z)}{H_0} \times \begin{cases} \frac{1}{\sqrt{\Omega_k}} \sin\left[\sqrt{\Omega_k} D(z)\right] & \Omega_k > 0 \\ D(z) & \Omega_k = 0 \\ \frac{1}{\sqrt{|\Omega_k|}} \sinh\left[\sqrt{|\Omega_k|} D(z)\right] & \Omega_k < 0 \end{cases} \tag{34}$$

and

$$D(z) = H_0 \int_0^z \frac{1}{H(z')}. \tag{35}$$

However, since M is not known with sufficient accuracy from theoretical arguments, Union2.1 data are only reliable up to a normalization,

$$\mu_{\mathrm{obs}}(z; \mathcal{M}) = \mu_{\mathrm{Union2.1}}(z) + \mathcal{M}, \tag{36}$$

where $\mu_{\mathrm{Union2.1}}$ is the value reported in the Union2.1 compilation, and $\mathcal{M}$ depends both on the absolute magnitude of supernovae Ia and on H_0, but it does not affect the expansion history $H(z)/H_0$. It has to be treated as a *nuisance parameter*, fitted along with the other cosmological parameters and marginalized over.

The distance modulus for a given redshift and set of parameters is computed via numerical integration using (27). The total χ^2_{SN} is computed analogously to (32),

$$\chi^2 = \sum_i \frac{[\mu_{\mathrm{th}}(z_i) - \mu_{\mathrm{obs}}(z_i; \mathcal{M})]^2}{\sigma_i^2}. \tag{37}$$

C. BAO

BAO are observed in large scale structure (LSS), and are the result of sound waves propagating in the early Universe. In recent years, they have provided us with another relevant data set for cosmological fits. Measuring the position of the BAO peak in the LSS correlation function corresponds to measuring a combination of angular distance and luminosity distance, namely

$$D_V^3(z) \equiv \frac{d_L^2(z) z}{(1+z)^2 H(z)}. \tag{38}$$

This quantity tracks the comoving volume variation at a given redshift.

We consider the two BAO observables

$$A(z) \equiv \frac{H_0 D_V(z)\sqrt{\Omega_m}}{z}, \qquad d_z(z) \equiv \frac{r_s(z_{\mathrm{drag}})}{D_V(z)}, \tag{39}$$

where $r_s(z_{\mathrm{drag}})$ is the comoving sound horizon at the baryon drag epoch. This quantity needs to be calibrated with cosmic microwave background data assuming a fiducial cosmological model, with the latest data giving [33,34]

$$z_{\mathrm{drag}} = \begin{cases} 1020.7 \pm 1.1 & \mathrm{WMAP9} \\ 1059.62 \pm 0.31 & \mathrm{Planck} \end{cases} \quad r_s(z_{\mathrm{drag}}) = \begin{cases} 152.3 \pm 1.3 \\ 147.41 \pm 0.30 \end{cases} \tag{40}$$

In this sense, BAO data are slightly model dependent, since acoustic scales depend on the redshift (drag time redshift), inferred from first order perturbation theory assuming a given cosmology. However, the same data would still be reliable when studying any realistic cosmology which differs from ΛCDM only at relatively low redshifts. For the fits, we thus use a Gaussian prior using the Planck best value.

As for the case of SN data, the theoretical values of $A(z)$ of d_z are computed via numerical integration using the

[3]However, it should be stressed that SNIa absolute magnitudes can be neither directly measured nor inferred from theoretical considerations with arbitrary accuracy. See also below.

TABLE I. BAO data used in the analysis. For each experiment, we quote the observable more suitable for the analysis.

Experiment	z	$d_z \pm \sigma_d$	$A(z) \pm \sigma_A$	References
6dFGS	0.106	0.336 ± 0.015		[30]
SDSS-III	0.32	0.1181 ± 0.0023		[31]
	0.57	0.0726 ± 0.0007		
WiggleZ	0.44		0.474 ± 0.034	[32]
	0.6		0.442 ± 0.020	
	0.73		0.424 ± 0.021	

exact expression for the Hubble parameter (27). We use the BAO data shown in Table I. Not all data are uncorrelated. In fact, the covariance matrix (symmetric, we only quote the upper diagonal) for the WiggleZ data at $z = (0.44, 0.6, 0.73)$ is [32]

$$C^{-1} = \begin{pmatrix} 1040.3 & -807.5 & 336.8 \\ & 3720.3 & -1551.9 \\ & & 2914.9 \end{pmatrix}. \quad (41)$$

The total χ^2 for BAO data is

$$\chi^2_{\rm BAO} = \chi^2_{\rm 6dFGS} + \chi^2_{\rm SDSS-III} + \chi^2_{\rm WiggleZ}, \quad (42)$$

with

$$\chi^2_{\rm BAO,SDSS-III} = \sum \left[\frac{d_z^{\rm obs} - d_z^{\rm th}(z_i)}{\sigma_d} \right]^2,$$

$$\chi^2_{\rm WiggleZ} = (\mathbf{A}^{\rm obs} - \mathbf{A}^{\rm th})^T C^{-1} (\mathbf{A}^{\rm obs} - \mathbf{A}^{\rm th}). \quad (43)$$

D. Results

Here we use flat priors on the fitting parameters, Gaussian priors on $r_s(z_{\rm drag})$ as mentioned above, and we marginalize over the nuisance parameter $\mathcal{M}$. We show results for the broad prior $\Omega_k = \text{Uniform}(-1, 1)$ and $\Omega_k = 0$ in Fig. 1. The corresponding means, 95% confidence levels and best fits are shown in Table II. We can see that results are compatible with $\gamma = 0$, which corresponds to the ΛCDM or dark fluid [26] solution, and with $\Omega_k = 0$ which is assumed in many cosmological analyses. As expected, γ is constrained to small values, roughly $|\gamma| \lesssim 0.25$ at 2σ.

Considering only the χ^2 value, our modified cosmology appears to be only slightly preferred over the standard one, as one can see comparing the left and right tables in Table II. These considerations suggest that it would be useful to consider a a discussion on model selection criteria for our approach and the concordance model. See the next section for details.

V. MODEL SELECTION CRITERIA

For much of the community, the ΛCDM paradigm is the favorite framework to fit cosmic data, due to its simplicity

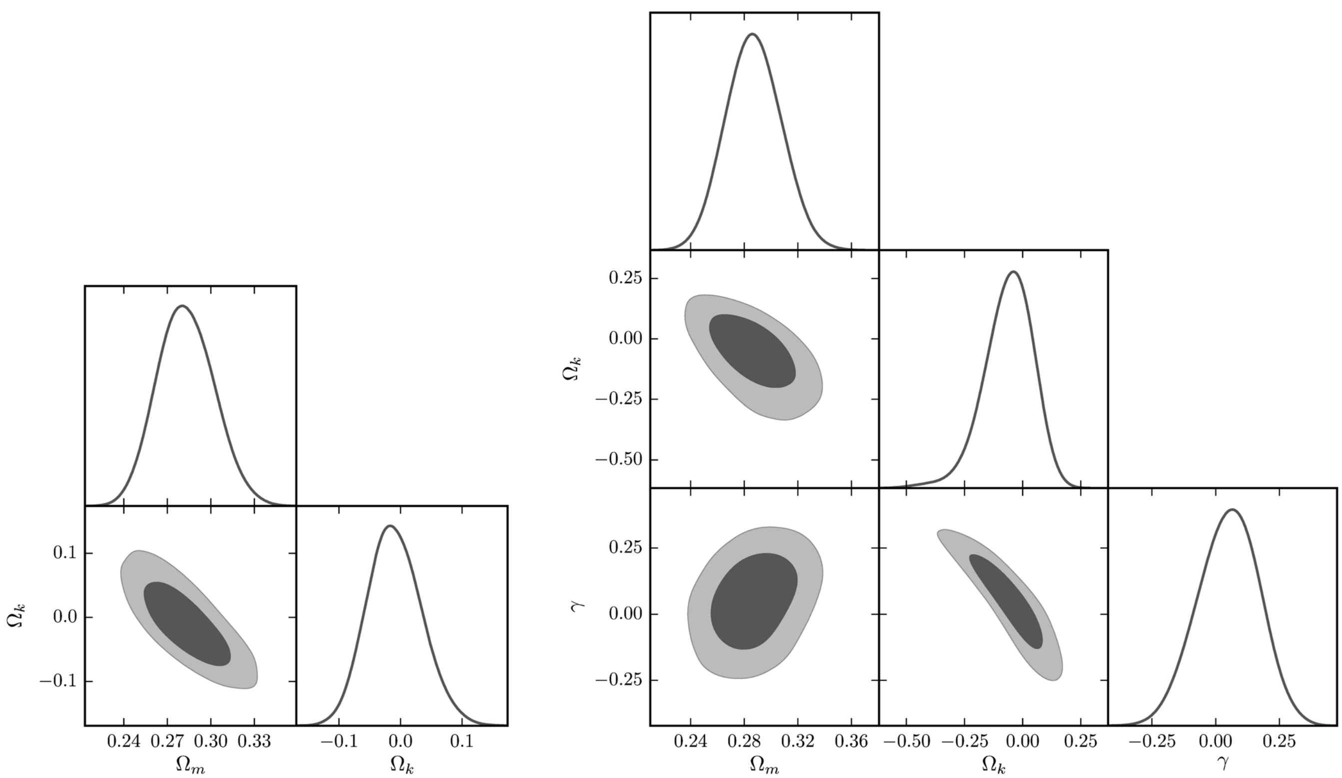

FIG. 1. 1σ and 2σ contours corresponding to $\gamma = 0$ (ΛCDM) (left), and to our model with γ free (right). See also Table II.

TABLE II. Results for our model for ΛCDM, i.e., $\gamma = 0$ (left) and γ generic (right). See also Fig. 1.

Parameter	Prior	95% limits	Best fit
Ω_m	Uniform(0,1)	$0.283^{+0.040}_{-0.037}$	0.2843
Ω_k	Uniform(-1,1)	$-0.009^{+0.090}_{-0.082}$	−0.0174
γ	$= 0$		
$\chi^2_{\rm tot}$			593.270
	ΛCDM ($\gamma = 0$)		
Parameter	Prior	95% limits	Best fit
Ω_m	Uniform(0,1)	$0.287^{+0.041}_{-0.039}$	0.2865
Ω_k	Uniform(-1,1)	$-0.06^{+0.20}_{-0.22}$	−0.05483
γ	Uniform(-1,1)	$0.05^{+0.23}_{-0.24}$	0.05877
$\chi^2_{\rm tot}$			592.984
	$\gamma \neq 0$		

and the fact that the only free parameters are Ω_m and Ω_k. However, a large number of different possibilities go beyond this choice and therefore one needs to determine methods able to compare the range of different competing cosmological models. Two statistical model-independent methods are offered by the so-called selection criteria, aimed at determining the best model by considering the combination of χ^2 and degrees of freedom. This turns out to be important, since it is possible that viable models with higher orders of parameters provide higher chi squares as well, which do not have to be considered as disfavored at all compared with the standard model.

Three suitable selection criteria are the Akaike information criterion (AIC), the corrected AIC (AICc) and the Bayesian information criterion (BIC) [35]. These tests are a standard diagnostic tool [36] of regression models [37]. They are defined as

$$\mathrm{AIC} \equiv -2\ln\mathcal{L} + 2d, \tag{44a}$$

$$\mathrm{AICc} \equiv \mathrm{AIC} + \frac{2d(d+1)}{N-d-1}, \tag{44b}$$

$$\mathrm{BIC} \equiv -2\ln\mathcal{L} + 2d\ln N, \tag{44c}$$

where

$$\mathcal{L} = \exp(-\chi^2/2) \tag{45}$$

is the chosen likelihood function, d is the number of model parameters and N is the number of data points, which in our case is

$$N = N_{\rm SN} + N_H + N_{\rm BAO} = 580 + 28 + 6 = 614. \tag{46}$$

The basic requirement is to essentially postulate two distribution functions, namely $f(x)$ and $g(x|\theta)$. Here, $f(x)$ is taken to be the *exact* distribution function, whereas $g(x|\theta)$ approximates the former. The way of approximating this makes use of a set of parameters which has been denoted by θ. Thus, there exists only a set of $\theta_{\rm min}$, which minimizes the difference between $g(x,\theta)$ and $f(x)$ [38].

TABLE III. Comparison between ΛCDM and our model using three criteria: AIC, AICc, and BIC. See the text for details.

Model	$\chi^2_{\rm best\ fit}$	Δd	$\Delta_{\rm AIC}$	$\Delta_{\rm AICc}$	$\Delta_{\rm BIC}$
ΛCDM	593.270	0	0	0	0
γ	592.984	1	1.714	1.721	12.554

It follows that the AIC, AICc and BIC values for a single model are meaningless since the exact model function $f(x)$ is unknown. For those reasons, one is only interested in the differences

$$\Delta X = X_\gamma - X_{\Lambda\rm CDM} \qquad X = \mathrm{AIC}, \mathrm{AICc}, \mathrm{BIC}. \tag{47}$$

These quantities must be evaluated for the whole set of models involved in the analysis.

Results are shown in Table III. The AIC(c) tests indicate a slight preference for ΛCDM, whereas the BIC test suggests a very strong preference. Indeed, the BIC has a much stronger penalty for extra parameters, although a pure Bayesian evidence analysis sometimes gives results more in line with the AIC(c).

VI. CONSEQUENCES ON EARLY-TIME COSMOLOGY

Let us now investigate how the correction due to our dark energy model affects the Universe's dynamics at high redshifts. To do so, we study the impact of the modified background evolution on density perturbations, which likely represent the most suitable framework in which one can naively check the goodness of any cosmological model at high redshifts. The perturbation equations, in their coarse-grained form, simply read

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0. \tag{48}$$

The so-called *growth evolution*, intimately related to δ, may be easily handled by means of the scale variable $\ln a$. One can parametrize the dark energy effects using the growth variable

$$D(a) = \frac{\delta}{a} \tag{49}$$

which satisfies

$$D'' + D'\left[\frac{5}{a} + \frac{(\ln E^2)'}{2}\right] + \frac{D}{a}\left[\frac{3}{a}\left(1 - \frac{\Omega_m}{2E^2a^3}\right) + \frac{(\ln E^2)'}{2}\right] = 0, \tag{50}$$

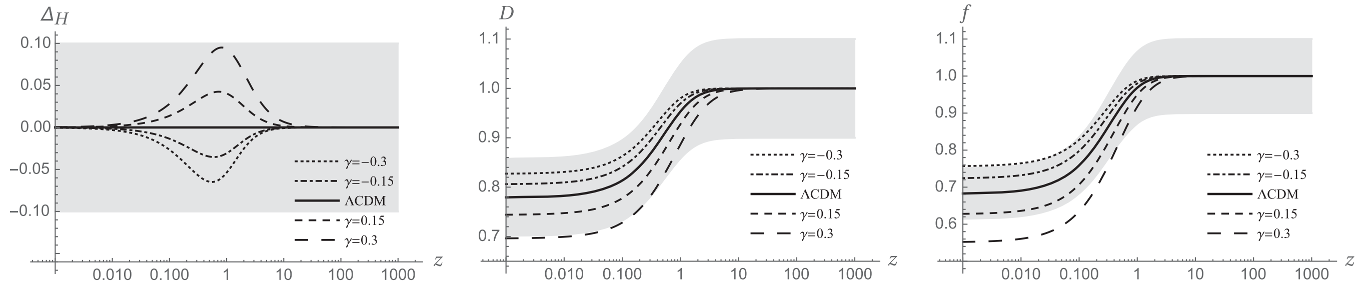

FIG. 2. Comparison between ΛCDM (black solid lines) and our model, for a few values of γ: −0.3 (dotted), −0.15 (dot-dashed), 0.15 (dashed, small) and 0.3 (dashed, large). The grey bands correspond to ±10% departures from ΛCDM. The plotted quantities are Δ_H [Eq. (53)], D [Eq. (49)], and f [Eq. (52)]. For definiteness, we have taken $\Omega_m = 0.3$ and $\Omega_k = 0$.

where a prime denotes differentiation with respect to the scale factor a, and $E \equiv H/H_0$. We assume the boundary conditions $D(a_{LSS}) = 1$ and $D'(a_{LSS}) = 0$, with $a_{LSS} = (1 + z_{LSS})^{-1}$, i.e., the last scattering surface scale factor, approximated by $z_{LSS} \approx 1089$.

The growth history for a given γ can be compared with the standard model, i.e., ΛCDM, keeping in mind that any viable cosmological models should not yield deviations exceeding about 10%.

Analogously, one can define the growth index as

$$f = \frac{d \ln \delta}{d \ln a}, \tag{51}$$

which enables one to rewrite the perturbation equations as

$$f' + \frac{f^2}{a} + \left[\frac{2}{a} + \frac{(\ln E^2)'}{2}\right] f - \frac{3\Omega_m}{2E^2 a^4} = 0, \tag{52}$$

with the boundary condition $f(a_{LSS}) = 1$. Furthermore, to corroborate the results on the shift parameters, we also plot the relative deviation

$$\Delta_H \equiv \frac{H_\Lambda - H_{\rm model}}{H_\Lambda} \tag{53}$$

between the Hubble rates of the ΛCDM model ($\gamma = 0$) and that corresponding to a general γ.

Plots of D, f and Δ_H for a few values of γ, compared with the ΛCDM predictions, are shown in Fig. 2. As we can see from numerical results, Δ_H differs substantially from 0 only at intermediate redshifts ($0.1 \lesssim z \lesssim 10$), peaking around $z \sim 1$. At large redshifts, the dark energy component is completely negligible, and our model is indistinguishable from ΛCDM. Naturally, we also have $\Delta_H \to 0$ as $z \to 0$ because we require that $H(z \to 0) = H_0$ for any model.

D and f start departing from the standard ΛCDM solution around $z \sim 1$, after which they follow the standard evolution but with a normalization factor with respect to the standard cosmological scenario.

All curves for Δ_H, D and f fit within the ±10% bands with the exception of the $\gamma = 0.3$ solution in the plot for f. Notice that the 95% limits from the numerical fits of Table II constrain $|\gamma| \lesssim 0.25$ so this result does not effectively reduce the allowed range of values for γ.

VII. DISCUSSION AND CONCLUSIONS

We have developed a simple dark energy model starting from an adiabatic fluid which evolves following rather standard thermodynamic considerations. The adiabatic index γ is expressed as a specific function of redshift and the Hubble parameter. In a homogeneous and isotropic universe, solutions give $\gamma =$ constant, but we would expect this to change if one or more of the assumptions are relaxed.

In the simplest case of $P \propto V^{-\gamma}$, the resulting fluid is the combination of two effective fluids evolving differently throughout the history of the Universe: a component scaling as $(1+z)^3$, i.e. as nonrelativistic matter, and another scaling as $(1+z)^{3\gamma}$, which corresponds to a dynamical dark energy term. The ΛCDM paradigm is included in our model, by taking $\gamma = 0$. We are free to choose P_0, ρ_0 and w_0 according to the desired amount of dark energy/dark matter and the redshift evolution of dark energy.

Although we do need two different numbers, e.g. ρ_0, γ, in order to specify the relative abundances of dark energy/dark matter, they are manifestations of the *same* adiabatic fluid, about whose cosmological evolution we have made no *a priori* assumption.

We fitted the model using SNIa, $H(z)$, and BAO data, and found the corresponding constraints on the model parameters. As expected, the ΛCDM solutions fit perfectly within the observational bounds, and are actually preferred by model selection criteria despite the slight improvement in terms of total χ^2 for our model.

We also performed an analysis of the evolution of density perturbations at high redshifts and found that reasonable values of γ are well within the allowed discrepancies from the standard cosmological scenario derived from the late universe constraints.

In future works, it would be interesting to relax the hypothesis $P \propto V^{-\gamma}$ and explore the consequences on cosmology. Moreover, it would be useful to constrain the heat capacities together with the adiabatic index, giving a possible explanation of the role played by the temperature.

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