

Observational constraints on new exact inflationary scalar-field solutionsJohn D. Barrow^{1,*} and Andronikos Paliathanasis^{2,†}¹*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*²*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile*

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An algorithm is used to generate new solutions of the scalar-field equations in homogeneous and isotropic universes. Solutions can be found for pure scalar fields with various potentials in the absence and presence of spatial curvature and other perfect fluids. A series of generalizations of the Chaplygin gas and bulk viscous cosmological solutions for inflationary universes are found. Furthermore other closed-form solutions which provide inflationary universes are presented. We also show how the Hubble slow-roll parameters can be calculated using the solution algorithm and we compare these inflationary solutions with the observational data provided by the Planck 2015 collaboration in order to constrain and rule out some of these models.

DOI: [10.1103/PhysRevD.94.083518](https://doi.org/10.1103/PhysRevD.94.083518)**I. INTRODUCTION**

Recent cosmological data indicate that the Universe has undergone two acceleration phases: an early acceleration phase called “inflation,” prior to a radiation-dominated era, and a more recent era of accelerated expansion which appears to continue today [1–7]. The gravitationally repulsive stress that is responsible for the current acceleration of the Universe is called “dark energy” and must possess sufficient negative pressure to exert gravitational repulsion. Its specific identity is still unknown and it may result from a modification of general relativity when gravity is very weak or the presence of a specific unknown matter field.

While a range of “exotic” fluids and modifications of the gravitational action can provide cosmological acceleration, scalar fields are the simplest candidates to explain the acceleration phases of the Universe. Moreover, scalar fields also have various applications in the inflationary phase of the Universe, for instance in driving chaotic inflation [8]. While the same scalar field might explain both of the periods of accelerated expansion, no convincing cosmological model has been found with this as a natural feature. In a scalar-field cosmology the field equations are of second order where the scalar field is introduced as an extra degree of freedom, with a corresponding conservation equation. These equations display unexpected complexity. Simple power-law potentials for the scalar field can create finite-time singularities during inflation [9] and lead to chaotic dynamics [10], or singularity avoidance [11] if the Universe is closed.

Very few exact scalar-field solutions in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with spatial curvature are known [12,13]. In a spatially flat FLRW spacetime closed-form solutions with or without sources for different scalar-field potentials, or scalar fields which

mimic other fluids, such as the Chaplygin gas, can be found in [14–30] while some other classes of integrable scalar-field models are also given [31–34]. Some solutions for three-dimensional FLRW spacetimes are given in [35–37]. However, scalar-field cosmology is conformally equivalent to other scalar-tensor theories, like Brans-Dicke or $f(R)$ -gravity. [38,39]. Hence, closed-form solutions of the conformally equivalent theories (see [40,41] and references therein) can be used to construct closed-form solutions or to find new integrable systems for the nonminimally coupled scalar-field model.

Recently, with the use of nonlocal conservation laws in [42], the general analytical solution has been expressed for an arbitrary scalar field with an arbitrary number of independent perfect fluids possessing constant equation of state parameters in spatially flat or nonflat FLRW universes. These general results are applied in this paper to derive precise forms of the scalar-field potential for various simple time-dependent forms for the expansion scale factor, or for particular equation of state parameters for the scalar field. Finally, the Hubble slow-roll parameters are studied for these closed-form solutions so that we can compare the inflationary parameters with the observable constraints provided by the Planck 2015 observations [7].

The plan of this paper is as follows. In Sec. II, the basic properties and definitions of scalar-field models are introduced. The cosmological metric we consider is the four-dimensional FLRW spacetime, while the gravitational action is that of general relativity with a minimally coupled scalar field. We review previous results in the literature and present the general analytical solution for the cosmological field equations for arbitrary scalar-field potential. Exact closed-form solutions obtained by using these general results are presented in Secs. III and IV. Specific closed-form solutions are derived for spatially flat FLRW universes when only one scalar field and a perfect fluid with constant equation of state parameter are present. For

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specific values of the barotropic parameter for the matter source, these results give closed-form solutions in the case of a nonflat FLRW universe. In Sec. V, we derive the Hubble slow-roll parameters for our models and compare them with those of the Planck 2015 data to isolate observationally allowed parameter ranges. Finally, in Sec. VI, we discuss our results and draw conclusions.

II. SCALAR-FIELD COSMOLOGY

We consider the gravitational action to be

$$S = S_{\text{EH}} + S_\phi + S_m, \quad (1)$$

in which $S_{\text{EH}} = \int dx^4 \sqrt{-g} R$ is the Einstein-Hilbert action, R is the Ricci scalar of the underlying spacetime geometry with metric tensor $g_{\mu\nu}$, $S_m = \int dx^4 \sqrt{-g} L_m$ is the matter action, and S_ϕ is the action for the scalar field, with

$$S_\phi = \int dx^4 \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + V(\phi) \right], \quad (2)$$

where $V(\phi)$ is the self-interaction potential of the scalar field ϕ . Variation of S with respect to $g_{\mu\nu}$ gives the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}, \quad (3)$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor of baryonic matter and radiation, and $T_{\mu\nu}(\phi)$ is the energy-momentum tensor associated with ϕ . Furthermore, variation with respect to ϕ gives

$$-g^{\mu\nu} \phi_{;\mu\nu} + V_{,\phi} = 0, \quad (4)$$

where we have considered that $\frac{\partial S_m}{\partial \phi} = 0$, so there is no interaction between the matter source, S_m , and the scalar field, ϕ , in the action integral (1).

Using the Bianchi identity for (3) we have that $T^{(\phi)\mu\nu}_{;\nu} + T^{(m)\mu\nu}_{;\nu} = 0$, which gives

$$T^{(\phi)\mu\nu}_{;\nu} = 0, \quad T^{(m)\mu\nu}_{;\nu} = 0. \quad (5)$$

These are the equations of motion for the matter sources S_m and the field ϕ . It can be seen that (5) is just Eq. (4).

By assuming that the Universe is spatially isotropic and homogeneous we select the four-dimensional spacetime to be that of FLRW:

$$ds^2 = -dt^2 + a^2(t) \frac{(dx^2 + dy^2 + dz^2)}{(1 + \frac{K}{4} \mathbf{x}^2)^2}. \quad (6)$$

Furthermore, we assume that ϕ inherits the symmetries of the metric (6). Therefore $\phi(t)$ and consequently

$\phi_{;\nu} = \dot{\phi} \delta_\nu^0$, where $\dot{\phi} = \frac{d\phi}{dt}$. Consider the comoving observer $u_\mu = \delta_t^\mu$ ($u^\mu u_\mu = -1$). In the 1 + 3 decomposition the energy-momentum tensor becomes

$$T_{\mu\nu}^{(\phi)} = (\rho_\phi + P_\phi) u_\mu u_\nu + P_\phi g_{\mu\nu}, \quad (7)$$

$$T_{\mu\nu}^{(m)} = (\rho_m + P_m) u_\mu u_\nu + P_m g_{\mu\nu}, \quad (8)$$

where

$$\rho_\phi \equiv \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad P_\phi \equiv \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (9)$$

are the energy density and the pressure of the scalar field and ρ_m , p_m are the components that correspond to the matter source S_m which we have assumed to be a perfect fluid. This follows also from (6).

Therefore, the field equations (3) for the line element (6) become

$$H^2 = \frac{1}{3} (\rho_m + \rho_\phi) - \frac{K}{a^2} \quad (10)$$

$$3H^2 + 2\dot{H} = - (P_m + P_\phi) - \frac{K}{a^2}, \quad (11)$$

where $H(t) \equiv \frac{\dot{a}}{a}$ is the Hubble function. Equations (5) become

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0 \quad (12)$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0 \quad (13)$$

while the corresponding equation of state (EoS) parameters are given by $w_m = P_m/\rho_m$ and

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{(\dot{\phi}^2/2) - V(\phi)}{(\dot{\phi}^2/2) + V(\phi)} \quad (14)$$

which means that $w_\phi < -\frac{1}{3}$ when $\dot{\phi}^2 < V(\phi)$. On the other hand, if the kinetic term of the scalar field is negligible with respect to the potential energy, i.e., $\frac{\dot{\phi}^2}{2} \ll V(\phi)$, then the equation of state parameter is $w_\phi \simeq -1$.

Substituting (9) into (13), we find Eq. (4) which for the line element (6) takes the form

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \quad (15)$$

The set of equations, (10), (11) and (15), provides us with the cosmological evolution, i.e. the scale factor $a(t)$, where a potential $V(\phi)$ and an equation of state parameter w_m have been defined.

There is a simple recipe [14–16,43,44] for finding exact solutions in the flat FLRW universes containing only the

scalar field ($\rho_m = p_m = 0 = K$), where the defining equations simplify to

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (16)$$

$$2\dot{H} = -\dot{\phi}^2. \quad (17)$$

The third equation (15) is a consequence of these equations. The recipe is to pick a physically realistic function $\phi(t)$, solve (17) to find $H(t)$, use $H(t)$ and $\phi(t)$ to find $V(t)$ from (16) and convert this to $V(\phi)$ using the initial $\phi(t)$. This completes the solution so long as the intermediate integrals can be performed and appropriate positivity conditions hold (for example, $H > 0$ and $V \geq 0$). However, with a perfect fluid or 3-curvature (which is just another perfect fluid) this method is not efficient and we must look to a more systematic version. To this method we now turn.

A. General analytical solution

In the line element (6) we use the comoving proper time, t , by putting $dt = N(\tau)d\tau$. From the action integral (1), we can now define

$$L(N, a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{N} \left(-3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 \right) - Na^3V(\phi) - N\rho_{m0}a^{-3(\gamma-1)} + 3NKa, \quad (18)$$

where for the matter source, S_m , we have put $w_m = \gamma - 1$. Hence the gravitational field equations follow from the Euler-Lagrange equations with respect to the variables $\{N, a, \phi\}$. However, as it can be seen, the field equations in the space of variables $\{N, a, \phi\}$ form a singular dynamical system with constraint equation $\frac{\partial L}{\partial N} = 0$. Hence, using [42] with the application of the results of [45], it has been shown that the gravitational field equations which follow from (18) admit an infinite number of (nonlocal) conservation laws. Specifically, every conformal Killing vector of the minisuperspace $\{a, \phi\}$ provides a conservation law and, as the minisuperspace has dimension two, the dimension of the conformal algebra is infinite and consequently we have an infinite number of conservation laws. Here, it is important to note that these conservation laws are not necessarily in involution. For more details see [45].

With the use of a specific (nonlocal) conservation law, in [42] it was proved that the field equations form an integrable system and for a specific lapse, ω , such as $dt = e^{F(\omega)/2}d\omega$, where $a(\omega) \equiv e^{\omega/6}$; that is, the line element is now

$$ds^2 = -e^{F(\omega)}d\omega^2 + e^{\omega/3} \frac{(dx^2 + dy^2 + dz^2)}{(1 + \frac{K}{4}\mathbf{x}^2)^2}, \quad (19)$$

and the solution is expressed in terms of the unknown function $F(\omega)$, which is directly related to the potential $V(\phi)$.

In the case of a spatially flat Universe, $K = 0$, and without matter source, $\rho_{m0} = 0$, it has been found that¹

$$\phi(\omega) = \pm \frac{\sqrt{6}}{6} \int \sqrt{F'(\omega)}d\omega, \quad (20)$$

where

$$V(\omega) = \frac{1}{12} e^{-F(\omega)}(1 - F'(\omega)) \quad (21)$$

and effective fluid components for the scalar field are

$$\begin{aligned} \rho_\phi(\omega) &= \frac{1}{12} e^{-F(\omega)}, \\ P_\phi(\omega) &= \frac{1}{12} e^{-F(\omega)}(2F'(\omega) - 1). \end{aligned} \quad (22)$$

Furthermore, in the case of a spatially flat universe with a perfect fluid the solution is generalized as follows:

$$\phi(\omega) = \pm \frac{\sqrt{6}}{6} \int [(F'(\omega) - 6\gamma\rho_{m0}e^{F-\frac{\gamma}{2}\omega})]^{1/2}d\omega, \quad (23)$$

where now

$$V(\omega) = \frac{1}{12} e^{-F(\omega)}(1 - F'(\omega)) + \frac{\gamma}{2}\rho_{m0}e^{-\frac{\gamma}{2}\omega} \quad (24)$$

and the fluid components become

$$\rho_\phi = \frac{1}{12} e^{-F(\omega)} - \rho_{m0}e^{-\frac{\gamma}{2}\omega} \quad (25)$$

and

$$P_\phi = \frac{1}{12} e^{-F(\omega)}(2F'(\omega) - 1) - (\gamma - 1)\rho_{m0}e^{-\frac{\gamma}{2}\omega}. \quad (26)$$

In the latter case, the total fluid stress, $T_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}$, can be described by a new field, Φ , which follows from (20)–(22). Also in the latter, if we assume that $\gamma = \frac{2}{3}$ (to mimic a curvature term in the Friedmann equations) and $\rho_{m0} = -3K$, then the solution of the scalar-field model in a nonflat FLRW spacetime is recovered.

The aim of this work is to derive specific closed-form solutions of the field equations using these results by assuming special inflationary functions for the scalar factor,

¹Where a prime, i.e. F' , denotes derivative with respect to ω .

or special equation of state parameters for the scalar field, which consequently combine to define the scalar-field potential.

III. CLOSED-FORM SOLUTIONS: SPATIALLY FLAT FLRW SPACETIME WITHOUT MATTER SOURCE

Consider the simplest scenario for a spatially flat FLRW spacetime containing only a scalar field. If we assume that the scalar field has a constant equation of state parameter, say $w_\phi = (-1 + \frac{2}{3q})$, where q is a constant, then from (22) we find that

$$F(\omega) = 2 \ln \left[\frac{e^{\frac{\omega}{6q}}}{6q(a_0)^{\frac{1}{q}}} \right], \quad (27)$$

and hence we have

$$\phi(\omega) = \frac{\sqrt{2}}{6} q^{-\frac{1}{2}} \omega, \quad V(\omega) = V_0 e^{-\frac{1}{3q}\omega}. \quad (28)$$

Therefore, $V(\phi) = V_0 e^{-\sqrt{\frac{2}{3}}\phi} = (a_0)^{\frac{2}{3}q} q(3q-1) e^{-\sqrt{\frac{2}{3}}\phi}$ in which, if we apply the transformation $d\omega \rightarrow dt$ to write (19) in the form of (6), we find the well-known power-law solution [which we can verify directly in (16)–(17)]

$$a(t) = a_0 t^q, \quad \phi = \phi_0 + \sqrt{2q} \ln(t). \quad (29)$$

However, this is only a particular solution of the exponential scalar-field potential problem [46]. The general solution can be found in [47], while some special solutions are given in [48,49].

We continue with the determination of the closed-form solution for some specific equation of state parameters for the scalar field.

A. Perfect fluid with cosmological constant

Assume that the scalar field satisfies the simple equation of state parameter

$$p_\phi = (\gamma - 1)\rho_\phi - 3\gamma\Lambda H_0^2. \quad (30)$$

Then, it follows from (22) that

$$2F' + 36\gamma\Lambda e^F - \gamma = 0. \quad (31)$$

We observe that, for $\Lambda = 0$, $F(\omega)$ is linear as above. Hence, for nonzero Λ and $\gamma \neq 0$, we find that

$$F(\omega) = -\ln(36\Omega_{m0}H_0^2 e^{-\frac{\gamma}{2}\omega} + 36\Omega_\Lambda H_0^2), \quad (32)$$

where $36\Omega_{m0}H_0^2$ is the constant of integration and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$. The Hubble function is

$$\frac{H^2(\omega)}{H_0^2} = \Omega_{m0} e^{-\frac{\gamma}{2}\omega} + \Omega_\Lambda, \quad (33)$$

which is equivalent to a cosmological model containing a perfect fluid and a cosmological constant. We can see that for $\gamma = 1$, Λ CDM cosmology is recovered.

Furthermore, using (32) we find

$$\phi(\omega) = -\frac{2}{\sqrt{3}\gamma} \operatorname{arctanh} \left(\sqrt{\frac{\Omega_{m0} + \Omega_\Lambda e^{\frac{\gamma}{2}\omega}}{\Omega_{m0}}} \right) \quad (34)$$

and

$$V(\omega) = \frac{3}{2} H_0^2 e^{-\frac{\gamma}{2}\omega} (\Omega_m(2-\gamma) + 2\Omega_\Lambda e^{\frac{\gamma}{2}\omega}), \quad (35)$$

where the equation of state parameter is

$$w_\phi(\omega) = -1 + \gamma \frac{\Omega_{m0}}{\Omega_{m0} + \Omega_\Lambda e^{\frac{\gamma}{2}\omega}}. \quad (36)$$

Finally, we find the potential

$$V(\phi) = 3\Lambda H_0^2 + \frac{3(2-\gamma)}{2} \Lambda H_0^2 \cosh^2 \left(\frac{\sqrt{3}\gamma}{2} \phi \right), \quad (37)$$

from which we observe that, for $\gamma = 2$, the potential is constant and the perfect-fluid term is that of stiff matter as we expect for the kinetic part of the scalar field. Furthermore, for $\gamma = 1$, we have the unified dark matter (UDM) scalar-field potential which has been found before [50,51]. The difference between this solution and that of [33] is that the free parameters have been selected so that the stiff fluid of the kinetic part of the field is eliminated. The transformation between the two line elements (19) and (6) is

$$\omega = \frac{4}{\gamma} \ln \left(\frac{1}{\Lambda} \exp \left(\frac{9}{2} \gamma^2 \Lambda H_0^2 t^2 \right) - \frac{\Omega_{m0}}{2} \right) - 9\gamma \Lambda H_0^2 t^2 - \frac{4 \ln(2)}{\gamma}. \quad (38)$$

B. Exponential function

Assume now that $F(\omega)$ is an exponential function, $F(\omega) = 2F_0 e^{F_1 \omega}$, which gives that

$$H^2(a) = \frac{1}{6} \exp(-F_0 a^{6F_1}) \quad (39)$$

while for the scalar field we find that

$$\begin{aligned}\phi(\omega) &= \frac{2\sqrt{3F_0F_1}}{3} e^{\frac{F_1\omega}{2}}, \\ V(\omega) &= \frac{\exp(-2F_0e^{F_1\omega})}{12} (1 - 2F_0F_1e^{F_1\omega}),\end{aligned}\quad (40)$$

which gives the potential

$$V(\phi) = \frac{1}{24} e^{-\frac{3}{2}F_1\phi^2} (2 - 3(F_1)^2\phi^2). \quad (41)$$

Finally, the parameter of the equation of state for the scalar field is

$$w_\phi = -1 + 4F_0F_1e^{F_1\omega}, \quad (42)$$

and after the transformation $d\omega \rightarrow dt$ gives this in terms of the inverse function of the exponential integral.

C. Chaplygin gas

Suppose that the scalar field satisfies the barotropic equation for the Chaplygin gas [52], that is,²

$$p_\phi = \frac{A_0}{144} (\rho_\phi)^{-1}. \quad (43)$$

When we substitute from (22) and solve the first-order differential equation, we find

$$F(\omega) = -\ln(\sqrt{A_1e^{-\omega}} - A_0), \quad (44)$$

where A_1 is a constant of integration. Therefore, we have

$$\phi^2(\omega) = \frac{1}{3} \operatorname{arctanh}^2\left(\sqrt{1 - \frac{A_0}{A_1}e^\omega}\right) \quad (45)$$

and

$$V(\omega) = \frac{1}{24} \frac{(2A_0e^{\omega/2} - A_1e^{-\omega/2})}{\sqrt{A_1 - A_0e^\omega}}, \quad (46)$$

which gives

$$V(\phi) = \frac{\sqrt{A_0}}{24} \sinh(\sqrt{3}\phi)(2 - \coth(\sqrt{3}\phi)). \quad (47)$$

Furthermore, for the parameter of the equation of state, we have

²Note that in a flat FLRW universe the Chaplygin gas is simply a bulk viscous stress for a pressure-free fluid with a bulk viscous coefficient proportional to $\rho^{-3/2}$. Similarly, the generalized Chaplygin gas with $p \propto \rho^\mu$ is simply a bulk viscous stress proportional to $\rho^{\mu+1/2}$. The bulk viscous solutions that correspond to all the Chaplygin gas models can therefore be found in Ref. [53].

$$w_\phi(\omega) = \frac{A_0e^\omega}{A_1 - A_0e^\omega}, \quad (48)$$

while the transformation $d\omega \rightarrow dt$ now gives this in terms of the inverse hypergeometric function.

D. Generalized Chaplygin gas I

The first generalization of the Chaplygin gas is by a modification of the equation of state to [54,55]

$$p_\phi = 12^\mu A_0 (\rho_\phi)^{(\mu+1)}, \quad (49)$$

where for $\mu = 0$ we are in the limit of a perfect fluid, and for $\mu = -2$ we have a Chaplygin gas (43).

For the function $F(\omega)$ we find

$$F(\omega) = \frac{1}{\mu} \ln(A_1 e^{\frac{\mu\omega}{2}} - A_0). \quad (50)$$

For the scalar field it follows that

$$\phi(\omega) = \frac{2\sqrt{3}}{3\mu} \operatorname{arctanh}\left(\sqrt{1 - \frac{A_0}{A_1}e^{-\frac{\mu\omega}{2}}}\right) \quad (51)$$

and

$$V(\omega) = \frac{1}{24} (A_1 e^{\frac{\mu\omega}{2}} - 2A_0)(A_1 e^{\frac{\mu\omega}{2}} - A_0)^{-1-\frac{1}{\mu}}. \quad (52)$$

From (51) and (52) the potential $V(\phi)$ is given by the following closed-form expression:

$$\begin{aligned}V(\phi) &= \frac{(A_0)^{-\frac{1}{\mu}}}{24} \left(\cosh^2\left(\frac{\sqrt{3}}{2}\mu^2\phi\right) - 2 \right) \\ &\times \left(\sinh^2\left(\frac{\sqrt{3}}{2}\mu^2\phi\right) \right)^{-1-\frac{1}{\mu}}.\end{aligned}\quad (53)$$

Furthermore, for the equation of state parameter

$$w_\phi(\omega) = \frac{A_0}{A_1 e^{\frac{\mu\omega}{2}} - A_0}, \quad (54)$$

while the transformation $d\omega \rightarrow dt$ is expressed in terms of the inverse hyperbolic function.

E. Generalized Chaplygin gas II

In [14] a generalized Chaplygin gas was proposed with barotropic equation

$$p_\phi = \gamma \rho_\phi^\lambda - \rho_\phi, \quad (55)$$

from which we can see that for $\lambda = 1$ a perfect fluid is recovered, while for $\lambda = 0$ expression (55) reduces to a special form of (30). Again, by substitution of (22) into (55)

we find that the solution of the first-order differential equation is

$$F(\omega) = -\frac{1}{1-\lambda} \ln(\bar{\gamma}\omega + \gamma_1), \quad (56)$$

where $\bar{\gamma} = 2^{1-2\lambda} 3^{1-\lambda} (\lambda-1)\gamma$, and γ_1 is a constant of integration. In what follows we assume that $\lambda \neq 0, 1$.

Hence, it follows that

$$\phi^2(\omega) = \frac{2}{3\bar{\gamma}} \frac{(\bar{\gamma}\omega + \gamma_1)}{(\lambda-1)} \quad (57)$$

and

$$V(\omega) = \frac{(\bar{\gamma}\omega + \gamma_1)^{-1+\frac{1}{1-\lambda}} ((\lambda-1)(\bar{\gamma}\omega + \gamma_1) - \bar{\gamma})}{12(\lambda-1)}. \quad (58)$$

This gives

$$V(\phi) = \frac{1}{36(\lambda^2-1)^2} \left(\frac{3}{2} \bar{\gamma}(\lambda-1) \right)^{\frac{1}{1-\lambda}} \times \phi^{-2+\frac{1}{1-\lambda}} (3(\lambda-1)^2 \phi^2 - 2). \quad (59)$$

Also, from (22), it follows that $\rho_\phi(\omega) = \frac{1}{12} (\bar{\gamma}\omega + \gamma_1)^{\frac{1}{1-\lambda}}$ and

$$w_\phi(\omega) = -1 + \frac{2\bar{\gamma}}{\lambda-1} (\bar{\gamma}\omega + \gamma_1)^{-1}. \quad (60)$$

For the scale factor $a(t)$ in the line element (6), we find that $\omega = -\frac{\gamma_1}{\bar{\gamma}} + \left(\left(\frac{\bar{\gamma} 2\lambda-1}{2\lambda-1} \right)^2 \right)^{-\frac{1}{2\lambda-1}} t^{-\frac{2-2\lambda}{2\lambda-1}}$ for $\lambda \neq \frac{1}{2}$, while

for $\lambda = \frac{1}{2}$ it follows that $\omega = -\frac{\gamma_1}{\bar{\gamma}} + \frac{1}{\bar{\gamma}} e^{\bar{\gamma}t}$. Therefore, for the cosmological scale factor we have

$$a(t) \approx \exp(a_1 t^N), \quad \lambda \neq \frac{1}{2} \quad (61)$$

and

$$a(t) \approx \exp(a_1 e^{\bar{\gamma}t}), \quad \lambda = \frac{1}{2}. \quad (62)$$

F. Generalized Chaplygin gas III

Consider now a third modification of the Chaplygin gas in which the pressure and the energy density for the scalar field satisfy the nonlinear relation

$$p_\phi = A\rho_\phi^{(2+\lambda)} + B\rho_\phi^{-\lambda}, \quad (63)$$

from which, for $\lambda = -1$, we have that $p_\phi = (A+B)\rho_\phi$. Equation (63) differs from that of [56] by a term ρ_ϕ . Using (22), we find that

$$F(\omega) = -\frac{1}{1+\lambda} \ln \left[F_0 \left(F_1 \tanh \left(\frac{(1+\lambda)}{4} F_1 \omega \right) - 1 \right) \right], \quad (64)$$

where $F_0 = 6^{-\frac{1}{1-\lambda}} 12^\lambda A^{-1}$, $F_1 = \sqrt{1-4AB}$.

Hence, we have that

$$\phi(\omega) = \frac{\sqrt{6}}{6(1+\lambda)} \sqrt{\frac{F_1}{1-F_1}} \sqrt{\cosh^2 \left(\frac{(1+\lambda)}{4} F_1 \omega \right) - \sinh^2 \left(\frac{(1+\lambda)}{4} F_1 \omega \right)} \times F_e \left(\left(\frac{F_1}{1+F_1} \left(1 - \tanh \left(\frac{(1+\lambda)}{4} F_1 \omega \right) \right) \right) \right), -\sqrt{\frac{1+F_1}{1-F_1}}, \quad (65)$$

where $F_e(\omega, x)$ is the incomplete elliptic integral.

Furthermore, for the potential we find

$$V(\omega) = \frac{(F_0)^{\frac{1}{1-\lambda}}}{48} \left(F_1 + 2 \sinh \left(\frac{F_1(1+\lambda)}{2} \omega \right) \right) \left(F_1 \tanh \left(\frac{(1+\lambda)}{4} F_1 \omega \right) - 1 \right)^{\frac{1}{1-\lambda}-1} \left(\frac{F_1 - 4 \cosh^2 \left(\frac{(1+\lambda)}{4} F_1 \omega \right)}{\cosh^2 \left(\frac{(1+\lambda)}{4} F_1 \omega \right)} \right). \quad (66)$$

Finally, the transformation $\omega \rightarrow t$ is given now in terms of hypergeometric functions. However, in the limit of large ω , expression (64) becomes constant and the solution approaches the de Sitter universe.

G. Generalized Chaplygin gas IV

We now consider another generalization of the basic Chaplygin gas, with equation of state

$$p_\phi = \frac{1}{6B - 12\rho_\phi} A - \rho_\phi, \quad (67)$$

which for $A = 0$, reduces to the cosmological constant, and for $B = 0$, to the Chaplygin gas II model, above, with $\lambda = -1$. On the other hand, for $B = 0$ and $\rho_\phi \rightarrow 0$, the behavior is that of the basic Chaplygin gas (43).

From (67), we have the two solutions:

$$F_\pm(\omega) = -\ln(B \pm \sqrt{B^2 - 2A\omega}). \quad (68)$$

Without loss of generality we work with the F_+ solution, so for the scalar field we find

$$\phi(\omega) = \frac{B\sqrt{\bar{\omega}(\bar{\omega} - B)} \ln(B + 2(\bar{\omega} - B + \sqrt{\bar{\omega}(\bar{\omega} - B)})) - 2\bar{\omega}(\bar{\omega} - B)}{\sqrt{24A\bar{\omega}(\bar{\omega} - B)}}, \quad (69)$$

$$V(\omega) = \frac{\bar{\omega}(\bar{\omega} - B) - A}{12(\bar{\omega} - B)}, \quad (70)$$

where $2A\omega = 2B\bar{\omega} - \bar{\omega}^2$.

For the equation of state parameter we have

$$w_\phi = \frac{\bar{\omega}(\bar{\omega} - B) - 2A}{\bar{\omega}(\bar{\omega} - B)}, \quad (71)$$

and the transformation $\omega \rightarrow t$ is the real solution of the algebraic equation, $2(3B - \bar{\omega})\sqrt{\bar{\omega}} = 3At$; that is,

$$\ln(a(t)) = \frac{(2B + (9A^2t^2 - 8B^3 + 3\sqrt{A^2t^2(9A^2t^2 - 16B^3)})^{1/3})^2}{3(9A^2t^2 - 8B^3 + 3\sqrt{A^2t^2(9A^2t^2 - 16B^3)})^{1/3}}, \quad (72)$$

for $(9A^2t^2 - 16B^3) > 0$. From this we can see that for large time $a(t) \approx \exp(t^2)$, which is a solution of the form (61) for the generalized Chaplygin gas I. This asymptotic behavior leads to a strong curvature singularity as $t \rightarrow \infty$.

H. Generalized Chaplygin gas V

Let the expression for the equation of state parameter now be

$$p = A\rho_\phi^\lambda + B\rho_\phi, \quad (73)$$

where, for $B = -1$ and $A = \gamma$, relation (55) is recovered. For (73) and for $B \neq -1$, we find

$$F(\omega) = \frac{1}{\lambda - 1} \ln\left(-\frac{A}{B} + \frac{\exp(\frac{\lambda-1}{2}\bar{B}\omega)}{B}\right), \quad (74)$$

where $B = \bar{B} - 1$. Therefore, the scalar field is

$$(\phi(\omega))^2 = \frac{4}{3\bar{B}(\lambda - 1)^2} \operatorname{arcsinh}^2\left(\frac{\exp(\frac{\lambda-1}{4}\bar{B}\omega)}{\sqrt{-A}}\right), \quad (75)$$

and the potential is

$$V(\omega) = \frac{2A + (\bar{B} - 2)e^{\frac{\lambda-1}{2}\bar{B}\omega}}{24(A - e^{\frac{\lambda-1}{2}\bar{B}\omega})} e^{-F(\omega)}. \quad (76)$$

From (75), we have that

$$\exp\left(\frac{\lambda - 1}{2}\bar{B}\omega\right) = \sinh^2\left(\frac{\sqrt{3\bar{B}(\lambda - 1)}}{2}\phi\right), \quad (77)$$

so the potential is

$$V(\phi) = -B^{\frac{1}{\lambda-1}} \left(2A + (\bar{B} - 2) \sinh^2\left(\frac{\sqrt{3\bar{B}(\lambda - 1)}}{2}\phi\right) \right) \times \left(\sinh^2\left(\frac{\sqrt{3\bar{B}(\lambda - 1)}}{2}\phi\right) - A \right)^{\frac{\lambda-2}{\lambda-1}}. \quad (78)$$

For the equation of state parameter we have

$$w_\phi = -1 + \frac{2}{B} \frac{\exp(\frac{\lambda-1}{2}\bar{B}\omega)}{\exp(\frac{\lambda-1}{2}\bar{B}\omega) - A}. \quad (79)$$

The transformation linking $\omega \rightarrow t$ is given in terms of the inverse hyperbolic function, except when $\lambda = \frac{1}{2}$, which yields $\omega = \frac{4}{B} \ln\left(\frac{1+e^{\frac{A}{4}}}{A}\right)$, and so

$$a(t) = \left(\frac{1 + e^{-4t}}{A} \right)^{\frac{2}{3b}}, \quad \lambda = \frac{1}{2}. \quad (80)$$

1. Bulk viscosity

The standard kinetic model for bulk viscosity in an isotropic and homogeneous cosmology (see for example Ref. [57]) replaces the pressure p by p' , where

$$p' = p - 3H\eta, \quad (81)$$

and if we have a bulk viscous coefficient η with $\eta = \alpha\rho^n$, with $\alpha > 0$ constant, then

$$p' = p - \sqrt{3\alpha}\rho^{n+1/2}. \quad (82)$$

In a spatially flat FLRW universe the solutions are found by solving equation $3H^2 = \rho$ together with the conservation equation

$$0 = \dot{\rho} + 3H(\rho + p') = \dot{\rho} + \sqrt{3}\rho^{1/2}(\rho + p - \sqrt{3\alpha}\rho^{n+1/2}). \quad (83)$$

Picking $p = (\gamma - 1)\rho$ we can see that there are special de Sitter solutions with $H = H_0$, except for the special case $n = 1/2$ where the solutions for $a(t)$ are power law in t . The exact solutions for the field equations are given in [53]. When $n > 1/2$ the solution starts as de Sitter at past infinity and approaches FRW $a = t^{2/3\gamma}$ as $t \rightarrow \infty$. The behavior displaying approach to de Sitter as $t \rightarrow -\infty$ does not persist when curvature, anisotropy or another nonviscous fluid is added to the Friedmann equation. Finally, to set up the correspondence with Eq. (73) we have to identify $B\rho$ with $(\gamma - 1)\rho$ and A with $-\sqrt{3\alpha}$ and λ with $n + 1/2$.

I. Lambert function I

Suppose that $F(\omega)$ is given by a function of the Lambert function, $W(\omega)$, specifically,

$$F(\omega) = 2 \ln \left(\frac{1}{6p} \frac{W(e^{\frac{\omega}{6p}})}{W(e^{\frac{\omega}{6p}}) + 1} \right), \quad (84)$$

which gives the scale factor in the line element (6) as the simple function

$$a(t) = (t \exp(t))^p. \quad (85)$$

From (20)–(22) we find that

$$\phi(\omega) = \sqrt{2p} \ln \left[W(e^{\frac{\omega}{6p}}) \right] \sqrt{1 + W(e^{\frac{\omega}{6p}})}, \quad (86)$$

$$V(\omega) = 3p^2 + p \frac{1 - 3p + 6pW(e^{\frac{\omega}{6p}})}{(W(e^{\frac{\omega}{6p}}))^2} \quad (87)$$

and

$$w_\phi(\omega) = -1 + \frac{2}{3p(W(e^{\frac{\omega}{6p}}) + 1)^2}. \quad (88)$$

Expressions (86) and (87) give

$$V(\phi) = p(3p(e^{-\frac{\sqrt{2}\phi}{p}} + 1)^2 - e^{-\frac{\sqrt{2}\phi}{p}}) \quad (89)$$

while for $\phi(t)$ we have

$$\phi(t) = \sqrt{2p} \sqrt{(1 + t^p)} \ln t. \quad (90)$$

J. Lambert function II

We select a universe (6) in which the scale factor is given from the following formula,

$$a(t) = t^q \exp(pt), \quad (91)$$

where in general $q \neq p$, while for $p = q$ we reduce to the previous case.

We perform the transformation $t \rightarrow \omega$ in order to write the line element in the form of (19) and find that

$$F(\omega) = 2 \ln \left(\frac{W(\frac{p}{q} e^{\frac{\omega}{6q}})}{6p(W(\frac{p}{q} e^{\frac{\omega}{6q}}) + 1)} \right); \quad (92)$$

that is,

$$\phi(\omega) = \sqrt{2q} \ln \left(W \left(\frac{p}{q} e^{\frac{\omega}{6q}} \right) \right) \sqrt{1 + W \left(\frac{p}{q} e^{\frac{\omega}{6q}} \right)}, \quad (93)$$

$$V(\omega) = 3p^2 + \frac{p^2 3q - 1 + 6qW(\frac{p}{q} e^{\frac{\omega}{6q}})}{q [W(\frac{p}{q} e^{\frac{\omega}{6q}})]^2} \quad (94)$$

and

$$w_\phi(\omega) = -1 + \frac{2}{3q} \left(1 + W \left(\frac{p}{q} e^{\frac{\omega}{6q}} \right) \right)^{-2}. \quad (95)$$

Hence, we have

$$V(\phi) = 3p^2 + \frac{p^2 6q e^{\frac{\phi}{\sqrt{2q}}} + (3q - 1)}{q e^{\frac{2\phi}{\sqrt{2q}}}}, \quad (96)$$

where easily we observe that for $q = p$ expression (89) is recovered.

We should mention that scale factors (85) and (91) can be constructed under a rescaling transformation of scalar-field solutions of the field equations ($H \rightarrow H + \text{constant}$) from the power-law solution $a(t) = t^\beta$ found in Ref. [58]. This can also be a special solution of a two-scalar-field model [59].

K. Error function solution

Assume now that the pressure and the energy density for the scalar field satisfy the equation of state parameter

$$p_\phi = -\frac{A}{6} e^{-12B\rho_\phi} - \rho_\phi, \quad (97)$$

where for $B > 0$, when $\rho_\phi \rightarrow \infty$, $p_\phi = -\rho_\phi$. This gives

$$F(\omega) = -\ln\left(\frac{1}{B} \ln(AB\omega)\right), \quad (98)$$

which is a real function when $A\omega < 0$. Therefore, for the scalar field, we find

$$\phi(\omega) = -\frac{4}{3} D(\sqrt{2^{-1} \ln(AB\omega)}), \quad (99)$$

where $D(x)$ is the Dawson integral.³

For the scalar-field potential, it follows that

$$V(\omega) = \frac{1}{12B} \left(\frac{1}{\omega} + \ln(AB\omega) \right), \quad (100)$$

which has a minimum at $\omega = 1$ and $B > 0$. Finally the equation of state parameter is

$$w_\phi(\omega) = -1 - \frac{2}{\omega \ln(AB\omega)}. \quad (101)$$

The expansion scale factor $a(t)$ is given in terms of the inverse error function.

IV. SPATIALLY FLAT FLRW SPACETIME WITH MATTER SOURCE

We continue our analysis by assuming that a perfect fluid with constant equation of state parameter, $p_m = (\gamma - 1)\rho_m$, is added to the scalar field. Now, in order to find closed-form solutions for the scalar field, equations (23)–(26) have to be solved.

A. Solution I

First, we consider the special case in which the scalar field has a constant equation of state parameter equal with that of the perfect fluid, i.e., $w_\phi = (\gamma - 1)$. From (25)–(26) it follows that $2F' - \gamma = 0$, which gives

$$F(\omega) = \frac{\gamma}{2} \omega + F_0, \quad (102)$$

and using (23) we have that $\phi(\omega)$ is a linear function.

³The Dawson integral function is $D(x) = \frac{\sqrt{\pi}}{2} \exp[-x^2] \operatorname{erfi}(x) = \exp[-x^2] \int_0^x \exp[y^2] dy$.

For the scalar-field potentials, we derive

$$V(\phi) = V_0 \exp\left(-\frac{\sqrt{3}\gamma}{\sqrt{\gamma(1-12\rho_0 e^{F_0})}} \phi\right), \quad (103)$$

which is just the exponential potential, as expected [60]. In addition we have $V_0 = V_0(\gamma, \rho_{m0}, F_0)$ or, specifically,

$$V_0 = (2 - \gamma) \exp(e^{-F_0}) + \frac{1}{2} \gamma \rho_{m0}. \quad (104)$$

B. Solution II

Let the scalar field have a constant equation of state parameter $w_\phi = \gamma_\phi - 1$, but in contrast to the above, $\gamma_\phi \neq \gamma$. This scaling solution has been studied before in [24]. Therefore, for this ansatz we find that the unknown function, $F(\omega)$, of the line element (19) has the form

$$F(\omega) = \frac{\gamma_\phi}{2} \omega + \frac{\gamma - \gamma_\phi}{2} \bar{F}_0 - \ln\left(12\rho_{m0} \exp\left(\frac{\gamma - \gamma_\phi}{2} (\bar{F}_0 - \omega)\right) - 1\right), \quad (105)$$

where we see that the linear function (102) is recovered for $\gamma = \gamma_\phi$.

For the scalar field, we find that

$$\phi(\omega) = -\frac{2\sqrt{3}\gamma_\phi}{3} \operatorname{arctanh}\left(\sqrt{12\rho_{m0} \exp\left(\frac{\gamma - \gamma_\phi}{2} (\bar{F}_0 - \omega)\right) - 1}\right); \quad (106)$$

that is,

$$\omega = -\bar{F}_0 + \frac{2}{\gamma - \gamma_\phi} \ln\left(\frac{1 + \tanh^2\left(\frac{1}{2}\sqrt{3}\gamma_\phi(\gamma - \gamma_\phi)\phi\right)}{12\rho_{m0}}\right). \quad (107)$$

Furthermore, for the potential of the scalar field we find that in terms of ω it is expressed as

$$V(\omega) = V_0(\gamma, \gamma_\phi, \rho_{m0}, \bar{F}_0) \exp\left(-\frac{\gamma_\phi}{2} \omega\right) \quad (108)$$

or in terms of ϕ with the use (107)

$$V(\phi) = V_0(\gamma, \gamma_\phi, \rho_{m0}, \bar{F}_0) (12\rho_{m0})^{\frac{\gamma_\phi}{\gamma - \gamma_\phi}} \times e^{\bar{F}_0 \frac{\gamma_\phi}{\gamma - \gamma_\phi}} \left(1 + \tanh^2\left(\frac{1}{2}\sqrt{3}\gamma_\phi(\gamma - \gamma_\phi)\phi\right)\right)^{\frac{\gamma_\phi}{\gamma - \gamma_\phi}} \quad (109)$$

in which

$$V_0(\gamma, \gamma_\phi, \rho_{m0}, \bar{F}_0) = \left(\rho_{m0} + \frac{1}{24}(\gamma_\phi - 2) \exp\left(\frac{\gamma_\phi - \gamma}{2} \bar{F}_0\right) \right). \quad (110)$$

C. Solution III

We consider that the scalar provides two fluid terms: a fluid with constant equation of state parameter $\bar{\gamma}$, and a component which mimics the perfect fluid ρ_m . That means that we assume the Hubble function to be

$$H(a) = H_0 \sqrt{\Omega_1 a^{-3\gamma} + \Omega_2 a^{-3\bar{\gamma}}}, \quad (111)$$

and so

$$F(\omega) = -\ln(36\Omega_1 H_0^2 e^{-\frac{\gamma}{2}\omega} + 36\Omega_2 H_0^2 e^{-\frac{\bar{\gamma}}{2}\omega}). \quad (112)$$

Hence, for the scalar field it follows that

$$\phi(\omega) = \sqrt{6} \int \sqrt{\frac{(\Omega_1 - 36\gamma\Omega_{m0})e^{-\frac{\gamma}{2}\omega} + \Omega_2 H_0^2 a^{-\frac{\bar{\gamma}}{2}\omega}}{\Omega_1 H_0^2 e^{-\frac{\gamma}{2}\omega} + \Omega_2 H_0^2 a^{-\frac{\bar{\gamma}}{2}\omega}}} d\omega \quad (113)$$

and

$$V(\omega) = \frac{3}{2}(\bar{\gamma} - 2)\Omega_2 H_0^2 e^{-\frac{\bar{\gamma}}{2}\omega} + \frac{12\gamma\rho_{m0} + 36(2 - \gamma)\Omega_1 H_0^2}{24} e^{-\frac{\gamma}{2}\omega}, \quad (114)$$

where $\Omega_{m0} = \frac{\rho_{m0}}{3H_0^2}$. In order to specify the exact form of $V(\phi)$, the inverse function $\omega(\phi)$ has to be determined from the integral (113). However, for the specific case in which $\bar{\gamma} = 0$, where the extra fluid term is that of the cosmological constant, we find that the scalar-field potential has the form

$$V(\phi) = V_1 + \bar{V}_0(1 + V_3 \tanh^2(\bar{V}_2 \phi))^{-\frac{4}{\bar{\gamma}}}, \quad (115)$$

where $V_{0-2} = V_{0-3}(\Omega_1, \Omega_2, H_0, \gamma, \rho_{m0})$. This is different from the exponential term and differs from (110).

For example, if we assume that $\gamma = 1$, i.e., we are in the Λ -cosmology, with $\Omega_1 + \Omega_2 = 1$, we have that

$$V(\phi) = 3(1 - \Omega_1)H_0^2 + \frac{3}{2}(\Omega_1 + \rho_{m0})e^{-\frac{\omega}{2}} \quad (116)$$

and

$$V(\phi) = 3(1 - \Omega_1)H_0^2 + \frac{3}{8} \left(1 + \frac{\rho_{m0}}{\Omega_1}\right) \times (e^{-\lambda\Delta\phi} - (1 - \Omega_1)e^{-2\lambda\Delta\phi})^2, \quad (117)$$

where $\lambda = \frac{\sqrt{3\Omega_1 H_0}}{\sqrt{\Omega_1 H_0^2 - 12\rho_{m0}}}$. This is nothing other than a special case of the UDM model [50]; that is, the UDM provides a dust component in the field equations. This property for the UDM potential has been found earlier [27] and also for a class of scalar-field potentials of the form (115) in [29]. On the other hand, the exponential behavior of the potential is expected according to the results of [51] because a scaling solution is an attractor in scalar-field models when the potentials have asymptotically exponential terms.

D. De Sitter universe

As a final case consider that the line element (6) is that of the de Sitter universe, $a(t) = a_0 e^{H_0 t}$, which means that $F(\omega)$ in (19) is a constant function, $F(\omega) = F_0$. That can be seen as a special case of the previous model that we studied in which the scalar field eliminates the perfect fluid, i.e., $\Omega_1 = 0$.

Therefore, we find the potential to be

$$V(\phi) = \frac{1}{12} e^{-F_0} \left(1 - \frac{\gamma^2}{3} \phi^2\right). \quad (118)$$

We end our analysis here and we recall that, if we set $\gamma = \frac{2}{3}$ and $\rho_{m0} = -3K$, then the solutions that have been presented in this section hold also for the scalar-field model in a nonflat FLRW universe without a matter source.

V. INFLATIONARY SLOW-ROLL PARAMETERS

In scalar-field cosmology, the parameters

$$\varepsilon_V = \left(\frac{V_{,\phi}}{2V}\right)^2, \quad \eta_V = \frac{V_{,\phi\phi}}{2V}, \quad (119)$$

are called the potential slow-roll (PSR) parameters [61] and provide us with an inflationary universe when $\varepsilon_V \ll 1$. The condition $\eta_V \ll 1$ is also important for the duration of the inflation phase.

Alternatively, more accurate parameters which describe the inflationary phase of the Universe are provided by the so-called Hubble slow-roll (HSR) parameters [62]

$$\varepsilon_H = -\frac{d \ln H}{d \ln a} = \left(\frac{H_{,\phi}}{H}\right)^2, \quad (120)$$

and

$$\eta_H = -\frac{d \ln H_{,\phi}}{d \ln a} = \frac{H_{,\phi\phi}}{H}. \quad (121)$$

The PSR parameters and HSR parameters are related exactly through the relations

$$\varepsilon_V = \varepsilon_H \left(\frac{3 - \eta_H}{3 - \varepsilon_H}\right)^2 \quad (122)$$

and

$$\eta_H = \frac{\sqrt{\varepsilon_H}}{3 - \varepsilon_H} \eta_{H,\phi} + \left(\frac{3 - \eta_H}{3 - \varepsilon_H} \right) (\varepsilon_H + \eta_H) \quad (123)$$

or approximately through $\varepsilon_V \simeq \varepsilon_H$ and $\eta_V \simeq \varepsilon_H + \eta_H$ when ε_H, η_H are both very small. Therefore, when the closed-form solution of the field equations is known, it is more accurate to work with the HSR parameters in order to study the inflationary phase of the model rather than with the PSR parameters.

The analytical solution which was presented in Sec. II A can be used to write the slow-roll parameters in terms of the function, ω , or the number of e -folds, N_e . Recall that the number of e -folds is given by the formula

$$N_e = \int_{t_i}^{t_f} H(t) dt = \ln \frac{a_f}{a_i} = \frac{1}{6} (\omega_f - \omega_i), \quad (124)$$

where $a_f = a(t_f)$ is the moment at which inflation ends, $\varepsilon_H(t_f) = 1$, while $a_i = a(t_i)$ is the moment at which inflation starts. It is assumed that N_e lies in the interval $N_e \in [50, 60]$.

Therefore, the HSR parameters are

$$\varepsilon_H = 3F', \quad \eta_H = 3 \frac{(F')^2 - F''}{F'}, \quad (125)$$

where either from (122) and (123) or directly from with the use of (20) and (21) the PSR parameters can be derived in terms of ω . Here we comment that the PSR parameters depend always upon a higher derivative of F , in contrast to the HSR parameters, $\varepsilon_V = \varepsilon_V(F', F'')$ and $\eta_V = \eta_V(F', F'', F''')$.

In a similar way, the HSR expansion parameters [61] can be expressed in terms of the function $F(\omega)$ and its derivative. For example, the third-order HSR parameter is

$$\xi_H \equiv \frac{H_\phi H_{\phi\phi\phi}}{H^2} = -\frac{9\sqrt{6}}{4(F')^{\frac{5}{2}}} [(F')^4 - (3F'^2 + 2F'')F'' + 2F'F''']. \quad (126)$$

Note that the spectral indices for the density perturbations, and for the gravitational waves in the first approximation, are given in terms of the HSR parameters by

$$n_s = 1 - 4\varepsilon_H + 2\eta_H, \quad n_g = -2\varepsilon_H, \quad (127)$$

while the tensor to scalar ratio is $r = 10\varepsilon_H$. Finally, the range of the scalar spectral index is given by

$$n'_s = 2\varepsilon_H \eta_H - 2\xi_H. \quad (128)$$

From the Planck 2015 Collaboration [7], we have that the above parameters are $n_s = 0.968 \pm 0.006$ and $n'_s = -0.003 \pm 0.007$, while the tensor to scalar ratio has

a value smaller than 0.11, i.e., $r < 0.11$. From these values some intervals for the HSR parameters can be determined. For instance from r it follows that $\varepsilon_H \lesssim 0.01$.

In what follows, we determine the HSR parameters for some of the solutions of Sec. III and compare them with the Planck data constraints. Specifically, we study the following models: generalized Chaplygin gases I–V; the Lambert function II model, and the error function solution.

A. Generalized Chaplygin gas I

For the generalized Chaplygin gas model, (49), the HSR parameters are given by

$$\varepsilon_H = \frac{3}{2} \frac{1}{1 - \frac{A_0}{A_1} e^{-\frac{\mu}{\gamma}\omega}}, \quad \eta_H = \frac{1}{2} (2\varepsilon_H(1 + \mu) - 3\mu), \quad (129)$$

and

$$\xi_H = -\frac{3\sqrt{2}}{8} \sqrt{\varepsilon_H} [(1 + \mu)(1 + 2\mu)\sqrt{\varepsilon_H} - 3\mu(3 + 2\mu)], \quad (130)$$

where, for $\mu = -2$, the parameters reduce to those of the basic Chaplygin gas, (43). Moreover, inflation ends when $\omega_f = -\frac{2}{\mu} \ln(-\frac{1}{2} \frac{A_1}{A_0})$, from which we find that

$$\varepsilon_H(\omega_i) = \frac{3}{2 + e^{3N\mu}}. \quad (131)$$

As above, the spectral indices can be expressed in terms of ε_H by

$$n_s = 1 + 2\varepsilon_H(\mu - 1) - 3\mu \quad (132)$$

and

$$n'_s = 2(1 + \mu)\varepsilon_H^2 - \frac{3\sqrt{2}}{4} (2\sqrt{2}\mu - 1 - 3\mu - 2\mu^2)\varepsilon_H - \frac{9\sqrt{2}}{4} (3 + 2\mu)\sqrt{\varepsilon_H}. \quad (133)$$

From (131), we observe that in order for $\varepsilon_H < 1$, μ should be positive. In Fig. 1 the $n_s - r$ and $n_s - n'_s$ diagrams are presented. They reveal that for $N_e = 60$ we have $\max n_s \simeq 0.887 < 0.968$ while at the same time $r \simeq 0.08$ and $n'_s = -0.02$, which corresponds to $\mu \simeq 0.033$; that is, a small deviation from a perfect fluid. Also, we mention that for smaller values of N_e the maximum of n_s is smaller, n'_s is smaller, although the scalar ratio has a similar value. For $N_e = 55$, we have $\max n_s = 0.877$, $r \simeq 0.09$ and $n'_s = -0.022$ for $\mu \simeq 0.032$.

B. Generalized Chaplygin gas II

Consider now the generalized Chaplygin gas II model, (55). For this model the HSR parameters are calculated to be

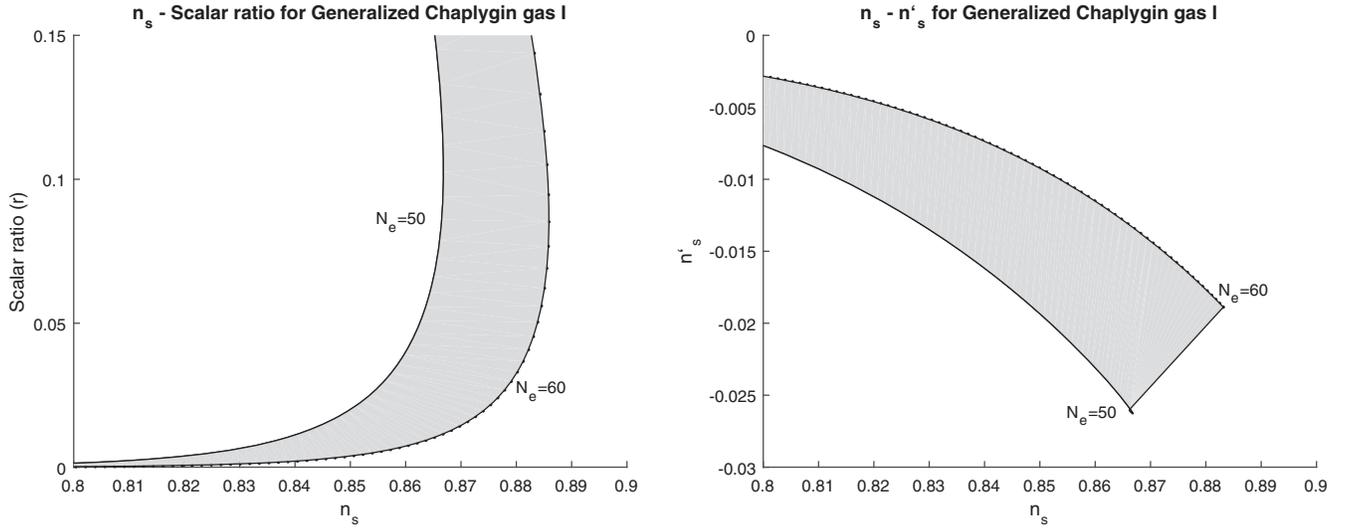


FIG. 1. Diagrams $r = r(n_s)$ (left) and $n'_s = n'_s(n_s)$ (right) for the generalized Chaplygin gas I model for the number of e -folds in the range $N_e \in [50, 60]$.

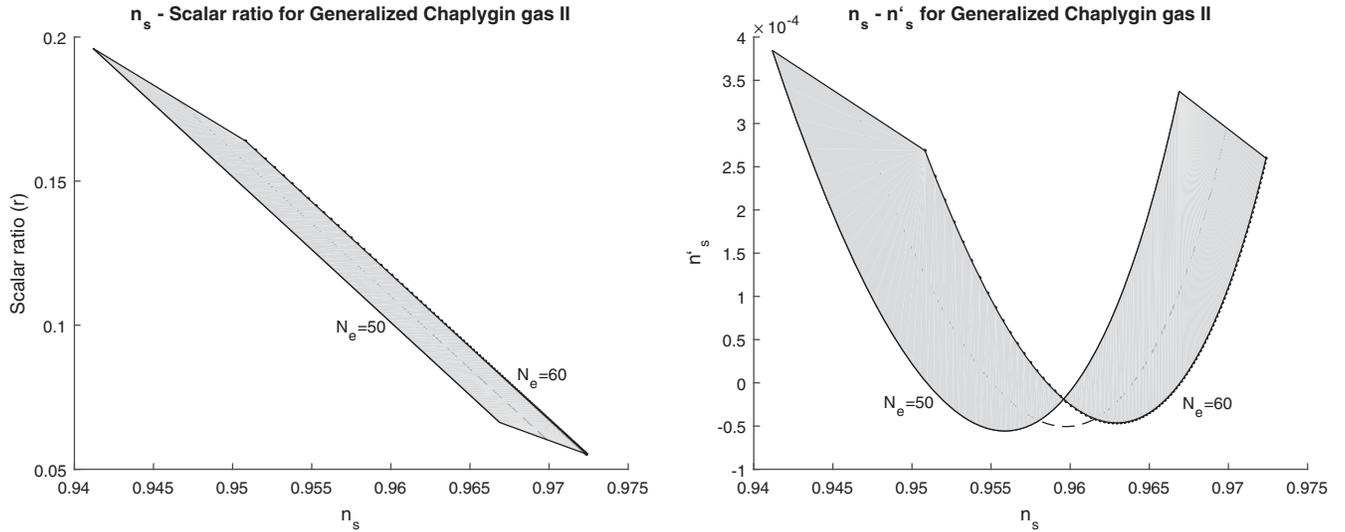


FIG. 2. Left: $n_s - r$ diagram for the generalized Chaplygin gas II model. Right: $n_s - n'_s$ diagram for the same model. The plots are for $N_e \in [50, 60]$ and $\lambda \in [-0.5, 0.5]$.

$$\varepsilon_H = \frac{3\bar{\gamma}}{\lambda - 1} (\bar{\gamma}\omega + \gamma_1)^{-1}, \quad (134)$$

and

$$\eta_H = \lambda\varepsilon_H, \quad \xi_H = -\frac{\sqrt{2}}{4}\lambda(2\lambda - 1)(\varepsilon_H)^{\frac{3}{2}}, \quad (135)$$

where for $\lambda = \frac{1}{2}$ the parameter ξ_H becomes zero. Furthermore, we find that

$$\begin{aligned} n_s &= 1 - 4\varepsilon_H + 2\lambda\varepsilon_H, \\ n'_s &= 2\lambda(\varepsilon_H)^2 + \frac{\sqrt{2}}{2}\lambda(2\lambda - 1)(\varepsilon_H)^{\frac{3}{2}} \end{aligned} \quad (136)$$

and inflation ends at the point $\omega_f = \frac{3}{\lambda - 1} - \frac{\gamma_1}{\bar{\gamma}}$. Therefore, we have that

$$\varepsilon_H(\omega_i) = (1 + 2N(1 - \lambda))^{-1}, \quad (137)$$

where $\varepsilon_H < 1$ and $\varepsilon_H \geq 0$ for $\lambda < 1$, while for $\lim_{\lambda \rightarrow -\infty} \varepsilon_H(\omega_i) = 0$. In Fig. 2 we give the $n_s - r$ and $n_s - n'_s$ diagrams for various values of the parameter λ in the range $\lambda \in [-0.5, 0.5]$. From these diagrams, we observe that for $N_e = 55$ (dashed lines), for $n_s = 0.968$, we have $r \approx 0.073$ and $n'_s = 2 \times 10^{-4}$. These are values inside the range of values consistent with the Planck 2015

Collaboration, in contrast to the situation for the generalized Chaplygin gas I.

C. Generalized Chaplygin gas III

For the equation of state parameter (63), that is for the solution (64), we find that

$$\varepsilon_H = \frac{3(F_1)^2}{4} \left(1 - \tanh \left(\frac{(1+\lambda)}{4} F_1 \omega \right) \right)^{-1} \times \cosh^{-2} \left(\frac{(1+\lambda)}{4} F_1 \omega \right) \quad (138)$$

and

$$\eta_H = \frac{1}{2} (2\varepsilon_H + (1+\lambda) \sqrt{(\varepsilon_H)^2 - 3\varepsilon_H + 9(F_1)^2}), \quad (139)$$

$$\xi_H = \frac{-3\sqrt{\varepsilon_H}}{4\sqrt{2}} (\varepsilon_H(3+4\lambda+2\lambda^2) - 3(1+\lambda)^2 + 3(1+\lambda) \sqrt{4(\varepsilon_H)^2 - 12\varepsilon_H + 9(F_1)^2}). \quad (140)$$

Therefore, we have

$$n_s = 1 - 2\varepsilon_H + (1+\lambda) \sqrt{(\varepsilon_H)^2 - 3\varepsilon_H + 9(F_1)^2}. \quad (141)$$

From (139), we see that $\eta_H \rightarrow 0$, when $\varepsilon_H \rightarrow 0$, if and only if $3(1+\lambda)|F_1| \rightarrow 0$, while at the same time $\xi_H \rightarrow 0$; that is $n_s \rightarrow 1$ and $n'_s \rightarrow 0$. Furthermore, from (138), we find that inflation ends at

$$e^{\omega_f} = 2^{-\frac{2}{(1+\lambda)F_1}} \left(\frac{3(F_1)^2 - 8\sqrt{9(F_1)^2 - 8} - 2}{F_1 - 1} \right)^{\frac{2}{(1+\lambda)F_1}}, \quad (142)$$

which requires $(F_1)^2 \geq \frac{8}{9}$; that is, from the above, λ should be very close to -1 .

In Fig 3, we give the evolution of the $n_s - r$ and $n_s - n'_s$ diagrams. We observe that n_s reaches the observed value 0.986 when the number of e -folds N_e exceeds 60.

D. Generalized Chaplygin gas IV

The HSR parameters for the generalized Chaplygin gas IV model (67) are found to be

$$\varepsilon_H = \frac{3A}{B^2 - 2A\omega + B\sqrt{B^2 - 2A\omega}},$$

$$\eta_H = \frac{3A}{2A\omega - B^2}, \quad (143)$$

$$\xi_H = -\sqrt{\frac{3}{2}} \frac{9A^{\frac{3}{2}}(4B + 3\sqrt{B^2 - 2A\omega})}{2(B^2 - 2A\omega)^{\frac{3}{2}}(B(B + \sqrt{B^2 - 2A\omega}) - 2A\omega)^{\frac{1}{2}}}, \quad (144)$$

from which we find $\omega_f^{\pm} = \frac{B^2 - 6A \pm \sqrt{B^2 + 12AB^2}}{4A}$. In Fig. 4, the $n_s - r$ and $n_s - n'_s$ diagrams are given for ω_f^- , and for various values of the parameter B ; we have assumed that $A = 1$. From the diagrams it is easy to see that this model can fit the Planck 2015 data quite well. Specifically, we find that for $n_s \approx 0.968$ and for $N_e = 55$, $r \approx 0.054$ and $n'_s \approx -1.510^{-3}$.

E. Generalized Chaplygin gas V

For function (74), the HSR parameters are now calculated to be

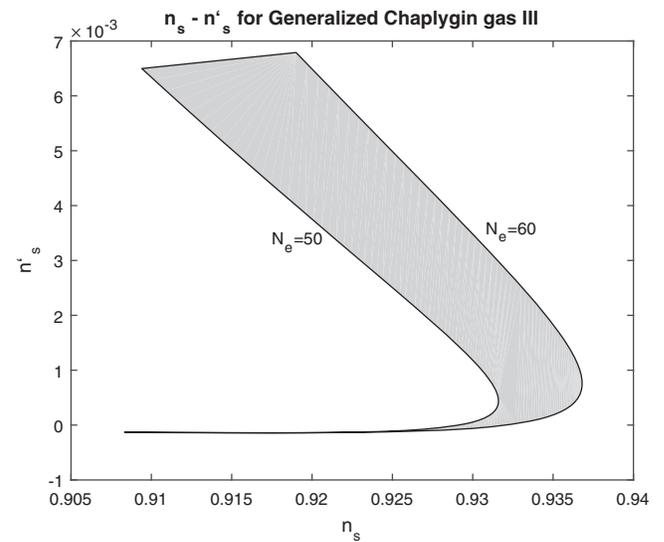
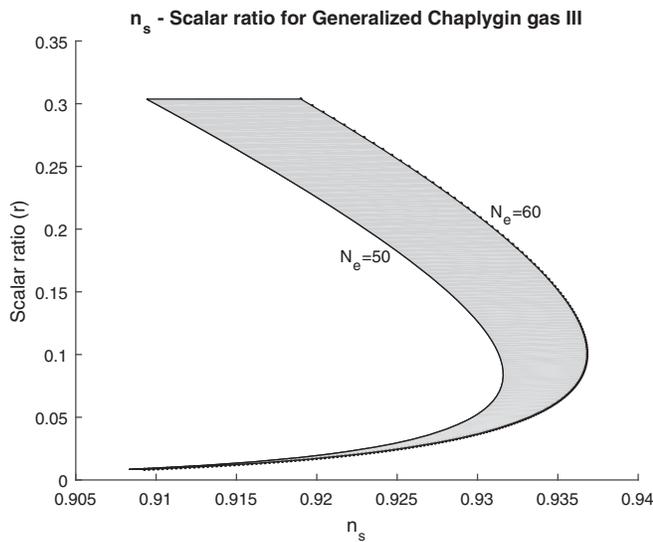


FIG. 3. Left: $n_s - r$ diagram for the generalized Chaplygin gas III model. Right: $n_s - n'_s$ diagram for the same model. The plots are for $N_e \in [50, 60]$ and $\lambda \in [-1.03, -1.01]$, with $F_1 = -1.0001$.

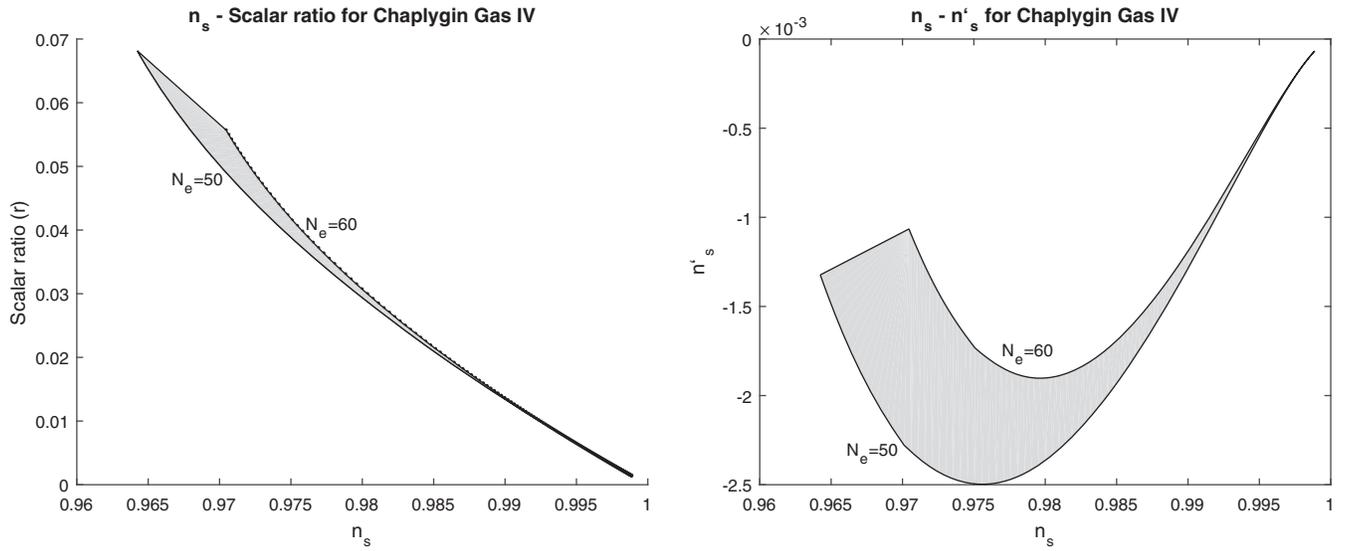


FIG. 4. Diagrams $r = r(n_s)$ (left) and $n'_s = n'_s(n_s)$ (right) for the generalized Chaplygin gas IV model for the number of e -folds in the range $N_e \in [50, 60]$, for $A = 1$ and free parameter B in the range $B \in [-10, 100]$.

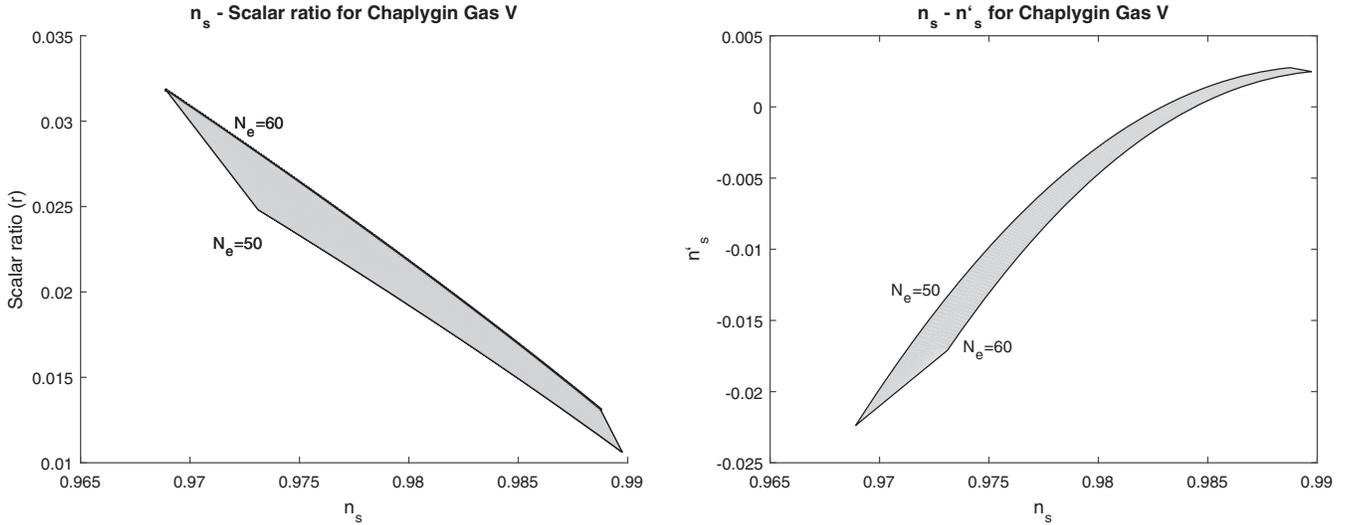


FIG. 5. Left: $n_s - r$ diagram for the model generalized Chaplygin gas V. Right: $n_s - n'_s$ diagram for the same model. The plots are for $N_e \in [50, 60]$, $\bar{B} = -0.002$ and $\lambda \in [-1, 0)$.

$$\begin{aligned} \varepsilon_H &= \frac{3\bar{B}}{2} (1 - A e^{-\frac{\lambda-1}{2}\bar{B}\omega}), \\ \eta_H &= -\frac{3}{2}\bar{B}(\lambda-1) + \lambda\varepsilon_H, \end{aligned} \quad (145)$$

and

$$\begin{aligned} \xi_H &= -\frac{9\sqrt{2\varepsilon_H}}{8} ((\lambda-1)^2(6\bar{B}-4\varepsilon_H) \\ &\quad + (\lambda-1)(9\bar{B}-6\varepsilon_H)-2\varepsilon_H), \end{aligned} \quad (146)$$

from which we find that inflation ends when

$$\omega_f = \frac{2(\ln 2A - \ln(2-3\bar{B}))}{\bar{B}(\lambda-1)}. \quad (147)$$

From (145) we observe that when $\varepsilon_H \rightarrow 0$, $\eta_H \simeq \bar{B}(\lambda-1)$; hence $\eta_H \rightarrow 0$ when $\bar{B} \rightarrow 0$, or $\lambda \rightarrow 1$. Recall that $\bar{B} = 0$ means that we are in the case of the generalized Chaplygin gas II model. On the other hand, by replacing $\omega_i = \omega_f - 6N_e$ in (145) and (146) using (147), it follows that the HSR parameters are independent on the constant A , and are functions of \bar{B} , λ and the number of e -folds N_e . We choose $\bar{B} = -0.002$ and for the ranges $N_e \in [50, 60]$ and $\lambda \in [-1, 0)$, we present the $n_s - r$ and $n_s - n'_s$ diagrams in Fig 5. We can see that this differs from that of the Chaplygin gas II model.

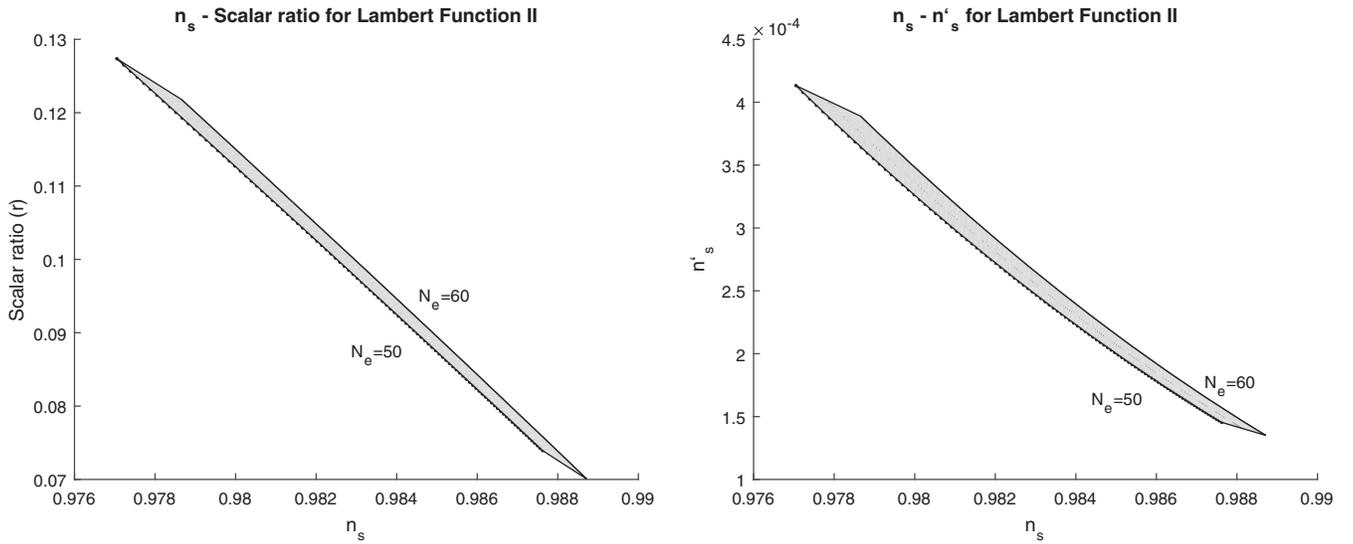


FIG. 6. Left: $n_s - r$ diagram for the model “Lambert function II.” Right: $n_s - n'_s$ diagram for the same model. The plots are for $N_e \in [50, 60]$ and $q \in [65, 100]$.

F. Lambert function II

For the scale factor (91) we find that the HSR parameters are

$$\varepsilon_H = \frac{1}{q} \left(1 + W \left(\frac{p}{q} e^{\frac{p}{6q}} \right) \right)^{-2} \quad (148)$$

and

$$\eta_H = \sqrt{\frac{\varepsilon_H}{q}}, \quad \xi_H = -\frac{3\sqrt{2}}{4q^2} \sqrt{\varepsilon_H}, \quad (149)$$

while the spectral indices become

$$\begin{aligned} n_s &= 1 - 4\varepsilon_H + 2\sqrt{\frac{\varepsilon_H}{q}}, \\ n'_s &= \frac{2}{\sqrt{q}} (\varepsilon_H)^{\frac{3}{2}} + \frac{3\sqrt{2}}{2q^2} \sqrt{\varepsilon_H}. \end{aligned} \quad (150)$$

From (148), we find that inflation ends at $\omega_f = 6 \ln \left(\frac{\sqrt{q}(1-\sqrt{q})}{p} \right) + \frac{1-\sqrt{q}}{\sqrt{q}}$. It is important to mention here that in order for (148) to be positive we need $q > 0$, while ω_f is real when $\frac{1-\sqrt{q}}{p} > 0$. Furthermore, we find

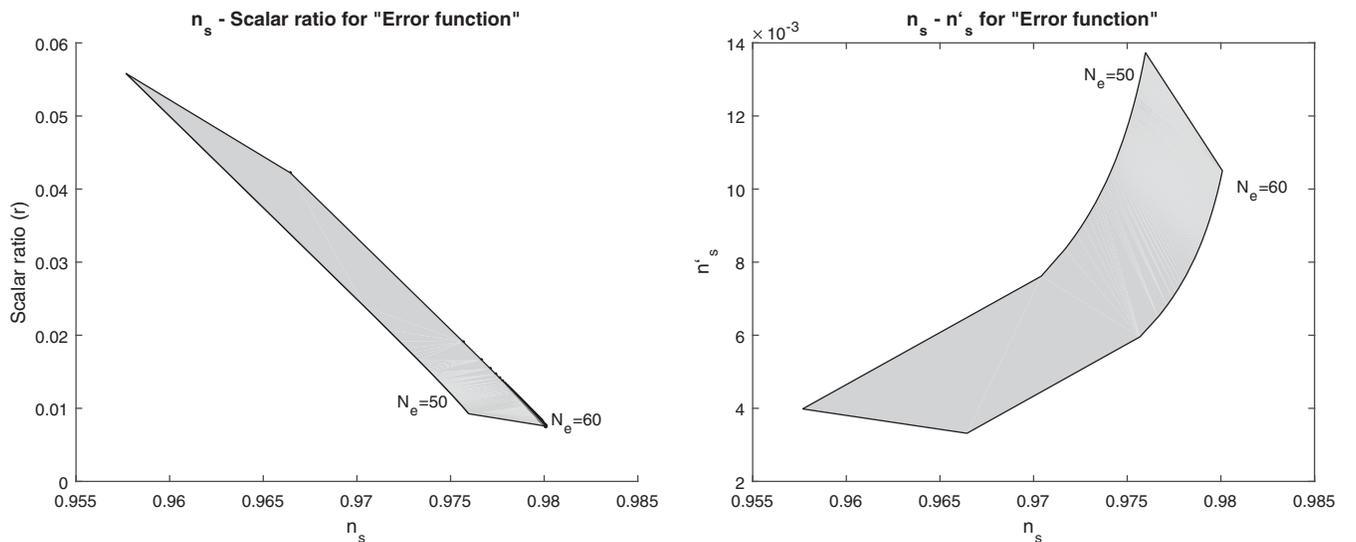


FIG. 7. Left: $n_s - r$ diagram for the model error function. Right: $n_s - n'_s$ diagram for the same model. The plots are for $N_e \in [50, 60]$ and $AB \in [-100, -0.02]$.

$$\varepsilon_H = \frac{1}{q} \left(1 + W \left(\frac{(1 - \sqrt{q})}{\sqrt{q}} e^{\frac{1-\sqrt{q}}{\sqrt{q}}} \exp \left(-\frac{N}{q} \right) \right) \right)^{-2}. \quad (151)$$

In Fig. 6 the $n_s - r$ and $n_s - n'_s$ diagrams are given for $q \in [65, 100]$, where we observe that $n'_s \rightarrow 0$ as $n_s \rightarrow 1$, while the relation $r(n_s)$ is linear. However, for $N_e \in [50, 60]$ we see that in the range of n_s given by the Planck data we need to have $r > 0.11$.

G. Error function solution

For the ‘‘error function’’ solution for the equation of state (97), the HSR parameters are given by

$$\begin{aligned} \varepsilon_H &= -\frac{3}{\omega \ln(AB\omega)}, \\ \eta_H &= \frac{3}{\omega}, \\ \xi_H &= \frac{3}{2\sqrt{2}} \sqrt{\varepsilon_H} \eta_H (2 \ln(AB\omega) - 1), \end{aligned} \quad (152)$$

and so inflation ends when ω reaches the value

$$\omega_f = -\frac{3}{W(-3AB)} \quad (153)$$

where W is the Lambert function. We can easily see that in order for ω_f to be real, $AB < 0$. The $n_s - r$ and $n_s - n'_s$ diagrams for this model are given in Fig. 7 for the range of e -folds $N_e \in [50, 60]$ and for $AB \in [-100, -0.02]$. From the plots we observe that for $r < 0.06$, we have $n_s \in [0.957, 0.981]$ and $n'_s \in [3, 10] \times 10^{-3}$. These values are compared with the best-fit values from the Planck Collaboration.

VI. CONCLUSIONS

In this work we studied exact solutions in scalar-field cosmology using a new mathematical approach, and with an emphasis on inflationary models. We have found new

closed-form solutions for spatially flat FLRW universes with or without an extra matter source. For the latter cosmological scenario, we determined exact solutions for the case in which the scalar field mimics the perfect fluid, the scalar field has a constant equation of state parameter different from that of the perfect fluid, and when the scalar field provides two perfect-fluid terms in the field equations. The first solution is the well-known special solution of the exponential potential, while in the other two solutions the scalar-field potentials are expressed in hypertrigonometric functions and the unified cold dark matter potential is recovered. Furthermore, these expressions can be applied in order to construct other solutions in a FLRW spacetime with spatial curvature.

In the cosmological scenario in which the Universe is dominated by the scalar field we determined the scalar-field model in which the equivalent equation of state parameter is that of the Chaplygin gas, or some generalizations of the Chaplygin gas which have been proposed in the literature. We also considered solutions in which the Hubble function is expressed in terms of the Lambert function or by logarithmic function. These models provide exact inflationary universe solutions.

We compared these solutions with the constraints on inflation from the Planck 2015 Collaboration. In order to perform this analysis we expressed the Hubble slow-roll parameters in terms of the expansion scale factor in the variables defined by our solution-generating functions. For every specific model and solution we calculated the HSR parameters and we derived the spectral indices in the first approximation. The diagrams for the density perturbations (n_s) with the scalar ratio (r) and the variation n'_s have been derived and the subset of models which are compatible with the Planck 2015 data set have been delineated.

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