

Doubling strong lensing as a cosmological probe

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Strong gravitational lensing provides a geometric probe of cosmology in a unique manner through distance ratios involving the source and lens. This is well-known for the time delay distance derived from measured delays between lightcurves of the images of variable sources such as quasars. Recently, double source plane lens systems involving two constant sources lensed by the same foreground lens have been proposed as another probe, involving a different ratio of distances measured from the image positions and fairly insensitive to the lens modeling. Here we demonstrate that these two different sets of strong lensing distance ratios have strong complementarity in cosmological leverage. Unlike other probes, the double source distance ratio is actually more sensitive to the dark energy equation of state parameters w_0 and w_a than to the matter density Ω_m , for low redshift lenses. Adding double source distance ratio measurements can improve the dark energy figure of merit by 40% for a sample of fewer than 100 low redshift systems, or even better for the optimal redshift distribution we derive.

DOI: [10.1103/PhysRevD.94.083510](https://doi.org/10.1103/PhysRevD.94.083510)**I. INTRODUCTION**

The concept of strong gravitational lensing, the significant impact of gravity on the propagation of light, dates back to Michell in 1783 [1]. Michell used this to predict black holes (then called dark stars) and gravitational redshift. In modern cosmology, strong lensing can be used as a geometric probe of spacetime and the expansion of the Universe. Refsdal [2] was a pioneer in this, going beyond the Hubble constant to address the expansion deceleration parameter and jerk. The distance ratios, similar to focal lengths, entering in strong lensing effects such as multiple image separations and time delays provide complementary information to the expansion history mapping from single distances such as the luminosity distance of supernovae or angular distance of baryon acoustic oscillations.

Three main aspects of strong lensing have been used as cosmological probes: the statistical abundance of strongly lensed images, whether arcs or multiple images, the angular separation of multiple images in individual systems, and the time delay between multiple images of a variable source (see [3] for a comprehensive introduction). Each involves different ratios of the distances between the observer and lens, observer and source, and lens and source. The sensitivity of these to cosmology, especially the dark energy equation of state, and their unique properties relative to single distances, was discussed by [4]. In particular, the last two observables can exhibit positive correlation between the dark energy equation of state value today w_0 and a measure of its time variation w_a , unlike the classic cosmology probes. This offers the potential for complementarity with standard, single distance measurements and hence greater leverage on estimation of cosmological parameters.

Each strong lensing technique has its specific dependence on other ingredients besides cosmographic distances, e.g. selection effects and the growth of cosmic structure in the case of abundances, or the mass profile of the lens and line of sight structure in the last two methods. Time delay cosmography has made the greatest advances in the last several years, with improved lens modeling [5–11], clear understanding of the cosmological leverage [12,13], high accuracy time delay estimation [14–20], and actual application to derive cosmological constraints on geometry and dark energy [8,9,21]. For an up to date review, see [22].

In this article we return to investigation of the second strong lensing technique, using image separations. This is also a geometric probe of cosmology, and recent developments have increased its potential. The sensitivity to the lensing mass distribution poses an obstacle to its use for precision cosmology, but [23–26] proposed using a ratio of ratios technique for canceling much of this dependence. Double source plane lensing, where two independent sources are each lensed into multiple images by the same foreground mass, involves a ratio of distance ratios where the lens model nearly cancels out. Moreover, examples have recently been observed and the number of such systems is likely to increase with wide field surveys underway such as the Dark Energy Survey, HyperSuprime Cam, and KiloDegree Survey.

In Sec. II we briefly summarize the concept of double source plane lensing (DSPL) and investigate the cosmological sensitivity of its central quantity, the ratio of distance ratios. We demonstrate the significant complementarity of DSPL measurements with time delay lens observations in Sec. III, in terms of constraints on dark energy and cosmology. Section IV examines the impact of the redshift distribution of the lens and source sample,

relevant to future surveys such as Euclid, LSST, and WFIRST, and Sec. V presents the conclusions.

II. DOUBLE SOURCE PLANE LENSING DISTANCES

A. Introduction to DSPL

A double source plane lensing system occurs when two background sources are sufficiently aligned behind a common foreground lens that both are split into multiple images. The angular separation between images depends on the ratio of the lens-source distance D_{ls} and observer-source distance D_s , and the mass of the lens (feeding into its Einstein radius). In the ratio of angular separations for images from source 1 and images from source 2, the lens mass will cancel out (in the ideal case), leaving a purely geometric distance measure.

There are subtleties to this picture, treated in [25–27], and these should be carefully read. For example, the ratio comes fundamentally from the deflection angles scaled to the source plane, and this involves the surface mass density of the lens: the lens model cancellation is exact only for a singular isothermal sphere distribution or light rays passing through the same radius of the lens—but in the general case the lens model uncertainty is still significantly suppressed. A form of the mass sheet degeneracy common to lensing systems, where mass apart from the lens can be degenerate with the observables, exists for DSPL [28] and requires dynamical information (e.g. lens velocity dispersion) to break. We concentrate on galaxy scale lensing where the mass distribution is expected to be cleaner. Lensing of the further source by the nearer one must also be taken into account.

[Note that weak lensing shear ratios for multiple source planes [29–31] involve a similar distance ratio, but in a statistical ensemble sense. It is not clear to what extent the lens mass profile cancels out when averaging over many (thousands or millions) of galaxy lenses. There is also a form of the mass sheet degeneracy for the multiplane shear ratio [32]. Nevertheless, it is exciting to see that the weak lensing shear ratio method has just been used to combine galaxy sources and the cosmic microwave background as a second source plane to give distance constraints [33].]

Finally, double source lensing is rarer than single source lensing so there are fewer systems to use in cosmological constraints, but on the other hand there is no requirement for the sources to be time variable and monitored for extended periods of time as for time delay lensing.

Just as has been done for time delay cosmography, all these sources of systematic uncertainty will need to be addressed through a program of observations, data simulation challenges, and theoretical work. To motivate that such effort is worthwhile, here we investigate the cosmological constraint impact if DSPL becomes a new, mature probe. This is particularly relevant with the approach of

next generation wide field surveys such as Euclid, LSST, and WFIRST that should find abundant samples of DSPL. For example, [27] estimates that the Euclid satellite will find of order ~ 2000 DSPL and the ground based Large Synoptic Survey Telescope (LSST) will find a similar number. Using the technique of [34,35] estimates that using only the best seeing exposures to get the highest quality will deliver ~ 160 DSPL from Euclid and ~ 120 from LSST. The WFIRST satellite will find fewer but has the advantage of excellent spatial resolution for more precise image separation measurements.

Initial work on cosmological parameter estimation appears in [25,26], concentrating on a flat universe with constant dark energy equation of state w and a restricted sample of lenses. We expand on this by focusing on the more general time varying dark energy equation of state $w(z) = w_0 + w_a z/(1+z)$ in common use, and exploring the influence of distributions in the redshifts of lens, source 1, and source 2, and briefly examining the case of nonflat universes. (Note [25] did in one section consider $w_0 - w_a$ constraints for a fixed redshift distribution.) Most importantly, we identify strong complementarity between the two strong lensing probes of DSPL and time delay lensing.

B. Cosmological sensitivity

The central quantity for DSPL is the ratio of distance ratios,

$$\beta = \frac{D_{ls}(z, z_1)}{D_s(z_1)} \frac{D_s(z_2)}{D_{ls}(z, z_2)}, \quad (1)$$

where the lens is at redshift z , the nearer source is at z_1 , and the further source at z_2 . We begin by examining its sensitivity to cosmological parameters.

Most obviously, it is dimensionless and so independent of the Hubble constant H_0 . This alone indicates some complementarity with the dimensional time delay distance, which is the ratio

$$D_{\Delta t} = \frac{D_l(z)D_s(z_1)}{D_{ls}(z, z_1)}(1+z), \quad (2)$$

and so is inversely proportional to H_0 . To probe the sensitivity of β to the time varying dark energy equation of state parameters, we use the Fisher information formalism. Note that $\beta = \beta(z, z_1, z_2)$ so for visual ease of presentation we exhibit the results for a series of lens redshifts z , and set $z_1/z = 2$, $z_2/z_1 = 1.5$. This is motivated by the lensing kernel peaking when the lens is roughly halfway between the source and observer, but we analyze various other cases both in this section and in Sec. IV.

Figure 1 illustrates the cosmological sensitivity of β as a function of z through the Fisher derivatives $\partial\beta/\partial p_i$, for the parameters p_i of the dimensionless matter density Ω_m , the

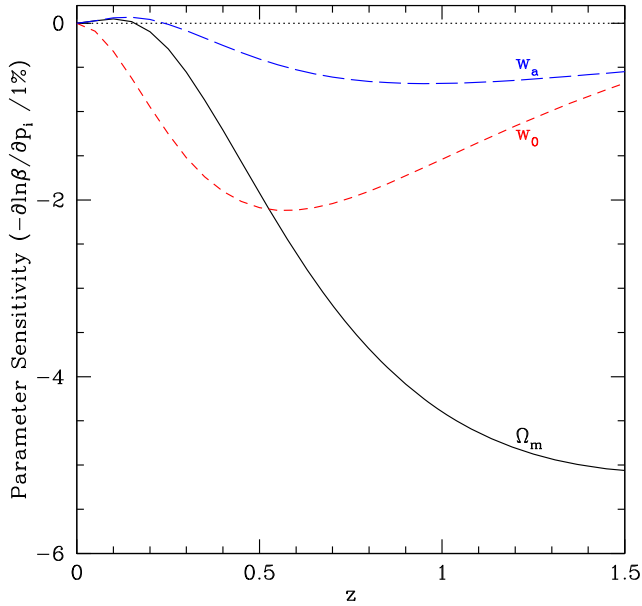


FIG. 1. The sensitivity of measurements of the double source lensing distance ratio β for constraining cosmological parameters is plotted as a function of the lens redshift z . The magnitude of the sensitivity is for a 1% measurement of β , but the more interesting aspects come from the shape of the curves: the null of the Ω_m curve at $z \approx 0.15$ and the opposite signs for w_0 and w_a sensitivities for $z \lesssim 0.23$, indicating a distinct behavior from single distance probes.

dark energy equation of state today w_0 , and the dark energy equation of state time variation w_a . The fiducial cosmology is a flat Λ CDM universe with $\Omega_m = 0.3$. (We assume spatial flatness throughout the article, except for part of Sec. III that examines the free curvature case.) The larger the absolute value of the sensitivity, and the greater the distinction between the shapes of the derivative curves, the more information exists and the tighter are the cosmological constraints.

We identify several interesting properties. Classic cosmology probes are more sensitive to the matter density Ω_m than the dark energy equation of state [for example, low redshift supernova distances probe the deceleration parameter $q_0 = [1 + 3w(1 - \Omega_m)]/2$, so the sensitivity derivative with respect to Ω_m is ~ 1.4 times larger than that for w , and this only gets larger at higher redshift]. DSPL, however, is more sensitive to both w_0 and w_a than Ω_m at low redshift, and indeed has a null in the dependence on Ω_m at $z \approx 0.15$. This means that degeneracy with the matter density is strongly suppressed there. Furthermore, w_0 and w_a have opposite signs for $z < 0.23$, meaning they have positive correlation, unlike single distance and growth probes. The sensitivity to dark energy parameters is relatively strong out to $z \approx 0.6$, before the matter density dominates for higher redshift systems. This means that the systems with the greatest leverage are at observationally benign low redshifts, easing follow-up requirements such as redshift or

velocity dispersion measurements. Thus, DSPL appears at first glance to be quite promising.

Taking into account the covariances between the cosmological parameters, we can use Fisher information analysis to estimate constraints on the cosmology for a series of measurements. To emphasize the unique aspects of positive correlation between w_0 and w_a , and reduced sensitivity to the matter density degeneracy, we show an illustrative calculation (omitting the axis values due to the idealized precision) in Fig. 2.

The solid ellipses highlight the evolution of the degeneracy direction between the dark energy equation of state parameters by fixing $\Omega_m = 0.3$. We see that indeed the case with the lens redshift at $z = 0.23$, where the w_a curve in Fig. 1 crosses zero, gives a vertical parameter estimation contour (we actually use lenses at $z, z \pm 0.05$ to obtain finite contours). Lower redshift lenses, where the w_0 and w_a sensitivities are positively correlated, give ellipses leaning to the right, the opposite of standard single distance and growth probes, while higher redshift lenses give negative correlation. The dotted lines show the elongation of the contours when marginalizing over Ω_m , as should be done. The expansion of the contours due to the matter density covariance is most severe at higher redshift, and changes the contour direction and size less at lower redshift. Interestingly, the marginalization over Ω_m actually tilts the ellipses to the right, so positive correlation in the $w_0 - w_a$ plane occurs for lenses at $z \lesssim 0.4$, not just $z < 0.23$.

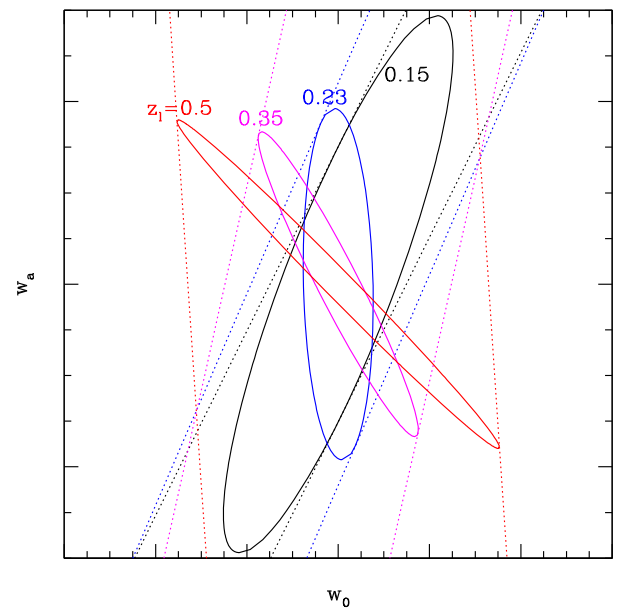


FIG. 2. The leverage of measurements of the double source lensing distance ratio β for constraining the dark energy equation of state value today w_0 and time variation w_a is plotted for different lens redshifts z . This shows the evolution of covariance direction. Solid ellipses fix $\Omega_m = 0.3$ while dotted ellipses marginalize over Ω_m .

C. Analytic redshift dependence

DSPL seems an interesting possibility as a new complementary cosmological probe therefore. Before proceeding further, let us return to the point about the source redshifts. Consider moving the further source closer to the nearer one, say $z_2 = z_1 + 0.1$. This dramatically reduces the cosmological sensitivity, by a factor ~ 10 according to numerical computation. We can derive this analytically. Let $z_2 = z_1 + \delta$. Then

$$\begin{aligned} D_s(z_2) &= (1+z_2)^{-1} \int_0^{z_2} dz/H(z) \\ &= (1+z_1+\delta)^{-1} [(1+z_1)D_s(z_1) + \delta/H(z_1)] \\ &\approx D_s(z_1) \left[1 + \frac{\delta}{1+z_1} \left(\frac{1}{H_1 D_1} - 1 \right) \right]. \end{aligned} \quad (3)$$

Calculating $D_{ls}(z, z_2)$ similarly, this gives

$$\beta \approx 1 + \frac{\delta}{1+z_1} \left[\frac{1}{H_1 D_1} - \frac{1}{H_1 D_{ls}(z, z_1)} \right]. \quad (4)$$

This makes sense since as $z_2/z_1 \rightarrow 1$, then $\beta \rightarrow 1$ and a constant has no cosmological sensitivity. Thus the sensitivity is suppressed by the small factor $\delta/(1+z_1)$.

What if we take $z_2/z_1 \gg 1$? Very high redshift sources are harder to observe since their images will generally be fainter; also, the follow-up resources needed to measure their redshift will be more expensive. Thus, we take the somewhat conservative case of using $z_2/z_1 = 1.5$ as a baseline, though we revisit this in Sec. IV.

If we were to move the closer source nearer to the lens, we reduce the value of β . Instead of it being near unity, it can become much smaller. Again, this reflects that the peak of the lensing kernel is where the lens is roughly midway between the source and observer. For $z_1 = z + \delta$,

$$\beta \approx \frac{\delta}{1+z} \frac{1}{H(z)D(z)} \frac{D_s(z_2)}{D_{ls}(z, z_2)}. \quad (5)$$

We see this is suppressed by $\delta/(1+z)$. However, an interesting point is that we also have sensitivity not just to distances but to the Hubble parameter $H(z)$ directly. Which one wins out in its influence on the cosmological parameter estimation depends on the measurement assumptions. If one says the measurements are at constant relative precision, e.g. 1% measurements, then we find that the extra information from the direct $H(z)$ dependence is more significant. Note though that since β may be smaller by a factor 10, say, this means that the measurement must have $\sigma_\beta \sim 0.001$ rather than $\sigma_\beta \sim 0.01$, for a 1% measurement of β . This would be highly challenging, and therefore we conservatively take $z_1/z = 2$, although again we revisit this in Sec. IV.

III. COSMOLOGICAL CONSTRAINTS

While DSPL has interesting cosmological sensitivity properties, it does not have much raw sensitivity magnitude. Rather it is the complementarity with time delay lensing and a high redshift probe such as the cosmic microwave background (CMB) that is of value. We consider time delay lensing as in [12]—1% precision on $D_{\Delta t}$ in six redshift bins from $z = 0.1$ – 0.6 —and CMB measurements of the distance to last scattering and the physical matter density $\Omega_m h^2$ (where $h = H_0/100$ km/s/Mpc) of the quality of Planck satellite measurements. For DSPL we adopt as the baseline 1% measurements, basically of the image separations, of 96 systems in the range $z = 0.1$ – 0.6 . Recall that this range was identified as most theoretically promising, as well as observationally tractable, in Sec. II B.

The addition of this new probe improves the dark energy figure of merit $\text{FOM} = \sqrt{\det F_{w_0 w_a}}$, a measure of the uncertainty in dark energy equation of state estimation, by 43%. Figure 3 shows the confidence contours in the $w_0 - w_a$ plane, marginalized over the other cosmological parameters. The matter density Ω_m is determined to 0.0047, w_0 to 0.072, w_a to 0.25, and the reduced Hubble constant h to 0.0047.

Thus both strong lensing probes work well together. Furthermore, the combination remains complementary with a single distance probe like supernovae. The dashed curve shows the addition of a midterm supernova sample of the rough quality, including systematics, expected by the end of the Dark Energy Survey, approximately

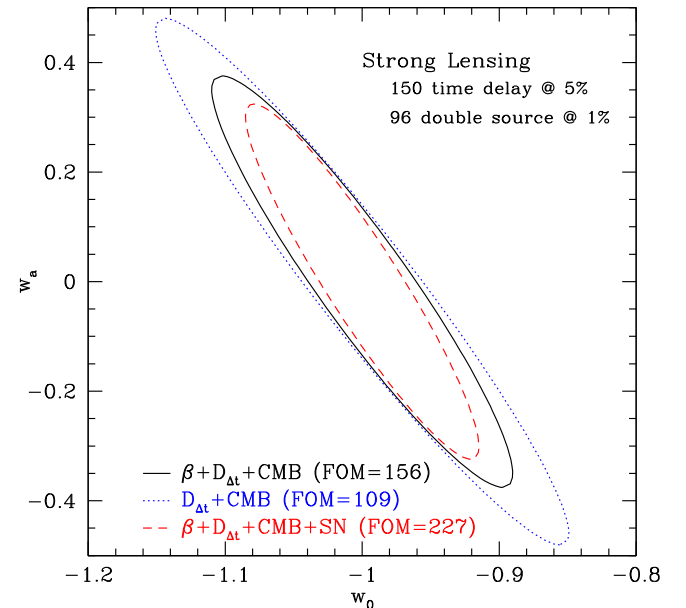


FIG. 3. Cosmological parameter estimation uncertainty is plotted in the dark energy equation of state plane for the case of strong lensing time delays (dotted curve), time delays plus double source plane strong lensing (solid), and the two strong lensing probes plus supernova distances (dashed). DSPL brings significant complementarity.

equivalent to 150 supernovae at $z < 0.1$, 900 at $z = 0.1-1$, and 42 at $z = 1-1.7$ (as used in [12]), with a systematic of $0.02(1+z)/2.7$ mag.

Degrading the DSPL precision to 2% reduces the FOM by 22%. We can also examine the impact of the redshift range used for the lenses. Cutting out the low ($z = 0.1$) or high ($z = 0.6$) part of the sample reduces the FOM by 14%, while extending it to $z = 0.7$ improves it by 13%. As expected, estimation of h suffers most when removing the low redshift systems, while predominantly the dark energy parameter constraints loosen when removing the high redshift systems. Thus the range of lens system redshifts between $z = 0.1-0.6$ seems a happy medium, especially as higher redshift systems become more expensive for follow-up to obtain spectroscopic redshifts and lens velocity dispersions. (Note that [36] describes code for optimizing the science return under constrained follow-up resources by a merit vs cost weighting.)

Recalling that [12] found that time delay lensing greatly immunized supernova distances against the degeneracy due to spatial curvature, we consider the effect of doubling strong lensing. Figure 4 shows that combining DSPL with time delay lensing similarly reduces the constraint loss due to spatial curvature density Ω_k . While allowing Ω_k to float blows up the w_0-w_a contour area by a factor 20, relative to the flat case, for time delay lensing plus CMB, this factor is only 4.5 for the DSPL plus time delay lensing plus CMB combination. Moreover, in the curvature free case,

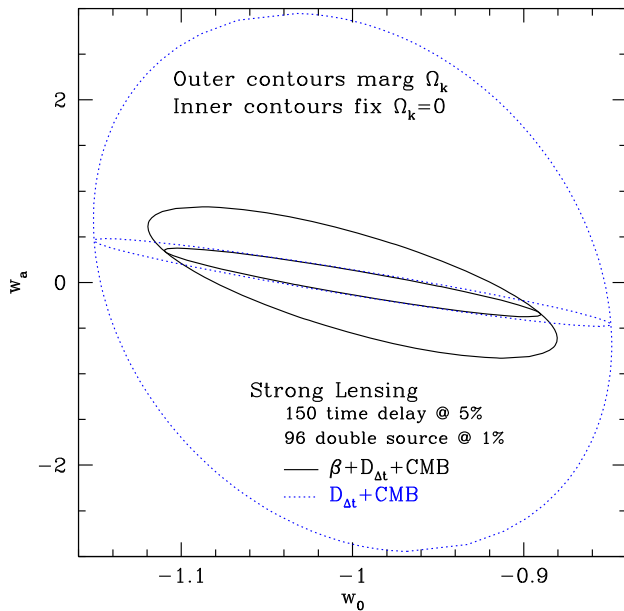


FIG. 4. Cosmological parameter estimation uncertainty, allowing for (outer contours) and fixing (inner contours) spatial curvature, is plotted in the dark energy equation of state plane. The curvature induced degeneracy in the case of strong lensing time delays (dotted curves) is substantially tamed by the combination of double source plane lensing with the lensing time delay measurements (solid curves).

including DSPL tightens the constraint on the curvature by a factor 4.6, to $\sigma(\Omega_k) = 0.0072$, and tightens the w_a determination by a factor 3.6.

IV. REDSHIFT SENSITIVITY

Although we gave good rationales in Sec. II for why the choices $z_1/z = 2$ and $z_2/z_1 = 1.5$ were reasonable, from both theoretical sensitivity and observational follow-up points of view, let us revisit the question of optimal redshift distributions. We vary these two redshift ratios and study the impact on dark energy figure of merit.

Figure 5 shows the results for the case of constant relative precision, i.e. 1% per DSPL system. Recall from Sec. II that we showed analytically that as z_1/z gets close to unity, β involves the Hubble parameter $H(z)$ itself. This would be expected to bring in extra cosmological information and indeed that is exactly what we find: the figure of merit improves as we move to the left in the figure. As we increase z_2/z_1 and move up in the figure, we have an increased lever arm in distance, allowing for greater complementarity in the measurements, and again the FOM increases. For one source close to the lens and the other much further away, the FOM from the combination of DSPL, time delay lensing, and CMB can reach 373.

However, this obscures some difficulties. Pushing the ratios in these directions lowers β , and it is only the assumption of constant relative precision that enables such gains. Indeed in the extreme case leading to a FOM of 373, we have $\beta = 0.13$ (rather independent of lens redshift) and

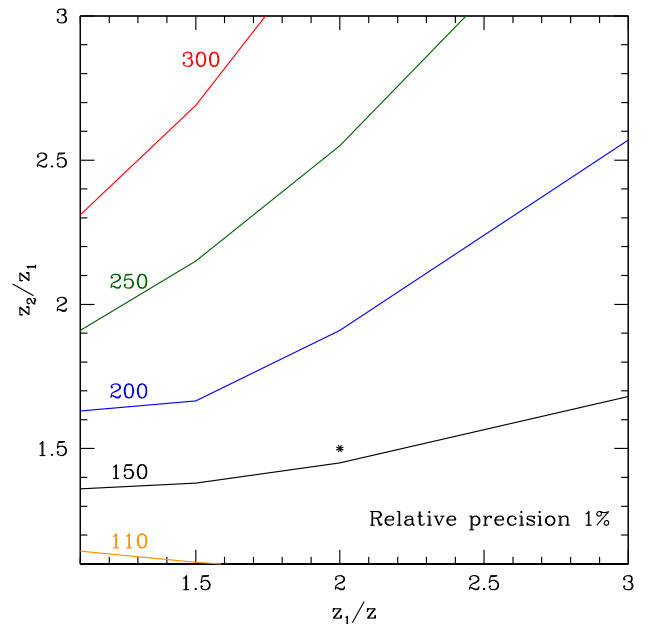


FIG. 5. Isocontours of dark energy figure of merit are plotted for a range of source-source and source-lens redshift ratios, assuming constant relative precision of 1% on β . The baseline case used in Sec. III, justified in Sec. II, with $z_1/z = 2$ and $z_2/z_1 = 1.5$, is indicated by the star.

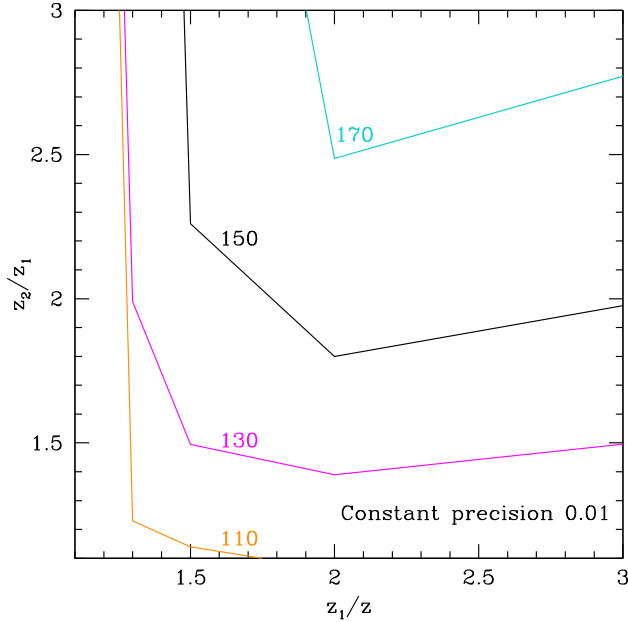


FIG. 6. Isocontours of dark energy figure of merit are plotted for a range of source-source and source-lens redshift ratios, as in Fig. 5, but now assuming a more realistic constant absolute precision of 0.01 on β (corresponding to $\sim 1\%$ relative precision for well spaced sources and lens).

so are positing an extremely small measurement uncertainty $\sigma_\beta = 0.0013$. Low β also means the images sample a wider range of lens radii, reducing the insensitivity to mass model. Moreover, the further source will be at a high redshift possibly more difficult for precision follow-up. (These challenges are not wholly beyond practicality: angular separations such as Einstein radii have been measured to better than 0.1% under good circumstances and the double source plane system SDSSJ0946 + 1006 does have $z_2/z_1 = 3.8$ [26].)

Instead let us consider constant absolute precision $\sigma_\beta = 0.01$. This will penalize redshift distributions that give low β , since then the signal to noise is lower. Figure 6 bears this out exactly. Indeed we see that the optimum source 1 to lens redshift ratio $z_1/z = 2$, in order to get the highest FOM for fixed source 2 to source 1 redshift ratio z_2/z_1 . This justifies our previous adoption of $z_1/z = 2$. As one pushes the farther source to higher redshift, β slowly declines from unity with increasing z_2/z_1 , but the longer cosmological lever arm wins out and the FOM increases. The price of this, however, is more difficult observations: the images of the farther source will be fainter and of lower signal to noise and hence harder to measure well, plus follow-up time for, e.g., the source redshift will be more expensive. We therefore have used the conservative assumption that $z_2/z_1 = 1.5$. Further work, using specific exposure time calculations for a given survey instrument, may eventually indicate that this source-source redshift ratio can be increased; this would bring a further gain in FOM.

V. CONCLUSIONS

Cosmography is an important element of the quest to understand the nature of cosmic acceleration, especially to the extent that it is free of uncertainties in the growth of structure (e.g. the mapping between galaxies or clusters and the underlying dark matter field). A new geometrical probe would be a valued addition to the cosmic toolbox. While it is very early days yet, double source plane lensing (DSPL) possesses several intriguing properties that motivate further development of theoretical, simulation, and observational studies.

Uniquely, DSPL for low redshift systems is more sensitive to the dark energy equation of state than to the matter density, and indeed there is a nulling of the matter density degeneracy. Furthermore, it is one of the rare probes, involving distance ratios like time delay lensing, that has a positive correlation between dark energy equation of state parameters w_0 and w_a , thus offering special complementarity with single distance measures like supernovae and baryon acoustic oscillations. Finally, we have demonstrated that double source plane lensing has complementarity with time delay lensing, doubling strong gravitational lensing as a cosmological probe. This holds in both flat and, especially, spatial curvature free cosmologies.

We have identified the optimum redshift distributions of lens and sources under various measurement assumptions, both analytically and numerically, and quantified the dark energy figures of merit. Improvement of the figure of merit by 40% with the addition of DSPL is found under reasonable assumptions. Issues of systematics, and observational and follow-up practicalities, certainly remain to be addressed, but the calculations here show the worth of undertaking such efforts. Moreover, we have been somewhat conservative in not using second sources at much higher redshift, which have the potential to lead to a FOM over 200.

Double source plane lensing has become an observational reality, with two galaxy scale systems currently known, dozens more likely to be found by the current generation of wide field surveys, and hundreds to thousands expected in the next generation by Euclid, LSST, and WFIRST. Time delay lensing is similarly poised for a cornucopia of data. Doubling strong lensing may prove to be a fruitful path forward in understanding the geometry and accelerated expansion of the Universe.

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