

Relativistic dynamics of superfluid-superconducting mixtures in the presence of topological defects and an electromagnetic field with application to neutron stars

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The relativistic dynamic equations are derived for a superfluid-superconducting mixture coupled to an electromagnetic field. For definiteness, and bearing in mind possible applications of our results to neutron stars, it is assumed that the mixture is composed of superfluid neutrons, superconducting protons, and normal electrons. We analyze the proton superconductivity of both types I and II and allow for the possible presence of neutron and proton vortices (or magnetic domains in the case of type-I proton superconductivity). The derived equations neglect all dissipative effects except for the mutual friction dissipation and are valid for arbitrary temperatures (i.e., they do not imply that all nucleons are paired), which is especially important for magnetar conditions. It is demonstrated that these general equations can be substantially simplified for typical neutron stars, for which a kind of magneto-hydrodynamic approximation is justified. Our results are compared to the nonrelativistic formulations existing in the literature, and a number of discrepancies are found. In particular, it is shown that, generally, the electric displacement \mathbf{D} does not coincide with the electric field \mathbf{E} , contrary to what is stated in previous works. The relativistic framework developed here is easily extendable to account for more sophisticated microphysics models, and it provides the necessary basis for realistic modeling of neutron stars.

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I. INTRODUCTION

Assume that we have a relativistic magnetized finite-temperature plasma (possibly in the strong gravitational field) composed of superfluid neutral particles, superconducting positively charged particles, and normal (nonsuperconducting) negatively charged particles. Depending on the density, the positively charged particles may form either a type-I or type-II superconductor, and the plasma may contain topological defects—Feynman-Onsager and/or Abrikosov vortices. What are the macroscopic dynamic equations describing such a system?

The question is not as far fetched as it may seem at first glance. For example, the neutron-proton-electron (*npe*) mixture in the outer neutron-star cores meets all the conditions formulated above. First, it is relativistic and magnetized. The typical surface magnetic field is $B \sim 10^8 \div 10^{15}$ G [1,2] and is likely to be larger in the deeper layers [3]; the surface gravitation acceleration is also huge, $g_s \sim 2 \times 10^{14}$ cm s⁻² [2], electrons are ultrarelativistic, while neutrons can be moderately relativistic. Second, according to microscopic calculations [4,5], confirmed (to some extent) by observations of cooling and glitching neutron stars [6–8], neutrons and protons in their interiors become superfluid/superconducting at temperatures $T \lesssim T_{ci}$, where $T_{ci} \sim 10^8 \div 10^{10}$ K is the nucleon critical temperature ($i = n, p$).

Third, in a rotating magnetized neutron star, it can be energetically favorable to form Feynman-Onsager/Abrikosov vortices [9] (the latter are formed only if the protons are type-II superconductor; if, instead, they are type I, different structures appear; see Sec. V for more details).

Thus, it is not surprising that the dynamic properties of magnetized superfluid-superconducting neutron-star plasma have been the subject of numerous studies in the past, both in nuclear matter (see, e.g., Refs. [10–20]) and in quark matter (e.g., Refs. [21–24]). In particular, Vardanyan and Sedrakyan [12] were the first who generalized hydrodynamics of a mixture of two superfluids [25,26] to charged superfluids coupled to the electromagnetic field. These equations were further extended by Holm and Kupersmidt [13] to N charged superfluids, who derived these equations from the Hamiltonian formalism. Finally, the most general *nonrelativistic* finite-temperature equations, describing charged superfluids and accounting for the mutual friction forces [27,28] between various liquid components, were formulated by Mendell and Lindblom [14], who used in their work the ideas of Refs. [13,25,29]. This important work was subsequently used by Mendell [15,16] who applied the equations of Ref. [14] to neutron stars, assuming that all neutrons and protons are paired (i.e., $T \ll T_{ci}$). (A little later, Sedrakian and Sedrakian [17]

did similar work by extending the results of Ref. [12] to include dissipation and mutual friction forces in their equations.) In his work, Mendell formulated a set of simplified magnetohydrodynamic equations, but, unfortunately, incorrectly identified the magnetic field \mathbf{H} with the magnetic induction \mathbf{B} and the electric displacement \mathbf{D} with the electric field \mathbf{E} . The first of these inaccuracies (identification of \mathbf{H} with \mathbf{B}) was noticed in Ref. [30] and corrected by Glampedakis, Andersson, and Samuelsson [19] (hereafter GAS11); the second inaccuracy (identification of \mathbf{D} with \mathbf{E}) is discussed here (see Appendix G 2). Except for the corrected inaccuracy, the GAS11 version of magnetohydrodynamics is equivalent (up to notations) to that of Mendell [15] and is the most advanced treatment of superfluid-superconducting mixtures in neutron stars to date. It is derived using the variational framework [20,31] and assuming $T = 0$.

All the works discussed by us so far were performed in the nonrelativistic approximation. This is a rather serious shortcoming because, as we have already mentioned, neutron stars are essentially relativistic objects. The extension of the magnetohydrodynamics of GAS11 (as well as more general equations of Ref. [14]) to the relativistic case is not trivial. For uncharged one-component superfluids, this problem has been addressed in Refs. [32–39] and has recently been “solved” in Ref. [40] (hereafter G16). We are aware of only one attempt [41] to consider charged mixtures in full relativity. This reference neglected all dissipation effects (including mutual friction) and studied only the low-temperature case $T \ll T_{ci}$; unfortunately, it did not provide a nonrelativistic limit for the derived equations so that it is hard to compare them with the formulations available in the literature. Note that Ref. [41] adopted the variational approach similar to that developed in Ref. [36] in application to uncharged superfluids. This approach was criticized in G16 (see Appendix F there), where it was argued that it does not reproduce the well-established nonrelativistic Hall-Vinen-Bekarevich-Khalatnikov superfluid hydrodynamics [25,29]. We believe the same conclusion applies also to the results of Ref. [41].

The aim of the present study is to fill the existing gaps and derive a set of relativistic finite-temperature equations describing superfluid-superconducting mixtures, bearing in mind application of these results to magnetized rotating neutron stars. As in Refs. [29] and G16, our derivation rests on the consistency between the conservation laws and the entropy generation equation. For definiteness, in this paper we consider a liquid composed of superfluid neutrons (n), superconducting protons (p), and normal electrons (e). Extension of our results to more complicated compositions is straightforward (see, e.g., Refs. [24,38,42,43]). Here we are mostly interested in the nondissipative equations (but we allow for mutual friction dissipation; see remark 1 in Sec. VI). Correspondingly, we assume that neutron and

proton thermal excitations as well as electrons move with one and the same “normal” four-velocity u^μ . In what follows all thermodynamic quantities are defined in the frame comoving with the normal (nonsuperfluid) liquid component, in which $u^\mu = (1, 0, 0, 0)$. By default, any three-dimensional vector appearing in the text (e.g., magnetic induction \mathbf{B}) is written in that frame.

The paper is organized as follows. Section II introduces Maxwell’s equations in the medium written both in the standard and explicitly Lorentz-covariant form. Section III considers uncharged and charged mixtures in the absence of vortices and other magnetic domain structures. In Sec. IV we discuss the strategy for generalization of equations of Sec. III in order to allow for the topological defects and related bound charges and currents in the mixture. In Sec. V this strategy is applied to derive the corresponding dynamic equations under assumption of type-I superconductivity of protons. Section VI is devoted to considering type-II proton superconductivity and accounting for the possible presence of both neutron (Feynman-Onsager) and proton (Abrikosov) vortices. Section VII proves that the energy-momentum tensors obtained in Secs. V and VI are symmetric, and expresses them through a set of phenomenological coefficients which can be calculated by specifying a microscopic model for the energy density of the mixture. The general dynamic equations of Sec. VI are simplified for typical neutron-star conditions in Sec. VIII. Finally, we sum up in Sec. IX.

The paper also contains a number of appendixes, where we present technical, more model-dependent, or less important results. In particular, Appendix A introduces some basic notation used throughout the paper. Appendix B provides a correspondence table between our notation and that adopted in G16. Appendix C contains an example of the energy density transformation used in Secs. V and VI. Appendix D reveals the relation between the energy-momentum tensor of Sec. V and the well-known Abraham tensor. Appendix E discusses some general relations characterizing isolated neutron or proton vortices. Appendix F demonstrates that there exist some bound charges associated with each moving vortex. Appendix G presents two simple microscopic models allowing one to determine the phenomenological coefficients from Sec. VII. Finally, Appendix H contains the full set of dynamic equations derived in Secs. V and VI, and Appendix I analyzes the nonrelativistic limit of simplified equations of Sec. VIII.

Unless otherwise stated, in all sections except for Sec. II and Appendixes E, F, G, and I, the speed of light c , the Planck constant \hbar , and the Boltzmann constant k_B are set to unity, $c = \hbar = k_B = 1$. Throughout the paper, we assume that the spacetime metric is flat, $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Generalization of our results to arbitrary $g_{\mu\nu}$ is straightforward and can be achieved by replacing ordinary derivatives in all equations with their covariant counterparts.

II. MAXWELL'S EQUATIONS IN THE MEDIUM

A. Standard form of Maxwell's equations

Maxwell's equations in the medium take the form

$$\operatorname{div} \mathbf{D} = 4\pi \rho_{\text{free}}, \quad (1)$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (2)$$

$$\operatorname{div} \mathbf{B} = 0, \quad (3)$$

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{free}} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad (4)$$

where \mathbf{E} and \mathbf{B} are the electric field and magnetic induction, respectively; \mathbf{D} and \mathbf{H} are the electric displacement and magnetic field, respectively; ρ_{free} and \mathbf{J}_{free} are macroscopic averages of the free charge and current densities in the medium (e.g., Ref. [44]). In the absence of bound charges and currents one has $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$.

Equations (1)–(4) contain the continuity equation for the electric charge,

$$\frac{\partial \rho_{\text{free}}}{\partial t} + \operatorname{div} \mathbf{J}_{\text{free}} = 0, \quad (5)$$

and the energy equation,

$$\frac{\partial \varepsilon_{\text{EM}}}{\partial t} = -\mathbf{E} \mathbf{J}_{\text{free}} + \frac{c}{4\pi} \operatorname{div} [\mathbf{H} \times \mathbf{E}], \quad (6)$$

where

$$d\varepsilon_{\text{EM}} = \frac{1}{4\pi} \mathbf{E} d\mathbf{D} + \frac{1}{4\pi} \mathbf{H} d\mathbf{B} \quad (7)$$

is the differential of the electromagnetic energy density ε_{EM} .

B. Relativistic representation

Maxwell's equations (1)–(4) can be rewritten in a manifestly Lorentz-covariant form [45,46]. To see this let us introduce the tensors $F^{\alpha\beta}$ and $G^{\alpha\beta}$ such that

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \quad (8)$$

$$G^{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}, \quad (9)$$

where $A^\alpha = (\phi, \mathbf{A})$ is the electromagnetic four-potential.¹ Using the definitions (8)–(9), Maxwell's equations (1)–(4) can be represented as

$$\partial_\alpha {}^* F^{\alpha\beta} = 0, \quad (10)$$

$$\partial_\alpha G^{\alpha\beta} = -4\pi J_{(\text{free})}^\beta, \quad (11)$$

where $J_{(\text{free})}^\alpha = (\rho_{\text{free}}, \mathbf{J}_{\text{free}}/c)$ is the four-current density of free charges and ${}^* F^{\mu\nu}$ is the tensor dual to $F^{\mu\nu}$ (see Appendix A).

C. Four-vectors E^μ , B^μ , D^μ , and H^μ

As is shown in Appendix A, for any antisymmetric tensor it is possible to introduce the corresponding ‘‘electric’’ and ‘‘magnetic’’ four-vectors [see Eqs. (A3) and (A4)]. In the case of electromagnetic tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ we shall use the following (standard) notation for these vectors,

$$E^\mu \equiv F_{(\text{E})}^\mu = u_\nu F^{\mu\nu}, \quad (12)$$

$$D^\mu \equiv G_{(\text{E})}^\mu = u_\nu G^{\mu\nu}, \quad (13)$$

$$B^\mu \equiv F_{(\text{M})}^\mu = u_\nu {}^* F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\eta} u_\nu F_{\lambda\eta}, \quad (14)$$

$$H^\mu \equiv G_{(\text{M})}^\mu = u_\nu {}^* G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\eta} u_\nu G_{\lambda\eta} \quad (15)$$

instead of, respectively, the universal notations $F_{(\text{E})}^\mu$, $G_{(\text{E})}^\mu$, $F_{(\text{M})}^\mu$, and $G_{(\text{M})}^\mu$ suggested in Appendix A. In the comoving frame, in which the four-velocity of normal liquid component is $u^\mu = (1, 0, 0, 0)$ these vectors reduce to $E^\mu = (0, \mathbf{E})$, $B^\mu = (0, \mathbf{B})$, $D^\mu = (0, \mathbf{D})$, and $H^\mu = (0, \mathbf{H})$.

III. NO VORTICES, BOUND CHARGES, OR BOUND CURRENTS

In order to establish notations and get some insight into the problem, we start with the simplest possible situation and discuss relativistic equations for the superfluid-superconducting *npe* mixture without vortices, bound charges, or bound currents. The latter assumption means that we set $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$ in all equations in this section.

A. General structure

The relativistic equations describing the *npe* mixture consist of the energy-momentum conservation,

¹We remind that in a given coordinate system:

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi,$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}.$$

$$\partial_\mu T^{\mu\nu} = 0 \quad (16)$$

and continuity equations for particle species j (here and hereafter index $j = n, p$, and e),²

$$\partial_\mu j_{(j)}^\mu = 0. \quad (17)$$

In Eqs. (16) and (17) $T^{\mu\nu}$ is the total energy-momentum tensor, which is a sum of fluid and electromagnetic contributions,

$$T^{\mu\nu} = T_{(\text{fluid})}^{\mu\nu} + T_{(\text{EM})}^{\mu\nu}, \quad (18)$$

and $j_{(j)}^\mu$ is the current density for particle species j . These equations should be supplemented by the second law of thermodynamics, Maxwell's equations (see Sec. II), as well as by a number of additional equations and constraints describing superfluid degrees of freedom (see below).

B. Uncharged mixtures

Assume for a moment that all the mixture components (n , p , and e) are uncharged. The corresponding non-dissipative hydrodynamics has been extensively studied, e.g., in Refs. [47–50]. It consists of the second law of thermodynamics

$$d\varepsilon_{\text{fluid}} = TdS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}) \quad (19)$$

and Eqs. (16), (17), in which the energy-momentum tensor, $T^{\mu\nu} = T_{(\text{fluid})}^{\mu\nu}$, is given by

$$T_{(\text{fluid})}^{\mu\nu} = (P_{\text{fluid}} + \varepsilon_{\text{fluid}})u^\mu u^\nu + P_{\text{fluid}} g^{\mu\nu} + Y_{ik}(w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu), \quad (20)$$

and the particle four-currents are

$$j_{(i)}^\mu = n_i u^\mu + Y_{ik} w_{(k)}^\mu, \quad (21)$$

$$j_{(e)}^\mu = n_e u^\mu. \quad (22)$$

Here and below, the subscripts i and k refer to nucleons: $i, k = n, p$. Unless otherwise stated, a summation is assumed over repeated spacetime indices μ, ν, \dots (Greek letters) and nucleon species indices i and k (Latin letters).

In Eqs. (19)–(22), $\varepsilon_{\text{fluid}}$ and S are the fluid energy density and entropy density, respectively; T is the temperature; μ_j and n_j are the relativistic chemical potential and number density for particles $j = n, p$, and e , respectively; P_{fluid} is the pressure given by the standard formula

$$P_{\text{fluid}} \equiv -\frac{\partial(\varepsilon_{\text{fluid}} V)}{\partial V} = -\varepsilon_{\text{fluid}} + \mu_e n_e + \mu_i n_i + TS, \quad (23)$$

where V is the system volume and the partial derivative is taken at fixed total number of particles $n_j V$ ($j = n, p, e$) total entropy SV , and fixed scalars $w_{(i)\mu} w_{(k)}^\mu$ [25,26,40].

Further, Y_{ik} in Eqs. (19)–(21) is the relativistic entrainment matrix [47,51–53], analogue of the superfluid or mass-density matrix ρ_{ik} of the nonrelativistic theory [26,54–57]. In the nonrelativistic limit both matrices are related by the formula [47]: $Y_{ik} = \rho_{ik}/(m_i m_k c^2)$, where m_i is the bare nucleon mass ($i = n$ or p). Finally, the normal four-velocity u^μ is normalized by the condition

$$u_\mu u^\mu = -1; \quad (24)$$

and the four-vectors $w_{(i)}^\mu$ in Eqs. (19)–(21) describe the superfluid degrees of freedom and are subject to condition

$$u_\mu w_{(i)}^\mu = 0, \quad (25)$$

which ensures that all the thermodynamic quantities are indeed defined (measured) in the comoving frame in which $u^\mu = (1, 0, 0, 0)$ [see G16 for a detailed discussion]. In particular, using Eq. (25) one finds from Eqs. (20) and (21)

$$u_\mu u_\nu T_{\text{fluid}}^{\mu\nu} = \varepsilon_{\text{fluid}}, \quad (26)$$

$$u_\mu j_{(i)}^\mu = -n_i. \quad (27)$$

To close the system of hydrodynamic equations we need two additional conditions relating the four-vectors $w_{(i)}^\mu$ with the wave function phases Φ_i of the nucleon Cooper-pair condensates. These conditions are ($i = n, p$)

$$w_{(i)}^\mu \equiv \partial^\mu \phi_i - \mu_i u^\mu, \quad (28)$$

where the scalar $\phi_i = \Phi_i/2$. Equations (28) can be reformulated exclusively in terms of $w_{(i)}^\mu$ as

$$\partial_\mu [w_{(i)\nu} + \mu_i u_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu] = 0. \quad (29)$$

It is simply a statement that $\partial_\mu \partial_\nu \phi_i - \partial_\nu \partial_\mu \phi_i = 0$ (or, equivalently, $\partial_\mu \partial_\nu \Phi_i - \partial_\nu \partial_\mu \Phi_i = 0$).

The system of hydrodynamic equations is now closed and contains, in particular, the entropy generation equation, which can be obtained by composing a vanishing combination, $u_\nu \partial_\mu T^{\mu\nu} = 0$, and following the same derivation as that discussed in G16. Ignoring for the moment the “superfluid” equations (28) [or (29)], one obtains

$$T \partial_\mu (S u^\mu) = u^\nu Y_{ik} w_{(k)}^\mu \{ \partial_\mu [w_{(i)\nu} + \mu_i u_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu] \}. \quad (30)$$

²We neglect, for clarity, possible sources in these equations due to beta-processes, thus assuming that the latter are effectively frozen. They can be easily accounted for if necessary.

The right-hand side of this equation vanishes in view of Eq. (29), so that the system entropy does not increase³ and is carried with the same velocity u^μ as the normal (nonsuperfluid) liquid component.

C. Charged mixtures

How should equations of the previous section be modified for charged mixtures? Concerning the continuity equations (17), the corresponding particle current densities are still given by Eqs. (21) and (22), and should be considered as definitions of the four-vectors u^μ and $w_{(i)}^\mu$ [u^μ is still being normalized by Eq. (24)]. The condition (25) also remains unchanged since it directly follows from the comoving frame definition (see Sec. II A of G16 for a thorough discussion of this issue). Next, the second law of thermodynamics (19) and the pressure definition (23) retain their form, because they are written for the fluid energy density and fluid pressure, and hence should not include field contributions.⁴ In contrast, the energy-momentum tensor $T^{\mu\nu}$ in Eq. (16) should be modified in order to account for the electromagnetic field contribution. It is now given by Eq. (18) with

$$T_{(\text{EM})}^{\mu\nu} = \frac{1}{4\pi} \left(F^\mu{}_\gamma F^{\nu\gamma} - \frac{1}{4} g^{\mu\nu} F_{\gamma\delta} F^{\gamma\delta} \right). \quad (31)$$

This standard [58] electromagnetic tensor is obtained under assumption $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$ [and hence $G^{\alpha\beta} = F^{\alpha\beta}$, see Eqs. (8) and (9)]. It does not include any ‘‘mixed’’ terms depending on both fluid and field degrees of freedom because of the same reason as that discussed in the footnote 4. A more general situation, in which such a decoupling is ambiguous (not well-defined), is considered in Secs. V and VI.

It remains to find out how the presence of charges affects the superfluid equations (28) and/or (29). For that, it is instructive to repeat the derivation of the entropy generation equation, now taking into account the electromagnetic contribution (31). Again, composing a vanishing combination $u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu T_{(\text{fluid})}^{\mu\nu} + u_\nu \partial_\mu T_{(\text{EM})}^{\mu\nu} = 0$ and noting that $\partial_\mu T_{(\text{EM})}^{\mu\nu} = -F^{\nu\mu} J_{(\text{free})\mu}$ on account of Maxwell’s equations (10) and (11) (see, e.g., § 8, Chapter 2 of Ref. [58]), one gets

$$T \partial_\mu (S u^\mu) = u^\nu Y_{ik} w_{(k)}^\mu \{ \partial_\mu [w_{(i)\nu} + \mu_i u_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu] \} - u^\nu F_{\nu\mu} J_{(\text{free})}^\mu, \quad (32)$$

³We remind the reader that in this work we are mainly interested in the nondissipative dynamics.

⁴We remind the reader that the situation considered in this section (superfluid-superconducting mixture in the absence of vortices and not in the intermediate state) allows us to separate fluid and field degrees of freedom.

where the four-current density of free charges is given by the formula [we use Eqs. (21) and (22)],

$$J_{(\text{free})}^\mu = e_j J_{(j)}^\mu = J_{(\text{norm})}^\mu + e_i Y_{ik} w_{(k)}^\mu, \quad (33)$$

in which e_j is the charge of particle j and

$$J_{(\text{norm})}^\mu = e_j n_j u^\mu = e_p (n_p - n_e) u^\mu \quad (34)$$

is the normal (nonsuperconducting) part of the four-current density. Correspondingly, noticing that $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ [see Eq. (8)] and $E^\mu = u_\nu F^{\mu\nu}$ [Eq. (12)], Eq. (32) can be rewritten as

$$T \partial_\mu (S u^\mu) = u^\nu Y_{ik} w_{(k)}^\mu \{ \partial_\mu [w_{(i)\nu} + \mu_i u_\nu + e_i A_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu + e_i A_\mu] \} + E_\mu J_{(\text{norm})}^\mu. \quad (35)$$

The last term in the rhs of this equation equals zero,

$$E_\mu J_{(\text{norm})}^\mu = 0, \quad (36)$$

in view of the definitions (12), (34), and the equality

$$u_\mu u_\nu F^{\mu\nu} = 0, \quad (37)$$

following from the antisymmetry property of the tensor $F^{\mu\nu}$. Equation (35) then becomes

$$T \partial_\mu (S u^\mu) = u^\nu Y_{ik} w_{(k)}^\mu \{ \partial_\mu [w_{(i)\nu} + \mu_i u_\nu + e_i A_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu + e_i A_\mu] \}. \quad (38)$$

The rhs of this equation must vanish identically because, by assumption, there should be no entropy generation in the system (we disregard all the dissipative corrections). Using this requirement, it is tempting to conclude that the new form of the superfluid equation in the presence of the electromagnetic field is

$$\partial_\mu [w_{(i)\nu} + \mu_i u_\nu + e_i A_\nu] - \partial_\nu [w_{(i)\mu} + \mu_i u_\mu + e_i A_\mu] = 0 \quad (39)$$

or, equivalently,

$$w_{(i)}^\mu = \partial^\mu \phi_i - \mu_i u^\mu - e_i A^\mu, \quad (40)$$

where, again, the scalar $\phi_i = \Phi_i/2$. This is indeed the correct equation that could be obtained immediately from the requirement of gauge invariance of the resulting superfluid hydrodynamics (see, e.g., Ref. [47]). As follows from the microscopic theory [59], the wave function phase Φ_i and the four-potential A^μ transform as

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi, \quad (41)$$

$$\Phi_i \rightarrow \Phi_i + 2e_i\chi \quad (42)$$

under gauge transformations (χ is an arbitrary scalar function). The four-vectors $w_{(i)}^\mu$ and hence Eqs. (39), (40), and other equations in this section are thus manifestly gauge invariant.⁵ The system of relativistic equations formulated here reduces to the vortex-free equations of Mendell [15] and Sedrakian *et al.* [17] in the nonrelativistic limit (see also Ref. [60]).

Remark 1. As noted above, a simple problem considered by us here allows to decouple the fluid and field degrees of freedom. In this approach $\varepsilon_{\text{fluid}}$ and P_{fluid} are, respectively, the *fluid* energy density and pressure, while field contributions are treated separately. Such a decoupling is hampered in more general situations (see Secs. V and VI). To facilitate comparison with the results of Secs. V and VI it is worth to reformulate the equations discussed here in terms of the total energy density ε ,

$$\varepsilon = \varepsilon_{\text{fluid}} + \varepsilon_{\text{EM}} \quad (43)$$

and the “pressure” P , defined as [cf. Eq. (23)]

$$P \equiv -\frac{\partial(\varepsilon V)}{\partial V} = -\varepsilon + \mu_e n_e + \mu_i n_i + TS, \quad (44)$$

where the partial derivative is taken at constant $n_j V$ ($j = n, p, e$), SV , $w_{(i)\mu} w_{(k)}^\mu$, B^μ , and D^μ ($= E^\mu$ in this section). In Eq. (43) ε_{EM} is the energy density of the electromagnetic field measured in the comoving frame,

$$\varepsilon_{\text{EM}} = \frac{\mathbf{E}^2}{8\pi} + \frac{\mathbf{B}^2}{8\pi} = \frac{E_\alpha E^\alpha}{8\pi} + \frac{B_\alpha B^\alpha}{8\pi}, \quad (45)$$

where the four-vectors E^α and B^α are given by Eqs. (12) and (14); in the comoving frame they equal, respectively, $(0, \mathbf{E})$ and $(0, \mathbf{B})$. Using Eq. (45), it follows from Eq. (44) that

$$P = P_{\text{fluid}} - \frac{1}{8\pi}(E_\alpha E^\alpha + B_\alpha B^\alpha). \quad (46)$$

Before reformulating the dynamic equations it is instructive to note that the energy-momentum tensor $T_{(\text{EM})}^{\mu\nu}$ of the electromagnetic field can generally be rewritten as

$$T_{(\text{EM})}^{\mu\nu} = -\frac{1}{8\pi}(E_\alpha E^\alpha + B_\alpha B^\alpha)g^{\mu\nu} + T_{(\text{E})}^{\mu\nu} + T_{(\text{M})}^{\mu\nu}, \quad (47)$$

where the “electric” part of the tensor equals

⁵The four-vectors $w_{(i)}^\mu$ ($i = n, p$) are observables (i.e., must be gauge invariant) since they define the particle current density $j_{(i)}^\mu$ in the comoving frame [see Eq. (21)].

$$T_{(\text{E})}^{\mu\nu} = -\frac{1}{4\pi}(E^\mu E^\nu - \perp^{\mu\nu} E_\alpha E^\alpha) \quad (48)$$

and the “magnetic” part is

$$T_{(\text{M})}^{\mu\nu} = \frac{1}{4\pi}(\perp_{\delta\alpha} F^{\mu\delta} F^{\nu\alpha} - u^\mu u^\nu u^\gamma u_\beta F^{\alpha\beta} F_{\alpha\gamma}). \quad (49)$$

Here $\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ is the projection operator. Using Eqs. (43)–(49) the second law of thermodynamics takes the form [cf. Eq. (19)]

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}) + \frac{1}{4\pi} E_\alpha dE^\alpha + \frac{1}{4\pi} B_\alpha dB^\alpha, \quad (50)$$

while the tensor $T^{\mu\nu}$ becomes [cf. Eq. (18)]

$$T^{\mu\nu} = (P + \varepsilon)u^\mu u^\nu + P g^{\mu\nu} + Y_{ik}(w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu) + T_{(\text{E})}^{\mu\nu} + T_{(\text{M})}^{\mu\nu}. \quad (51)$$

Because

$$u_\mu u_\nu T_{(\text{E})}^{\mu\nu} = 0, \quad (52)$$

$$u_\mu u_\nu T_{(\text{M})}^{\mu\nu} = 0, \quad (53)$$

it satisfies the condition

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon. \quad (54)$$

All other hydrodynamic equations remain unchanged.

IV. SETTING UP THE PROBLEM

Simple examples considered in the previous section suggest a possible general approach to the problem of formulation of the macroscopic (smooth-averaged) dynamic equations in various interesting situations (e.g., in the system with vortices or in the system with small-scale domain structure of the magnetic field). The approach is based on using the entropy generation equation to constrain the dynamics of superfluid-superconducting mixtures; it has been applied recently in G16 (see also Ref. [29]) and we refer the interested reader to those references for more details. All the quantities in this and subsequent sections are assumed to be averaged over the volume containing large amount of inhomogeneities (vortices or magnetic domains).

Assume that the second law of thermodynamics takes the form

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}) + d\varepsilon_{\text{add}}, \quad (55)$$

where ε is the *total* energy density of the system. All the terms in the rhs of this equation except for the last one are the standard terms of superfluid hydrodynamics [see Eq. (19) and Refs. [40,47,48,50]]; an additional term $d\varepsilon_{\text{add}}$ contains vortex or electromagnetic contribution to $d\varepsilon$, or both.

Accounting for this term in Eq. (55) should not affect most of the dynamic equations due to the very same reasons as those discussed in the beginning of Sec. III C (see also Sec. III B in G16, where a similar problem is discussed in detail). In particular, the expressions (21) and (22) for the free four-current densities $j_{(n)}^\mu$, $j_{(p)}^\mu$, and $j_{(e)}^\mu$ [which satisfy the continuity equation (17)] should be considered as the *definitions* of the four-vectors $w_{(n)}^\mu$, $w_{(p)}^\mu$, and the four-velocity u^μ , normalized by the condition (24). Thus, they remain unchanged. Next, the requirement that all the thermodynamic quantities are measured in the (comoving) frame, in which $u^\mu = (1, 0, 0, 0)$,⁶ unambiguously leads to the same constraints (25). Finally, the free-charge four-current density $J_{(\text{free})}^\mu$ and the pressure P will still be defined by Eqs. (33) and (44), respectively; Maxwell's equations (1)–(4) or (10)–(11) will also, of course, retain their form.

The only equations that should be modified are the expression for the total energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = (P + \varepsilon)u^\mu u^\nu + P g^{\mu\nu} + Y_{ik}(w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu) + \Delta T^{\mu\nu}. \quad (56)$$

[which still satisfies Eq. (16)], and the “superfluid” equations for neutrons and protons [Eqs. (39) or (40) in the simple example of Sec. III C]. The correction $\Delta T^{\mu\nu}$ in Eq. (56) must be *symmetric*; it includes vortex and/or electromagnetic contributions to $T^{\mu\nu}$ and is absent in the standard superfluid hydrodynamics [see Eq. (20)]. Because in the comoving frame the component T^{00} of the tensor $T^{\mu\nu}$ equals, by definition, ε ,⁷ one should have there $\Delta T^{00} = 0$, or, in an arbitrary frame,

$$u_\mu u_\nu \Delta T^{\mu\nu} = 0. \quad (57)$$

To determine the correction $\Delta T^{\mu\nu}$ and the form of superfluid equations, we, as was already mentioned, utilize the entropy generation equation. It can be derived using the equations discussed above in this section. The result is [cf. Eq. (38) and also equation (65) in G16]

⁶Mathematically, this requirement is expressed by the condition (27).

⁷In an arbitrary frame this requirement translates into $u_\mu u_\nu T^{\mu\nu} = \varepsilon$, see Eq. (54).

$$T\partial_\mu(Su^\mu) = u_\nu Y_{ik} w_{(k)\mu} \tilde{\mathcal{V}}_{(i)}^{\mu\nu} - u^\mu \partial_\mu \varepsilon_{\text{add}} + u_\nu \partial_\mu \Delta T^{\mu\nu}, \quad (58)$$

where

$$\tilde{\mathcal{V}}_{(i)}^{\mu\nu} \equiv \partial^\mu [w_{(i)}^\nu + \mu_i u^\nu] - \partial^\nu [w_{(i)}^\mu + \mu_i u^\mu]. \quad (59)$$

As one sees, Eq. (58) depends on ε_{add} , which is assumed to be specified, and on $\Delta T^{\mu\nu}$, which is unknown. Because entropy is conserved in the absence of dissipation, the rhs of this equation should vanish identically. As shown in Secs. V and VI, this requirement is sufficient to fully reconstruct dynamics of superfluid-superconducting *npe* mixture.

V. RELATIVISTIC DYNAMIC EQUATIONS FOR THE *NPE* MIXTURE: TYPE-I PROTON SUPERCONDUCTIVITY

In this section, we consider a nonrotating superfluid-superconducting *npe* mixture in the absence of Feynman-Onsager and Abrikosov (single flux quantum) vortices. However, in contrast to Sec. III C we formally assume that the magnetic field \mathbf{H} does not necessarily coincide with the magnetic induction \mathbf{B} ; i.e., there are some bound currents in the system. One can imagine that these currents can be generated either due to (very weak, in reality) magnetic response of particles in the mixture (e.g., electrons) to an applied external magnetic field (case 1), or due to appearance of various inhomogeneous structures of the (microscopic) magnetic field in the mixture similar to those appearing in the intermediate state of ordinary type-I superconductors (see, e.g., Ref. [61] and Sec. V B below; case 2). The dynamic equations in this latter case are a bit more complicated since the proton phase winding around such structures can be nonzero. Thus, for pedagogical reasons we start with the simplest (but unrealistic) situation of a homogeneous *npe* mixture with well-behaved phases Φ_i and $\mathbf{B} \neq \mathbf{H}$ (case 1). In what follows we, for generality, assume also that the electric displacement \mathbf{D} is not equal to the electric field \mathbf{E} (although we set $\mathbf{D} = \mathbf{E}$ in the final equations, see Sec. VII A).

A. Case 1: Homogeneous *npe* mixture with $\mathbf{B} \neq \mathbf{H}$

The starting point of our consideration is the expression for the electromagnetic contribution $d\varepsilon_{\text{add}}$ to the second law of thermodynamics,

$$d\varepsilon_{\text{add}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu. \quad (60)$$

This formula is specialized to the comoving frame, which is, generally, noninertial, because u^μ changes in time and space. In the absence of bound charges and currents one has $D^\mu = E^\mu$ and $H^\mu = B^\mu$, so that Eq. (60) reduces to the last two electromagnetic terms in the rhs of Eq. (50). In the

special case when the comoving frame is inertial, Eq. (60) is transformed to the standard form (7) [see the definitions (12)–(15)]. Using Eqs. (12)–(15), the last term in Eq. (60) can be rewritten as

$$\begin{aligned} \frac{1}{4\pi} H_\mu dB^\mu &= \frac{1}{4\pi} H_\mu \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu dF_{\alpha\beta} + \frac{1}{4\pi} H_\mu \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} du_\nu \\ &= \frac{1}{8\pi} \perp G^{\alpha\beta} dF_{\alpha\beta} \\ &\quad + \frac{1}{8\pi} (H_\mu \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} + \underline{u^\nu G_{\alpha\beta} F^{\alpha\beta}}) du_\nu \\ &= \frac{1}{8\pi} (\perp G^{\alpha\beta} dF_{\alpha\beta} + 2F_{\alpha\beta} G^{\alpha\gamma} u^\beta du_\gamma), \end{aligned} \quad (61)$$

where the added underlined term vanishes on account of normalization condition (24) (hence $u^\nu du_\nu = 0$) and we used the notation from Appendix A. Similarly,

$$\begin{aligned} \frac{1}{4\pi} E_\mu dD^\mu &= \frac{1}{4\pi} [d(D_\mu E^\mu) - D_\mu dE^\mu] \\ &= \frac{1}{4\pi} [d(D_\mu E^\mu) - D_\mu u_\nu dF^{\mu\nu} - D_\mu F^{\mu\nu} du_\nu] \\ &= \frac{1}{4\pi} \left[d(D_\mu E^\mu) - \frac{1}{2} (D_\mu u_\nu - D_\nu u_\mu) dF^{\mu\nu} - D_\mu F^{\mu\nu} du_\nu \right] \\ &= \frac{1}{4\pi} \left[d(D_\alpha E^\alpha) + \frac{1}{2} (\perp G^{\alpha\beta} dF_{\alpha\beta} - 2D_\alpha F^{\alpha\gamma} du_\gamma) \right] \\ &= \frac{1}{4\pi} \left[d(D_\alpha E^\alpha) + \frac{1}{2} (\perp G^{\alpha\beta} dF_{\alpha\beta} - 2G_{\alpha\beta} F^{\alpha\gamma} u^\beta du_\gamma) \right]. \end{aligned} \quad (62)$$

Equations (61) and (62) can be further transformed as described in Appendix C. For that, we identify

$$\begin{aligned} O^{\alpha\beta} &= \frac{1}{4\pi} \perp G^{\alpha\beta}, \\ \mathcal{F}^{\alpha\beta} &= F^{\alpha\beta}, \\ \mathcal{B}^{\alpha\beta} &= F^{\alpha\beta}, \\ \mathcal{A}^{\alpha\beta} &= \frac{1}{4\pi} G^{\alpha\beta} \end{aligned}$$

in case of Eq. (61) and

$$\begin{aligned} O^{\alpha\beta} &= \frac{1}{4\pi} \perp G^{\alpha\beta}, \\ \mathcal{F}^{\alpha\beta} &= F^{\alpha\beta}, \\ \mathcal{B}^{\alpha\beta} &= -\frac{1}{4\pi} G^{\alpha\beta}, \\ \mathcal{A}^{\alpha\beta} &= F^{\alpha\beta} \end{aligned}$$

in case of Eq. (62). As a result, the second term in the rhs of Eq. (58) can be presented as [see Eq. (C7)]

$$\begin{aligned} -u^\mu \partial_\mu \varepsilon_{\text{add}} &= u^\nu F_{\mu\nu} \partial_\alpha \left(\frac{1}{4\pi} \perp G^{\mu\alpha} + \frac{1}{4\pi} \perp G^{\mu\alpha} \right) \\ &\quad - \partial_\mu [u^\nu (T_{(E)\nu}^\mu + T_{(M)\nu}^\mu)] \\ &\quad + \partial_\mu u^\nu (T_{(E)\nu}^\mu + T_{(M)\nu}^\mu), \end{aligned} \quad (63)$$

where the “electric” and “magnetic” tensors are given, respectively, by

$$T_{(E)\nu}^\mu = \frac{1}{4\pi} (\perp G^{\mu\alpha} F_{\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} F^{\alpha\beta} G_{\alpha\gamma} + g^\mu{}_\nu D_\alpha E^\alpha), \quad (64)$$

$$T_{(M)\nu}^\mu = \frac{1}{4\pi} (\perp G^{\mu\alpha} F_{\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} G^{\alpha\beta} F_{\alpha\gamma}). \quad (65)$$

It can be verified that if $G^{\mu\nu} = F^{\mu\nu}$ then these tensors reduce to the tensors $T_{(E)}^{\mu\nu}$ and $T_{(M)}^{\mu\nu}$ from Sec. III C [see Eqs. (48) and (49) there], $T_{(E)}^{\mu\nu} = T_{(E)}^{\mu\nu}$ and $T_{(M)}^{\mu\nu} = T_{(M)}^{\mu\nu}$. For actual calculations, it is convenient to represent the tensors (64) and (65) in the form

$$T_{(E)}^{\mu\nu} = \frac{1}{4\pi} (\perp^{\mu\nu} D^\alpha E_\alpha - D^\mu E^\nu), \quad (66)$$

$$T_{(M)}^{\mu\nu} = \frac{1}{4\pi} (\perp G^{\mu\alpha} \perp F_{\nu\alpha} + u^\nu \perp G^{\mu\alpha} E_\alpha + u^\mu \perp G^{\nu\alpha} E_\alpha). \quad (67)$$

The first term in the rhs of Eq. (63) can be further simplified by making use of Eqs. (A9), (11), (33), (34), and (36),

$$\begin{aligned} u^\nu F_{\mu\nu} \partial_\alpha \left(\frac{1}{4\pi} \perp G^{\mu\alpha} + \frac{1}{4\pi} \perp G^{\mu\alpha} \right) &= u^\nu F_{\mu\nu} \partial_\alpha \left(\frac{1}{4\pi} G^{\mu\alpha} \right) \\ &= u^\nu F_{\mu\nu} J_{(\text{free})}^\mu \\ &= u^\nu F_{\mu\nu} e_i Y_{ik} W_{(k)}^\mu. \end{aligned} \quad (68)$$

Substituting now Eq. (63) into Eq. (58), one gets

$$\begin{aligned} T \partial_\mu (S u^\mu) &= u_\nu Y_{ik} W_{(k)\mu} (\tilde{\mathcal{V}}_{(i)}^{\mu\nu} + e_i F^{\mu\nu}) \\ &\quad - \partial_\mu [u^\nu (T_{(E)\nu}^\mu + T_{(M)\nu}^\mu - \Delta T_{\nu}^\mu)] \\ &\quad + \partial_\mu u^\nu (T_{(E)\nu}^\mu + T_{(M)\nu}^\mu - \Delta T_{\nu}^\mu), \end{aligned} \quad (69)$$

from which one can conclude that⁸

$$\tilde{\mathcal{V}}_{(i)}^{\mu\nu} + e_i F^{\mu\nu} = 0 \quad (70)$$

and, correspondingly, in order to vanish identically the rhs of Eq. (69),

⁸Equation (70) coincides with the superfluid Eq. (39) from Sec. III C, see the definition (59).

$$\Delta T^{\mu\nu} = \mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu}. \quad (71)$$

Note that $\Delta T^{\mu\nu}$ automatically satisfies the condition (57); the fact that the tensor $\Delta T^{\mu\nu}$ is symmetric will be proven in Sec. VII A. The physical meaning of Eq. (70) is transparent. Using the definition (59) it can be rewritten in the form of the superfluid Eq. (39) from Sec. III C, or as a gauge-invariant expression (40) for the four-vector $w_{(i)}^\mu$, $w_{(i)}^\mu = \partial^\mu \phi_i - \mu_i u^\mu - e_i A^\mu$, where the scalar $\phi_i = \Phi_i/2$ and Φ_i is the smooth-averaged wave function phase of the Cooper-pair condensate.⁹ Equation (70) thus states that

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \Phi_i = 0, \quad (72)$$

which is quite natural since we assume in this section that there are no vortices and nonsuperconducting domains in the system (the phases Φ_i are well defined everywhere in the mixture).

B. Case 2: The *npe* mixture in the intermediate state

Now let us discuss how the equations of the previous section should be modified in order to apply them to *npe* mixture in the intermediate state. But first let us clarify what we mean by the term “intermediate.”

According to some estimates (e.g., GAS11), protons in the inner cores of neutron stars can form a type-I superconductor. Upon neutron star cooling the superconducting region expands, but it is generally believed that this process is not accompanied by the magnetic flux expulsion (the Meissner effect) because of the huge electric conductivity of the outer core and crust (see Refs. [62,63] and a comment 8 in Ref. [22]). As a result, it becomes energetically favorable for *npe* mixture to find itself in the “intermediate” state, consisting of alternating domains of superconducting (field-free) regions and nonsuperconducting regions hosting the magnetic field. The topology of these domains can be very diverse and depends, in particular, on their nucleation history [61,64–66]. This complicates substantially the problem of calculation of the total energy density ε for such matter. However, we neglect below the relatively small surface and boundary contributions to ε [45,67]. In this approximation the actual domain structure is not important for the energy calculation. We further assume that the produced magnetic structures have a closed topology; i.e., normal domains are completely surrounded by the superconducting phase [61]. This assumption seems reasonable since the magnetic field of a typical neutron star, $B \sim 10^{12}$ G, is much smaller

than the critical thermodynamic field, $H_c \sim 10^{14} - 10^{15}$ G [66], while it is well known [45,61,65–67] that it is advantageous for a relatively weak field to penetrate the superconductor in the form of flux tubes, each containing many flux quanta. For definiteness, this very form of normal domains (flux tubes) will be assumed by us in what follows. Note, however, that the actual form of normal domains is not really important for the subsequent consideration (what is important is the closed topology assumption).

The distance between the neighboring flux tubes can be estimated as [66,67] $b \sim \sqrt{R\delta}$, where R is the typical size of the intermediate-state region and δ is the typical width of the normal-superconducting boundary [67]. Taking $\delta \sim \xi_p \sim 10^{-11}$ cm (ξ_p is the proton coherence length) and $R \sim 5$ km, one obtains $b \sim 2 \times 10^{-3}$ cm. Then the flux tube radius is $a \approx b(B/H_c)^{1/2} \sim 6 \times 10^{-5}$ cm and the number of flux quanta in a single flux tube $N_\phi \approx \pi a^2 H_c / \hat{\phi}_{p0} \approx 6 \times 10^{13}$, where $\hat{\phi}_{p0}$ is given by Eq. (G13), and we choose $B = 10^{12}$ G and $H_c = 10^{15}$ G.

From these estimates, one can conclude that the flux tubes are rather large objects that should interact efficiently with the surrounding normal matter (electrons and nucleon Bogoliubov excitations), and hence should move (at least, in the nondissipative limit) with the normal liquid component. In the terminology of the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) hydrodynamics, one can say that the system is in the “strong-drag” regime (see G16). Using the strong-drag assumption, one can try to derive the dynamic equations for the *npe* mixture in the intermediate state. First, note that all consideration of Sec. VA up to and including Eq. (69) is applicable to the intermediate state as well since it only uses, as a starting point, the expression (60) for the energy density, which remains correct. From Eq. (69), one then deduces the same Eq. (70) for neutrons (by assumption, there are no Feynman-Onsager vortices in the system) and Eq. (71) for the electromagnetic correction to the energy-momentum tensor. However, for *protons*, Eq. (70) cannot be applied and must be modified. The reason is, as suggested by the London argument (e.g., Ref. [59]), there is a nonzero proton phase winding $\oint \partial_\mu \Phi_p dx^\mu$ around each flux tube; i.e., the phase Φ_p , averaged over the volume containing many flux tubes, does not satisfy the “potentiality condition” (72). This situation is reminiscent of that observed in the HVBK-hydrodynamics (see, e.g., Ref. [25] and G16). In particular, in G16 it is shown that the strong-drag regime we are interested in, is realized if one replaces Eq. (70) for protons with the less restrictive condition,¹⁰

⁹The fact that $w_{(i)}^\mu$ (and hence all other dynamic equations) appears to be gauge-invariant, is not trivial and is directly related to the adopted expression (60) for $d\varepsilon_{\text{add}}$, in particular, to the assumption that H^μ in this expression is indeed the magnetic field four-vector given by Eq. (15).

¹⁰Equation (73) is a special case of the more general Eq. (93) from the next section, which, although describes a different system (*npe* mixture with type-II proton superconductivity), has some mathematical resemblance to what is studied here.

$$u_\nu(\tilde{\mathcal{V}}_{(p)}^{\mu\nu} + e_p F^{\mu\nu}) = 0. \quad (73)$$

It is easily verified that, with this equation, the rhs of Eq. (69) is still zero, as it should be. Summarizing, we find that, to model the npe mixture in the intermediate state, one should use superfluid Eqs. (70) for neutrons and (73) for protons; the correction $\Delta T^{\mu\nu}$ to the energy-momentum tensor $T^{\mu\nu}$ is given by Eq. (71). The last thing to do in order to close the system of dynamic equations discussed here is to specify the relation between the tensors $G^{\mu\nu}$ and $F^{\mu\nu}$, and to prove that the resulting tensor $\Delta T^{\mu\nu}$ is indeed symmetric. This is done in Sec. VII A. A complete system of equations is summarized in Appendix H.

Remark 1. It may be noted that exactly the same derivation of the tensor (71) can be made also for *normal* (nonsuperfluid) matter if we put $Y_{ik} = 0$ in all relevant equations. This indicates that the tensor $\Delta T^{\mu\nu}$ must be well known in the electrodynamics of continuous media. As shown in Appendix D, this is indeed the case and it is directly related to the so-called Abraham electromagnetic tensor in the medium (see, e.g., Refs. [46,68,69]).

VI. RELATIVISTIC DYNAMIC EQUATIONS FOR NPE MIXTURE WITH NEUTRON AND PROTON VORTICES: TYPE-II PROTON SUPERCONDUCTIVITY

In this section, we consider a region of densities where protons form a type-II superconductor and allow for the possible presence of neutron and proton vortices in the system. Since our consideration is very similar to that in G16 we will be brief here and refer the interested reader to this reference for more details.¹¹ In the system with vortices, the condition $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Phi_i = 0$ is not satisfied at the vortex lines. Hence, as in Sec. V B, the macroscopic (smooth-averaged) superfluid Eq. (39) [or (70)] should be replaced by a weaker constraint [see Eq. (93) below]. In what follows, it will be convenient to use the *vorticity* tensor $\mathcal{V}_{(i)}^{\mu\nu}$,

$$\begin{aligned} \mathcal{V}_{(i)}^{\mu\nu} &\equiv \tilde{\mathcal{V}}_{(i)}^{\mu\nu} + e_i F^{\mu\nu} \\ &= \partial^\mu [w_{(i)}^\nu + \mu_i u^\nu + e_i A^\nu] - \partial^\nu [w_{(i)}^\mu + \mu_i u^\mu + e_i A^\mu], \end{aligned} \quad (74)$$

with the obvious property [cf. Eq. (10)]

$$\partial_\mu \star \mathcal{V}_{(i)}^{\mu\nu} = 0. \quad (75)$$

The tensor $\mathcal{V}_{(i)}^{\mu\nu}$ is equivalent to $m_i \text{curl} \mathbf{V}_{si} + e_i \mathbf{B}$ of the nonrelativistic HVBK-hydrodynamics (\mathbf{V}_{si} is the superfluid

velocity).¹² The geometrical meaning of this tensor is quite transparent. Assume we have a surface spanned by some closed contour. Then $\mathcal{V}_{(i)}^{\mu\nu}$ is related to the number N_i of neutron ($i = n$) or proton ($i = p$) vortices piercing the surface by the formula (see G16 for more details)¹³

$$\frac{1}{2} \int df^{\mu\nu} \mathcal{V}_{(i)\mu\nu} = \pi N_i, \quad (76)$$

where an integral is taken over the surface area. In the absence of vortices, one has $\mathcal{V}_{(i)}^{\mu\nu} = 0$ [see Eqs. (39) and (74)]. With the tensor $\mathcal{V}_{(i)}^{\mu\nu}$, one can construct, using Eqs. (A3) and (A4), the “electric” and “magnetic” four-vectors $\mathcal{V}_{(Ei)}^\mu$ and $\mathcal{V}_{(Mi)}^\mu$, respectively,

$$\mathcal{V}_{(Ei)}^\mu \equiv u_\nu \mathcal{V}_{(i)}^{\mu\nu}, \quad (77)$$

$$\mathcal{V}_{(Mi)}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{V}_{(i)\alpha\beta}. \quad (78)$$

In addition to modifying the superfluid equation, vortices affect also the second law of thermodynamics (55), because a certain amount of energy is associated with each vortex. This energy should be accounted for in Eq. (55) together with the electromagnetic contribution. The expression for $d\epsilon_{\text{add}}$, that includes the vortex contribution, reads

$$\begin{aligned} d\epsilon_{\text{add}} &= \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}_{(Ei)}^\mu d\mathcal{W}_{(Ei)\mu} \\ &\quad + \mathcal{W}_{(Mi)\mu} d\mathcal{V}_{(Mi)}^\mu, \end{aligned} \quad (79)$$

where the four-vectors $\mathcal{W}_{(Ei)}^\mu$ and $\mathcal{W}_{(Mi)}^\mu$ are analogous to D^μ and H^μ , respectively. As shown in Sec. VII B (see below), they can generally be presented as

$$\mathcal{W}_{(Ei)}^\mu \equiv u_\nu \mathcal{W}_{(i)}^{\mu\nu}, \quad (80)$$

$$\mathcal{W}_{(Mi)}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{W}_{(i)\alpha\beta}. \quad (81)$$

Here $\mathcal{W}_{(i)}^{\mu\nu}$ is some auxiliary antisymmetric tensor, which plays the same role with respect to $\mathcal{V}_{(i)}^{\mu\nu}$ as the tensor $G^{\mu\nu}$ with respect to $F^{\mu\nu}$. It is easy to see that the third and fourth terms in the rhs of Eq. (79) are written in full analogy with, respectively, the first and second electromagnetic terms. This coincidence is not accidental. As detailed in Appendix G 2 the fourth term here is basically responsible for the vortex energy (including its magnetic energy), while

¹²To be more precise, the vector $\mathcal{V}_{(Mi)}^\mu$ [see Eq. (78) below], constructed with the help of this tensor, is equivalent to $m_i \text{curl} \mathbf{V}_{si} + e_i \mathbf{B}$.

¹³Note that the factor 1/2 was inadvertently omitted in the corresponding equation (42) in that reference.

¹¹Note that G16 uses somewhat different notation. The correspondence table between our notation and that of G16 is provided in Appendix B.

the third term comes into play if one takes into account the electric field generated by moving vortices. Note that in G16 only the fourth term has been allowed for, since that reference analyzed uncharged superfluids. In addition, in that reference it was from the very beginning assumed that $\mathcal{W}_{(Mi)}^\mu$ is directly proportional to $\mathcal{V}_{(Mi)}^\mu$, $\mathcal{W}_{(Mi)}^\mu \propto \mathcal{V}_{(Mi)}^\mu$, which is the only viable option in the absence of other magnetic vectors in the problem [cf. Eq. (115) in Sec. VII B].

Our next step will be to transform the energy $d\varepsilon_{\text{add}}$ in a way similar to how it was done in Sec. V. The first two terms in the rhs of Eq. (79) are transformed exactly as in Sec. V, the result is given by Eq. (63). Let us analyze the fourth term. It reads [cf. Eq. (61)]

$$\begin{aligned} \mathcal{W}_{(Mi)\mu} d\mathcal{V}_{(Mi)}^\mu &= \mathcal{W}_{(Mi)\mu} \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} u_\nu d\mathcal{V}_{(i)\alpha\beta} \\ &+ \mathcal{W}_{(Mi)\mu} \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \mathcal{V}_{(i)\alpha\beta} du_\nu \\ &= \frac{1}{2} \perp \mathcal{W}_{(i)}^{\alpha\beta} d\mathcal{V}_{(i)\alpha\beta} + \frac{1}{2} (\mathcal{W}_{(Mi)\mu} \varepsilon^{\mu\nu\alpha\beta} \mathcal{V}_{(i)\alpha\beta} \\ &+ \underline{u^\nu \mathcal{W}_{(i)\alpha\beta} \mathcal{V}_{(i)}^{\alpha\beta}}) du_\nu \\ &= \frac{1}{2} (\perp \mathcal{W}_{(i)}^{\alpha\beta} d\mathcal{V}_{(i)\alpha\beta} + 2\mathcal{V}_{(i)\alpha\beta} \mathcal{W}_{(i)}^{\alpha\gamma} u^\beta du_\gamma), \end{aligned} \quad (82)$$

where the underlined term vanishes because $u^\nu du_\nu = 0$; $\perp \mathcal{W}_{(i)}^{\alpha\beta}$ is defined by Eq. (A5). To further transform this

equation we make use of Appendix C. Comparing Eq. (82) with (C1) allows us to identify

$$\begin{aligned} \mathcal{O}^{\alpha\beta} &= \perp \mathcal{W}_{(i)}^{\alpha\beta}, \\ \mathcal{F}^{\alpha\beta} &= \mathcal{V}_{(i)}^{\alpha\beta}, \\ \mathcal{B}^{\alpha\beta} &= \mathcal{V}_{(i)}^{\alpha\beta}, \\ \mathcal{A}^{\alpha\beta} &= \mathcal{W}_{(i)}^{\alpha\beta}, \end{aligned}$$

hence,

$$\begin{aligned} -u^\mu \mathcal{W}_{(Mi)\alpha} \partial_\mu \mathcal{V}_{(Mi)}^\alpha &= u^\nu \mathcal{V}_{(i)\mu\nu} \partial_\alpha \perp \mathcal{W}_{(i)}^{\mu\alpha} \\ &- \partial_\mu [u^\nu (\perp \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} \mathcal{W}_{(i)}^{\alpha\beta} \mathcal{V}_{(i)\alpha\gamma})] \\ &+ \partial_\mu u^\nu (\perp \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} \mathcal{W}_{(i)}^{\alpha\beta} \mathcal{V}_{(i)\alpha\gamma}). \end{aligned} \quad (83)$$

Looking at Eqs. (82) and (83), it may be noted that the transformation of the fourth and second terms in Eq. (79) are identical provided that one replaces $\mathcal{W}_{(Mi)}^\mu \rightarrow H^\mu/(4\pi)$ and $\mathcal{V}_{(Mi)}^\mu \rightarrow B^\mu$ [compare Eqs. (61) and (82)]. Similarly, the transformation of the third and first terms in Eq. (79) can be obtained from one another by replacing $\mathcal{W}_{(Ei)}^\mu \rightarrow D^\mu/(4\pi)$ and $\mathcal{V}_{(Ei)}^\mu \rightarrow E^\mu$. With these replacements, one can use Eq. (62) to transform the third term in Eq. (79). The result is

$$\begin{aligned} -u^\mu \mathcal{V}_{(Ei)\alpha} \partial_\mu \mathcal{W}_{(Ei)}^\alpha &= u^\nu \mathcal{V}_{(i)\mu\nu} \partial_\alpha \parallel \mathcal{W}_{(i)}^{\mu\alpha} - \partial_\mu [u^\nu (\parallel \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} \mathcal{V}_{(i)}^{\alpha\beta} \mathcal{W}_{(i)\alpha\gamma} + g^\mu{}_\nu \mathcal{W}_{(Ei)\alpha} \mathcal{V}_{(Ei)}^\alpha)] \\ &+ \partial_\mu u^\nu (\parallel \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} \mathcal{V}_{(i)}^{\alpha\beta} \mathcal{W}_{(i)\alpha\gamma} + g^\mu{}_\nu \mathcal{W}_{(Ei)\alpha} \mathcal{V}_{(Ei)}^\alpha). \end{aligned} \quad (84)$$

Collecting together the electromagnetic terms (63) and the vortex terms (83) and (84), one obtains

$$\begin{aligned} -u^\mu \partial_\mu \varepsilon_{\text{add}} &= u^\nu F_{\mu\nu} \partial_\alpha \left(\frac{1}{4\pi} \parallel G^{\mu\alpha} + \frac{1}{4\pi} \perp G^{\mu\alpha} \right) \\ &+ u^\nu \mathcal{V}_{(i)\mu\nu} \partial_\alpha (\parallel \mathcal{W}_{(i)}^{\mu\alpha} + \perp \mathcal{W}_{(i)}^{\mu\alpha}) \\ &- \partial_\mu [u^\nu (\mathcal{T}_{(E)\nu}^\mu + \mathcal{T}_{(M)\nu}^\mu + \mathcal{T}_{(VE)\nu}^\mu + \mathcal{T}_{(VM)\nu}^\mu)] \\ &+ \partial_\mu u^\nu (\mathcal{T}_{(E)\nu}^\mu + \mathcal{T}_{(M)\nu}^\mu + \mathcal{T}_{(VE)\nu}^\mu + \mathcal{T}_{(VM)\nu}^\mu), \end{aligned} \quad (85)$$

where the tensors $\mathcal{T}_{(E)\nu}^\mu$ and $\mathcal{T}_{(M)\nu}^\mu$ are given by Eqs. (64) and (65), and

$$\mathcal{T}_{(VE)\nu}^\mu = \parallel \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} \mathcal{V}_{(i)}^{\alpha\beta} \mathcal{W}_{(i)\alpha\gamma} + g^\mu{}_\nu \mathcal{W}_{(Ei)\alpha} \mathcal{V}_{(Ei)}^\alpha, \quad (86)$$

$$\mathcal{T}_{(VM)\nu}^\mu = \perp \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} \mathcal{W}_{(i)}^{\alpha\beta} \mathcal{V}_{(i)\alpha\gamma} \quad (87)$$

are, respectively, the ‘‘electric’’ and ‘‘magnetic’’ vortex contributions to the energy-momentum tensor (note a summation over $i = n, p$ here). Similarly to tensors (64) and (65), these tensors can be represented as [cf. Eq. (66) and (67)]

$$\mathcal{T}_{(VE)}^{\mu\nu} = \perp^{\mu\nu} \mathcal{W}_{(Ei)}^\alpha \mathcal{V}_{(Ei)\alpha} - \mathcal{W}_{(Ei)}^\mu \mathcal{V}_{(Ei)}^\nu, \quad (88)$$

$$\mathcal{T}_{(VM)}^{\mu\nu} = \perp \mathcal{W}_{(i)}^{\mu\alpha} \perp \mathcal{V}_{(i)\alpha}^\nu + u^\nu \perp \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(Ei)\alpha} + u^\mu \perp \mathcal{W}_{(i)}^{\nu\alpha} \mathcal{V}_{(Ei)\alpha}. \quad (89)$$

Using Eq. (85), as well as Eq. (68), the definition (74), and the equality $\mathcal{W}_{(i)}^{\mu\alpha} = \parallel \mathcal{W}_{(i)}^{\mu\alpha} + \perp \mathcal{W}_{(i)}^{\mu\alpha}$ [see Eq. (A9)], the entropy generation equation (58) becomes

$$\begin{aligned}
T\partial_\mu(Su^\mu) &= u^\nu \mathcal{V}_{(i)\mu\nu} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}] \\
&\quad - \partial_\mu [u^\nu (\mathcal{T}_{(E)\nu}^\mu + \mathcal{T}_{(M)\nu}^\mu + \mathcal{T}_{(VE)\nu}^\mu \\
&\quad + \mathcal{T}_{(VM)\nu}^\mu - \Delta T_{\nu}^\mu)] + \partial_\mu u^\nu (\mathcal{T}_{(E)\nu}^\mu + \mathcal{T}_{(M)\nu}^\mu \\
&\quad + \mathcal{T}_{(VE)\nu}^\mu + \mathcal{T}_{(VM)\nu}^\mu - \Delta T_{\nu}^\mu). \quad (90)
\end{aligned}$$

The rhs of this equation has the same structure as Eq. (77) in G16. Correspondingly, its analysis and the resulting equations are very similar. Using the argumentation of that reference, one finds that, in order for the entropy to be conserved, it is necessary to have

$$u^\nu \mathcal{V}_{(i)\mu\nu} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}] = 0, \quad (91)$$

$$\Delta T^{\mu\nu} = \mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu} + \mathcal{T}_{(VE)}^{\mu\nu} + \mathcal{T}_{(VM)}^{\mu\nu}. \quad (92)$$

Note that $\Delta T^{\mu\nu}$ satisfies the required constraint (57) and is symmetric (see Sec. VII B). As demonstrated in G16, the condition (91) is equivalent to the following equation, which replaces the superfluid Eq. (39) [or (70)] of the vortex-free system,

$$u^\nu \mathcal{V}_{(i)\mu\nu} = \mu_i n_i f_{(i)\mu}, \quad (93)$$

where

$$f_{(i)}^\mu = \alpha_i \perp^{\mu\nu} \mathcal{V}_{(i)\nu\lambda} W_{(i)\delta} \perp^{\lambda\delta}, \quad (94)$$

$$W_{(i)}^\mu \equiv \frac{1}{n_i} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}], \quad (95)$$

and α_i is a nondissipative mutual friction coefficient [note that there are no summation over i in Eqs. (93)–(95)]. The lhs of Eq. (93) is simply the four-vector $\mathcal{V}_{(Ei)}^\mu$, so that it can be rewritten as (now in the dimensional form)

$$\mathcal{V}_{(Ei)}^\mu = \frac{\mu_i n_i}{c^3} f_{(i)}^\mu. \quad (96)$$

Equations (92) and (93) [or (96)] are the main results of this section. They show how the energy-momentum tensor and superfluid equation should be modified in the presence of vortices. These equations depend on the tensors $G^{\mu\nu}$ and $\mathcal{W}_{(i)}^{\mu\nu}$, which will be found in Sec. VII B. The symmetry of the tensor $\Delta T^{\mu\nu}$ will be demonstrated in the same section. The whole system of dynamic equations in the presence of vortices is summarized in Appendix H.

Remark 1. In this work we are mainly interested in the nondissipative dynamic equations. In particular, we assumed that normal (nonsuperfluid) components of all particle species move with one and the same velocity u^μ . In principle, this condition does not guarantee that there is no dissipation in the system. Indeed, the entropy can be produced, e.g., because of scattering of nucleon thermal

excitations and/or electrons off the vortex cores. This mechanism is known as the ‘‘mutual friction’’ [15,17,25,27–29,70]. Only this dissipative mechanism has been taken into account in GAS11. To include mutual friction dissipation into consideration, we start with Eq. (90) and require positive definiteness of its right-hand side. Then, following the consideration of G16 [see the text after Eq. (77) in that reference], we find that Eq. (91) should be replaced with the inequality

$$u^\nu \mathcal{V}_{(i)\mu\nu} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}] \geq 0, \quad (97)$$

from which one obtains the same superfluid equation (93), but with $f_{(i)}^\mu$ given by

$$\begin{aligned}
f_{(i)}^\mu &= \alpha_i \perp^{\mu\nu} \mathcal{V}_{(i)\nu\lambda} W_{(i)\delta} \perp^{\lambda\delta} \\
&\quad + \frac{\beta_i - \gamma_i}{\mathcal{V}_{(Mi)}} \perp^{\mu\eta} \perp^{\nu\sigma} \mathcal{V}_{(i)\eta\sigma} \mathcal{V}_{(i)\lambda\nu} W_{(i)\delta} \perp^{\lambda\delta} \\
&\quad + \gamma_i \mathcal{V}_{(Mi)} W_{(i)\delta} \perp^{\mu\delta}, \quad (98)
\end{aligned}$$

where α_i is the same nondissipative coefficient as in Eq. (94); $\beta_i \geq 0$ and $\gamma_i \geq 0$ are the positive dissipative mutual friction coefficients and

$$\mathcal{V}_{(Mi)} \equiv \sqrt{\mathcal{V}_{(Mi)}^\mu \mathcal{V}_{(Mi)\mu}}. \quad (99)$$

Note that Eq. (98) is not the most general form of $f_{(i)}^\mu$ satisfying the inequality (97). In principle, there could be cross terms depending on both $\mathcal{V}_{(n)}^{\mu\nu}$ and $\mathcal{V}_{(p)}^{\mu\nu}$ (see, e.g., Ref. [15] for a nonrelativistic analogue of such terms). These terms are ignored in Eq. (98) since we do not see any plausible physical interpretation behind them. Anyway, one should bear in mind the possibility that Eq. (98) is not complete. In the nonrelativistic limit a more general expression for $f_{(i)}^\mu$ is contained in the Appendix of Ref. [15]. Generalization of that result to the relativistic case is straightforward.

Remark 2. Expression (98) for $f_{(i)}^\mu$ can be rewritten in terms of the magnetic four-vector $\mathcal{V}_{(Mi)}^\mu$ as [see a similar formula (53) in G16]

$$f_{(i)}^\mu = -\alpha_i X_{(i)}^\mu - \beta_i \epsilon^{\mu\nu\lambda\eta} u_\nu e_{(i)\lambda} X_{(i)\eta} + \gamma_i e_{(i)}^\mu W_{(i)}^\lambda \mathcal{V}_{(Mi)\lambda}, \quad (100)$$

where $e_{(i)}^\mu \equiv \mathcal{V}_{(Mi)}^\mu / \mathcal{V}_{(Mi)}$ and $X_{(i)}^\mu \equiv \epsilon^{\mu\nu\lambda\eta} u_\nu \mathcal{V}_{(Mi)\lambda} W_{(i)\eta}$.

Remark 3. As it is argued in Refs. [17,25], the coefficients γ_i ($i = n$ or p) in Eq. (98) are most likely very small and the corresponding terms can be neglected. Assume that it is indeed the case and that the tensor $\mathcal{V}_{(i)}^{\mu\nu}$ satisfies Eq. (93) with $f_{(i)}^\mu$ defined by Eq. (98). Then it can be shown (see

remark 2 in Sec. III A of G16) that a four-vector $v_{(Li)}^\mu$ exists, given by,

$$v_{(Li)}^\mu = u^\mu - \mu_i n_i \alpha_i W_{(i)\nu} \perp^{\mu\nu} + \frac{\mu_i n_i \beta_i}{\mathcal{V}_{(Mi)}} \perp^{\mu\alpha} \perp^{\nu\beta} \mathcal{V}_{(i)\alpha\beta} W_{(i)\nu}, \quad (101)$$

such that the combination $v_{(Li)}^\nu \mathcal{V}_{(i)\mu\nu}$ is identically zero,

$$v_{(Li)}^\nu \mathcal{V}_{(i)\mu\nu} = 0 \quad (102)$$

(no summation over i here). Equation (102) is analogous to the vorticity conservation equation of the nonrelativistic HVBK-hydrodynamics (see Appendix A of G16) and the four-vector $v_{(Li)}^\mu$ has the meaning of (non-normalized) vortex velocity.

Using Eqs. (96), (100), and (101), it is straightforward to show that the spatial components \mathbf{v}_{Li}/c , \mathcal{V}_{Ei} , and \mathcal{V}_{Mi} of the four-vectors $v_{(Li)}^\mu$, $\mathcal{V}_{(Ei)}^\mu$, and $\mathcal{V}_{(Mi)}^\mu$ are related, in the comoving frame, by the condition

$$\mathcal{V}_{Ei} = \frac{1}{c} \mathcal{V}_{Mi} \times \mathbf{v}_{Li}. \quad (103)$$

For future convenience the latter equation is written in the dimensional form.

Remark 4. It is notable that the vortex energy-momentum tensors (86) and (87) are obtained in the same way and have exactly the same structure as, respectively, the electromagnetic tensors (64) and (65). This is a direct consequence of the striking similarity of the electromagnetic and vortex contributions to the energy density $d\epsilon_{\text{add}}$ in Eq. (79).

VII. SYMMETRY OF THE ENERGY-MOMENTUM TENSOR

The symmetry of the energy-momentum tensors obtained in Secs. V and VI is not manifest. In this section, we prove that they are indeed symmetric. To do this it is necessary to express the tensors $G^{\mu\nu}$ and $\mathcal{W}_{(i)}^{\mu\nu}$ in Eqs. (66), (67), and (88), (89) through the tensors $F^{\mu\nu}$ and $\mathcal{V}_{(i)}^{\mu\nu}$. This can be done by specifying the expression for the energy density $d\epsilon_{\text{add}}$ [see Eqs. (60) and (79)], which is different for the situations considered in Secs. V and VI.

A. The *npe* mixture in the intermediate state (type-I proton superconductivity)

We start with the intermediate state model of Sec. VB. Generally, since there are no vortices in the system, the energy density ϵ can be a function of S , n_i , n_e , $w_{(i)\mu} w_{(k)}^\mu$, and various invariants composed of the four-vectors D^α

and B^α in combination with the four-vectors u^μ and $w_{(i)}^\mu$ characterizing the system in the field-free case.¹⁴ In what follows, we assume that there are no bound charges in the system (i.e., nonsuperconducting domains move with the normal liquid component), so that $E^\mu = D^\mu$, that is ϵ depends on D^μ through the term $D_\mu D^\mu / (8\pi)$. Concerning magnetic contribution, the simplest (and largest) invariant allowed by the symmetry¹⁵ is $x \equiv B_\mu B^\mu / (8\pi)$ [the factor $1/(8\pi)$ is introduced for further convenience]. We, thus, have for $d\epsilon$ the same equation (55) with

$$d\epsilon_{\text{add}} = \frac{1}{4\pi} D_\mu dD^\mu + \frac{1}{4\pi} \frac{\partial \epsilon}{\partial x} B_\mu dB^\mu, \quad (104)$$

where the partial derivative is taken at constant S , n_i , n_e , $w_{(i)\mu} w_{(k)}^\mu$, and D^μ . Comparing this equation with Eq. (60), one finds that, indeed,

$$E^\mu = D^\mu \quad (105)$$

and

$$H^\mu = \gamma B^\mu \quad \text{with} \quad \gamma \equiv \frac{\partial \epsilon}{\partial x}. \quad (106)$$

Equations (105) and (106) completely determine the tensors $\parallel G^{\mu\nu} = \parallel F^{\mu\nu}$ and $\perp G^{\mu\nu} = \gamma \perp F^{\mu\nu}$, and hence the tensor $G^{\mu\nu} = \parallel G^{\mu\nu} + \perp G^{\mu\nu}$ [see Eqs. (A6), (A5), and (A9)]. Using them, the electromagnetic tensor (71) can be presented in the manifestly symmetric form,

$$\Delta T^{\mu\nu} = -\frac{1}{4\pi} (E^\mu E^\nu - \perp^{\mu\nu} E_\alpha E^\alpha) + \frac{\gamma}{4\pi} (\perp_{\delta\alpha} F^{\mu\delta} F^{\nu\alpha} - u^\mu u^\nu u^\lambda u_\beta F^{\alpha\beta} F_{\alpha\lambda}). \quad (107)$$

The phenomenological coefficient γ is calculated for a simple model in Appendix G 1.

B. The *npe* mixture with neutron and proton vortices (type-II proton superconductivity)

In this case ϵ can depend on additional invariants composed of the four-vectors D^μ , B^μ , $\mathcal{W}_{(En)}^\mu$, $\mathcal{W}_{(Ep)}^\mu$, $\mathcal{V}_{(Mn)}^\mu$, and $\mathcal{V}_{(Mp)}^\mu$ [see Eq. (79)]. One can construct the

¹⁴We remind the reader that ϵ is a scalar defined in the comoving frame; it is thus invariant under Lorentz transformations.

¹⁵Other possible invariants, for example, $B_\mu w_{(i)}^\mu B_\nu w_{(k)}^\nu$, $\epsilon_{\alpha\beta\gamma\delta} u^\beta w_{(i)}^\gamma B^\delta \epsilon^{\alpha bcd} u_b w_{(i)c} B_d$ etc. are small, because the four-vector $w_{(i)}^\mu$ is proportional to the generally small difference between the normal and superfluid velocities, see, e.g., G16. Note also that the invariant $u_\mu B^\mu$, which could be used as a building brick for constructing other invariants, is zero, $u_\mu B^\mu = 0$; see Eq. (14).

following invariants from these vectors¹⁶: $z = D_\mu D^\mu / (8\pi)$, $z_i = D_\mu \mathcal{W}_{(Ei)}^\mu$, $z_{ik} = \mathcal{W}_{(Ei)\mu} \mathcal{W}_{(Ek)}^\mu / 2$, $x = B_\mu B^\mu / (8\pi)$, $x_i = B_\mu \mathcal{V}_{(Mi)}^\mu$, and $x_{ik} = \mathcal{V}_{(Mi)\mu} \mathcal{V}_{(Mk)}^\mu / 2$ ($i, k = n$ or p). Correspondingly, the differential of the energy density $\varepsilon(S, n_i, n_e, w_{(i)\mu} \mathcal{W}_{(k)}^\mu, z, z_i, z_{ik}, x, x_i, x_{ik})$ is given by Eq. (55), in which

$$\begin{aligned} d\varepsilon_{\text{add}} &= \frac{1}{4\pi} \frac{\partial \varepsilon}{\partial z} D_\mu dD^\mu + \frac{\partial \varepsilon}{\partial z_i} d[D_\mu \mathcal{W}_{(Ei)}^\mu] + \frac{\partial \varepsilon}{\partial z_{ik}} \mathcal{W}_{(Ek)\mu} d\mathcal{W}_{(Ei)}^\mu + \frac{1}{4\pi} \frac{\partial \varepsilon}{\partial x} B_\mu dB^\mu + \frac{\partial \varepsilon}{\partial x_i} d[B_\mu \mathcal{V}_{(Mi)}^\mu] + \frac{\partial \varepsilon}{\partial x_{ik}} \mathcal{V}_{(Mk)\mu} d\mathcal{V}_{(Mi)}^\mu \\ &= \frac{1}{4\pi} [\gamma^{(E)} D_\mu + 4\pi \Gamma_i^{(E)} \mathcal{W}_{(Ei)\mu}] dD^\mu + [\Gamma_{ik}^{(E)} \mathcal{W}_{(Ek)\mu} + \Gamma_i^{(E)} D_\mu] d\mathcal{W}_{(Ei)}^\mu \\ &\quad + \frac{1}{4\pi} [\gamma^{(M)} B_\mu + 4\pi \Gamma_i^{(M)} \mathcal{V}_{(Mi)\mu}] dB^\mu + [\Gamma_{ik}^{(M)} \mathcal{V}_{(Mk)\mu} + \Gamma_i^{(M)} B_\mu] d\mathcal{V}_{(Mi)}^\mu, \end{aligned} \quad (108)$$

where

$$\gamma^{(E)} \equiv \frac{\partial \varepsilon}{\partial z}, \quad \gamma^{(M)} \equiv \frac{\partial \varepsilon}{\partial x}, \quad (109)$$

$$\Gamma_i^{(E)} \equiv \frac{\partial \varepsilon}{\partial z_i}, \quad \Gamma_i^{(M)} \equiv \frac{\partial \varepsilon}{\partial x_i}, \quad (110)$$

$$\Gamma_{ik}^{(E)} = \Gamma_{ki}^{(E)} \equiv \frac{\partial \varepsilon}{\partial z_{ik}}, \quad \Gamma_{ik}^{(M)} = \Gamma_{ki}^{(M)} \equiv \frac{\partial \varepsilon}{\partial x_{ik}}. \quad (111)$$

Comparing Eqs. (108) and (79), one identifies

$$E^\mu = \gamma^{(E)} D^\mu + 4\pi \Gamma_i^{(E)} \mathcal{W}_{(Ei)}^\mu, \quad (112)$$

$$\mathcal{V}_{(Ei)}^\mu = \Gamma_{ik}^{(E)} \mathcal{W}_{(Ek)}^\mu + \Gamma_i^{(E)} D^\mu, \quad (113)$$

$$H^\mu = \gamma^{(M)} B^\mu + 4\pi \Gamma_i^{(M)} \mathcal{V}_{(Mi)}^\mu, \quad (114)$$

$$\mathcal{W}_{(Mi)}^\mu = \Gamma_{ik}^{(M)} \mathcal{V}_{(Mk)}^\mu + \Gamma_i^{(M)} B^\mu. \quad (115)$$

The system of Eqs. (112) and (113) can be inverted and the four-vectors D^μ and $\mathcal{W}_{(Ei)}^\mu$ can be presented as

$$D^\mu = \tilde{\gamma}^{(E)} E^\mu + 4\pi \tilde{\Gamma}_i^{(E)} \mathcal{V}_{(Ei)}^\mu, \quad (116)$$

$$\mathcal{W}_{(Ei)}^\mu = \tilde{\Gamma}_{ik}^{(E)} \mathcal{V}_{(Ek)}^\mu + \tilde{\Gamma}_i^{(E)} E^\mu, \quad (117)$$

where the quantities $\tilde{\gamma}^{(E)}$, $\tilde{\Gamma}_i^{(E)}$, and $\tilde{\Gamma}_{ik}^{(E)}$ can easily be expressed through $\gamma^{(E)}$, $\Gamma_i^{(E)}$, and $\Gamma_{ik}^{(E)}$ using Eqs. (112) and (113). From Eqs. (115) and (117) one now sees that the four-vectors $\mathcal{W}_{(Ei)}^\mu$ and $\mathcal{W}_{(Mi)}^\mu$ indeed have the form assumed in Eqs. (80) and (81).

Using Eqs. (114)–(117), as well as Eqs. (A5), (A6), and (A9), one can find the tensors $G^{\mu\nu}$ and $\mathcal{W}_{(i)}^{\mu\nu}$,¹⁷ and to present the tensor $\Delta T^{\mu\nu}$ (92) in the manifestly symmetric form,

$$\begin{aligned} \Delta T^{\mu\nu} &= -\frac{\tilde{\gamma}^{(E)}}{4\pi} (E^\mu E^\nu - \perp^{\mu\nu} E_\alpha E^\alpha) - \tilde{\Gamma}_i^{(E)} [(\mathcal{V}_{(Ei)}^\mu E^\nu + E^\mu \mathcal{V}_{(Ei)}^\nu) - 2\perp^{\mu\nu} E_\alpha \mathcal{V}_{(Ei)}^\alpha] - \frac{\tilde{\Gamma}_{ik}^{(E)}}{2} [(\mathcal{V}_{(Ei)}^\mu \mathcal{V}_{(Ek)}^\nu + \mathcal{V}_{(Ek)}^\mu \mathcal{V}_{(Ei)}^\nu) \\ &\quad - 2\perp^{\mu\nu} \mathcal{V}_{(Ei)\alpha} \mathcal{V}_{(Ek)}^\alpha] + \frac{\gamma^{(M)}}{4\pi} (\perp_{\delta\alpha} F^{\mu\delta} F^{\nu\alpha} - u^\mu u^\nu u^\gamma u_\beta F^{\alpha\beta} F_{\alpha\gamma}) + \Gamma_i^{(M)} [\perp_{\delta\alpha} (\mathcal{V}_{(i)}^{\mu\delta} F^{\nu\alpha} + F^{\mu\delta} \mathcal{V}_{(i)}^{\nu\alpha}) - 2u^\mu u^\nu u^\gamma u_\beta \mathcal{V}_{(i)}^{\alpha\beta} F_{\alpha\gamma}] \\ &\quad + \frac{\Gamma_{ik}^{(M)}}{2} [\perp_{\delta\alpha} (\mathcal{V}_{(i)}^{\mu\delta} \mathcal{V}_{(k)}^{\nu\alpha} + \mathcal{V}_{(k)}^{\mu\delta} \mathcal{V}_{(i)}^{\nu\alpha}) - 2u^\mu u^\nu u^\gamma u_\beta \mathcal{V}_{(i)}^{\alpha\beta} \mathcal{V}_{(k)\alpha\gamma}]. \end{aligned} \quad (118)$$

In the absence of vortices this tensor reduces to that in Sec. VII A [see Eq. (107)]. In another limiting case of only one neutral superfluid particle species (e.g., $i = n$) it reproduces the tensor presented in G16 if one sets all

¹⁶Of course, the number of possible invariants is much larger. Here we only write out those invariants whose physical meaning is clear to us (see Appendix G 2), but one should bear in mind that it is straightforward to consider other possibilities.

the coefficients except for $\Gamma_{nn}^{(M)}$ to zero [see equation (79) in that reference]. A simple microscopic model allowing to calculate the phenomenological coefficients $\tilde{\gamma}^{(E)}$, $\gamma^{(M)}$, $\tilde{\Gamma}_i^{(E)}$, $\Gamma_i^{(M)}$, $\tilde{\Gamma}_{ik}^{(E)}$, and $\Gamma_{ik}^{(M)}$ in Eq. (118) is considered in

¹⁷In particular, $\mathcal{W}_{(i)}^{\mu\nu} = \perp \mathcal{W}_{(i)}^{\mu\nu} + \parallel \mathcal{W}_{(i)}^{\mu\nu}$, where $\perp \mathcal{W}_{(i)}^{\mu\nu} = \epsilon^{\alpha\beta\mu\nu} u_\beta (\Gamma_{ik}^{(M)} \mathcal{V}_{(Mk)\alpha} + \Gamma_i^{(M)} B_\alpha) = \Gamma_{ik}^{(M)} \perp \mathcal{V}_{(k)}^{\mu\nu} + \Gamma_i^{(M)} \perp F^{\mu\nu}$ and $\parallel \mathcal{W}_{(i)}^{\mu\nu} = \tilde{\Gamma}_{ik}^{(E)} \parallel \mathcal{V}_{(k)}^{\mu\nu} + \tilde{\Gamma}_i^{(E)} \parallel F^{\mu\nu}$.

Appendix G 2. This model is analogous to the model discussed in detail in GAS11.

VIII. “MAGNETOHYDRODYNAMIC” APPROXIMATION FOR *NPE* MIXTURE WITH NEUTRON AND PROTON VORTICES (TYPE-II PROTON SUPERCONDUCTIVITY)

General equations of Secs. VI and VII B can be substantially simplified if the magnetic induction \mathbf{B} is much larger than the fields \mathbf{E} , \mathbf{D} , and \mathbf{H} in the comoving frame (hereafter the *magnetohydrodynamic* approximation).¹⁸ As it is discussed in Appendix G 2 (see remark 1 there), as well as in GAS11, this is a typical situation in real neutron stars. Note also that in the comoving frame $\mathcal{V}_{(Ei)} \equiv (\mathcal{V}_{(Ei)\mu} \mathcal{V}_{(Ei)}^\mu)^{1/2} \sim (1/c) \mathcal{V}_{(Mi)}$ and can be neglected in comparison to $\mathcal{V}_{(Mi)}$ [this follows from the analysis of the superfluid Eq. (96) and its nonrelativistic counterpart in Appendix I]. In addition, one can neglect the neutron-related four-vector $\mathcal{V}_{(Mn)}^\mu$ in comparison to the proton four-vector $\mathcal{V}_{(Mp)}^\mu$ in Eq. (114), because the lengths of these vectors are proportional to the vortex density N_{Vi} [see Eq. (G7)], which is larger for protons by more than ten orders of magnitude. Using these facts, a number of simplifications are possible:

- (1) One can omit H^μ (and $\mathcal{V}_{(Mn)}^\mu$, as we have already mentioned) in Eq. (114). This leads to the condition relating B^μ and $\mathcal{V}_{(Mp)}^\mu$ (here and below in this section we, for definiteness, use the parameters $\gamma^{(M)}$, $\Gamma_i^{(M)}$, $\Gamma_{ik}^{(M)}$, etc. calculated for a simple microscopic model of Appendix G 2),

$$\mathcal{V}_{(Mp)}^\mu \approx -\frac{\gamma^{(M)}}{4\pi\Gamma_p^{(M)}} B^\mu = \frac{\pi}{\hat{\phi}_{p0}} B^\mu = e_p B^\mu. \quad (119)$$

Physically, this condition means that almost all the magnetic induction is produced by the proton vortices. Note that, from Eq. (74) it follows

$$\mathcal{V}_{(Mp)}^\mu = \tilde{\mathcal{V}}_{(Mp)}^\mu + e_p B^\mu. \quad (120)$$

Comparing this equation with Eq. (119) one sees that in the adopted approximation the vector $\tilde{\mathcal{V}}_{(Mp)}^\mu$, which reduces to $(0, m_p \text{curl} \mathbf{V}_{sp})$ in the nonrelativistic limit (see footnote 12), should be neglected in comparison to $e_p B^\mu$.

¹⁸In what follows we assume that the relative velocity between the normal and superfluid components is much smaller than the speed of light c . As is argued in G16 (see Appendix D there), this is not a very restrictive requirement.

- (2) Because \mathbf{H} and \mathbf{D} are small by assumption, one can discard Maxwell’s equation (11), setting to zero the four-current density $J_{(\text{free})}^\mu$ in all other equations,

$$J_{(\text{free})}^\mu = e_j j_{(j)}^\mu = e_p (n_p - n_e) u^\mu + e_i Y_{ik} w_{(k)}^\mu = 0, \quad (121)$$

that is, since $u_\mu w_{(i)}^\mu = 0$ [see Eq. (25)],

$$n_e = n_p, \quad (122)$$

$$e_i Y_{ik} w_{(k)}^\mu = 0. \quad (123)$$

- (3) One can ignore the first three terms in the rhs of the expression (79) for the energy density $d\varepsilon_{\text{add}}$, because they depend on small four-vectors E^μ , D^μ , H^μ , $\mathcal{V}_{(Ei)}^\mu$, and $\mathcal{W}_{(Ei)}^\mu$.¹⁹ The last term in Eq. (79) is large in comparison to the neglected terms, since it is independent of these small vectors, as is shown below. Using Eq. (114) with $H^\mu = 0$, as well as Eqs. (115), and (G29)–(G32), one obtains (no summation over i here)²⁰

$$\mathcal{W}_{(Mi)}^\mu = \frac{\lambda_i}{m_i \mathcal{V}_{(Mi)}} \mathcal{V}_{(Mi)}^\mu, \quad (124)$$

so that this last term can be approximately presented as

$$d\varepsilon_{\text{add}} \approx \sum_{i=n,p} \frac{\lambda_i}{m_i \mathcal{V}_{(Mi)}} \mathcal{V}_{(Mi)\mu} d\mathcal{V}_{(Mi)}^\mu, \quad (125)$$

where λ_p and λ_n are given, respectively, by Eqs. (G10) and (G11).²¹ Note that the proton-related term ($i = p$) in Eq. (125) reduces to $e_p \lambda_p / (m_p B) B_\mu dB^\mu$ in view of Eq. (119) [here $B \equiv (B_\mu B^\mu)^{1/2}$].

- (4) One can repeat the derivation of Sec. VI with $d\varepsilon_{\text{add}}$ from Eq. (125). As a result, one will derive Eqs. (92) and (93) with the following modifications: (i) The

¹⁹The four-vector $\mathcal{W}_{(Ei)}^\mu$ is expressed through $\mathcal{V}_{(Ei)}^\mu$ and E^μ by Eq. (117) and hence is small. Note that the tensor $\parallel \mathcal{W}_{(i)}^{\mu\nu}$ is also small since it is in turn related to $\mathcal{W}_{(Ei)}^\mu$ by Eq. (80).

²⁰We emphasize once again that the relation (124) is only valid in the (usually adopted) approximation of noninteracting vortices, see Appendix G 2.

²¹In principle, the term with $i = n$ in Eq. (125) could be neglected in comparison to the $i = p$ term, since in neutron stars $\mathcal{V}_{(Mn)} \ll \mathcal{V}_{(Mp)}$ and $\lambda_n \sim \lambda_p$. However, we prefer to retain this term here in order to describe situations when protons are normal and $i = p$ term is absent.

first three tensors in the rhs of Eq. (92) will not appear in the approximation adopted here, since they are smaller than the fourth term (in principle, this can be independently checked by direct comparison of the elements of four these tensors). We thus left with

$$\Delta T^{\mu\nu} = \mathcal{T}_{(\text{VM})}^{\mu\nu}. \quad (126)$$

(ii) The four-vector $W_{(i)}^\mu$, entering the definition of $f_{(i)}^\mu$ in Eq. (93), will be modified (no summation over i is assumed),

$$\begin{aligned} W_{(i)}^\mu &= \frac{1}{n_i} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}] \\ &= \frac{1}{n_i} \{Y_{ik} w_{(k)}^\mu + \partial_\alpha [\perp \mathcal{W}_{(i)}^{\mu\alpha} + \parallel \mathcal{W}_{(i)}^{\mu\alpha}]\} \\ &\approx \frac{1}{n_i} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \perp \mathcal{W}_{(i)}^{\mu\alpha}] \\ &= \frac{1}{n_i} \{Y_{ik} w_{(k)}^\mu + \partial_\alpha [\epsilon^{\gamma\beta\mu\alpha} u_\beta \mathcal{W}_{(Mi)\gamma}]\} \end{aligned} \quad (127)$$

[see the footnote 19, Eqs. (A5), (A6), and (A9) and note that we neglect the small term depending on $\mathcal{W}_{(Ei)}^\mu$ here]. All other equations remain exactly the same.

Summarizing, the system of simplified ‘‘magnetohydrodynamic’’ equations for npe mixture consists of the energy-momentum and particle conservation laws (16) and (17) with $j_{(j)}^\mu$ given by Eqs. (21), (22) and $T^{\mu\nu}$ given by Eq. (56) with $\Delta T^{\mu\nu}$ from Eq. (126). When calculating $\Delta T^{\mu\nu}$ one should express $\perp \mathcal{W}_{(i)}^{\mu\alpha}$ through $\mathcal{W}_{(Mi)}^\mu$, which in turn should be found from Eq. (124). These equations should be supplemented by Maxwell’s equation (10), the second law of thermodynamics (55) with $d\mathcal{E}_{\text{add}}$ defined in Eq. (125), and by the conditions (24), (25), (122), and (123). Finally, the system is closed by the neutron and proton superfluid equations (93) [or (96)], in which $f_{(i)}^\mu$ is defined by Eq. (98) [or, equivalently, by Eq. (100)] and $W_{(i)}^\mu$ is given by Eq. (127). The nonrelativistic version of some of these equations is presented in Appendix I.

Remark 1. It is interesting that, using Eqs. (100) and (119), the proton four-vector $f_{(p)}^\mu$ can be represented in terms of B^μ ,

$$f_{(p)}^\mu \approx -\alpha_p X_{(p)}^\mu - \beta_p \epsilon^{\mu\nu\lambda\eta} u_\nu e_{(p)\lambda} X_{(p)\eta} + e_p \gamma_p e_{(p)}^\mu W_{(p)}^\lambda B_\lambda, \quad (128)$$

where $e_{(p)}^\mu \approx B^\mu/B$ and $X_{(i)}^\mu \approx e_p \epsilon^{\mu\nu\lambda\eta} u_\nu B_\lambda W_{(i)\eta}$.

Remark 2. Note that the proton four-vector $w_{(p)}^\mu$ can be found from the condition (123). The proton superfluid equation can thus be used to express the electric four-vector

E^μ . Using Eq. (74) in order to present $\mathcal{V}_{(Ei)}^\mu$ as $\mathcal{V}_{(Ei)}^\mu = \tilde{\mathcal{V}}_{(Ei)}^\mu + e_i E^\mu$, and substituting this expression (for $i = p$) into Eq. (96), one finds

$$E^\mu = -\frac{1}{e_p} \tilde{\mathcal{V}}_{(Ep)}^\mu + \frac{\mu_p n_p}{e_p} f_{(p)}^\mu, \quad (129)$$

where $f_{(p)}^\mu$ is given by Eq. (128). Together with Maxwell’s equation (10), this equation allows one, in principle, to exclude E^μ and obtain a closed equation for B^μ only (see remark 1 in Appendix I, where such an equation is derived in the nonrelativistic limit).

IX. SUMMARY AND CONCLUSIONS

This paper is devoted to studying the dynamic properties of superfluid-superconducting mixtures in neutron stars accounting for the possible presence of electric and magnetic fields, as well as neutron (Feynman-Onsager) and proton (Abrikosov) vortices. Our results and main conclusions are summarized as follows:

- (1) Using the method and ideas from Refs. [29] and G16, we derived a set of fully *relativistic* equations (see Appendix H) describing a charged mixture composed of superfluid neutrons, superconducting protons, and electrons (the simplest neutron-star composition). Generalization of these equations to more exotic compositions (including, e.g., muons, hyperons, etc.) is straightforward [24,38,42,43].
- (2) The proposed equations can be used at *finite* temperatures; i.e., they allow for the possible presence of neutron and proton (Bogoliubov) thermal excitations. This is especially important for a sufficiently hot neutron stars, such as magnetars, whose internal temperatures can be $\sim 10^8$ K, i.e., of the order of the nucleon critical temperatures T_{ci} [1,71] (we remind that at $T > T_{ci}$ nucleon species $i = n, p$ is completely nonsuperfluid).
- (3) The derived dynamic equations are ‘‘nondissipative’’ in a sense that to obtain them we assume that normal (nonsuperfluid) liquid components (electrons, nucleon thermal excitations, and entropy) move with one and the same velocity (i.e., diffusion effects are ignored). However, we do take into account the *mutual friction dissipation* [see Eqs. (97) and (98)]. Extension of our results to a fully dissipative problem is rather easy and will be reported elsewhere.
- (4) Estimates show that protons form type-II superconductor in the outer neutron-star core, but become of type-I in the inner core (e.g., GAS11, [66,72–74]). The dynamic equations are derived and analyzed in both these cases with the special emphasis on the more elaborated type-II case. It seems that the dynamics of the type-I superconductor is discussed

for the first time (in the astrophysical context), but the analysis presented is rather brief and simplified and should be considered as a first step towards the solution of this complex problem.

- (5) Our main results include the “electromagnetic” energy-momentum tensors $\mathcal{T}_{(E)}^{\mu\nu}$ (H20) and $\mathcal{T}_{(M)}^{\mu\nu}$ (H21), and the nucleon “vortex” energy-momentum tensors $\mathcal{T}_{(VE)}^{\mu\nu}$ (H27) and $\mathcal{T}_{(VM)}^{\mu\nu}$ (H28), as well as the “superfluid” equations for the cases of type-I (H23), (H24) and type-II (H30) proton superconductivities. Remarkably, the vortex energy-momentum tensors have the same structure and are obtained exactly in the same way as the electromagnetic tensors (H20) and (H21) (see remark 4 in Sec. VI).
- (6) As a byproduct of our work, it is shown that for normal matter the sum $\mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu}$ of the electromagnetic energy-momentum tensors is directly related to the so-called Abraham tensor $T_{\text{Abraham}}^{\mu\nu}$ of the standard electrodynamics of continuous media [46,68,69]. Thus, our results can be considered as one more derivation of this tensor based on the conservation laws and the requirement that the entropy of a non-dissipative closed system remains constant.
- (7) The equations derived in this paper [in particular, the expressions for the electromagnetic and vortex energy-momentum tensors $\mathcal{T}_{(E)}^{\mu\nu}$, $\mathcal{T}_{(M)}^{\mu\nu}$, $\mathcal{T}_{(VE)}^{\mu\nu}$, and $\mathcal{T}_{(VM)}^{\mu\nu}$] depend on the four-vectors E^μ , B^μ , $\mathcal{V}_{(Ei)}^\mu$, $\mathcal{V}_{(Mi)}^\mu$ and the complementary four-vectors D^μ , H^μ , $\mathcal{W}_{(Ei)}^\mu$, $\mathcal{W}_{(Mi)}^\mu$. The physical meaning of these four-vectors is described in detail in the text. For example, the spatial components of E^μ , B^μ , D^μ , and H^μ reduce, respectively, to the electric field, magnetic induction, electric displacement, and magnetic field in the comoving frame moving with the normal liquid component (see Sec. II C); the other four-vectors are related to vortices.

The four-vectors mentioned above are not all independent. To express the quantities D^μ , H^μ , $\mathcal{W}_{(Ei)}^\mu$, $\mathcal{W}_{(Mi)}^\mu$ through E^μ , B^μ , $\mathcal{V}_{(Ei)}^\mu$, $\mathcal{V}_{(Mi)}^\mu$, one should specify, as in the usual electrodynamics of continuous media, the microphysics model for the mixture. This is done, for two simple models, in Appendix G (in particular, one of these models analyzes the system of noninteracting vortices). However, it is important to point out that the general equations obtained here will likely remain unchanged if one considers more complex models. The only thing that should be modified in the latter case is the relations between the fields D^μ , H^μ , $\mathcal{W}_{(Ei)}^\mu$, $\mathcal{W}_{(Mi)}^\mu$ and E^μ , B^μ , $\mathcal{V}_{(Ei)}^\mu$, $\mathcal{V}_{(Mi)}^\mu$.
- (8) It is instructive to compare our results with the most advanced nonrelativistic magnetohydrodynamics of GAS11, describing superfluid-superconducting

- mixtures. In comparison to GAS11, we (i) take into account the relativistic and finite-temperature effects, (ii) provide a general framework allowing one to easily incorporate new physics into the existing dynamic equations, and (iii) demonstrate that the electric displacement field \mathbf{D} is *not* generally equal to the electric field \mathbf{E} , contrary to what was assumed in GAS11 and some other papers starting from the work by Mendell [15] (see also Ref. [17]).
- (9) The rather complex general system of equations derived in this work can be substantially simplified for typical neutron-star conditions, for which a kind of “magnetohydrodynamic” approximation is justified. This approximation is analogous to the usual magnetohydrodynamic approximation for ordinary stars. The corresponding equations are derived and analyzed for a simple model of Appendix G 2 in Sec. VIII; their nonrelativistic limit is presented in Appendix I, where we also derive a “magnetic field evolution equation” (I24). It is shown that the latter equation coincides with that proposed in Ref. [75], but differs from the evolution equation derived in Ref. [76] using magnetohydrodynamics of GAS11.

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APPENDIX A: SOME USEFUL DEFINITIONS

Assume we have an arbitrary antisymmetric tensor $\mathcal{A}^{\mu\nu}$, which can be represented in the matrix form as

$$\mathcal{A}^{\mu\nu} = \begin{pmatrix} 0 & \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{A}_{03} \\ -\mathcal{A}_{01} & 0 & \mathcal{A}_{12} & \mathcal{A}_{13} \\ -\mathcal{A}_{02} & -\mathcal{A}_{12} & 0 & \mathcal{A}_{23} \\ -\mathcal{A}_{03} & -\mathcal{A}_{13} & -\mathcal{A}_{23} & 0 \end{pmatrix}. \quad (\text{A1})$$

Here and below all matrix representations of tensors/vectors are given in the comoving frame, i.e. in the frame in which the normal four-velocity is $u^\mu = (1, 0, 0, 0)$.

The tensor $\star\mathcal{A}^{\mu\nu}$, dual to the tensor $\mathcal{A}^{\mu\nu}$, is

$$\begin{aligned} \star\mathcal{A}^{\mu\nu} &\equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \mathcal{A}_{\alpha\beta} \\ &= \begin{pmatrix} 0 & \mathcal{A}_{23} & -\mathcal{A}_{13} & \mathcal{A}_{12} \\ -\mathcal{A}_{23} & 0 & -\mathcal{A}_{03} & \mathcal{A}_{02} \\ \mathcal{A}_{13} & \mathcal{A}_{03} & 0 & -\mathcal{A}_{01} \\ -\mathcal{A}_{12} & -\mathcal{A}_{02} & \mathcal{A}_{01} & 0 \end{pmatrix}. \quad (\text{A2}) \end{aligned}$$

Using these tensors, one can construct the “electric” $\mathcal{A}_{(E)}^\mu$ and “magnetic” $\mathcal{A}_{(M)}^\mu$ four-vectors [77]

$$\mathcal{A}_{(E)}^\mu \equiv u_\nu \mathcal{A}^{\mu\nu} = (0, \mathcal{A}_{01}, \mathcal{A}_{02}, \mathcal{A}_{03}), \quad (\text{A3})$$

$$\mathcal{A}_{(M)}^\mu \equiv u_\nu \star \mathcal{A}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{A}_{\alpha\beta} = (0, \mathcal{A}_{23}, -\mathcal{A}_{13}, \mathcal{A}_{12}), \quad (\text{A4})$$

and two additional tensors

$$\perp \mathcal{A}^{\mu\nu} = \epsilon^{\alpha\beta\mu\nu} u_\beta \mathcal{A}_{(M)\alpha} = \perp^{\mu\alpha} \perp^{\nu\beta} \mathcal{A}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_{12} & \mathcal{A}_{13} \\ 0 & -\mathcal{A}_{12} & 0 & \mathcal{A}_{23} \\ 0 & -\mathcal{A}_{13} & -\mathcal{A}_{23} & 0 \end{pmatrix}, \quad (\text{A5})$$

$$\parallel \mathcal{A}^{\mu\nu} = -u^\nu \mathcal{A}_{(E)}^\mu + u^\mu \mathcal{A}_{(E)}^\nu = -u^\nu u_\alpha \mathcal{A}^{\mu\alpha} + u^\mu u_\alpha \mathcal{A}^{\nu\alpha} = \begin{pmatrix} 0 & \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{A}_{03} \\ -\mathcal{A}_{01} & 0 & 0 & 0 \\ -\mathcal{A}_{02} & 0 & 0 & 0 \\ -\mathcal{A}_{03} & 0 & 0 & 0 \end{pmatrix} \quad (\text{A6})$$

with the properties

$$u_\nu \perp \mathcal{A}^{\mu\nu} = 0, \quad (\text{A7})$$

$$\perp_{\mu\nu} \parallel \mathcal{A}^{\mu\nu} = 0, \quad (\text{A8})$$

where $\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator and $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor, $\epsilon^{0123} = 1$. One can see that the tensor $\mathcal{A}^{\mu\nu}$ can be decomposed as

$$\mathcal{A}^{\mu\nu} = \perp \mathcal{A}^{\mu\nu} + \parallel \mathcal{A}^{\mu\nu}. \quad (\text{A9})$$

APPENDIX B: COMPARISON OF NOTATION USED IN THIS PAPER AND IN G16

Some of the parameters introduced in G16 and in the present paper differ only by the index i , since here we have two superfluid/superconducting particle species [neutrons ($i = n$) and protons ($i = p$)], whereas G16 deals with one particle species. Such parameters are not provided in the table below.

G16	This work	Parameter name
$F^{\mu\nu}$	$\mathcal{V}_{(i)}^{\mu\nu}$	Vorticity tensor
$O^{\mu\nu}$	$\perp \mathcal{V}_{(i)}^{\mu\nu}$	“Magnetic” part of the vorticity tensor
H^μ	$\mathcal{V}_{(Mi)}^\mu$	“Magnetic” vorticity-related vector
$-E^\mu$	$\mathcal{V}_{(Ei)}^\mu$	“Electric” vorticity-related vector
$V_{(L)}^\mu$	$v_{(Li)}^\mu$	Vortex four-velocity (non-normalized)
H	$\mathcal{V}_{(Mi)}$	Length of the four-vector $\mathcal{V}_{(Mi)}^\mu$ (or H^μ)
\mathbf{H}	\mathcal{V}_{Mi}	Spatial part of the four-vector $\mathcal{V}_{(Mi)}^\mu$ (or H^μ)
$-\mathbf{E}$	\mathcal{V}_{Ei}	Spatial part of the four-vector $\mathcal{V}_{(Ei)}^\mu$ (or $-E^\mu$)
V_L	v_{Li}	Spatial part of the vortex four-velocity $v_{(Li)}^\mu$ [or $V_{(L)}^\mu$]

APPENDIX C: ENERGY DENSITY TRANSFORMATION

Assume we have a term in the expression for the energy density which takes the form

$$d\varepsilon_{\text{part}} = \frac{1}{2}(O^{\alpha\beta}d\mathcal{F}_{\alpha\beta} + 2\mathcal{B}_{\alpha\beta}A^{\alpha\gamma}u^\beta du_\gamma), \quad (\text{C1})$$

where $O^{\alpha\beta}$, $\mathcal{A}^{\alpha\beta}$, and $\mathcal{B}^{\alpha\beta}$ are some arbitrary *antisymmetric* tensors; $\mathcal{F}^{\alpha\beta}$ is the antisymmetric tensor satisfying the condition²²

$$\partial_\alpha \star \mathcal{F}^{\alpha\beta} = 0; \quad (\text{C2})$$

and u^μ is the four-velocity of normal liquid component. Our aim will be to transform the expression $-u^\mu \partial_\mu \varepsilon_{\text{part}}$ to some standard form [see Eq. (C7) in what follows]; this transformation is used several times in the main text of the paper (see also G16). Using (C1), one has

$$-u^\mu \partial_\mu \varepsilon_{\text{part}} = -\frac{1}{2}u^\mu O^{\alpha\beta} \partial_\mu \mathcal{F}_{\alpha\beta} - u^\mu u^\delta \mathcal{B}_{\alpha\delta} A^{\alpha\gamma} \partial_\mu u_\gamma. \quad (\text{C3})$$

The first term in the rhs of Eq. (C3) can be transformed as

$$-\frac{1}{2}u^\mu O^{\alpha\beta} \partial_\mu \mathcal{F}_{\alpha\beta} = u^\nu \mathcal{F}_{\mu\nu} \partial_\alpha O^{\mu\alpha} - \partial_\mu (u^\nu O^{\mu\alpha} \mathcal{F}_{\nu\alpha}) + \partial_\mu u^\nu (O^{\mu\alpha} \mathcal{F}_{\nu\alpha}). \quad (\text{C4})$$

To obtain this expression, we used Eq. (C2), which is equivalent to

$$\partial_\mu \mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{F}_{\mu\beta} + \partial_\beta \mathcal{F}_{\alpha\mu}, \quad (\text{C5})$$

and the fact that both tensors $\mathcal{F}^{\mu\nu}$ and $O^{\mu\nu}$ are antisymmetric.

The second term in the rhs of Eq. (C3) can be rewritten as

$$\begin{aligned} & -u^\mu u^\delta \mathcal{B}_{\alpha\delta} A^{\alpha\gamma} \partial_\mu u_\nu \\ &= -[u^\mu u^\delta \mathcal{B}_{\alpha\delta} A^{\alpha\gamma} + \underline{u^\mu u^\nu u^\beta u_\gamma \mathcal{B}_{\alpha\beta} A^{\alpha\gamma}}] \partial_\mu u_\nu \\ &= -u^\mu u^\gamma \perp_{\nu\beta} A^{\alpha\beta} \mathcal{B}_{\alpha\gamma} \partial_\mu u^\nu \\ &= -u^\mu u^\gamma \perp_{\nu\beta} A^{\alpha\beta} \mathcal{B}_{\alpha\gamma} \partial_\mu u^\nu + \underline{\partial_\mu (u^\nu u^\mu u^\gamma \perp_{\nu\beta} A^{\alpha\beta} \mathcal{B}_{\alpha\gamma})}, \end{aligned} \quad (\text{C6})$$

where the underlined terms equal zero (because $u_\nu \partial_\mu u^\nu = 0$ and $u^\nu \perp_{\nu\beta} = 0$); they are added here in order to symmetrize the corresponding energy-momentum tensor $T^{\mu\nu}$ and to satisfy the condition $u_\mu u_\nu \Delta T^{\mu\nu} = 0$ (see the main text). Combining Eqs. (C4) and (C6), one obtains

²²For example, it can be the electromagnetic tensor $F^{\alpha\beta}$ or the vorticity tensor $\mathcal{V}_{(i)}^{\mu\nu}$, see Eqs. (10) and (75).

$$\begin{aligned} -u^\mu \partial_\mu \varepsilon_{\text{part}} &= u^\nu \mathcal{F}_{\mu\nu} \partial_\alpha O^{\mu\alpha} \\ &\quad - \partial_\mu [u^\nu (O^{\mu\alpha} \mathcal{F}_{\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} A^{\alpha\beta} \mathcal{B}_{\alpha\gamma})] \\ &\quad + \partial_\mu u^\nu (O^{\mu\alpha} \mathcal{F}_{\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} A^{\alpha\beta} \mathcal{B}_{\alpha\gamma}). \end{aligned} \quad (\text{C7})$$

APPENDIX D: ENERGY-MOMENTUM TENSOR (71) AND ITS RELATION TO THE ABRAHAM TENSOR

As mentioned in Sec. V, the derivation of the energy-momentum tensor (71) can also be applied to ordinary (nonsuperfluid) matter. In other words, this tensor should have a well-known counterpart in the literature. Here we explore this issue in more detail.

We consider a normal (isotropic and homogeneous in the comoving frame) dielectric “fluid” with the energy-momentum tensor

$$T^{\mu\nu} = (P + \varepsilon)u^\mu u^\nu + P g^{\mu\nu} + \Delta T^{\mu\nu} \quad (\text{D1})$$

and the second law of thermodynamics

$$d\varepsilon = Tds + \mu dn + \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu. \quad (\text{D2})$$

In Eqs. (D1) and (D2) $\Delta T^{\mu\nu}$ is given by Eq. (71)²³; n is the “particle” number density [it can be composite particles; in the case of a few particle species j the second term in Eq. (D2) should be replaced with $\sum_j \mu_j dn_j$]; μ is the relativistic chemical potential; and P is the pressure,

$$P = -\varepsilon + \mu n + TS. \quad (\text{D3})$$

Since the medium is isotropic and homogeneous, the displacement vector \mathbf{D} and magnetic induction \mathbf{B} can be presented, in the comoving frame, as

$$\mathbf{D} = \hat{\varepsilon} \mathbf{E}, \quad (\text{D4})$$

$$\mathbf{B} = \hat{\mu} \mathbf{H}, \quad (\text{D5})$$

where $\hat{\varepsilon}$ and $\hat{\mu}$ are the corresponding permeabilities (scalars). We assume, in addition, that the permeabilities are *field-independent*, but can generally be functions of n and S . Because the time components of the four-vectors D^μ , E^μ , B^μ , and H^μ all vanish in the comoving frame, it follows from Eqs. (D4) and (D5) that

²³Note that for a dielectric fluid the free-charge four-current density $J_{(\text{free})}^\mu$ in Eq. (11) equals zero, $J_{(\text{free})}^\mu = 0$, hence the first line in the rhs of Eq. (63) is zero too and the derivation of Sec. VA can indeed be used to obtain $\Delta T^{\mu\nu}$ in the form (71).

$$D^\mu = \hat{\varepsilon} E^\mu, \quad (\text{D6})$$

$$B^\mu = \hat{\mu} H^\mu. \quad (\text{D7})$$

Using Eqs. (D6) and (D7), Eq. (D2) can be readily integrated and presented as

$$\begin{aligned} \varepsilon &= \varepsilon_{\text{fluid}}(n, S) + \frac{1}{8\pi} (E_\alpha D^\alpha + H_\alpha B^\alpha) \\ &= \varepsilon_{\text{fluid}}(n, S) + \frac{1}{8\pi} (\hat{\varepsilon} E_\alpha E^\alpha + \hat{\mu} H_\alpha H^\alpha), \end{aligned} \quad (\text{D8})$$

where $\varepsilon_{\text{fluid}}(n, S)$ is the fluid energy density, the same function of n and S as in the absence of the electromagnetic field. Combining Eqs. (D3) and (D8), one obtains

$$P = -\varepsilon_{\text{fluid}}(n, S) + \mu n + TS - \frac{1}{8\pi} (E_\alpha D^\alpha + H_\alpha B^\alpha). \quad (\text{D9})$$

The chemical potential μ and temperature T in this equation still depend on the fields D^α and B^α . As follows from Eqs. (D2) and (D8),

$$\begin{aligned} \mu(n, S, D_\alpha D^\alpha, B_\alpha B^\alpha) &= \frac{\partial \varepsilon(n, S, D_\alpha D^\alpha, B_\alpha B^\alpha)}{\partial n} \\ &= \frac{\partial \varepsilon_{\text{fluid}}(n, S)}{\partial n} - \frac{1}{8\pi} \left(\frac{\partial \hat{\varepsilon}(n, S)}{\partial n} E_\alpha E^\alpha + \frac{\partial \hat{\mu}(n, S)}{\partial n} H_\alpha H^\alpha \right) \\ &\equiv \mu_{\text{fluid}}(n, S) + \delta\mu, \end{aligned} \quad (\text{D10})$$

where $\mu_{\text{fluid}}(n, S) = \partial \varepsilon_{\text{fluid}}(n, S) / \partial n$ is the same function of n and S as in the system without the electromagnetic field and $\delta\mu$ is

$$\delta\mu = -\frac{1}{8\pi} \left(\frac{\partial \hat{\varepsilon}(n, S)}{\partial n} E_\alpha E^\alpha + \frac{\partial \hat{\mu}(n, S)}{\partial n} H_\alpha H^\alpha \right). \quad (\text{D11})$$

Similar formulas can also be written for the temperature, $T = T_{\text{fluid}}(n, S) + \delta T$, where

$$\delta T = -\frac{1}{8\pi} \left(\frac{\partial \hat{\varepsilon}(n, S)}{\partial S} E_\alpha E^\alpha + \frac{\partial \hat{\mu}(n, S)}{\partial S} H_\alpha H^\alpha \right). \quad (\text{D12})$$

Substituting Eq. (D10) and similar equation for T into Eq. (D3), we arrive at

$$P = P_{\text{fluid}} + \delta\mu n + \delta TS - \frac{1}{8\pi} (E_\alpha D^\alpha + H_\alpha B^\alpha), \quad (\text{D13})$$

where $P_{\text{fluid}} = -\varepsilon_{\text{fluid}} + \mu_{\text{fluid}} n + T_{\text{fluid}} S$. Now, using equations derived above one can present Eq. (D1) in the form

$$T^{\mu\nu} = T^{\mu\nu}_{(\text{fluid})} + T^{\mu\nu}_{(\text{EM})}, \quad (\text{D14})$$

where $T^{\mu\nu}_{(\text{fluid})} = (P_{\text{fluid}} + \varepsilon_{\text{fluid}}) u^\mu u^\nu + P_{\text{fluid}} g^{\mu\nu}$ is the fluid energy-momentum tensor (the same as in the absence of electromagnetic field) and $T^{\mu\nu}_{(\text{EM})}$ is the electromagnetic tensor in the *medium*,

$$\begin{aligned} T^{\mu}_{(\text{EM})\nu} &= \perp^\mu{}_\nu (\delta\mu n + \delta TS) - \frac{1}{8\pi} g^\mu{}_\nu (E_\alpha D^\alpha + H_\alpha B^\alpha) \\ &\quad + T^{\mu}_{(\text{E})\nu} + T^{\mu}_{(\text{M})\nu} \\ &= \perp^\mu{}_\nu (\delta\mu n + \delta TS) + \frac{1}{8\pi} g^\mu{}_\nu (E_\alpha D^\alpha - H_\alpha B^\alpha) \\ &\quad + \frac{1}{4\pi} [G^{\mu\alpha} F_{\nu\alpha} + u^\mu u^\nu \perp_{\nu\beta} (F^{\alpha\beta} G_{\alpha\gamma} - G^{\alpha\beta} F_{\alpha\gamma})]. \end{aligned} \quad (\text{D15})$$

It is easily checked that this tensor equals to the so-called Abraham tensor, $T^{\mu\nu}_{(\text{Abraham})}$ [46,68,69],

$$T^{\mu\nu}_{(\text{EM})} \equiv T^{\mu\nu}_{(\text{Abraham})} = T^{\mu\nu}_{(\text{Minkowski})} + (g^{\mu}_{(\text{A})} - g^{\mu}_{(\text{M})}) u^\nu, \quad (\text{D16})$$

where $T^{\mu\nu}_{(\text{Minkowski})}$ is the Minkowski tensor [46],

$$\begin{aligned} T^{\mu\nu}_{(\text{Minkowski})} &\equiv \perp^{\mu\nu} (\delta\mu n + \delta TS) \\ &\quad + \frac{1}{4\pi} \left(F^\mu{}_\gamma G^{\nu\gamma} - \frac{1}{4} g^{\mu\nu} F_{\gamma\delta} G^{\gamma\delta} \right). \end{aligned} \quad (\text{D17})$$

and the four-vectors $g^{\mu}_{(\text{A})}$ and $g^{\mu}_{(\text{M})}$ are

$$g^{\mu}_{(\text{A})} = \frac{1}{4\pi} \varepsilon^{\mu\nu\alpha\beta} u_\nu E_\alpha H_\beta, \quad (\text{D18})$$

$$g^{\mu}_{(\text{M})} = \frac{1}{4\pi} \varepsilon^{\mu\nu\alpha\beta} u_\nu D_\alpha B_\beta. \quad (\text{D19})$$

The latter four-vectors reduce, in the comoving frame, to

$$(0, \mathbf{g}_A) = \left(0, \frac{\mathbf{E} \times \mathbf{H}}{4\pi} \right), \quad (\text{D20})$$

$$(0, \mathbf{g}_M) = \left(0, \frac{\mathbf{D} \times \mathbf{B}}{4\pi} \right), \quad (\text{D21})$$

where \mathbf{g}_A is the so-called Abraham momentum density (it coincides with the energy flux density) and \mathbf{g}_M is the Minkowski momentum density. In the comoving frame the tensor $T^{\mu\nu}_{(\text{EM})}$ [= $T^{\mu\nu}_{(\text{Abraham})}$] can be schematically presented as

$$T^{\mu\nu}_{(\text{EM})} = T^{\mu\nu}_{(\text{Abraham})} = \begin{pmatrix} \varepsilon_{\text{EM}} & \mathbf{g}_A \\ \mathbf{g}_A & -\sigma_{lm} \end{pmatrix}, \quad (\text{D22})$$

where ε_{EM} is the energy density and σ^{lm} is the stress tensor of the electromagnetic field ($l, m = 1, 2, 3$),

$$\varepsilon_{EM} = \frac{1}{8\pi}(\hat{\varepsilon}E^2 + \hat{\mu}H^2), \quad (D23)$$

$$\sigma^{lm} = \frac{1}{4\pi}(E^l D^m + H^l B^m) - \left[\frac{E^2}{8\pi} \left(\hat{\varepsilon} - n \frac{\partial \hat{\varepsilon}}{\partial n} - S \frac{\partial \hat{\varepsilon}}{\partial S} \right) + \frac{H^2}{8\pi} \left(\hat{\mu} - n \frac{\partial \hat{\mu}}{\partial n} - S \frac{\partial \hat{\mu}}{\partial S} \right) \right] \delta^{lm}, \quad (D24)$$

and δ^{lm} is the Kronecker symbol. To obtain Eq. (D24), we express $\delta\mu$ and δT with the help of Eqs. (D11) and (D12). Usually, one accounts only for the dependence of $\hat{\varepsilon}$ and $\hat{\mu}$ on n [46]. In the latter case Eq. (D22) reduces to the standard equation for Abraham tensor (see, e.g., Refs. [46,69]).

APPENDIX E: GENERAL FORMULAS FOR ISOLATED NEUTRON AND PROTON VORTICES

Here we briefly review the properties of isolated neutron and proton vortices taking into account the entrainment effect [26] and closely following Refs. [11,15], GAS11, and G16. Note, however, that our consideration differs from that in Refs. [11,15] and GAS11 in three aspects: (i) we use a bit different (but equivalent) formulation of superfluid hydrodynamics; (ii) we consider the *relativistic npe* mixture, and thus employ relativistic entrainment matrix instead of its nonrelativistic counterpart [26]; (iii) we do not assume the zero-temperature approximation. Although below we make use of the London equations, one should bear in mind that it is not a very good approximation when the particle coherence length becomes comparable to their London penetration depth [59,67].

1. London equations and their solution

Assume that a neutron $i = n$ or proton $i = p$ vortex is at rest in the chosen coordinate frame and there are no external (superfluid and normal) particle currents and magnetic field at the spatial infinity. We also assume that all the velocities generated by the vortex are nonrelativistic (but, at the same time, equation of state is relativistic), so that one can use nonrelativistic expressions for, e.g., particle current densities. All equations below are written in dimensional units.

Consider, for example, a proton vortex ($i = p$; the case $i = n$ can then be obtained by exchanging $p \rightleftharpoons n$ in all formulas). In the presence of the vortex p the gradient of the scalar ϕ_p , which is proportional to the wave-function phase Φ_p of the Cooper-pair condensate ($\phi_p = \Phi_p/2$), is given by (e.g., G16)

$$\partial^a \phi_p = \frac{e_\varphi}{2r}, \quad (E1)$$

where e_φ is the unit vector in the azimuthal direction (φ is the polar angle); r is the distance from the vortex; and $a = 1, 2, 3$ is the space index. Using Eq. (40) one then has

$$w_{(p)}^a = \hbar c \partial^a \phi_p - e_p A^a, \quad (E2)$$

where we make use of the fact that $u^a = (0, 0, 0)$. Similarly, for neutrons one has

$$\partial^a \phi_n = 0, \quad (E3)$$

$$w_{(n)}^a = -e_n A^a, \quad (E4)$$

(we do not set $e_n = 0$ in order to rewrite easily these formulas for neutron vortex if necessary), so that the total electric current density is [see Eq. (33)]

$$\mathbf{J}_{\text{free}} = c e_i Y_{ik} w_{(k)}^a = a_1 A^a + a_2 \partial^a \phi_p, \quad (E5)$$

where the parameters a_1 and a_2

$$a_1 = -c(e_n^2 Y_{nn} + 2e_n e_p Y_{np} + e_p^2 Y_{pp}), \quad (E6)$$

$$a_2 = \hbar c^2(e_n Y_{np} + e_p Y_{pp}) \quad (E7)$$

are constants since we neglect small dependence of Y_{ik} on r (see, e.g., Ref. [25] and G16 where a similar approximation is discussed). Now, using Maxwell's equations (3) and (4) with $\mathbf{H} = \mathbf{B}$, one arrives at the following equation for the vortex magnetic field \mathbf{B}

$$-\Delta \mathbf{B} = \frac{4\pi}{c} [a_1 \mathbf{B} + \pi a_2 e_z \delta(r)], \quad (E8)$$

or

$$\Delta \mathbf{B} - \frac{1}{\delta_p^2} \mathbf{B} = -\frac{\hat{\phi}_{p0}}{\delta_p^2} e_z \delta(r), \quad (E9)$$

where $\delta(r)$ is the two-dimensional delta-function in polar coordinate system (r, ϕ) ; e_z is the unit vector along the vortex axis; and

$$\frac{1}{\delta_p^2} \equiv -\frac{4\pi a_1}{c} = 4\pi(e_n^2 Y_{nn} + 2e_n e_p Y_{np} + e_p^2 Y_{pp}), \quad (E10)$$

$$\hat{\phi}_{p0} \equiv -\frac{\pi a_2}{a_1} = \frac{\pi \hbar c (e_n Y_{np} + e_p Y_{pp})}{e_n^2 Y_{nn} + 2e_n e_p Y_{np} + e_p^2 Y_{pp}}. \quad (E11)$$

Here δ_p is the London penetration depth and $\hat{\phi}_{p0}$ is the magnetic flux associated with the vortex (see below). The nonrelativistic limit of these equations can be reproduced if one takes into account that then $Y_{ik} \rightarrow \rho_{ik}/(m_i m_k c^2)$, where ρ_{ik} is the entrainment (or mass-density) matrix

[26,54–57]. Equation (E9) can easily be solved [59], the result is

$$\mathbf{B}(r) = \frac{\hat{\phi}_{p0}}{2\pi\delta_p^2} K_0\left(\frac{r}{\delta_p}\right) \mathbf{e}_z, \quad (\text{E12})$$

where $K_0(r)$ is the MacDonald function. One can verify that, indeed, $\hat{\phi}_{p0}$ is the total vortex magnetic flux, $\int_0^\infty B(r)2\pi r dr = \hat{\phi}_{p0}$. Using (E12), one finds: $\text{curl } \mathbf{B} = \hat{\phi}_{p0}/(2\pi\delta_p^3)K_1(r/\delta_p)\mathbf{e}_\varphi$, and hence from Eqs. (4) and (E5)

$$\mathbf{A}(r) = \frac{\hat{\phi}_{p0}}{2\pi} \left[\frac{1}{r} - \frac{1}{\delta_p} K_1\left(\frac{r}{\delta_p}\right) \right] \mathbf{e}_\varphi, \quad (\text{E13})$$

so that Eqs. (E2) and (E4) can be rewritten as

$$w_{(p)}^a = \frac{\hbar c}{2r} \left(1 - \frac{e_p \hat{\phi}_{p0}}{\pi \hbar c} \right) \mathbf{e}_\varphi + \frac{e_p \hat{\phi}_{p0}}{2\pi \delta_p} K_1\left(\frac{r}{\delta_p}\right) \mathbf{e}_\varphi, \quad (\text{E14})$$

$$w_{(n)}^a = -\frac{e_n \hat{\phi}_{p0}}{2\pi} \left[\frac{1}{r} - \frac{1}{\delta_p} K_1\left(\frac{r}{\delta_p}\right) \right] \mathbf{e}_\varphi. \quad (\text{E15})$$

For neutron vortex similar formulas can be obtained by exchanging $p \rightleftharpoons n$ in Eqs. (E1)–(E15). Note that, in the case of protons, the first term in the rhs of Eq. (E14) equals zero.

2. Vortex energy

Neglecting a small contribution from the vortex core, the general expression for the vortex energy per unit length is

$$\hat{E}_V = \int \frac{1}{2} [Y_{nn} \mathbf{w}_n^2 + 2Y_{np} \mathbf{w}_n \mathbf{w}_p + Y_{pp} \mathbf{w}_p^2] r dr d\varphi + \int \frac{B^2}{8\pi} r dr d\varphi, \quad (\text{E16})$$

where $\mathbf{w}_i = [w_{(i)}^1, w_{(i)}^2, w_{(i)}^3]$. The first integral in this equation is the kinetic energy of superfluid currents [15,43]; the second integral is the magnetic energy, it is generally smaller (e.g., GAS11). Equations (E12), (E14), and (E15) allow one to calculate the integrals in Eq. (E16) and to obtain the following approximate expressions for, respectively, proton \hat{E}_{Vp} and neutron \hat{E}_{Vn} vortex energies per unit length,

$$\hat{E}_{Vp} \approx \frac{\pi}{4} \hbar^2 c^2 Y_{pp} \ln\left(\frac{\delta_p}{\xi_p}\right), \quad (\text{E17})$$

$$\hat{E}_{Vn} \approx \frac{\pi}{4} \hbar^2 c^2 \frac{(Y_{nn} Y_{pp} - Y_{np}^2)}{Y_{pp}} \ln\left(\frac{b_n}{\xi_n}\right). \quad (\text{E18})$$

In these formulas ξ_p and ξ_n are, respectively, the proton and neutron coherence lengths [15] (effective sizes of the

vortex cores) and b_n is some “external” radius of the order of the typical intervortex spacing (see, e.g., Refs. [25] and G16). In the nonrelativistic limit, these formulas reduce to the corresponding expressions (A12) and (A18) of Mendell [15].

Equations (E17) and (E18) are derived under assumption that a neutron (proton) vortex is at rest in the comoving frame [i.e., in the frame in which $u^\mu = (1, 0, 0, 0)$]. As it is argued in G16 in application to uncharged superfluids, the same equations also apply to moving vortices, provided that the difference between the macroscopic (smooth-averaged) normal and superfluid velocities in the system is much smaller than the speed of light c . The latter condition is always satisfied in neutron stars (see G16 for details). Thus, it is justifiable to assume that Eqs. (E17) and (E18) represent correct vortex energies, independently of whether vortices move or not.

APPENDIX F: BOUND CHARGES IN THE PRESENCE OF VORTICES

The aim of this appendix is to explain why the displacement field \mathbf{D} is not generally equal to the electric field \mathbf{E} in the system with vortices. In what follows, it is assumed that we sit in the comoving frame, i.e. the frame associated with the normal liquid component. Consider, for example, a single proton vortex directed along the axis z of the Cartesian coordinate system xyz and moving with the velocity $\mathbf{v}_L = v_{Lx} \mathbf{e}_x + v_{Lz} \mathbf{e}_z$, where \mathbf{e}_x and \mathbf{e}_z are the unit vectors along the axes x and z , respectively. In the rest frame of the vortex its magnetic field $\mathbf{B}(r)$ is given by Eq. (E12). Correspondingly, as follows from Eq. (2), in the comoving frame it generates the electric field (e.g., Ref. [78]),

$$\mathbf{E} = -\frac{1}{c} \mathbf{v}_L \times \mathbf{B}(r), \quad (\text{F1})$$

(we assume that $|\mathbf{v}_L| \ll c$, which is always the case [79]; the same formula can be obtained by making Lorentz transformation from the vortex rest frame to the comoving frame). An associated charge density, ρ_c , induced in that frame, is found from Maxwell’s equation $\text{div } \mathbf{E} = 4\pi \rho_c$,

$$\rho_c = \frac{v_{Lx}}{4\pi c} \frac{dB(r)}{dr} \sin \varphi, \quad (\text{F2})$$

where φ is the polar angle in the xy -plane. Correspondingly, the dipole moment of the vortex segment of length Δz is given by

$$\mathbf{P}_V = \int \mathbf{r} \rho_c dV = -\frac{v_{Lx}}{4\pi c} \hat{\phi}_{p0} \Delta z \mathbf{e}_y, \quad (\text{F3})$$

where $\hat{\phi}_{p0}$ is introduced in Eq. (G13) and \mathbf{e}_y is the unit vector along y . Now, assuming that there are many vortices

moving with one and the same velocity v_L , the dipole moment of the unit volume is

$$\mathbf{P} = \frac{\mathbf{P}_V N_{Vp}}{\Delta z} = -\frac{v_{Lx}}{4\pi^2 c b_p^2} \hat{\phi}_{p0} \mathbf{e}_y \quad (\text{F4})$$

[see Eq. (G6) for a definition of N_{Vp}]. It is easily checked that \mathbf{P} and the average electric field \mathbf{E} , generated by vortices, are related by the standard condition [45], $\mathbf{E} = -4\pi\mathbf{P}$, which should take place for any homogeneous system in which all currents are bound, so that $\mathbf{D} = 0$. We come to conclusion that the electric field of moving vortices should be considered as produced by bound charges, similar to how their magnetic field is produced by (vortex) bound currents. A further implication of this observation can be found in Appendix G 2.

APPENDIX G: DETERMINATION OF THE PHENOMENOLOGICAL COEFFICIENTS OF SEC. VII FOR TWO SIMPLE MICROSCOPIC MODELS

Our aim here will be to determine the exact form of Eq. (55) (or, equivalently, to find an expression for $d\varepsilon_{\text{add}}$) in two situations considered above (the intermediate state and “vortex” state of npe mixture). This aim can be achieved by specifying a microphysics model for the energy density of the system. Below, for illustration, we consider two very simple microphysics models (in particular, the model, considered in Sec. G 2 was studied in GAS11), but one should bear in mind that the very same approach can be used to formulate dynamic equations for more elaborated models.

1. Intermediate state of a nonrotating npe mixture (type-I proton superconductivity)

Assume we are sitting in the normal-liquid (comoving) frame in which nonsuperconducting domains (flux tubes) are at rest. Let us calculate the coefficient γ in Eq. (106), which allow us to determine $d\varepsilon_{\text{add}}$ from Eq. (104). In what follows, instead of ε it will be more convenient to deal with the (Helmholtz) free energy density, $F \equiv \varepsilon - TS$.

The magnitude of the field in a flux tube coincides with the critical thermodynamic field H_c [45], it is directed along the average magnetic induction \mathbf{B} , and can be found from the following approximate formula [67],

$$F_{\text{nonsp}} - F_{\text{sp}} \approx \frac{H_c^2}{8\pi}, \quad (\text{G1})$$

where F_{nonsp} is the free energy density of nonsuperconducting matter in the flux tube²⁴ and F_{sp} is the free energy density of the surrounding (superconducting) matter, it is

²⁴It does not include the energy of the magnetic field [67].

the same function of thermodynamic quantities as in the absence of the magnetic field.

Now, introducing the volume fraction occupied by nonsuperconducting domains, x_{nonsp} , and following the consideration of Refs. [45,67] (in particular, neglecting all striction effects), it is easy to obtain an expression for the macroscopically averaged free energy density F of the npe mixture in the intermediate state,

$$F \approx F_{\text{sp}} + \frac{H_c^2}{4\pi} x_{\text{nonsp}}. \quad (\text{G2})$$

On the other hand, magnetic flux conservation requires that the average magnetic induction \mathbf{B} to be given by $|\mathbf{B}| = H_c x_{\text{nonsp}}$. Hence, Eq. (G2) can be represented as [45]

$$F \approx F_{\text{sp}} + \frac{H_c |\mathbf{B}|}{4\pi} = F_{\text{sp}} + \frac{H_c}{4\pi} (B_\mu B^\mu)^{1/2}. \quad (\text{G3})$$

The latter equality is written in an explicitly Lorentz-invariant form; B^μ is given by Eq. (14). Now, using Eqs. (55), (104), (G3), and the definition $F \equiv \varepsilon - TS$, one can find that the macroscopic parameter H^μ of the phenomenological theory of Sec. VII A is

$$H^\mu = \gamma B^\mu, \quad (\text{G4})$$

where

$$\gamma = \frac{H_c}{(B_\mu B^\mu)^{1/2}}. \quad (\text{G5})$$

Remark 1. The model discussed here and in Sec. VII A is designed for describing the nonrotating npe mixture in the intermediate state. Generalization of the model to allow for rotation and neutron vortices is rather straightforward and can be done along the lines discussed in Appendix G 2.

2. The npe mixture with neutron and proton vortices (type-II proton superconductivity)

We follow here the approach similar to that described in Sec. IV. 2 of GAS11 and in Appendix D of G16. We work in the comoving frame and neglect vortex-vortex interactions in all calculations. Assume we have a bunch of parallel neutron or proton vortices with the intervortex spacing b_i ($i = n$ or p). The parameter b_i is related to the average number of vortices N_{Vi} per unit area by the formula (see, e.g., Ref. [25] and G16),

$$\pi b_i^2 = \frac{1}{N_{Vi}}. \quad (\text{G6})$$

On the other hand, as follows from Eqs. (76) and (78) [cf. Eq. (D9) of G16],

$$N_{Vi} = \frac{|\epsilon^{abc}\mathcal{V}_{(i)bc}|}{2\pi\hbar} = \frac{1}{\pi\hbar}|\mathcal{V}_{(Mi)}^a|, \quad (\text{G7})$$

where a , b , and c are the space indices and we use dimensional units. To obtain this formula we perform integration in Eq. (76) over the unit area in the plane perpendicular to vortex lines. The areal density N_{Vi} is *defined* in the comoving frame. It is thus a Lorentz invariant and it can be rewritten in an explicitly Lorentz-invariant form as

$$N_{Vi} = \frac{1}{\pi\hbar} \sqrt{\mathcal{V}_{(Mi)\mu}\mathcal{V}_{(Mi)}^\mu} = \frac{1}{\pi\hbar} \mathcal{V}_{(Mi)} \quad (\text{G8})$$

[see Eq. (99) for the definition of $\mathcal{V}_{(Mi)}$]. For an uncharged fluid $\mathcal{V}_{(Mi)}$ reduce, in the nonrelativistic limit, to $m_i|\text{curl}\mathbf{V}_{si}|$, where m_i is the mass of particle species i and \mathbf{V}_{si} is the superfluid velocity.

Using Eqs. (G6), (G8) and (E17), (E18), the vortex energy density $\epsilon_{\text{vortex}i}$ can be presented as²⁵

$$\epsilon_{\text{vortex}i} = \frac{\hat{E}_{Vi}}{\pi b_i^2} = \frac{\hat{E}_{Vi}}{\pi\hbar} \mathcal{V}_{(Mi)} \equiv \frac{\lambda_i}{m_i} \mathcal{V}_{(Mi)}, \quad (\text{G9})$$

where

$$\lambda_p = \frac{1}{4} \hbar c^2 m_p Y_{pp} \ln\left(\frac{\delta_p}{\xi_p}\right), \quad (\text{G10})$$

$$\lambda_n = \frac{1}{4} \hbar c^2 m_n \frac{(Y_{nn}Y_{pp} - Y_{np}^2)}{Y_{pp}} \ln\left(\frac{b_n}{\xi_n}\right). \quad (\text{G11})$$

In the absence of entrainment ($Y_{np} = 0$) or for a one-component liquid Eq. (G11) reduces to the parameter λ defined in Eq. (D10) of G16. This parameter is, in turn, the relativistic generalization of the parameter λ introduced in Refs. [25,29].

The contribution of vortex magnetic field \mathbf{B}_{Vi} to the total magnetic induction can be found the same way as $\epsilon_{\text{vortex}i}$ [cf. Eq. (50) of GAS11],

$$\mathbf{B}_{Vi} = \hat{\phi}_{i0} N_{Vi} \frac{\mathcal{V}_{(Mi)}^a}{\mathcal{V}_{(Mi)}} = \frac{\hat{\phi}_{i0}}{\pi\hbar} \mathcal{V}_{(Mi)}^a, \quad (\text{G12})$$

where $\mathcal{V}_{(Mi)}^a/\mathcal{V}_{(Mi)}$ is the unit vector along the local direction of vortex lines, while $\hat{\phi}_{p0}$ and $\hat{\phi}_{n0}$ are [see Eq. (E11)]

$$\hat{\phi}_{p0} = \frac{\pi\hbar c}{e_p}, \quad (\text{G13})$$

$$\hat{\phi}_{n0} = \frac{\pi\hbar c}{e_p} \frac{Y_{np}}{Y_{pp}}. \quad (\text{G14})$$

Similarly, the contribution of the vortex electric field \mathbf{E}_{Vi} , to the (averaged) electric field \mathbf{E} is (see Appendix F)

$$\mathbf{E}_{Vi} = -\frac{1}{c} \mathbf{v}_{Li} \times \mathbf{B}_{Vi} = \frac{\hat{\phi}_{i0}}{\pi\hbar c} \mathbf{v}_{Mi} \times \mathbf{v}_{Li} = \frac{\hat{\phi}_{i0}}{\pi\hbar} \mathbf{v}_{Ei}, \quad (\text{G15})$$

where \mathbf{v}_{Li} is the velocity of vortex species i . To obtain the last two equalities in the rhs of Eq. (G15) we made use of Eqs. (103) and (G12).

Having determined $\epsilon_{\text{vortex}i}$, our next step will be to write down the total energy density ϵ of the system in the comoving frame. As it is discussed in detail in GAS11, it is the sum of five²⁶ “noninterfering” terms (see also G16 for a similar discussion of ϵ in an uncharged fluid),

$$\begin{aligned} \epsilon = & \epsilon_{\text{fluid}}(n_n, n_p, n_e, S, w_{(i)\mu} w_{(k)}^\mu) + \epsilon_{\text{vortex}n} \\ & + \epsilon_{\text{vortex}p} + \frac{\mathbf{B}_L^2}{8\pi} + \frac{\mathbf{E}_L^2}{8\pi}. \end{aligned} \quad (\text{G16})$$

The first term here is the same as in the absence of vortices and magnetic field in the system; it consists of the internal energy of the fluid at rest plus kinetic energy of superfluid currents (i.e., terms depending on $w_{(i)\mu} w_{(k)}^\mu$). The differential of ϵ_{fluid} contributes only to the first four terms in Eq. (55) and does not affect $d\epsilon_{\text{add}}$. Thus, this term is not interesting for us here. The second and third terms account for the vortex energies, including the magnetic energy of vortices. Further, the fourth term represents the magnetic energy density of the so-called “London field,” which is not associated with vortices. The London field can be nonzero even far from vortices, and for our model it equals

$$\mathbf{B}_L = \mathbf{B} - \mathbf{B}_{Vn} - \mathbf{B}_{Vp} \quad (\text{G17})$$

[see Eq. (G12) for the definition of vortex contribution to magnetic induction]. Generally, this field is very small. For example, for a uniformly rotating one-component vortex-free superconductor $\mathbf{B}_L \approx -2mc\Omega/e = -2 \times 10^{-2} [\Omega/(100 \text{ s}^{-1})] \text{ G}$, where Ω is the spin frequency, and to make the estimate we take $m = m_p$ and $e = e_p$ (see, e.g., Ref. [15] and GAS11 for more details). Finally, the last term in Eq. (G16) is similar to the fourth term, but describes the *electric* energy density of matter, not associated with vortices. Similarly to Eq. (G17), it can be presented as

²⁵Strictly speaking, this is the vortex energy obtained under assumption that the vortex is at rest in the comoving frame. Thus, it neglects, for example, the contribution to the energy density from the electric field generated by a moving vortex (see Appendix F). All such contributions are small and can be ignored, as it is emphasized in the end of Appendix E 2.

²⁶GAS11 considered only the first four of these terms and ignored the last one since that reference assumed (incorrectly) that there are no bound charges in the system.

$$\mathbf{E}_L = \mathbf{E} - \mathbf{E}_{Vn} - \mathbf{E}_{Vp}. \quad (\text{G18})$$

The two last terms in the rhs of Eq. (G16) can be rewritten in the manifestly Lorentz-invariant form, $\mathbf{B}_L^2/(8\pi) = B_{(L)\mu} B_{(L)}^\mu/(8\pi)$ and $\mathbf{E}_L^2/(8\pi) = E_{(L)\mu} E_{(L)}^\mu/(8\pi)$, if we introduce the London field four-vectors $B_{(L)}^\mu$ and $E_{(L)}^\mu$,

$$B_{(L)}^\mu \equiv B^\mu - B_{(Vn)}^\mu - B_{(Vp)}^\mu, \quad (\text{G19})$$

$$E_{(L)}^\mu \equiv E^\mu - E_{(Vn)}^\mu - E_{(Vp)}^\mu, \quad (\text{G20})$$

where the corresponding vortex-related four-vectors are defined as

$$B_{(Vi)}^\mu \equiv \frac{\hat{\phi}_{i0}}{\pi\hbar} \mathcal{V}_{(Mi)}^\mu, \quad (\text{G21})$$

$$E_{(Vi)}^\mu \equiv \frac{\hat{\phi}_{i0}}{\pi\hbar} \mathcal{V}_{(Ei)}^\mu. \quad (\text{G22})$$

It is easily verified that in the comoving frame the time components of these four-vectors $B_{(L)}^\mu$, $B_{(Vi)}^\mu$, $E_{(L)}^\mu$, and $E_{(Vi)}^\mu$ are all zero, while their spatial components coincide with those of the three-dimensional vectors \mathbf{B}_L , \mathbf{B}_{Vi} , \mathbf{E}_L , and \mathbf{E}_{Vi} , respectively [see Eqs. (G12), (G15), (G17), and (G18)]²⁷

Using these definitions as well as Eqs. (G9) and (G16), the second law of thermodynamics (55) takes the form

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}) + d\varepsilon_{\text{add}}, \quad (\text{G23})$$

where

$$T = \frac{\partial \varepsilon_{\text{fluid}}}{\partial S} + \sum_{k=n,p} \left\{ \frac{1}{m_k} \frac{\partial \lambda_k}{\partial S} \mathcal{V}_{(Mk)} - \frac{1}{4\pi^2 \hbar} \frac{\partial \hat{\phi}_{k0}}{\partial S} B_{(L)\mu} \mathcal{V}_{(Mk)}^\mu \right\}, \quad (\text{G24})$$

$$\mu_i = \frac{\partial \varepsilon_{\text{fluid}}}{\partial n_i} + \sum_{k=n,p} \left\{ \frac{1}{m_k} \frac{\partial \lambda_k}{\partial n_i} \mathcal{V}_{(Mk)} - \frac{1}{4\pi^2 \hbar} \frac{\partial \hat{\phi}_{k0}}{\partial n_i} B_{(L)\mu} \mathcal{V}_{(Mk)}^\mu \right\}, \quad (\text{G25})$$

$$\mu_e = \frac{\partial \varepsilon_{\text{fluid}}}{\partial n_e}, \quad (\text{G26})$$

²⁷Note also that when protons are normal, one has $Y_{np} = 0$ [52], hence $\hat{\phi}_{n0} = 0$ and, consequently, $B_{(Vn)}^\mu = E_{(Vn)}^\mu = 0$.

$$Y_{ik} = 2 \frac{\partial \varepsilon_{\text{fluid}}}{\partial (w_{(i)}^\alpha w_{(k)\alpha})} + 2 \sum_{l=n,p} \left\{ \frac{1}{m_l} \frac{\partial \lambda_l}{\partial [w_{(i)}^\alpha w_{(k)\alpha}]} \mathcal{V}_{(Ml)} - \frac{1}{4\pi^2 \hbar} \frac{\partial \hat{\phi}_{l0}}{\partial [w_{(i)}^\alpha w_{(k)\alpha}]} B_{(L)\mu} \mathcal{V}_{(Ml)}^\mu \right\}, \quad (\text{G27})$$

$$d\varepsilon_{\text{add}} = \sum_{k=n,p} \frac{\lambda_k}{m_k \mathcal{V}_{(Mk)}} \mathcal{V}_{(Mk)\mu} d\mathcal{V}_{(Mk)}^\mu + \frac{1}{4\pi} B_{(L)\mu} \left[dB^\mu - \frac{\hat{\phi}_{n0}}{\pi\hbar} d\mathcal{V}_{(Mn)}^\mu - \frac{\hat{\phi}_{p0}}{\pi\hbar} d\mathcal{V}_{(Mp)}^\mu \right] + \frac{1}{4\pi} \left[E^\mu - \frac{\hat{\phi}_{n0}}{\pi\hbar} \mathcal{V}_{(En)}^\mu - \frac{\hat{\phi}_{p0}}{\pi\hbar} \mathcal{V}_{(Ep)}^\mu \right] dE_{(L)\mu}. \quad (\text{G28})$$

In Eqs. (G24)–(G28) the parameters $\varepsilon_{\text{fluid}}$, λ_i , and $\hat{\phi}_{i0}$ should be treated as the same functions of S , n_i , n_e , and $w_{(i)}^\alpha w_{(k)\alpha}$ as in the absence of vortices and the magnetic field. The underlined terms there are generally small and can be neglected. The terms underlined once are small because they depend on the tiny London field $B_{(L)}^\mu$ [see Eq. (G19)]; the term underlined twice is small because λ_i is a very weak function of $w_{(i)}^\alpha w_{(k)\alpha}$ in the regime when the dependence of Y_{ik} on the difference between the velocities of superfluid and normal liquid components can be neglected (e.g., G16). The second term in Eq. (G28) also depends on $B_{(L)}^\mu$ and can, in principle, be omitted. However, we keep it in what follows because it is this term which makes H^μ nonzero. Comparing (G28) with the general expression (108) for $d\varepsilon_{\text{add}}$ and using Eqs. (114)–(117), one finds

$$\gamma^{(M)} = \tilde{\gamma}^{(E)} = 1, \quad (\text{G29})$$

$$\Gamma_i^{(M)} = \tilde{\Gamma}_i^{(E)} = -\frac{\hat{\phi}_{i0}}{4\pi^2 \hbar}, \quad (\text{G30})$$

$$\Gamma_{ik}^{(M)} = \frac{\lambda_i}{m_i \mathcal{V}_{(Mi)}} \delta_{ik} + \frac{\hat{\phi}_{i0} \hat{\phi}_{k0}}{4\pi^3 \hbar^2}, \quad (\text{G31})$$

$$\tilde{\Gamma}_{ik}^{(E)} = \frac{\hat{\phi}_{i0} \hat{\phi}_{k0}}{4\pi^3 \hbar^2}. \quad (\text{G32})$$

The latter equation differs from its magnetic counterpart, Eq. (G31), because we neglected the electric field contribution to the vortex energy, $\varepsilon_{\text{vortex}i}$. From Eqs. (114), (116), (G19), and (G20) it then follows that $H^\mu = B_{(L)}^\mu$ and $D^\mu = E_{(L)}^\mu$. The first of these equalities was earlier discussed in GAS11.

Remark 1. The results obtained above allow us to make a few useful estimates. First of all, since the total number of neutron vortices in a star is by more than ten orders of magnitude smaller than the total number of proton vortices (for a typical neutron star with $B \sim 10^{12}$ G and a period

$P \sim 0.1$ s, see, e.g., GAS11), one can neglect \mathbf{B}_{Vn} and \mathbf{E}_{Vn} in comparison to, respectively, \mathbf{B}_{Vp} and \mathbf{E}_{Vp} in Eqs. (G17) and (G18), and write

$$\mathbf{B} = \mathbf{H} + \mathbf{B}_{Vn} + \mathbf{B}_{Vp} \approx \mathbf{B}_{Vp}, \quad (\text{G33})$$

$$\mathbf{E} = \mathbf{D} + \mathbf{E}_{Vn} + \mathbf{E}_{Vp} \approx \mathbf{D} + \mathbf{E}_{Vp}. \quad (\text{G34})$$

Here we also neglect \mathbf{H} in Eq. (G33) since typically $|\mathbf{H}| \sim 2 \times 10^{-2} [\Omega / (100 \text{ s}^{-1})] \text{ G} \ll |\mathbf{B}|$, as discussed in the text above. Second, note that for a static or very weakly perturbed neutron star [i.e., a star for which v_{Lp} is so small, that \mathbf{E}_{Vp} in Eq. (G34) can be neglected; see Eq. (G15)], one can estimate $|\mathbf{E}|$ (and $|\mathbf{D}|$) as $|\mathbf{D}| \approx |\mathbf{E}| \sim |\nabla \mu_e| / e_p \sim 1 \text{ g}^{1/2} \text{ cm}^{-1/2} \text{ s}^{-1}$. The latter estimate allows one to find an approximate proton vortex velocity v_{Lp0} at which $|\mathbf{E}_{Vp}|$ becomes comparable to $|\mathbf{D}|$. Using Eq. (G15), one finds $v_{Lp0} \sim c |\nabla \mu_e| / (e_p |\mathbf{B}_{Vp}|) \sim 3 \times 10^{-2} \text{ cm s}^{-1}$ (we take $|\mathbf{B}_{Vp}| \approx |\mathbf{B}| = 10^{12} \text{ G}$). Thus, for example, at $|v_{Lp}| \gg v_{Lp0}$ one has: $\mathbf{E} \approx -(1/c) \mathbf{v}_{Lp} \times \mathbf{B}$, so that $|\mathbf{H}| \lesssim |\mathbf{D}| \ll |\mathbf{E}| \ll |\mathbf{B}|$. Correspondingly, in the opposite limit $|\mathbf{H}| \lesssim |\mathbf{D}| \approx |\mathbf{E}| \ll |\mathbf{B}|$.

APPENDIX H: SUMMARY OF RESULTS: FULL SYSTEM OF RELATIVISTIC EQUATIONS DESCRIBING DYNAMICS OF SUPERFLUID-SUPERCONDUCTING NEUTRON STARS

Here we present the full system of dynamic equations discussed in the main text. For the reader's convenience, this appendix is self-contained. In the present paper, we are mainly interested in nondissipative equations (the only dissipative mechanism, which is accounted for, is the mutual friction, see below). Thus, we assume that neutron and proton thermal excitations (Bogoliubov quasiparticles) as well as electrons move with one and the same four-velocity u^μ , normalized by the condition $u_\mu u^\mu = -1$.

Superfluid degrees of freedom are characterized by the four-vectors $w_{(i)}^\mu$ ($i = n, p$), which are closely related to the superfluid velocities of the corresponding nonrelativistic theory (see Appendix I), and are orthogonal to u^μ ,

$$u_\mu w_{(i)}^\mu = 0. \quad (\text{H1})$$

Other important parameters of the theory include the vorticity tensors $\mathcal{V}_{(i)}^{\mu\nu}$,

$$\mathcal{V}_{(i)}^{\mu\nu} \equiv \partial^\mu [w_{(i)}^\nu + \mu_i u^\nu] - \partial^\nu [w_{(i)}^\mu + \mu_i u^\mu] + e_i F^{\mu\nu}, \quad (\text{H2})$$

and the electromagnetic tensors $F^{\alpha\beta}$ and $G^{\alpha\beta}$ [see Eqs. (8) and (9)], satisfying Maxwell's equations (10) and (11),

$$\partial_\alpha {}^* F^{\alpha\beta} = 0, \quad (\text{H3})$$

$$\partial_\alpha G^{\alpha\beta} = -4\pi J_{(\text{free})}^\beta. \quad (\text{H4})$$

In these formulas, e_i is the charge of nucleon species i ;

$$J_{(\text{free})}^\mu = e_p (n_p - n_e) u^\mu + e_i Y_{ik} w_{(k)}^\mu \quad (\text{H5})$$

is the four-current density of free charges [see Eqs. (33) and (34)]; ${}^* F^{\mu\nu}$ is the tensor dual to $F^{\mu\nu}$ (see Appendix A); the thermodynamic parameters n_e , n_p , and Y_{ik} are defined in what follows. In addition to the tensors $\mathcal{V}_{(i)}^{\mu\nu}$, $F^{\alpha\beta}$, and $G^{\alpha\beta}$ it is convenient to introduce the four-vectors [see Eqs. (12)–(15), (77), and (78)]

$$\mathcal{V}_{(\text{E}i)}^\mu \equiv u_\nu \mathcal{V}_{(i)}^{\mu\nu}, \quad (\text{H6})$$

$$\mathcal{V}_{(\text{M}i)}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{V}_{(i)\alpha\beta}, \quad (\text{H7})$$

$$E^\mu \equiv u_\nu F^{\mu\nu}, \quad (\text{H8})$$

$$D^\mu \equiv u_\nu G^{\mu\nu}, \quad (\text{H9})$$

$$B^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\eta} u_\nu F_{\lambda\eta}, \quad (\text{H10})$$

$$H^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\eta} u_\nu G_{\lambda\eta}. \quad (\text{H11})$$

In the *comoving frame* in which the normal liquid component is at rest [i.e., $u^\mu = (1, 0, 0, 0)$] the space components of the four-vectors E^μ , D^μ , B^μ , and H^μ reduce to the electric field, displacement field, magnetic induction, and magnetic field, respectively.

The equations describing the dynamics of the superfluid-superconducting npe mixture consist of (i) Maxwell's equations (H3) and (H4); (ii) the particle and energy-momentum conservations,

$$\partial_\mu j_{(j)}^\mu = 0, \quad (\text{H12})$$

$$\partial_\mu T^{\mu\nu} = 0 \quad (\text{H13})$$

with

$$j_{(i)}^\mu = n_i u^\mu + Y_{ik} w_{(k)}^\mu, \quad (\text{H14})$$

$$j_{(e)}^\mu = n_e u^\mu, \quad (\text{H15})$$

and

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P g^{\mu\nu} + Y_{ik} (w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu) + \Delta T^{\mu\nu}; \quad (\text{H16})$$

(iii) the second law of thermodynamics [note that all the thermodynamic quantities are measured in the comoving frame, where $u^\mu = (1, 0, 0, 0)$],

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \frac{Y_{ik}}{2} d(w_{(i)}^\alpha w_{(k)\alpha}) + d\varepsilon_{\text{add}}; \quad (\text{H17})$$

and (iv) the superfluid equations, which will be discussed a bit later. In Eqs. (H12)–(H17) n_j and μ_j are, respectively, the number density and relativistic chemical potential of particle species $j = n, p, e$; T, S, ε , and $P = -\varepsilon + \mu_e n_e + \mu_i n_i + TS$ are the temperature, entropy density, energy density, and pressure, respectively. Note that all the thermodynamic quantities are defined (measured) in the comoving frame. Finally, Y_{ik} is the relativistic entrainment matrix [47,48,51–53] and $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the metric tensor.

The corrections $\Delta T^{\mu\nu}$ and $d\varepsilon_{\text{add}}$ in Eqs. (H16) and (H17) appear due to the electromagnetic and vortex contributions to the energy-momentum tensor and energy density, and differ depending on the assumed type (I or II) of the proton superconductivity. The same is also true for superfluid equations, thus they should be discussed separately for each case.

1. Vortex-free *npe* mixture in the intermediate state (type-I proton superconductivity)

Assuming that protons in the *npe* mixture form a type-I superconductor in the intermediate state and that neutrons are superfluid, one has the following formulas for $\Delta T^{\mu\nu}$ and $d\varepsilon_{\text{add}}$ (see Sec. V)

$$\Delta T^{\mu\nu} = \mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu}, \quad (\text{H18})$$

$$d\varepsilon_{\text{add}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu, \quad (\text{H19})$$

where

$$\mathcal{T}_{(E)}^{\mu\nu} = \frac{1}{4\pi} (\perp^{\mu\nu} D^\alpha E_\alpha - D^\mu E^\nu), \quad (\text{H20})$$

$$\mathcal{T}_{(M)}^{\mu\nu} = \frac{1}{4\pi} (\perp^{\mu\alpha} G^{\nu\alpha} F^\nu + u^\nu \perp^{\mu\alpha} G^{\nu\alpha} E_\alpha + u^\mu \perp^{\nu\alpha} G^{\nu\alpha} E_\alpha), \quad (\text{H21})$$

and

$$\perp^{\mu\nu} G^{\mu\nu} = \varepsilon^{\alpha\beta\mu\nu} u_\beta H_\alpha \quad (\text{H22})$$

(see Appendix A). In turn, superfluid equations for protons and neutrons take the following form [see Eqs. (70) and (73); we assume that there are no neutron vortices in the system]:

$$\mathcal{V}_{(n)}^{\mu\nu} = 0, \quad (\text{H23})$$

$$u_\mu \mathcal{V}_{(p)}^{\mu\nu} = 0. \quad (\text{H24})$$

These equations should be supplemented by the two conditions relating the four-vectors D^μ with E^μ and H^μ with B^μ . These conditions are obtained in Sec. VII A and in Appendix G 1.

2. The *npe* mixture in the presence of neutron and proton vortices (type-II proton superconductivity)

Assume now that protons form a type-II superconductor and consider the *npe* mixture in the mixed state, allowing for the presence of both neutron and proton vortices. The corrections $\Delta T^{\mu\nu}$ and $d\varepsilon_{\text{add}}$ are then given by (see Sec. VI)

$$d\varepsilon_{\text{add}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}_{(Ei)}^\mu d\mathcal{W}_{(Ei)\mu} + \mathcal{W}_{(Mi)\mu} d\mathcal{V}_{(Mi)}^\mu, \quad (\text{H25})$$

$$\Delta T^{\mu\nu} = \mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu} + \mathcal{T}_{(VE)}^{\mu\nu} + \mathcal{T}_{(VM)}^{\mu\nu}, \quad (\text{H26})$$

where $\mathcal{W}_{(Ei)}^\mu$ and $\mathcal{W}_{(Mi)}^\mu$ are the four-vectors analogous to D^μ and H^μ , respectively; their relation to the four-vectors $\mathcal{V}_{(Ei)}^\mu$, E^μ , $\mathcal{V}_{(Mi)}^\mu$, and B^μ is explored in Sec. VII B and (for a particular model) in Appendix G 2. In Eq. (H26) $\mathcal{T}_{(E)}^{\mu\nu}$ and $\mathcal{T}_{(M)}^{\mu\nu}$ are given by Eqs. (H20) and (H21), respectively, while $\mathcal{T}_{(VE)}^{\mu\nu}$ and $\mathcal{T}_{(VM)}^{\mu\nu}$ are

$$\mathcal{T}_{(VE)}^{\mu\nu} = \perp^{\mu\nu} \mathcal{W}_{(Ei)}^\alpha \mathcal{V}_{(Ei)\alpha} - \mathcal{W}_{(Ei)}^\mu \mathcal{V}_{(Ei)}^\nu, \quad (\text{H27})$$

$$\mathcal{T}_{(VM)}^{\mu\nu} = \perp^{\mu\alpha} \mathcal{W}_{(i)}^{\nu\alpha} \mathcal{V}_{(i)\alpha} + u^\nu \perp^{\mu\alpha} \mathcal{W}_{(i)}^{\nu\alpha} \mathcal{V}_{(Ei)\alpha} + u^\mu \perp^{\nu\alpha} \mathcal{W}_{(i)}^{\nu\alpha} \mathcal{V}_{(Ei)\alpha}, \quad (\text{H28})$$

where

$$\perp^{\mu\nu} \mathcal{W}_{(i)}^{\mu\nu} = \varepsilon^{\alpha\beta\mu\nu} u_\beta \mathcal{W}_{(Mi)\alpha}. \quad (\text{H29})$$

The superfluid equations for neutrons ($i = n$) and protons ($i = p$) take the form

$$u^\nu \mathcal{V}_{(i)\mu\nu} = \mu_i n_i f_{(i)\mu}, \quad (\text{H30})$$

where

$$f_{(i)}^\mu = \alpha_i \perp^{\mu\nu} \mathcal{V}_{(i)\nu\lambda} \mathcal{W}_{(i)\delta} \perp^{\lambda\delta} + \frac{\beta_i - \gamma_i}{\mathcal{V}_{(Mi)}} \perp^{\mu\eta} \perp^{\nu\sigma} \mathcal{V}_{(i)\eta\sigma} \mathcal{V}_{(i)\lambda\nu} \mathcal{W}_{(i)\delta} \perp^{\lambda\delta} + \gamma_i \mathcal{V}_{(Mi)} \mathcal{W}_{(i)\delta} \perp^{\mu\delta}, \quad (\text{H31})$$

(see remark 1 in Sec. VI). In Eq. (H31) $\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$; α_i is a nondissipative mutual friction coefficient; $\beta_i \geq 0$ and $\gamma_i \geq 0$ are the positive dissipative mutual friction coefficients, and²⁸

$$W_{(i)}^\mu \equiv \frac{1}{n_i} [Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{W}_{(i)}^{\mu\alpha}], \quad (\text{H32})$$

$$\mathcal{V}_{(Mi)} \equiv \sqrt{\mathcal{V}_{(Mi)}^\mu \mathcal{V}_{(Mi)\mu}}. \quad (\text{H33})$$

Recalling the definition (H6), one sees that Eq. (H30) is simply the statement that

$$\mathcal{V}_{(Ei)}^\mu = \mu_i n_i f_{(i)}^\mu. \quad (\text{H34})$$

As in Appendix H 1, the dynamic equations formulated here should be supplemented with the expressions relating the vectors D^μ , H^μ , $\mathcal{W}_{(Ei)}^\mu$, $\mathcal{W}_{(Mi)}^\mu$ with E^μ , B^μ , $\mathcal{V}_{(Ei)}^\mu$, $\mathcal{V}_{(Mi)}^\mu$. These expressions are discussed in Sec. VII B and in Appendix G 2.

APPENDIX I: NONRELATIVISTIC LIMIT OF “MAGNETOHYDRODYNAMIC” EQUATIONS OF SEC. VIII

Here we present the nonrelativistic limit of the simplified dynamic equations discussed in Sec. VIII. In what follows, unless otherwise stated, all the three-dimensional vectors appearing in the text (shown in boldface) are defined in the *laboratory* frame. As in other parts of the paper, indices i and k refer to nucleons: $i, k = n, p$; other Latin letters are the space indices; we use dimensional units in this Appendix.

The four-vector u^μ is related to the normal velocity \mathbf{V}_{norm} of the nonrelativistic superfluid hydrodynamics by the standard formula,

$$u^\mu \equiv (u^0, \mathbf{u}) = \left(\frac{1}{\sqrt{1 - \mathbf{V}_{\text{norm}}^2/c^2}}, \frac{\mathbf{V}_{\text{norm}}}{c\sqrt{1 - \mathbf{V}_{\text{norm}}^2/c^2}} \right). \quad (\text{II})$$

Instead of the four-vector $w_{(i)}^\mu \equiv (w_{(i)}^0, \mathbf{w}_{(i)})$, it is convenient to introduce the superfluid four-velocity $V_{(si)}^\mu \equiv (V_{(si)}^0, \mathbf{V}_{(si)})$, such that

$$w_{(i)}^\mu = m_i c V_{(si)}^\mu - \mu_i u^\mu. \quad (\text{I2})$$

As shown in G16 (see also Ref. [47]), the spatial component $\mathbf{V}_{(si)}$ of this four-vector is the superfluid velocity of

²⁸The tensor $\mathcal{W}_{(i)}^{\mu\nu}$ in Eq. (H32) equals $\mathcal{W}_{(i)}^{\mu\nu} = \parallel \mathcal{W}_{(i)}^{\mu\nu} + \perp \mathcal{W}_{(i)}^{\mu\nu}$, where $\parallel \mathcal{W}_{(i)}^{\mu\nu} = -u^\mu \mathcal{W}_{(Ei)}^\mu + u^\nu \mathcal{W}_{(Ei)}^\nu$ (see Appendix A).

the nonrelativistic theory.²⁹ Using Eq. (25) and the definition (I2), one finds the following equation for $V_{(si)}^\mu$, $u_\mu V_{(si)}^\mu = -\mu_i/(m_i c)$, from which the time component $V_{(si)}^0$ is

$$V_{(si)}^0 = \frac{\mu_i}{m_i c u^0} + \frac{\mathbf{u} \cdot \mathbf{V}_{(si)}}{u^0}. \quad (\text{I3})$$

In terms of $V_{(si)}^\mu$, the vorticity tensor (74) can be rewritten as

$$\begin{aligned} \mathcal{V}_{(i)}^{\mu\nu} &\equiv \frac{1}{c} \{ \partial^\mu [w_{(i)}^\nu + \mu_i u^\nu] - \partial^\nu [w_{(i)}^\mu + \mu_i u^\mu] + e_i F^{\mu\nu} \} \\ &= m_i [\partial^\mu V_{(si)}^\nu - \partial^\nu V_{(si)}^\mu] + \frac{e_i}{c} F^{\mu\nu}, \end{aligned} \quad (\text{I4})$$

while the electric vector $\mathcal{V}_{(Ei)}^\mu$ is given by Eq. (96), and the magnetic vector $\mathcal{V}_{(Mi)}^\mu$ is [see Eq. (H7)]

$$\mathcal{V}_{(Mi)}^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu m_i [\partial_\alpha V_{(si)\beta} - \partial_\beta V_{(si)\alpha}] + \frac{e_i}{c} B^\mu. \quad (\text{I5})$$

and reduces to $\mathcal{V}_{(Mi)}^\mu = (0, m_i \boldsymbol{\omega}_i)$ in the comoving frame, where we defined

$$\boldsymbol{\omega}_i \equiv \text{curl} \mathbf{V}_{(si)} + \frac{e_i}{m_i c} \mathbf{B}. \quad (\text{I6})$$

To leading order in $\mathbf{V}_{\text{norm}}/c$ the same expression $\mathcal{V}_{(Mi)}^\mu = (0, m_i \boldsymbol{\omega}_i)$ is also valid in the laboratory frame (and this is also true for other “magnetic” vectors). It remains to express the relativistic entrainment matrix, Y_{ik} , through its nonrelativistic counterpart, ρ_{ik} . As shown, e.g., in Ref. [47], in the nonrelativistic limit they are related by the formula: $\rho_{ik} = m_i m_k c^2 Y_{ik}$.

Using these definitions and relations the nonrelativistic version of the superfluid equation (93) takes the form

$$\begin{aligned} \partial_t \mathbf{V}_{(si)} + (\mathbf{V}_{(si)} \cdot \nabla) \mathbf{V}_{(si)} + \nabla \left[\check{\mu}_i - \frac{1}{2} |\mathbf{V}_{(si)} - \mathbf{V}_{\text{norm}}|^2 \right] \\ = -\text{curl} \mathbf{V}_{(si)} \times (\mathbf{V}_{\text{norm}} - \mathbf{V}_{(si)}) \\ - n \mathbf{f}_i + \frac{e_i}{m_i} \left(\mathbf{E} + \frac{\mathbf{V}_{\text{norm}}}{c} \times \mathbf{B} \right), \end{aligned} \quad (\text{I7})$$

where $\check{\mu}_i \equiv (\mu_i - m_i c^2)/m_i$ and

$$\begin{aligned} \mathbf{f}_i = -\alpha_i m_i [\boldsymbol{\omega}_i \times \mathbf{W}_i] - \beta_i m_i \mathbf{e}_i \times [\boldsymbol{\omega}_i \times \mathbf{W}_i] \\ + \gamma_i m_i \mathbf{e}_i (\mathbf{W}_i \cdot \boldsymbol{\omega}_i). \end{aligned} \quad (\text{I8})$$

²⁹Note that $V_{(si)}^\mu$ is measured in cm/s while u^μ is dimensionless [see Eq. (II)].

In the latter formula $\mathbf{e}_i = \boldsymbol{\omega}_i/|\boldsymbol{\omega}_i|$; \mathbf{W}_i is the spatial part of the four-vector $W_{(i)}^\mu$, which is, in the dimensional form [see footnote 28 and Eqs. (H29), (95)],

$$W_{(i)}^\mu = \frac{1}{n_i} [cY_{ik}W_{(k)}^\mu + \partial_\alpha(\epsilon^{\delta\beta\mu\alpha}u_\beta\mathcal{W}_{(Mi)\delta} + \|\mathcal{W}_{(i)}^{\mu\alpha}\|)]. \quad (\text{I9})$$

We have not made yet any simplifying assumption about the value of the magnetic induction \mathbf{B} , so up until now our nonrelativistic equations are quite general. Now let us make full use of simplifications of Sec. VIII.³⁰ Employing Eqs. (119) and (120), Eq. (I6) can be presented as

$$\boldsymbol{\omega}_n = \text{curl}\mathbf{V}_{sn}, \quad (\text{I10})$$

$$\boldsymbol{\omega}_p \approx \frac{e_p}{m_p c} \mathbf{B}. \quad (\text{I11})$$

In turn, Eq. (124) becomes

$$\mathcal{W}_{(Mn)}^\mu \equiv (0, \mathcal{W}_{Mn}) = \frac{\lambda_n}{m_n \mathcal{V}_{(Mn)}} \mathcal{V}_{(Mn)}^\mu = \frac{\lambda_n}{m_n \boldsymbol{\omega}_n} (0, \boldsymbol{\omega}_n), \quad (\text{I12})$$

$$\mathcal{W}_{(Mp)}^\mu \equiv (0, \mathcal{W}_{Mp}) \approx \frac{\lambda_p}{m_p \mathcal{V}_{(Mp)}} \mathcal{V}_{(Mp)}^\mu \approx \frac{\lambda_p}{m_p B} (0, \mathbf{B}). \quad (\text{I13})$$

As it was argued in Sec. VIII, the term depending on $\|\mathcal{W}_{(i)}^{\mu\alpha}\|$ in Eq. (I9) is small and can be omitted. Thus, the resulting nonrelativistic expression for $W_{(i)}^\mu$ is given by (see Appendix C of G16 for a similar equation)

$$\mathbf{W}_i = \frac{1}{n_i} \left[\sum_{k=n,p} \frac{\rho_{ik}}{m_i} (\mathbf{V}_{sk} - \mathbf{V}_{\text{norm}}) + \text{curl}\mathcal{W}_{Mi} \right], \quad (\text{I14})$$

where the vectors \mathcal{W}_{Mi} are defined in Eqs. (I12)–(I13). Equations (I10)–(I11) and (I14) should be used to calculate \mathbf{f}_i [see Eq. (I8)]. Equation (I14) can be further simplified in the case of protons ($i = p$) if we note that the conditions (122) and (123) can be rewritten as

$$n_e = n_p, \quad (\text{I15})$$

$$\sum_{k=n,p} \rho_{pk} (\mathbf{V}_{sk} - \mathbf{V}_{\text{norm}}) = 0. \quad (\text{I16})$$

Using Eqs. (I14) and (I16), one obtains

³⁰We remind the reader that Sec. VIII utilizes the model of noninteracting vortices discussed in Appendix G 2.

$$\mathbf{W}_p = \frac{1}{n_p} \text{curl}\mathcal{W}_{Mp}. \quad (\text{I17})$$

Next, within the magnetohydrodynamic approximation adopted here, the vortex-related corrections (125) and (126) to, respectively, the second law of thermodynamics (55) and the energy-momentum tensor (56) are given, in the nonrelativistic limit, by

$$\begin{aligned} d\epsilon_{\text{add}} &\approx \sum_{i=n,p} \frac{\lambda_i}{m_i \mathcal{V}_{(Mi)}} \mathcal{V}_{(Mi)\mu} d\mathcal{V}_{(Mi)}^\mu \\ &= \frac{\lambda_n}{|\boldsymbol{\omega}_n|} \boldsymbol{\omega}_n d\boldsymbol{\omega}_n + \frac{\lambda_p}{|\boldsymbol{\omega}_p|} \boldsymbol{\omega}_p d\boldsymbol{\omega}_p, \end{aligned} \quad (\text{I18})$$

$$\begin{aligned} \Delta T^{\mu\nu} &= \mathcal{T}_{(\text{VM})}^{\mu\nu} \\ &= \begin{pmatrix} 0 & \mathbf{g}_n \\ \mathbf{g}_n & \Pi_{(\text{Vn})}^{lm} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{g}_p \\ \mathbf{g}_p & \Pi_{(\text{Vp})}^{lm} \end{pmatrix}, \end{aligned} \quad (\text{I19})$$

where

$$\mathbf{g}_i = \frac{1}{c} [m_i n_i \mathbf{f}_i + (\mathcal{V}_{Mi} \times \mathbf{V}_{\text{norm}})] \times \mathcal{W}_{Mi}, \quad (\text{I20})$$

$$\Pi_{(\text{Vi})}^{lm} = \mathcal{V}_{Mi} \mathcal{W}_{Mi} \delta^{lm} - \mathcal{V}_{(Mi)}^l \mathcal{W}_{(Mi)}^m \quad (\text{I21})$$

and $\mathcal{V}_{Mi} = m_i \boldsymbol{\omega}_i$. Using the definition for the critical magnetic field H_{c1} , $H_{c1} \equiv 4\pi \hat{E}_{Vp} / \hat{\phi}_{p0}$ (see, e.g., Ref. [59]), as well as Eqs. (G9), (G14), (I11), (I13), and (I21), it is easily demonstrated that the proton tensor $\Pi_{(\text{Vp})}^{lm}$ can be represented as

$$\Pi_{(\text{Vp})}^{lm} = \frac{H_{c1}}{4\pi} \left(B \delta^{lm} - \frac{B^l B^m}{B} \right). \quad (\text{I22})$$

Note that $d\epsilon_{\text{add}}$ in Eq. (I18) can be considered as defined in the laboratory frame up to corrections $\sim \mathbf{V}_{\text{norm}}/c$. All other parameters and equations of the theory [e.g., continuity equations, the remaining parts of the second law of thermodynamics (55) and the energy-momentum tensor (56)] have the same form as in the standard (vortex-free) superfluid hydrodynamics (see, e.g., Refs. [25,60,80] and G16). However, it is very important to point out that the temperature T and chemical potential μ_i will be renormalized in the presence of vortices according to Eqs. (G24) and (G25).

Remark 1. Using the equations obtained above it is straightforward to derive the ‘‘magnetic evolution’’ equation. To this aim let us take a curl of Eq. (I7) written for protons ($i = p$). Then, using Maxwell’s equation (2) and neglecting the terms depending on $\text{curl}\mathbf{V}_{sp}$ in comparison to the similar terms depending on $e_p/(m_p c) \mathbf{B}$ [our magnetohydrodynamic approximation; see a note after Eq. (120)], one gets

$$\frac{\partial \mathbf{B}}{\partial t} + \text{curl} \left[\frac{m_p c}{e_p} n_p \mathbf{f}_p + \mathbf{B} \times \mathbf{V}_{\text{norm}} \right] = 0. \quad (\text{I23})$$

This equation can be further simplified if one neglects the small kinetic coefficient γ_p in Eq. (I8). Eq. (I23) can then be rewritten as (see also Ref. [75] for a similar equation)

$$\frac{\partial \mathbf{B}}{\partial t} + \text{curl}(\mathbf{B} \times \mathbf{v}_{Lp}) = 0, \quad (\text{I24})$$

where \mathbf{v}_{Lp} is the nonrelativistic velocity of proton vortices [spatial part of the four-vector $v_{(Lp)}^\mu$; see Eq. (I01)], given by

$$\mathbf{v}_{Lp} = \mathbf{V}_{\text{norm}} - \alpha_p m_p n_p \mathbf{W}_p - \frac{\beta_p}{B} m_p n_p \mathbf{B} \times \mathbf{W}_p \quad (\text{I25})$$

with

$$\mathbf{W}_p = \frac{1}{m_p n_p} \text{curl} \left(\frac{\lambda_p}{B} \mathbf{B} \right) \quad (\text{I26})$$

[see Eqs. (I13) and (I17)]. The physical meaning of Eq. (I24) is obvious: It describes transport of the magnetic field (produced by the proton vortices) with the vortices. A bit different equation has been recently obtained, in the approximation of vanishing temperature, in Ref. [76] [see Eq. (67) there].³¹ The magnetic field in that reference is transported with the velocity which *differs* from the vortex velocity \mathbf{v}_{Lp} . This is a puzzling result since Ref. [76] explicitly assumes that the magnetic field is confined to proton vortices [see Eq. (65) in that reference] and hence should be carried along with them.

Note, in passing, that the energy consideration of Ref. [76] does not look convincing. In particular, Eq. (76) in that reference disagrees with the result of Ref. [10] for the free magnetic energy density F_{mag} (which must coincide with the magnetic energy density in the limit of $T = 0$); see the formula after Eq. (16) in Ref. [10].

³¹The same equation follows from the Maxwell's equation (2) and Eqs. (161) and (162) of GAS11.

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