

Universal coefficient of the exact correlator of a large- N matrix field theory

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Exact expressions have been proposed for correlation functions of the large- N (planar) limit of the $(1+1)$ -dimensional $SU(N) \times SU(N)$ principal chiral sigma model. These were obtained with the form-factor bootstrap. The short-distance form of the two-point function of the scaling field $\Phi(x)$ was found to be $N^{-1} \langle \text{Tr} \Phi(0)^\dagger \Phi(x) \rangle = C_2 \ln^2 mx$, where m is the mass gap, in agreement with the perturbative renormalization group. Here we point out that the universal coefficient C_2 is proportional to the mean first-passage time of a Lévy flight in one dimension. This observation enables us to calculate $C_2 = 1/16\pi$.

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I. INTRODUCTION

The main problem of quantum chromodynamics is to understand quark confinement and the mass gap. An analogous problem with similar features, e.g., asymptotic freedom and nontrivial anomalous dimensions, is the principal chiral sigma model (PCSM) of a matrix field $U(x) \in SU(N)$, $N \geq 2$, where x^0 and x^1 are the time and space coordinates, respectively. The action is

$$S = \frac{N}{2g_0^2} \int d^2x \eta^{\mu\nu} \text{Tr} \partial_\mu U(x)^\dagger \partial_\nu U(x), \quad (1.1)$$

where $\mu, \nu = 0, 1$, $\eta^{00} = 1$, $\eta^{11} = -1$, $\eta^{01} = \eta^{10} = 0$, and g_0 is the coupling. This action does not change under the global transformation $U(x) \rightarrow V_L U(x) V_R$, for two matrices $V_L, V_R \in SU(N)$. The scaling or renormalized field operator $\Phi(x)$ is an average of $U(x)$ over a region of size b , where $\Lambda^{-1} < b \ll m^{-1}$, where Λ is the ultraviolet cutoff and m is the mass of the fundamental excitation. The normalization of Φ is determined by

$$\langle 0 | \Phi(0)_{b_0 a_0} | P, \theta, a_1, b_1 \rangle = N^{-1/2} \delta_{a_0 a_1} \delta_{b_0 b_1}, \quad (1.2)$$

where the ket on the right is a one-particle (hence the symbol P) state with rapidity θ (that is, with momentum components $p_0 = m \cosh \theta$, $p_1 = m \sinh \theta$), and we implicitly sum over left and right colors a_1 and b_1 , respectively.

In the bootstrap approach for some two-dimensional field theories, the exact S matrix [1] and form factors [2] can be found heuristically, using the powerful property of integrability. On the other hand, the bootstrap begins from

one's expectations about the mass spectrum, rather than proving these expectations. In particular, one must assume the existence of a mass gap m . In our opinion, the ultimate goal of the bootstrap should be to *reconstruct* the Lagrangian or Hamiltonian formulation of the quantum field theory (we will say more about this towards the end of the paper). This would provide a proof of the existence of the mass gap in the latter formulations. A more modest step forward [3] was to show that if $N \rightarrow \infty$ [4], $m|x| \ll 1$, the bootstrap expression of the two-point function of the scaling field $\Phi(x)$, in Euclidean space, has the behavior

$$N^{-1} \langle 0 | \text{Tr} \mathcal{T} \Phi(0)^\dagger \Phi(x) | 0 \rangle \simeq C_2 \ln^2(m|x|), \quad (1.3)$$

where \mathcal{T} denotes time ordering. This result was obtained from the exact expression for the Wightman (non-time-ordered) two-point function in Minkowski space [5,6]:

$$\begin{aligned} \mathcal{W}(x) &= N^{-1} \langle 0 | \text{Tr} \Phi(0)^\dagger \Phi(x) | 0 \rangle \\ &= \int_{-\infty}^{\infty} \frac{d\theta_1}{4\pi} e^{ip_1 \cdot x} + \frac{1}{4\pi} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} d\theta_1 \cdots \\ &\quad \times \int_{-\infty}^{\infty} d\theta_{2l+1} e^{i \sum_{j=1}^{2l+1} p_j \cdot x} \prod_{j=1}^{2l} \frac{1}{(\theta_j - \theta_{j+1})^2 + \pi^2}, \end{aligned} \quad (1.4)$$

where θ_j are rapidities and $p_j = m(\cosh \theta_j, \sinh \theta_j)$ are the corresponding momentum vectors, for $j = 1, \dots, 2l+1$.

In this paper, we will show how to evaluate the coefficient C_2 in (1.3). For pedagogical completeness, we will briefly review the derivation of (1.4) in Sec. II, and how this series was used to find (1.3) in Sec. III.

Standard saddle-point large- N methods fail for the PCSM. This is related to the fact that the Feynman

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diagrams in the large- N limit are planar [4], instead of linear. We emphasize that (1.3) is a significant departure from $n \rightarrow \infty$ results for simpler isovector quantum field theories, e.g., $O(n)$ nonlinear sigma models or $\mathbb{C}\mathbb{P}(n-1)$ models. In particular, Eq. (1.3) is not the correlation function of a free field, although a free *master* field does exist [5].

The result (1.3) is in perfect agreement with the perturbative renormalization group applied to the action (1.1). We give a brief summary here (for a more complete discussion, see Ref. [7]). Consider the regularized Euclidean correlation function (obtained after a Wick rotation, $x^0 \rightarrow ix^0$) is $G(|x|, \Lambda) = N^{-1} \langle 0 | \mathcal{T} \text{Tr} \Phi(x) \Phi(0)^\dagger | 0 \rangle$, defined with an ultraviolet cutoff Λ . The ultraviolet behavior of the correlation function may be found from the renormalization group equations:

$$\begin{aligned} \frac{\partial \ln G(R, \Lambda)}{\partial \ln \Lambda} &= \gamma(g_0^2) = \gamma_1 g_0^2 + \dots, \\ \frac{\partial g_0^2(\Lambda)}{\partial \ln \Lambda} &= \beta(g_0^2) = -\beta_1 g_0^4 + \dots \end{aligned} \quad (1.5)$$

The coefficients of the anomalous dimension $\gamma(g_0^2)$ and the beta function $\beta(g_0^2)$ are $\gamma_1 = (N^2 - 1)/(2\pi N^2)$ and $\beta_1 = 1/(4\pi)$. For large Λ , the dimensionless quantity $G(R, \Lambda)$ becomes a function of the product of the two variables $G(R\Lambda)$. Integrating (1.5) yields the leading behavior

$$G(R, \Lambda) \sim C [\ln(R\Lambda)]^{\gamma_1/\beta_1}. \quad (1.6)$$

The power of the logarithm is $\gamma_1/\beta_1 = 2 - 2/N^2$, which becomes 2 in the limit of infinite N .

Note that C_2 in Eq. (1.3) is a universal quantity, because the normalization of $\Phi(x)$ is set by (1.2). We will show that $C_2 = 1/16\pi$. In fact, this quantity has already been evaluated in the context of Lévy flights [8]; it is proportional to the mean first-passage time in one dimension.

In the next section, we briefly review the form factors and correlation functions of the scaling field. Then we explain how this expression leads to (1.3) and present an expression for C_2 in terms of the spectrum of the square root of the one-dimensional Laplacian $\Delta^{1/2} = \sqrt{-d^2/du^2}$, with $u \in [-1, 1]$, in Sec. III. In Sec. IV, we determine the value of C_2 . We conclude with a few remarks in Sec. V.

II. FORM FACTORS AND CORRELATION FUNCTIONS OF THE PRINCIPAL CHIRAL SIGMA MODEL

The expression (1.4) was found from the form factors of the scaling field in the large- N limit [5,6]. These form factors satisfy a set of axioms proposed by Smirnov [2] (motivated by the Lehman-Symanzik-Zimmerman formulation of field theory [9]), formulated with the S matrix of the PCSM (for finite N) discussed in Ref. [10]. Form

factors and correlation functions of other local operators in the PCSM may be found in Ref. [11]. Some of these results have been extended to a finite volume in Ref. [12].

The S matrix of the elementary excitations of the principal chiral model [10] depends upon the incoming rapidities θ_1 and θ_2 [as discussed in the Introduction, the momentum vectors are $(p_j)_0 = m \cosh \theta_j$, $(p_j)_1 = m \sinh \theta_j$], the outgoing rapidities θ'_1 and θ'_2 , and the rapidity difference $\theta = |\theta_{12}| = |\theta_1 - \theta_2|$. In the limit of large N we assume that m is fixed, as $N \rightarrow \infty$ (all available evidence indicates that this is the standard 't Hooft limit). The excitations which survive in the large- N limit are elementary particles and elementary antiparticles. The $1/N$ expansion of the two-particle S matrix is

$$\begin{aligned} S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} &= [1 + O(1/N^2)] \left[\delta_{a_2}^{c_2} \delta_{b_2}^{d_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N\theta} (\delta_{a_1}^{c_2} \delta_{b_2}^{d_2} \delta_{a_2}^{c_1} \delta_{b_1}^{d_1} \right. \\ &\quad \left. + \delta_{a_2}^{c_2} \delta_{b_1}^{d_2} \delta_{a_1}^{c_1} \delta_{b_2}^{d_1}) - \frac{4\pi^2}{N^2 \theta^2} \delta_{a_1}^{c_2} \delta_{b_1}^{d_2} \delta_{a_2}^{c_1} \delta_{b_2}^{d_1} \right]. \end{aligned} \quad (2.1)$$

The generalized S matrix is defined by replacing $\theta = |\theta_{12}|$ with $\theta = \theta_{12}$ in (2.1). The generalization is necessary to analytically continue rapidities into the complex plane. The S matrix of one particle and one antiparticle $S_{PA}(\theta)$ is obtained by crossing (2.1) from the s channel to the t channel:

$$\begin{aligned} S_{PA}(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} &= [1 + O(1/N^2)] \left[\delta_{b_2}^{d_2} \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N\hat{\theta}} (\delta_{a_1 a_2} \delta_{c_1 c_2} \delta_{b_2}^{d_2} \delta_{b_1}^{d_1} \right. \\ &\quad \left. + \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1 b_2} \delta_{d_1 d_2}) - \frac{4\pi^2}{N^2 \hat{\theta}^2} \delta_{a_1 a_2} \delta_{c_1 c_2} \delta_{b_1 b_2} \delta_{d_1 d_2} \right], \end{aligned} \quad (2.2)$$

where $\hat{\theta} = \pi i - \theta$ is the crossed rapidity difference.

For a set of particles with labels 1, 2, etc., we denote a particle's rapidity θ_j , left color a_j and right color b_j by $P_j = \{P, \theta_j, a_j, b_j\}$. For a set of antiparticles with labels 1, 2, etc., we denote an antiparticle's rapidity θ_j , right color b_j and left color a_j by $A_j = \{P, \theta_j, b_j, a_j\}$ (the reversal of color indices is a convention). A multiparticle in-state may be written as

$$\begin{aligned} |P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; \dots; P, \theta_k, a_k, b_k; A, \theta_{k+1}, \\ b_{k+1}, a_{k+1}; A, \theta_{k+2}, b_{k+2}, a_{k+2}; \dots; A, \theta_{k+j}, b_{k+j}, a_{j+1}\rangle_{\text{in}} \\ = |P_1; P_2; \dots P_k; A_{k+1}; A_{k+2}; \dots A_{k+j}\rangle_{\text{in}}. \end{aligned}$$

By ‘‘form factors,’’ we mean matrix elements of local operators. Using Smirnov's axioms [2], form factors of $\Phi(x)$ consistent with the S matrix [Eqs. (2.1) and (2.2)] can be found:

$$\begin{aligned}
 & \langle 0 | \Phi(0)_{b_0 a_0} | P_1; P_2; \dots; P_M; A_{M+1}; A_{M+2}; \dots; A_{2M-1} \rangle_{\text{in}} \\
 &= \frac{\sqrt{N}}{N^M} \sum_{\sigma, \tau \in S_M} F_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) \prod_{j=0}^{M-1} \delta_{a_j a_{\sigma(j)+M}} \delta_{b_j b_{\tau(j)+M}},
 \end{aligned} \quad (2.3)$$

where the leading part in the $1/N$ expansion of the function $F_{\sigma\tau} = F_{\sigma\tau}^0 + O(1/N)$ is

$$F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1}) = \frac{(-4\pi)^{M-1} K_{\sigma\tau}}{\prod_{j=1}^{M-1} [\theta_j - \theta_{\sigma(j)+M} + \pi i] [\theta_j - \theta_{\tau(j)+M} + \pi i]}, \quad (2.4)$$

where

$$K_{\sigma\tau} = \begin{cases} 1, & \sigma(j) \neq \tau(j), \text{ for all } j \\ 0, & \text{otherwise} \end{cases}. \quad (2.5)$$

We note that (1.2) agrees with (2.3), (2.4) and (2.5) for $M = 1$. The case of $M = 2$ was solved in Ref. [5], while the general case was solved in Ref. [6].

The Wightman function is found from the completeness relation:

$$\begin{aligned}
 \mathcal{W}(x) &= \frac{1}{N} \sum_{a_0, b_0} \sum_X \langle 0 | \Phi(0)_{b_0 a_0} | X \rangle_{\text{in}} \langle X | \Phi(0)_{b_0 a_0}^* | 0 \rangle e^{ip_X \cdot x} \\
 &= \frac{1}{N} \sum_{a_0, b_0} \sum_X |\langle 0 | \Phi(0)_{b_0 a_0} | X \rangle_{\text{in}}|^2 e^{ip_X \cdot x},
 \end{aligned} \quad (2.6)$$

where X denotes an arbitrary choice of particles, momenta and colors, and where p_X is the momentum eigenvalue of the state $|X\rangle$. Substitution of (2.3), (2.4) and (2.5) into the completeness relation yields Eq. (1.4).

For increasingly large separation x , states $|X\rangle_{\text{in}}$ with many excitations in (2.6) contribute minimally to $\mathcal{W}(x)$. Therefore, the leading large-distance behavior is exponential decay, as expected for a massive theory. Evaluating correlation functions for small x requires keeping all the terms of (2.6), which is considerably more subtle [3].

III. SHORT-DISTANCE BEHAVIOR AND THE FRACTIONAL LAPLACIAN

The expression for the Wightman function (1.4) can be studied at short distances, by a method similar to that of Ref. [13] for the Ising model, using that model's exact form factors [14]. We Wick-rotate the time variable to Euclidean space, setting $x^1 = 0$ and replacing x^0 by iR , $R > 0$. The phases in (1.4) change via $\exp(ip_j \cdot x) \rightarrow \exp(-mR \cosh \theta_j)$. We define $L = \ln \frac{1}{mR}$. As mR becomes small, $\exp(-mR \cosh \theta_j)$ becomes approximately the characteristic function of $(-L, L)$, equal to unity for $-L < \theta < L$ and zero everywhere else. The characteristic function appears the same way in the Feynman-Wilson gas [15]. The short-distance Euclidean two-point function is now

$$\begin{aligned}
 G(mR) &= \frac{L}{2\pi} + \frac{L}{4\pi} \sum_{l=1}^{\infty} \int_{-1}^1 du_1 \dots \int_{-1}^1 du_{2l+1} \\
 &\times \prod_{j=1}^{2l} \frac{1}{L[(u_j - u_{j+1})^2 + (\pi/L)^2]},
 \end{aligned} \quad (3.1)$$

where $\theta_j = Lu_j$.

The terms of (3.1) are related to the fractional-power Laplace operator $\Delta^{1/2} = \sqrt{-d^2/du^2}$ [16]. This operator acts on a function $f(u)$, vanishing for $u \notin (b, c)$, by [16]

$$\Delta^{1/2} f(u) = \frac{1}{\pi} \text{P} \int_b^c du' \frac{f(u') - f(u)}{(u' - u)^2}, \quad (3.2)$$

where P denotes the principal value. We set $b = -1$, $c = 1$. The operator $\Delta^{1/2}$ has an infinite set of discrete eigenvalues λ_n , of the eigenfunctions $\varphi_n(u)$, $\Delta^{1/2} \varphi_n = \lambda_n \varphi_n$, $n = 1, 2, \dots$, with $0 < \lambda_1 < \lambda_2 < \dots$, with $\varphi_n(\pm 1) = 0$. Here is the relation: for $u, u' \in (-1, 1)$, we define the operator $H(L)$ by

$$\frac{1}{L[(u - u')^2 + (\pi/L)^2]} = \langle u' | e^{-\frac{\pi}{L} H(L)} | u \rangle. \quad (3.3)$$

By (3.2), (3.3) and a straightforward calculation, we find that $H(L)$ is an approximation to $\Delta^{1/2}$, i.e., $H(L) = \Delta^{1/2} + O(1/L)$, with spectrum

$$\begin{aligned}
 H(L) \varphi_n(u, L) &= \lambda_n(L) \varphi_n(u, L), \\
 \int_{-1}^1 du |\varphi_n(u, L)|^2 &= 1, \quad \lambda_n(L) = \lambda_n + O(1/L), \\
 \varphi_n(u, L) &= \varphi_n(u) + O(1/L).
 \end{aligned} \quad (3.4)$$

Summing over l in Eq. (3.1) yields, from (3.4),

$$\begin{aligned}
 G(mR) &= \frac{L}{4\pi} \int_{-1}^1 du' \int_{-1}^1 du \left\langle u' \left| \frac{1}{1 - e^{-2\pi H(L)/L}} \right| u \right\rangle \\
 &= \frac{L}{4\pi} \sum_{n=1}^{\infty} \left| \int_{-1}^1 du \varphi_n(u, L) \right|^2 \frac{1}{1 - e^{-2\pi \lambda_n/L + O(1/L^2)}}.
 \end{aligned} \quad (3.5)$$

Expanding (3.5) in powers of $1/L$ (we are expanding within the sum, which requires justification; see Ref. [3] for a more careful discussion), we find

$$G(mR) = \frac{L^2}{8\pi^2} \sum_{n=1}^{\infty} \left| \int_{-1}^1 du \varphi_n(u) \right|^2 \lambda_n^{-1} + O(L).$$

This is (1.3), with the universal coefficient identified as

$$C_2 = \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \left| \int_{-1}^1 du \varphi_n(u) \right|^2 \lambda_n^{-1}. \quad (3.6)$$

IV. EVALUATION OF C_2

This section is the heart of this paper. An expression which is proportional to the right-hand side of (3.6) was evaluated in Ref. [8]. The result is

$$\sum_{n=1}^{\infty} \left| \int_{-1}^1 du \phi_n(u) \right|^2 \lambda_n^{-1} = \int_{-1}^1 du \sqrt{1-u^2} = \frac{\pi}{2}, \quad (4.1)$$

which is the mean first-passage time of a Lévy flight in one dimension, calculated via the corresponding continuum theory, namely the anomalous Laplace equation. Since the discussion in Ref. [8] may not be easily accessible to readers working in quantum field theory, we present a short derivation of (4.1) below.

We write the square root of the one-dimensional Laplacian (3.2) as (with $b = -1$, $c = 1$)

$$\begin{aligned} \Delta^{1/2} f(u) &= \frac{1}{2\pi} \int_{-1}^1 du' \left[\frac{1}{(u-u'-i\epsilon)^2} + \frac{1}{(u-u'+i\epsilon)^2} \right] f(u'). \end{aligned}$$

For a function $f(u)$ satisfying the Dirichlet boundary conditions $f(b) = f(c) = 0$, this may be integrated by parts to

$$\Delta^{1/2} f(u) = -\frac{1}{\pi} \int_{-1}^1 du' \frac{df(u')}{du'} \mathbf{P} \frac{1}{u'-u}, \quad (4.2)$$

where \mathbf{P} denotes the principal value. Detailed properties of the eigenfunctions $\phi_n(u)$ and eigenvalues λ_n can be found in Ref. [17].

Let us define the function $C(u)$ by

$$C(u) = \sum_{n=1}^{\infty} \lambda_n^{-1} \int_{-1}^1 du' \phi_n(u') \phi_n(u) \quad (4.3)$$

and note that

$$C_2 = \frac{1}{8\pi^2} \int_{-1}^1 du C(u). \quad (4.4)$$

This function satisfies

$$\Delta^{1/2} C(u) = 1 \quad (4.5)$$

by completeness. Now the spectrum of $\Delta^{1/2}$ in a finite interval is strictly positive. If there were two square-normalizable solutions to (4.5), their difference would be a square-normalizable function annihilated by this positive operator; but this is impossible unless the difference vanishes. Therefore, the square-normalizable solution $C(u)$ to (4.5) is unique, and must be equal to (4.3). After presenting the solution to Eqs. (4.2) and (4.5) for $C(u)$, we will integrate to find C_2 [8].

A solution of the integral equation

$$\int_{-1}^1 du' \frac{dC(u')}{du'} \mathbf{P} \frac{1}{(u'-u)^\alpha} = -1, \quad \alpha > 0, \quad (4.6)$$

which is square normalizable, is

$$C(u) = \frac{1}{\pi\alpha} (1-u^2)^{\alpha/2}. \quad (4.7)$$

This is easily checked: the derivative has two branch points at ± 1 . Taking the branch cut on the real axis from -1 to 1 , straightforward complex integration yields (4.6). Specializing to $\alpha = 1$, as in Eq. (4.2), we obtain the solution to Eq. (4.5). Upon integrating $C(u)$, we obtain (4.1) and

$$C_2 = \frac{1}{8\pi^2} \frac{\pi}{2} = \frac{1}{16\pi}, \quad (4.8)$$

which is the result we claimed for the universal coefficient.

V. REMARKS

Let us summarize the result of Ref. [3] and this paper. We have found that the correlation function of the scalar field has the Euclidean asymptotic behavior

$$N^{-1} \langle 0 | \text{Tr} T \Phi(0)^\dagger \Phi(x) | 0 \rangle \simeq \begin{cases} \int \frac{d^2 p}{(2\pi)^2} \frac{\exp(ip \cdot x)}{p^2 + m^2}, & |x| \gg m^{-1} \\ \frac{1}{16\pi} \ln^2(m|x|), & |x| \ll m^{-1} \end{cases}. \quad (5.1)$$

The normalization of the expressions on the right-hand side is completely determined by (1.2). The form for large separation x is the Wick rotation of the first term of (1.4). Perhaps the normalization of the short-distance form can be checked with a lattice Monte Carlo simulation at relatively large N , as has been done for the $O(3)$ nonlinear sigma model [18].

The agreement with the perturbative renormalization group is encouraging, but it is desirable to have a convincing demonstration that the canonical and bootstrap definitions of the PCSM are the same. We next discuss how it may be possible to determine the regularized Lagrangian directly from the bootstrap, using the trace anomaly.

Although the stress-energy-momentum tensor $T_{\mu\nu}$ of the classical field theory has a vanishing trace, $T_{\mu}^{\mu} = 0$, this property is broken in the quantum theory. The trace anomaly for PCSM, with a point-splitting cutoff $R = (R^0, R^1)$, is

$$\begin{aligned} T_{\mu}^{\mu} &= \frac{N}{2} \frac{dg(R)^{-2}}{d \ln R} [\text{Tr} \partial_{\alpha} U^{\dagger} \partial_{\beta} U |_{R} - \langle 0 | \text{Tr} \partial_{\alpha} U^{\dagger} \partial_{\beta} U |_{R} | 0 \rangle] \\ &= -\frac{N}{2g(R)^4} \beta(g) [\text{Tr} j^{\mu}(x+R/2) j_{\mu}(x-R/2) \\ &\quad - \langle 0 | \text{Tr} j^{\mu}(x+R/2) j_{\mu}(x-R/2) | 0 \rangle], \end{aligned} \quad (5.2)$$

where $\beta(g)$ is the beta function as before, and $j_\mu = i(\partial_\mu U)U^\dagger$ is a current. Notice that the right-hand side is proportional to the original Lagrangian; this is a general feature of the trace anomaly.

Now form factors of both the stress-energy-momentum tensor and current are known [11]. Matrix elements of the right-hand side of (5.2) can be calculated. These can be compared with matrix elements of the left-hand side. This can be done by working out the operator product expansion of currents, which should have the form (in Minkowski space)

$$\begin{aligned} & \frac{1}{N} \text{Tr} j^\mu(R/2) j_\mu(-R/2) \\ & \simeq \frac{1}{128\pi^2} \frac{1}{R^\alpha R_\alpha} - 2g(R)^4 \beta(g)^{-1} T_\mu^\mu(0) + \dots \end{aligned} \quad (5.3)$$

The first term on the right-hand side of (5.3) is the vacuum expectation value in (5.2). One of us (P. O.) has calculated this

term (not yet published) with methods similar to those of Ref. [3] and of this paper. The global symmetry implies that no logarithms can appear in this term, but in the context of our bootstrap method, it seems miraculous that they do not. By considering matrix elements of (5.2) between one-particle states, the second term of (5.3) may be determined.

We hope that the surprising connection between integrable quantum field theory and anomalous diffusion will yield further insights.

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- [1] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys. (N.Y.)* **120**, 253 (1979).
- [2] F. A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, *Advanced Series in Mathematical Physics Vol. 14* (World Scientific, Singapore, 1992); H. Babujian, A. Fring, M. Karowski, and A. Zapletal, *Nucl. Phys.* **B538**, 535 (1999).
- [3] P. Orland, *Phys. Rev. D* **90**, 125038 (2014).
- [4] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).
- [5] P. Orland, *Phys. Rev. D* **84**, 105005 (2011).
- [6] P. Orland, *Phys. Rev. D* **86**, 045023 (2012).
- [7] P. Rossi and E. Vicari, *Phys. Rev. D* **49**, 6072 (1994), Eq. (168); P. Rossi, M. Campostrini, and E. Vicari, *Phys. Rep.* **302**, 143 (1998), Eqs. (7.5) and (7.6).
- [8] S. V. Buldyrev, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, *Phys. Rev. E* **64**, 041108 (2001); S. V. Buldyrev, M. Gitterman, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, *Physica (Amsterdam)* **302A**, 148 (2001).
- [9] H. Lehmann, K. Symanzik, and W. Zimmermann, *Nuovo Cimento* **1**, 205 (1955); **6**, 319 (1957).
- [10] A. Polyakov and P. B. Wiegmann, *Phys. Lett.* **131B**, 121 (1983); P. B. Wiegmann, *Phys. Lett.* **141B**, 217 (1984); E. Abdalla, M. C. B. Abdalla, and M. Lima-Santos, *Phys. Lett.* **140B**, 71 (1984); P. Wiegmann, *Phys. Lett.* **142B**, 173 (1984); L. D. Faddeev and N. Yu. Reshetikhin, *Ann. Phys. (N.Y.)* **167**, 227 (1986).
- [11] A. Cortés Cubero, *Phys. Rev. D* **86**, 025025 (2012); A. Cortés Cubero and P. Orland, *Phys. Rev. D* **88**, 025044 (2013). There is an error in some of the higher-point form factors in these papers, which has been corrected in , *Phys. Rev. D* **90**, 065002 (2014).
- [12] A. Cortés Cubero, *Phys. Rev. D* **91**, 105025 (2015).
- [13] J. L. Cardy and G. Mussardo, *Nucl. Phys.* **B340**, 387 (1990); V. P. Yurov and Al. B. Zamolodchikov, *Int. J. Mod. Phys. A* **06**, 3419 (1991); O. A. Castro-Alvaredo and A. Fring, *Phys. Rev. D* **63**, 021701R (2000).
- [14] B. Berg, M. Karowski, and P. Weisz, *Phys. Rev. D* **19**, 2477 (1979); T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, *Phys. Rev. B* **13**, 316 (1976); M. Sato, T. Miwa, and M. Jimbo, *Proc. Japan Acad. Ser. A Math. Sci.* **53**, 6 (1977); *Publ. RIMS Kyoto Univ.* **16**, 531 (1980); B. Schroer and T. T. Truong, *Nucl. Phys.* **B144**, 80 (1978).
- [15] R. P. Feynman, *Phys. Rev. Lett.* **23**, 1415 (1969); K. G. Wilson, *Cornell preprint CLNS-131 (1970) in Proc. Fourteenth Scottish Universities Summer School in Physics, 1973*, edited by R. L. Crawford and R. Jennings (Academic Press, New York, 1974).
- [16] I. Podlubny, *Fractional Differential Equations* (Academic Press, London, 1999); S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integral and Derivatives* (Gordon and Breach, New York, 1993); B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators* (Springer-Verlag, New York, 2003).
- [17] E. Katzav and M. Adda-Bedia, *Europhys. Lett.* **83**, 30006 (2008); *J. Phys. A* **41**, 022002 (2008); M. Kwaśnicki, *J. Funct. Anal.* **262**, 2379 (2012).
- [18] J. Balog, F. Niedermayer, M. Pepe, P. Weisz, and U.-J. Wiese, *J. High Energy Phys.* **11** (2012) 140.