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Integral formalism for the construction of scheme transformations in quantum field theory

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We present an integral formalism for constructing scheme transformations in a quantum field theory. We apply this to generate several new useful scheme transformations. A comparative analysis is given of these scheme transformations in terms of their series expansion coefficients and their resultant effect on the interaction coupling, in particular at a zero of the beta function away from the origin in coupling-constant space.

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I. INTRODUCTION

The dependence of the interaction coupling in a quantum field theory on the Euclidean momentum scale, μ, where it is probed, is of basic importance. This is determined by the beta function of the theory [1]. For simplicity, we focus here on a theory (in four spacetime dimensions at zero temperature) with only one dimensionless interaction coupling. There has long been interest in a possible zero of the beta function away from the origin in coupling-constant space. For an infrared-free theory such as quantum electrodynamics or $\lambda\phi^4$ this would be an ultraviolet fixed point (UVFP) of the renormalization group (RG), while for an asymptotically free non-Abelian gauge theory, this would be an infrared fixed point (IRFP) of the renormalization group, calculated to a given order in perturbation theory, in both cases. Let us consider the latter case, of a non-Abelian gauge theory with a simple gauge group and hence a single gauge coupling. We shall denote the running gauge coupling as $g \equiv g(\mu)$ and define $\alpha(\mu) = g(\mu)^2/(4\pi)$. For technical simplicity, we take the fermions to be massless and avoid inclusion of any scalar fields, so that the theory involves only one dimensionless interaction coupling. With a given fermion content, the theory possesses an IRFP at the two-loop level if the two-loop coefficient in the beta function, b_2, has a sign opposite to that of the one-loop coefficient, b_1 [see Eq. (2.3) below]. At the two-loop (2ℓ) level, this IRFP occurs at the value $\alpha = \alpha_{\mathrm{IR},2\ell} = -4\pi b_1/b_2$. It is clearly desirable to calculate the value of this IR zero of the beta function to higher-loop order to achieve greater accuracy in its determination. However, while the one-loop and two-loop coefficients in the beta function are independent of the scheme used for regularization and renormalization, the coefficients at the level of three loops and higher depend on this scheme [2]. Indeed, this scheme dependence of higher-loop calculations is a general property of quantum field theories.

It is therefore incumbent upon one to assess how sensitive a given quantity is to the scheme used for the higher-loop calculation of this quantity. Here we concentrate on the calculation of the location of a zero of a beta function away from zero coupling but still at sufficiently small coupling that one can use perturbative methods. A procedure to assess the scheme dependence of the location of this zero in the beta function is to carry out the calculation first in a given scheme, obtain a result for the value of the IR zero at n-loop ($n\ell$) order, $\alpha_{\mathrm{IR},n\ell}$, then apply a scheme transformation, calculate the zero to this order in the transformed scheme, denoted $\alpha'_{\mathrm{IR},n\ell}$, and determine the fractional shift in the value. In a series of papers this program has been implemented [3–7]. References [3,4] pointed out that it is significantly more difficult to construct scheme transformations that can be applied away from the origin in coupling-constant space than it is to construct such transformations that are applicable in the vicinity of the origin, such as those used in quantum chromodynamics (QCD) calculations in the perturbative region, i.e., for small α_s. For example, consider the scheme transformation

$$\alpha = \frac{1}{2}\tanh(2\alpha'). \tag{1.1}$$

This is perfectly well-behaved near zero coupling, $\alpha = \alpha' = 0$, where it approaches the identity transformation, but is unacceptable at a generic zero of the beta function. This is clear from the inverse transformation, which is

$$\alpha' = \frac{1}{4}\ln\left(\frac{1+2\alpha}{1-2\alpha}\right). \tag{1.2}$$

As α approaches the value $1/2$ from below, $\alpha' \to \infty$, and for $\alpha > 1/2$, α' is complex.

Since, as was noted, coefficients in the beta function at the level of three loops and higher are scheme-dependent, it was anticipated that, in the vicinity of zero coupling, as in QCD, one could transform to a scheme where these coefficients vanish [8,9]. Another important result from [3–6] was an explicit construction of a scheme transformation that removes terms of loop level $n \geq 3$ in the beta function in the vicinity of $\alpha = \alpha' = 0$ and the demonstration that it is much more difficult to try to carry out this removal of higher-loop terms at a larger but still perturbative value of the coupling away from the origin. In [6] a generalized scheme transformation denoted S_{R,m,k_1} with $m \geq 2$ was presented with the property that it eliminates the n-loop terms in the beta function of a gauge theory from loop order $n = 3$ to order $n = m + 1$, inclusive and can be optimized to perform this removal in a substantial range of couplings away from the origin.

There is thus a need to construct and apply scheme transformations that are applicable not only near the origin in coupling-constant space (where they automatically reduce to the identity), but also at a zero of the beta function located away from the origin. The previous works [3–7] addressed this task and studied applications at an IR zero of the beta function in an asymptotically free non-Abelian gauge theory. Early interest in such a zero had made use of the scheme-independent one-loop and two-loop coefficients and had noted the associated behavior of scaling with anomalous dimensions [10,11]. Later, it was observed that in the region where the number of fermions approaches the maximum value allowed by asymptotic freedom (the value where the one-loop coefficient b_1 vanishes), this IR zero occurs at small coupling [12]. Moving away from this region toward larger values of $\alpha_{\mathrm{IR},2\ell}$ requires higher-order calculations [13–17] to achieve reasonable accuracy, whence the necessity of dealing with the issue of scheme dependence. These calculations made use of expressions for the three-loop and four-loop beta function coefficients, b_3 [17] and b_4 [18] that had been calculated in the $\overline{\mathrm{MS}}$ scheme [19]. References [14–16] carried out this analysis for a general gauge group and for fermions in both the fundamental representation and in the adjoint and rank-2 tensor representations. A particularly powerful approach uses scheme transformations that are dependent on an auxiliary parameter, r, with the property that as $r \to 0$, they approach the identity; by varying r continuously away from $r = 0$, one can thus study the scheme dependence as a function of this continuous variable [3–7]. A valuable result from this program of higher-order perturbative computations of the values of IR fixed points of asymptotically free non-Abelian gauge theories is improvement in the accuracy of calculations of anomalous dimensions, such as the anomalous dimension of the fermion bilinear operator, evaluated at the IR fixed point, $\gamma_{\mathrm{IR},n\ell} \equiv \gamma_{n\ell}(\alpha_{\mathrm{IR},n\ell})$. The result can then be compared with lattice calculations that are fully nonperturbative in the gauge coupling, although involving other approximations, such as finite lattice spacing, finite lattice volume, removal of fermion doubler modes, etc. [20]. For example for a (vectorial) SU(3) gauge theory with $N_f = 12$ massless Dirac fermions, the values of $\gamma_{\mathrm{IR},n\ell}$ at the two-loop, three-loop, and four-loop level were found to be 0.773, 0.312, and 0.253, respectively. The four-loop value is in good agreement with the lattice calculations $\gamma_{\mathrm{IR}} = 0.27 \pm 0.03$ [21], $\gamma_{\mathrm{IR}} \simeq 0.25$ [22], and $\gamma_{\mathrm{IR}} = 0.235 \pm 0.046$ [23]. This shows the value of calculating $\alpha_{\mathrm{IR},n\ell}$ to higher-loop order, since one evaluates $\gamma_{n\ell}$ at $\alpha = \alpha_{\mathrm{IR},n\ell}$ to obtain $\gamma_{\mathrm{IR},n\ell}$. Another approach to ascertaining the degree of scheme dependence is to use different schemes, such as the modified minimal subtraction $\overline{\mathrm{MS}}$ [19], momentum subtraction (MOM) [24], and RI′ [25], for the calculation of $\alpha_{\mathrm{IR},n\ell}$ and then compare the results [26,27] (see also [28]).

The program in [3–7] is complementary to work on optimized schemes to be applied in the neighborhood of the origin, as in perturbative QCD calculations [24,29,30]. Scheme transformations have also been used in recent studies of possible UV zeros in a beta function for several types of nonasymptotically free theories, including a U(1) gauge theory [31] and a globally invariant $\mathrm{O}(N)$ $\lambda|\vec{\phi}|^4$ theory [32]. We do not explicitly consider supersymmetric field theories here but note that scheme transformations have also been studied in such theories (e.g., [33–35]).

In this paper we report important further progress in this program of constructing scheme transformations that are acceptable for applications away from, as well as near, the origin in coupling-constant space. We present a method for generating scheme transformations based on an integral formalism. We demonstrate the usefulness of this integral formalism by utilizing it to construct several new scheme transformations that can be applied to determine the degree of scheme dependence of a higher-loop calculation of a zero of the beta function away from the origin in coupling-constant space. We also present a comparative analysis of scheme transformations in terms of the coefficients that enter in their Taylor series expansions in the relevant coupling and use this to infer how they shift a coupling that is in the perturbative region.

This paper is organized as follows. In Sec. II we discuss some relevant background on the beta function. In Sec. III we give a general discussion of scheme transformations, including the set of acceptable conditions that they must satisfy. In this section we derive a basic property concerning how a scheme transformation shifts the value of the coupling. In Sec. IV we present our new integral formalism for the construction of acceptable scheme transformation. In the subsequent sections we apply this formalism to generate a number of new useful scheme transformations for which explicit inverses can be calculated. Some comparative comments are included in Sec. IX, and our conclusions are given in Sec. X.

II. BETA FUNCTION

Here we briefly mention some necessary background for our later discussion. As noted before, although our results are more general, we shall focus in this paper on a non-Abelian gauge theory with a simple gauge group G and running gauge coupling $g(\mu)$, with a fermion content chosen such that the theory is asymptotically free. Such theories have the appeal that there is at least one regime, namely large Euclidean energy/momentum μ in the deep UV, where one can carry out reliable perturbative calculations. We define

$$a(\mu) \equiv \frac{g(\mu)^2}{16\pi^2} = \frac{\alpha(\mu)}{4\pi}. \tag{2.1}$$

The argument μ will often be suppressed in the notation. The beta function is $\beta_g = dg/dt$ or equivalently,

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = \frac{g}{2\pi}\beta_g, \tag{2.2}$$

where $dt = d\ln\mu$. This function has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell, \tag{2.3}$$

where an overall minus sign has been extracted in the prefactor. The n-loop ($n\ell$) beta function, denoted $\beta_{\alpha,n\ell}$, is given by (2.3) with the upper limit on the ℓ summation taken to be n rather than ∞. If the theory has an IR zero of the n-loop beta function $\beta_{\alpha,n\ell}$, we denote it by $\alpha_{\mathrm{IR},n\ell} = 4\pi a_{\mathrm{IR},n\ell}$.

As the reference scale μ decreases from large values in the deep UV to smaller scales toward the IR and $\alpha(\mu)$ increases, it approaches the value at the IR zero of the beta function, which we denote generically as α_{IR} in this paragraph. If the gauge group and fermion content are such that α_{IR} is sufficiently small, then the theory evolves to a chirally symmetric phase in the IR and $\alpha \to \alpha_{\mathrm{IR}}$ as $\mu \to 0$, so that α_{IR} is an exact IRFP of the renormalization group. If, on the other hand, α_{IR} is sufficiently large, then the gauge interaction produces bilinear fermion condensate(s) and associated spontaneous chiral symmetry breaking. In this case, the fermions pick up dynamical masses and are integrated out of the low-energy effective field theory that is operative at scales below the scale of the condensate formation. Hence, in this low-energy theory, the beta function changes form, and $\alpha(\mu)$ evolves away from α_{IR} toward stronger coupling. In this case, α_{IR} is only an approximate IRFP of the renormalization group.

III. SCHEME TRANSFORMATIONS

A scheme transformation is a mapping between α and α' or equivalently, between a and a', namely

$$a = a' f(a') \equiv F(a'). \tag{3.1}$$

Here it is convenient to introduce the notation $F(a')$ to emphasize the functional dependence of a on a'. In the limit where a and a' vanish, the theory becomes free, so a scheme transformation has no effect, i.e., it should approach the identity. This implies that

$$f(0) = 1. \tag{3.2}$$

The functions $f(a')$ that we consider have Taylor series expansions about $a = a' = 0$ of the form

$$f(a') = 1 + \sum_{s=1}^{s_{\max}} k_s (a')^s, \tag{3.3}$$

where the coefficients k_s are constants. Here, $s_{\max}$ may be finite or infinite. Thus, these functions $f(a')$ automatically satisfy the condition (3.2). Equivalently,

$$a = F(a') = a' + \sum_{s=1}^{s_{\max}} k_s (a')^{s+1}. \tag{3.4}$$

By using the method of reversion of series [36], one can calculate a Taylor series expansion for the inverse scheme transformation, $a' = F^{-1}(a)$, from the series (3.4). This series for the inverse may be written as

$$a' = F^{-1}(a) = a + \sum_{s=1}^{s_{\max}} \rho_s a^{s+1}. \tag{3.5}$$

In terms of the k_s coefficients, we have

$$\rho_1 = -k_1, \tag{3.6}$$

$$\rho_2 = 2k_1^2 - k_2, \tag{3.7}$$

$$\rho_3 = 5k_1 k_2 - k_3 - 5k_1^3, \tag{3.8}$$

$$\rho_4 = 6k_1 k_3 + 3k_2^2 + 14k_1^4 - k_4 - 21k_1^2 k_2, \tag{3.9}$$

and so forth for higher s.

Since a and a' are small in the perturbative region where these scheme transformations are applicable, it is of interest to consider truncations of the series (3.4) and (3.5). At the lowest order beyond the identity, Eq. (3.4) reduces to the equation $a = a'(1 + k_1 a')$. Although this is a quadratic equation for a', which has two formal solutions, only one is physical, as uniquely determined by the requirement that it must reduce to the identity as $k_1 \to 0$. This solution is

$$a' = \frac{1}{2k_1}[-1 + \sqrt{1 + 4k_1 a}]. \tag{3.10}$$

Similarly, to the same order, the series for the inverse transformation, Eq. (3.5), reduces to $a' = a(1 - k_1 a)$, and in the same way, although this is a quadratic equation in a with two formal solutions, one uniquely determines the physical solution by the requirement that as $k_1 \to 0$, it reduces to the identity. This solution is

$$a = \frac{1}{2k_1}[1 - \sqrt{1 - 4k_1 a'}]. \tag{3.11}$$

As is evident from either Eq. (3.10) or (3.11), for small a, if $k_1 > 0$, then $a > a'$, while if $k_1 < 0$, then $a < a'$.

We next give a general inequality that determines whether a scheme transformation increases or decreases the value of the coupling for small a, as a function of the sign of the lowest nonzero coefficient k_s in Eq. (3.4). This inequality applies even if this lowest nonzero coefficient is not k_1. It is useful, since some scheme transformations and their inverses have respective Taylor series (3.4) and (3.5) in which $k_1 = 0$. This is the case, for example, with the transformations (3.19) and (3.22) below. Let us denote the lowest-order nonzero coefficient k_s in Eqs. (3.3) and (3.4) as $k_{s_{\rm min}}$. Then we find the following general inequality for small a (and hence also small a'),

$$k_{s_{\rm min}} > 0 \Rightarrow a > a',$$
$$k_{s_{\rm min}} < 0 \Rightarrow a < a'. \tag{3.12}$$

A number of the scheme transformations studied in [3–7] depend on a parameter (denoted r in these works) and hence are actually one-parameter families of scheme transformations. Here and below, we shall often refer to a one-parameter family of scheme transformations as a single scheme transformation, with the dependence on the parameter r taken to be implicit. In accordance with the series expansion (3.4), $F(a')$ has the property

$$F'(0) \equiv \frac{dF(a')}{da'}\bigg|_{a'=0} = 1. \tag{3.13}$$

(No confusion should result from the prime used here for differentiation and the prime on a', which does not indicate any differentiation but just distinguishes a' from a.)

From (3.3), it follows that the Jacobian

$$J = \frac{da}{da'} = \frac{d\alpha}{d\alpha'} \tag{3.14}$$

can be expanded as

$$J = 1 + \sum_{s=1}^{s_{\rm max}} (s+1) k_s (a')^s \tag{3.15}$$

and therefore satisfies the condition

$$J = 1 \quad \text{at } a = a' = 0. \tag{3.16}$$

Since J is the derivative da/da', it is naturally expressed as a function of either a' or a.

The beta function in the transformed scheme is

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\beta_\alpha, \tag{3.17}$$

with the series expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell. \tag{3.18}$$

Owing to the fact that Eqs. (3.17) and (3.18) refer to the same function, one can solve for the b'_ℓ in terms of the b_ℓ and k_s. This yields the known results $b'_1 = b_1$ and $b'_2 = b_2$ for the one-loop and two-loop coefficients. In Refs. [3,4], explicit expressions were calculated and presented for higher-loop coefficients b'_ℓ with $\ell \geq 3$ in terms of the b_ℓ and k_s.

To be physically acceptable, a scheme transformation must satisfy several conditions, as was discussed in [4]. We state these for an asymptotically free gauge theory: (i) condition C_1: the scheme transformation must map a real positive α to a real positive α'; (ii) C_2: the scheme transformation should not map a moderate value of α, for which perturbation theory may be reliable, to a value of α' that is so large that perturbation theory is unreliable, or vice versa; (iii) C_3: the Jacobian J should not vanish (or diverge) or else the transformation would be singular; and (iv) C_4: since the existence of an IR zero of β is a scheme-independent property of an theory, a scheme transformation must satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero. Since $J = 1$ for $a = a' = 0$, the condition C_3 implies that J must be positive. Clearly, these apply both to a scheme transformation from a to a' and to the inverse from a' to a.

These four conditions $C_1 - C_4$ can always be satisfied by scheme transformations used to study the UV fixed point in an asymptotically free theory. This is clear from the fact that $f(a')$ approaches 1 as $a' \to 0$ in (3.3), so the transformation approaches the identity in this limit. However, as was pointed out in [3] and shown with a number of examples in [3–6], they are not automatically satisfied, and indeed, are quite restrictive conditions when one applies the scheme transformation at a zero of the beta function away from the origin, $\alpha = 0$, i.e., at an IR zero of the beta function for an asymptotically free theory or a possible UV zero of the beta function for an infrared-free theory.

Some further remarks on the applicability of a scheme transformation are appropriate here. Since a major application of scheme transformations is to determine how sensitive the value of a zero of the beta function, calculated to loop order $n = 3$ or higher, is to the scheme used for the calculation, and since such a calculation is only reliable if the coupling α is not too large, it follows that one need only impose the conditions C_1-C_4 in this range of values of α

that are not so large as to render perturbative calculations inapplicable. Nevertheless, it is valuable to have a scheme transformation that satisfies all of the conditions C_1-C_4 for arbitrary (physical, i.e., real, positive) values of α, so that one does not have to be concerned about trying to choose some nominal value of α beyond which it cannot be applied. To expand upon this point, we may compare and contrast two illustrative scheme transformations [4]. One of these satisfies the conditions C_1-C_4 for arbitrary values of α. This is the transformation

$$a = F(a') = \frac{1}{r}\sinh(ra') \tag{3.19}$$

with inverse

$$a' = \frac{1}{r}\ln\left[ra + \sqrt{1+(ra)^2}\right] \tag{3.20}$$

and Jacobian, expressed equivalently as a function of a' and a,

$$J = \cosh(ra') = \sqrt{1+(ra)^2}. \tag{3.21}$$

This is an example of a class of one-parameter families of scheme transformations whose members are invariant under reversal in sign of the auxiliary parameter r. Hence, for such transformations, we can, without loss of generality, take this parameter r to be nonnegative, and, as in [4], we shall do so. The application of this transformation in [4] to the IR zero in the beta function in an SU(N) theory with N_f fermions in the fundamental representation showed that for moderate r and for values of $\alpha_{\mathrm{IR},n\ell}$ for $n=3$ and $n=4$ loops that were not too large, these values were not sensitively dependent on the scheme used for their calculation.

In contrast, consider the scheme transformation

$$a = F(a') = \frac{1}{r}\tanh(ra') \tag{3.22}$$

with the inverse

$$a' = \frac{1}{2r}\ln\left(\frac{1+ra}{1-ra}\right) \tag{3.23}$$

and Jacobian, expressed as a function of a' and, equivalently, of a:

$$J = \frac{1}{\cosh^2(ra')} = (1-ra)(1+ra). \tag{3.24}$$

Again, we may, without loss of generality, take r to be nonnegative. Evidently, the inverse transformation (3.23) and the Jacobian are singular at $a = 1/r$, i.e., $\alpha = 4\pi/r$. The transformation (3.23) thus does not satisfy the conditions C_1-C_4 for arbitrary values of a. Indeed, a special case of this transformation with $r = 8\pi$ was given above in Eq. (1.1) and the singularity at $\alpha = 1/2$ in the inverse, Eq. (1.2) was noted. Hence, the scheme transformation (3.22) is not as well-behaved as (3.19) is. However, if one restricts the parameter r to sufficiently small values that the singularity at $\alpha = 4\pi/r$ occurs at a value of α substantially greater than unity, where one would not try to use perturbative methods, then this singularity would not prevent one from utilizing this transformation.

IV. INTEGRAL FORMALISM FOR CONSTRUCTION OF SCHEME TRANSFORMATIONS

Here we introduce and apply a general integral formalism for the construction of one-parameter families of scheme transformations. In this formalism, the starting point is a choice of a Jacobian $J(y)$ that will be used as the integrand of an integral representation of the function $F(a')$ defined in Eq. (3.1):

$$a = F(a') = \int_0^{a'} J(y)dy. \tag{4.1}$$

We choose $J(y)$ to be an analytic function of y satisfying the condition

$$J(0) = 1. \tag{4.2}$$

This guarantees that $J(a')$ and $f(a')$ have the respective Taylor series expansions (3.15) and (3.3) and hence that $f(a')$ satisfies the condition (3.2). As discussed above, the condition C_3 for an acceptable scheme transformation is that the Jacobian must not vanish, since otherwise the transformation is singular. The property $J(0) = 1$ together with analyticity of J imply that J must be positive for the ranges of couplings a and a' that are relevant for perturbative calculations for which these scheme transformations are applicable. Thus, we require that $J(y) > 0$ throughout the range of the integration variable y in Eq. (4.1). We can also include dependence of the scheme transformation on a (real) auxiliary parameter, denoted r. Differentiating Eq. (4.1) and using a basic theorem from calculus [Eq. (A2) in the Appendix] yields the relation $dF(a')/da' = da/da' = J(a')$, in agreement with Eq. (3.14). Using an appropriate choice for the Jacobian $J(z)$, we can also satisfy conditions C_1-C_4.

In addition to these general conditions for the acceptability of a scheme transformation, another important aspect of the analysis is the ease of inverting the transformation to solve for a' from a. As was evident in Refs. [4–6], for algebraic scheme transformations with finite values of $s_{\max}$ in Eq. (3.3), the inversion required the solution of an algebraic equation and a choice of which root to take for this solution. In contrast, for cases of algebraic or transcendental scheme transformations with $s_{\max} = \infty$, the inverse transformations were often simpler, in the sense that one did not have to make such a choice of which root of an algebraic equation to take.

To show the usefulness of this integral formalism for the construction of acceptable scheme transformations, we will employ it to generate a number of new scheme transformations which also have the advantage that their inverses can be calculated explicitly in closed form. Before doing this, we first illustrate how the method works with some scheme transformations that have already been studied in [3–7], which we showed to be acceptable for the analysis of a zero in a beta function located away from the origin in coupling constant space, in particular, an IR zero of the beta function of an asymptotically free non-Abelian gauge theory. Let us consider, for example, the scheme transformation (3.19) studied in [4]. To show how one could use our present integral formalism to construct this scheme transformation, we start with J and replace the variable a' by the integration variable y to get $J(y) = \cosh(ry)$. Substituting this function into Eq. (4.1), we obtain

$$a = F(a') = \int_0^{a'} \cosh(ry) dy = \frac{1}{r}\sinh(ra'), \qquad (4.3)$$

thereby rederiving the transformation (3.19).

Other examples are provided by the scheme transformations that we studied in [7]. One of these is

$$a = F(a') = \frac{1}{r}\ln(1 + ra') \qquad (4.4)$$

with inverse

$$a' = \frac{e^{ra} - 1}{r}. \qquad (4.5)$$

The Jacobian, expressed equivalently as a function of a' and a, is

$$J = \frac{1}{1 + ra'} = e^{-ra}. \qquad (4.6)$$

Again replacing the variable a' by y to get $J(y) = 1/(1 + ry)$ and then substituting this into Eq. (4.1), we reproduce the original transformation:

$$a = F(a') = \int_0^{a'} \frac{dy}{1 + ry} = \frac{1}{r}\ln(1 + ra'). \qquad (4.7)$$

Here, as we discussed in [7], the parameter r is restricted to lie in the range $r > -1/a'$ to avoid a singularity in the transformation and is further restricted by the condition that the scheme transformation satisfies conditions C_1-C_4. Similarly, if one uses $J(y) = 1/(1 - ry)^2$ in Eq. (4.1), one obtains another scheme transformation considered in [7], namely

$$a = F(a') = \frac{a'}{1 - ra'}. \qquad (4.8)$$

V. TRANSFORMATION WITH AN ALGEBRAIC $J(y)$

We next proceed to present new scheme transformations that we have constructed using our integral formalism. Recall that the starting point for the procedure is a choice of the Jacobian function $J(y)$ that serves as the integrand in Eq. (4.1) and that satisfies the requisite conditions that it is analytic and that $J(0) = 1$. For our first new transformation, we choose a $J(y)$ of algebraic form, namely

$$J(y) = (1 + ry)^p, \qquad (5.1)$$

where the power p is a positive real number. Then, calculating the integral in Eq. (4.1), we obtain the scheme transformation

$$a = F(a') = \frac{(1 + ra')^{p+1} - 1}{r(p + 1)}. \qquad (5.2)$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with

$$k_s = \frac{r^s}{(s + 1)}\binom{p}{s} = \frac{r^s}{(s + 1)!}\prod_{\ell=0}^{s-1}(p - \ell), \qquad (5.3)$$

where $\binom{a}{b} = a!/[b!(a - b)!]$ is the binomial coefficient. This is a finite series if p is an integer, and an infinite series otherwise. We list the coefficients k_s explicitly for the first few values of s for this scheme transformation and for others discussed in this paper in Table I. The series (5.3) has $s_{\min} = 1$ and

$$k_{s_{\min}} = k_1 = \frac{pr}{2}. \qquad (5.4)$$

The inverse transformation is

$$a' = \frac{1}{r}[\{(p + 1)ra + 1\}^{\frac{1}{p+1}} - 1]. \qquad (5.5)$$

Using this inverse transformation, one can express the Jacobian equivalently as a function of a':

$$J = (1 + ra')^p = [(p + 1)ra + 1]^{\frac{p}{p+1}}. \qquad (5.6)$$

There are two immediate restrictions on the parameter r arising from the requirement that $J > 0$ and that there not be any singularity in the scheme transformation (5.2), namely

TABLE I. Values of the coefficients k_s in Eq. (3.3) for scheme transformations discussed in the text. The transformation $F(a')$ is defined by Eq. (3.1): $a = a' f(a') = F(a')$. The equation numbers indicate where the given $F(a')$ is presented in the text.

$F(a')$	Eq.	k_1	k_2	k_3	k_4
$(1/r)\sinh(ra')$	(3.19)	0	$r^2/6$	0	$r^4/120$
$(1/r)\tanh(ra')$	(3.22)	0	$-r^2/3$	0	$2r^4/15$
$(1/r)\ln(1+ra')$	(4.4)	$-r/2$	$r^2/3$	$-r^3/4$	$r^4/5$
$a'/(1-ra')$	(4.8)	r	r^2	r^3	r^4
$\frac{(1+ra')^{p+1}-1}{r(p+1)}$	(5.2)	$\frac{r}{2}p$	$\frac{r^2}{3}\binom{p}{2}$	$\frac{r^3}{4}\binom{p}{3}$	$\frac{r^4}{5}\binom{p}{4}$
$a' + (1/r)\ln[\cosh(ra')]$	(6.2)	$r/2$	0	$-r^3/12$	0
$\ln[1+(1/r)\ln(1+ra')]$	(7.10)	$-(r+1)/2$	$(2r^2+3r+2)/6$	$-(6r^3+11r^2+12r+6)/24$	$(12r^4+25r^3+35r^2+30r+12)/60$
$\ln[1+(1/r)\sinh(ra')]$	(7.17)	$-1/2$	$(r^2+2)/6$	$-(2r^2+3)/12$	$(r^4+20r^2+24)/120$
$\ln[1+(1/r)\tanh(ra')]$	(7.21)	$-1/2$	$(1-r^2)/3$	$(4r^2-3)/12$	$(r^2-1)(2r^2-3)/15$
$\exp[(1/r)(e^{ra'}-1)]-1$	(8.8)	$(r+1)/2$	$(r^2+3r+1)/6$	$(r^3+7r^2+6r+1)/24$	$(r^4+15r^3+25r^2+10r+1)/120$
$\exp[(1/r)\sinh(ra')]-1$	(8.14)	$1/2$	$(r^2+1)/6$	$(4r^2+1)/24$	$(r^4+10r^2+1)/120$

$$r > -\frac{1}{a'} \quad \text{and} \quad r > -\frac{1}{(p+1)a}. \tag{5.7}$$

These restrictions are easily met, for example, by requiring that r be nonnegative. Moreover, the interval in the couplings α where one could use perturbative calculations reliably only extends up to values $\alpha \sim O(1)$, and since $a = \alpha/(4\pi)$, this interval only extends up to $a \sim O(0.1)$, so for moderate p, the lower bounds (5.7) evaluate to $r \gtrsim -O(10)$. This lower bound can easily be satisfied even with moderate negative values of r. With the restrictions (5.7) satisfied, the scheme transformation (5.2) satisfies the conditions C_1-C_4. If $r > 0$, then since $k_{s_{\min}} > 0$ (where $s_{\min} = 1$ here) it follows from our general result (3.12) above that $a > a'$ for small a, a'. If r is negative (in the range allowed by above restrictions) then $k_1 < 0$, so $a < a'$ for small a, a'.

VI. TRANSFORMATION WITH A TRANSCENDENTAL $J(y)$

For an application of our integral formalism using a Jacobian that is a transcendental function, we choose

$$J(y) = 1 + \tanh(ry). \tag{6.1}$$

Then, doing the integral in Eq. (4.1), we obtain

$$a = F(a') = a' + \frac{1}{r}\ln[\cosh(ra')]. \tag{6.2}$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with $k_s = 0$ for s even and

$$k_1 = \frac{r}{2}, \qquad k_3 = -\frac{r^3}{12}, \qquad k_5 = \frac{r^5}{45}, \qquad k_7 = -\frac{17r^7}{2520}, \tag{6.3}$$

etc. for higher values of s. For comparative purposes, we list these k_s for s up to 4 in Table I.

The Jacobian of this transformation, expressed as a function of a', is given by Eq. (6.1) with $y = a'$. The inverse of the scheme transformation has a simple form for certain values of r. For example, for $r = 1$, the inverse is

$$a' = \frac{1}{2}\ln(2e^a - 1). \tag{6.4}$$

Using Eq. (6.4), one can also express J as a function of a for this $r = 1$ case, obtaining $J = 2 - e^{-a}$.

The allowed range of the parameter r is determined by the requirement that the scheme transformation must satisfy the conditions C_1-C_4. If $r > 0$, then, since $k_{s_{\min}} > 0$ (where $s_{\min} = 1$ here), it follows that $a > a'$ for small a, a', while if $r < 0$, then $k_1 < 0$, so $a < a'$ for small a, a'.

VII. SCHEME TRANSFORMATIONS FOR WHICH $J(y) = (d/dy)\ln h(y)$

A. General

In order for the general integral formalism that we have presented above to be optimally useful, it is necessary that one should be able to do the integral (4.1) in closed form. It is therefore helpful to consider a class of Jacobian functions for which one is guaranteed to be able to calculate the integral (4.1). Clearly, if $J(y)$ is the derivative of another function, then one can always perform this integral. In this section we present one such class of Jacobian functions. These are functions that can be expressed as logarithmic derivatives (LDs) of smooth functions denoted $h(y)$:

$$J(y) = \frac{d}{dy}\ln h(y) = \frac{h'(y)}{h(y)}, \tag{7.1}$$

where $h'(y) \equiv dh(y)/dy$. To obtain acceptable scheme transformation functions, we require that $h(y)$ is positive for physical (nonnegative) values of the argument y and that

$$h(0) = h'(0). \tag{7.2}$$

The equality (7.2) guarantees that the present construction satisfies the condition (4.2) that $J(0) = 1$ and, as will be shown below, that it also satisfies the condition (3.2) that $f(0) = 1$. With $J(y)$ as specified in Eq. (7.1), we can perform the integral (4.1) immediately, obtaining the transformation function

$$a = F(a') = \int_0^{a'} \frac{h'(y)}{h(y)}dy = \ln\left[\frac{h(a')}{h(0)}\right]. \tag{7.3}$$

This shows why we required that $h(y)$ be positive for physical values of y, since otherwise $h(0)$ and/or $h(a')$ might vanish, rendering the logarithm singular. Since only the ratio $h(a')/h(0)$ enters in $F(a')$, it follows that $F(a')$ is invariant under a rescaling of $h(y)$. Consequently, we can, without loss of generality, rescale $h(y)$ so that $h(0) = 1$, and we shall do this. Combining this with Eq. (7.2), we have

$$h(0) = h'(0) = 1, \tag{7.4}$$

and combining Eq. (7.4) with Eq. (7.3), we obtain

$$F(a') = \ln[h(a')]. \tag{7.5}$$

Of course, for the J functions that we present in this section and the next, one could simply start with the resultant $F(a')$, but these examples are useful as additional illustrations of the integral formalism for which, furthermore, one can calculate the inverse transformations (7.8) as explicit closed-form expressions.

To prove that this construction satisfies the condition $f(0) = 1$, we use the definition (3.1) together with the analyticity of $h(a')$ at $a' = 0$. We write out the Taylor series expansion for $h(a')$ at the origin and use the property (7.2) that we have imposed:

$$h(a') = 1 + a' + \frac{1}{2!}h''(0)(a')^2 + \cdots \tag{7.6}$$

where here and below, the dots $\cdots$ denote higher powers of a'. Therefore,

$$\begin{aligned} f(0) &= \lim_{a'\to 0}\frac{F(a')}{a'} \\ &= \lim_{a'\to 0}\frac{1}{a'}\ln\left[1 + a' + \frac{1}{2}h''(0)(a')^2 + \cdots\right] \\ &= 1. \end{aligned} \tag{7.7}$$

Secondly, as noted above, this construction satisfies the condition $J(0) = 1$. Since $a = F(a')$ by Eq. (3.1), Eq. (7.5) is equivalent to $e^a = h(a')$, so the inverse transformation is given formally as

$$a' = h^{-1}(e^a), \tag{7.8}$$

where h^{-1} denotes the inverse of the function h. We have found several cases where this inverse can be calculated explicitly. We present some of these next.

B. LD function 1

Here we present our first function h to be used in Eq. (7.1) and (7.3) to generate a new scheme transformation. This is

$$h(y) = 1 + \frac{1}{r}\ln(1 + ry). \tag{7.9}$$

Hence, $h'(y) = 1/(1 + ry)$. By construction, this satisfies the condition that both $h(y)$ and $h'(y)$ are positive functions for physical (i.e., nonnegative) y and the condition that $h(0) = h'(0) = 1$. From (7.3), we have

$$a = F(a') = \ln\left[1 + \frac{1}{r}\ln(1 + ra')\right]. \tag{7.10}$$

We remark that for the families of scheme transformations studied so far in [3–6] that are dependent on an auxiliary parameter r, such as $a = (1/r)\sinh(ra')$ and the transformations studied in [7] such as $a = (1/r)\ln(1 + ra')$ and $a = a'/(1 - ra')$, setting $r = 0$ yields the identity transformation $a = F(a') = a'$. However, this is not the case for the transformation of Eq. (7.10). Instead, setting $r = 0$ in (7.10) yields the scheme transformation [37]

$$r = 0 \Rightarrow a = F(a') = \ln(1 + a'). \tag{7.11}$$

This property also holds for the transformations (7.17) and (7.21) discussed below. As is necessary, Eq. (7.11) obeys the requirement (3.2) that $f(0) = 1$, i.e., that the transformation becomes an identity $a = a'$ in the free-field limit $a \to 0$.

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with the k_s coefficients displayed in

Table I. In the special case $r = 0$, the coefficients k_s are given by the Taylor series expansion of $(1/a')\ln(1+a')$ around $a' = 0$, namely

$$r = 0 \Rightarrow k_s = \frac{(-1)^s}{s+1}. \tag{7.12}$$

This is also true of the coefficients k_s for the functions discussed in the next two subsections, VII C and VII D.

The inverse transformation is

$$a' = \frac{1}{r}[\exp[r(e^a - 1)] - 1]. \tag{7.13}$$

For the Jacobian, expressed in terms of a' and a, we calculate

$$J = \frac{1}{(1+ra')[1+\frac{1}{r}\ln(1+ra')]} = \exp[-a - r(e^a - 1)]. \tag{7.14}$$

The parameter r is restricted to the range

$$r > -\frac{1}{a'} \tag{7.15}$$

in order to avoid singularities in $h(y)$ and $F(a')$ and is further restricted by the requirement that this scheme transformation must satisfy the conditions C_1-C_4. These conditions can be satisfied for small positive r. With r positive (indeed with $r > -1$), $k_{s_{\min}} < 0$ (where $s_{\min} = 1$ here), so our general result (3.12) implies that $a < a'$ for small a, a'.

C. LD function 2

As an input for the construction of our next new scheme transformation, we use

$$h(y) = 1 + \frac{1}{r}\sinh(ry). \tag{7.16}$$

Thus, $h'(y) = \cosh(ry)$. Without loss of generality, the parameter r can be taken to be nonnegative, and we shall do this. Evidently, this function $h(y)$ satisfies the condition (7.4). From the general result (7.3), we obtain

$$a = F(a') = \ln\left[1 + \frac{1}{r}\sinh(ra')\right]. \tag{7.17}$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3), and we list the first few coefficients k_s in Table I. The invariance of the transformation $F(a')$ in Eq. (7.17) under the reversal in sign of the auxiliary parameter r is reflected in the property that the k_s involve only even powers of r. Here $s_{\min} = 1$ and $k_{s_{\min}} < 0$, so by our general result (3.12), it follows that $a < a'$ for small a, a'.

The inverse transformation is

$$a' = \frac{1}{r}\ln\left[r(e^a - 1) + \sqrt{1 + [r(e^a - 1)]^2}\right]. \tag{7.18}$$

For the Jacobian we calculate

$$J = \frac{\cosh(ra')}{1 + \frac{1}{r}\sinh(ra')}. \tag{7.19}$$

This transformation satisfies all of the conditions C_1-C_4.

D. LD function 3

Here we discuss a third function h for use in Eq. (7.1) and (7.3), namely

$$h(y) = 1 + \frac{1}{r}\tanh(ry). \tag{7.20}$$

Thus, $h'(y) = 1/\cosh^2(ry)$. This satisfies the condition (7.4). As with the previous $h(y)$ function in Eq. (7.16), we can, without loss of generality, take the parameter r to be nonnegative, and we shall do this. From the general result (7.3), we obtain

$$a = F(a') = \ln\left[1 + \frac{1}{r}\tanh(ra')\right]. \tag{7.21}$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with the first few k_s coefficients listed in Table I. Since $s_{\min} = 1$ and $k_1 < 0$, we infer that $a < a'$ for small a, a'.

The inverse transformation is

$$a' = \frac{1}{2r}\ln\left[\frac{1 + r(e^a - 1)}{1 - r(e^a - 1)}\right]. \tag{7.22}$$

The Jacobian, expressed as a function of a' and of a, is

$$J(a') = \frac{1}{\cosh^2(ra')[1 + \frac{1}{r}\tanh(ra')]} = e^{-a}[1 + r(e^a - 1)][1 - r(e^a - 1)]. \tag{7.23}$$

Although the transformation in Eq. (7.21) and the Jacobian in Eq. (7.23) are nonsingular for any r, the inverse transformation (7.22) does contain a singularity which restricts the range of r. Recalling that, without loss of generality, r has been taken to be nonnegative, this singularity occurs at $r = 1/(e^a - 1)$. Hence, we restrict r to be substantially less than $1/(e^a - 1)$ to avoid this singularity in the inverse transformation.

VIII. SCHEME TRANSFORMATIONS FOR WHICH $J(y) = (d/dy)e^{\phi(y)}$

A. General

Here we present another class of $J(y)$ functions that can be used in conjunction with our integral formalism to construct scheme transformations. As was true of the functions in Sec. VII, these function have the form of total derivatives, which guarantees that one can do the integral (4.1). We begin with an analytic function $\phi(y)$ that satisfies the conditions

$$\phi(0) = 0, \qquad \phi'(0) = 1. \tag{8.1}$$

We then set $J(y)$ equal to the derivative of the exponential of this function:

$$J(y) = \frac{d}{dy} e^{\phi(y)} = \phi'(y) e^{\phi(y)}. \tag{8.2}$$

Substituting this into the integral (4.1), we obtain

$$a = F(a') = e^{\phi(a')} - e^{\phi(0)} = e^{\phi(a')} - 1. \tag{8.3}$$

This yields $J(a') = dF(a')/da' = \phi'(a')e^{\phi(a')}$ so that, taking into account the property (8.1), it follows that $J(0) = 1$. Furthermore, this construction guarantees that the condition $f(0) = 1$ in Eq. (3.2) is satisfied. To prove this, we use the defining relation $a = a'f(a') = F(a')$ in Eq. (3.1) to obtain

$$f(a') = \frac{e^{\phi(a')} - 1}{a'}. \tag{8.4}$$

Expanding the numerator in a Taylor series around $a' = 0$, we get

$$\begin{aligned} f(a') &= \frac{1}{a'}[e^{\phi(0)} - 1 + \phi'(0)a' + O((a')^2)] \\ &= 1 + O(a'), \end{aligned} \tag{8.5}$$

from which it follows that $f(0) = 1$. The inverse is, formally,

$$a' = \phi^{-1}[\ln(a+1)], \tag{8.6}$$

where here ϕ^{-1} denotes the function that is the inverse of ϕ.

B. ϕ function 1

In order to show how Eqs. (8.2) and (8.3) can be used to construct new scheme transformations, we first take

$$\phi(y) = \frac{1}{r}(e^{ry} - 1). \tag{8.7}$$

For convenience, we may restrict r to be nonnegative. This function satisfies the condition (8.1). Substituting the resultant $J(y) = e^{ry}\exp[(1/r)(e^{ry} - 1)]$ into Eq. (4.1), we obtain the scheme transformation

$$a = F(a') = \exp\left[\frac{1}{r}(e^{ra'} - 1)\right] - 1. \tag{8.8}$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with the first few k_s coefficients listed in Table I. Note that in the limit as $r \to 0$, the scheme transformation (8.8) becomes

$$r = 0 \Rightarrow a = F(a') = e^{a'} - 1. \tag{8.9}$$

Hence, in this limit the coefficients are given by

$$r = 0 \Rightarrow k_s = \frac{1}{s!}. \tag{8.10}$$

These results also hold for the transformation (8.14) to be discussed below.

The inverse transformation is

$$a' = \frac{1}{r}\ln[1 + r\ln(a+1)]. \tag{8.11}$$

Using this, we may express the Jacobian in terms of a:

$$\begin{aligned} J &= e^{ra'}\exp\left[\frac{1}{r}(e^{ra'} - 1)\right] \\ &= (a+1)[1 + r\ln(a+1)]. \end{aligned} \tag{8.12}$$

This transformation satisfies conditions C_1-C_4.

C. ϕ function 2

As a second application of Eqs. (8.2) and (8.3), we use

$$\phi(y) = \frac{1}{r}\sinh(ry). \tag{8.13}$$

Without loss of generality, we take the auxiliary parameter r to be nonnegative. This function satisfies the condition (8.1). Substituting the resultant $J(y) = (d/dy)e^{(1/r)\sinh(ry)}$ into Eq. (4.1), we obtain the scheme transformation

$$a = F(a') = \exp\left[\frac{1}{r}\sinh(ra')\right] - 1. \tag{8.14}$$

The resultant series expansion for $f(a') = F(a')/a'$ has the form of Eq. (3.3) with the k_s coefficients listed in Table I. Because $k_{s_{\min}} > 0$ (with $s_{\min} = 1$ here), our general result (3.12) implies that $a > a'$ for small a, a'.

The inverse transformation is

$$a' = \frac{1}{r}\ln\left[r\ln(a+1) + \sqrt{1+[r\ln(a+1)]^2}\right]. \quad (8.15)$$

The Jacobian is

$$J = \cosh(ra')\exp\left[\frac{1}{r}\sinh(ra')\right]. \quad (8.16)$$

This transformation satisfies conditions C_1-C_4.

IX. COMPARATIVE ANALYSIS

In earlier work [3–7], a number of scheme transformations have been applied to ascertain the degree of scheme dependence of the value $\alpha_{\mathrm{IR},n\ell}$ of the IR zero, calculated up to four-loop order, of the beta function in an SU(N) gauge theory with various fermion contents. Comparisons have been made between results calculated in different schemes such as $\overline{\mathrm{MS}}$, MOM, and RI′ [26–28]. Scheme transformations have also been applied to study the possibility of a UV zero in the beta function of a U(1) gauge theory with N_f (charged) fermions and in a globally invariant O(N) $\lambda|\vec{\phi}|^4$ theory up to the five-loop level [31,32].

With the new scheme transformations generated by our integral formalism, we now have a reasonably large set of such transformations to use to study scheme dependence of the zero of a beta function away from the origin in coupling-constant space. In this section we include some remarks concerning the analytic structure of these transformations that are relevant to this application. First, in the perturbative regime of small to moderate values of α and hence also α' [which correspond to even smaller values of $a = \alpha/(4\pi)$ and $a' = \alpha'/(4\pi)$], the effect of the scheme transformation is largely determined by the values of the first few coefficients k_s in the Taylor series expansion of the transformation function $f(a')$ in Eq. (3.3) or equivalently, $F(a')$ in Eq. (3.4) for the first few values of s. Clearly, the same is true of the inverse scheme transformation, as is evident from the Taylor series (3.5) for this inverse, together with the coefficients ρ_s determined via series reversion from the coefficients k_s. Therefore, in this regime of moderately small couplings α and α', one can get a reasonably good determination of the shift in the value of $\alpha_{\mathrm{IR},n\ell}$ by examining the first few k_s coefficients. We have listed these for comparative purposes in Table I. The first four lines of this table describe scheme transformations from [3–7], while the next seven lines describe new scheme transformations presented and analyzed in the present paper.

Indeed, for small a and hence also small a', the question of whether a given scheme transformation increases or decreases the coupling is determined, using our general result (3.12), by the sign of the lowest nonzero coefficient, $k_{s_{\min}}$, in the Taylor series expansion (3.3) or equivalently, (3.4), using our general result. One can conveniently read this from our Table I for the new scheme transformations that we have presented in this paper.

Hence, by combining the numerical analyses in [3–7] with the analytic results for the first few k_s coefficients in Table I, we can infer the effects of our new scheme transformations. In particular, we may again infer that in an asymptotically free non-Abelian gauge theory with a two-loop IR zero at a value $\alpha_{\mathrm{IR},2\ell}$ that is not too large, and for moderate values of the auxiliary parameter r, the scheme dependence inherent in the calculation of $\alpha_{\mathrm{IR},n\ell}$ at $n=3$ and $n=4$ loops is moderately small.

X. CONCLUSIONS

In this paper we have presented an integral formalism for constructing scheme transformations. We have used this formalism to generate several new scheme transformations that are acceptable for the analysis of a zero of the beta function away from the origin in coupling-constant space. By performing Taylor-series expansions of these scheme transformations, we have formulated an analytic approach to their effect on the coupling. These results bolster previous numerical studies to show that in an asymptotically free gauge theory with a two-loop value of the IR zero of the beta function, $\alpha_{\mathrm{IR},2\ell}$, that is not too large, the scheme transformations presented here (with moderate values of the auxiliary parameter r) produce only relatively mild shifts in higher-loop values $\alpha_{\mathrm{IR},n\ell}$.

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APPENDIX: A RESULT FROM CALCULUS

We use the following result from multivariable calculus. Consider the integral

$$F(x) = \int_{y_1(x)}^{y_2(x)} \Phi(x,y)dy. \quad (A1)$$

Then,

$$\frac{dF(x)}{dx} = \frac{dy_2(x)}{dx}\Phi(x,y_2(x)) - \frac{dy_1(x)}{dx}\Phi(x,y_1(x)) + \int_{y_1(x)}^{y_2(x)} \frac{\partial\Phi(x,y)}{\partial x}dy. \quad (A2)$$

In particular, if $\Phi(x,y)$ does not depend on x, which we indicate by setting $\Phi(x,y) \equiv J(y)$ (which may depend on auxiliary parameters such as r), and if $y_2(x) = x$ and $y_1(x) = \mathrm{const.}$, then Eq. (A2) reduces to $dF/dx = J(x)$.

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