

Triple point in the $O(2)$ ghost model with higher-order gradient term

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(Received 9 August 2016; published 27 September 2016)

The phase structure and the infrared behavior of the Euclidean three-dimensional $O(2)$ symmetric ghost scalar field ϕ has been investigated in Wegner and Houghton's renormalization group framework, including higher derivatives in the kinetic term. It is pointed out that higher-derivative coupling provides three phases and leads to a triple point in that RG scheme. The types of the phase transitions have also been identified.

DOI: 10.1103/PhysRevD.94.065037

I. INTRODUCTION

It is well known that the existence of the triple point, the point of coexistence of three phases, is very common in condensed matter physics, generally realized as the coexistence of the gaseous, liquid, and solid phases of the same material, but also occurring in magnetic materials with more than one solid phase in equilibrium [1,2], as well as in ^4He as the equilibrium of two solid phases and a liquid, or that of the superfluid, normal fluid, and solid phases [3]. Now we shall show that in a particular approximation of the functional renormalization group approach, one finds that the three-dimensional Euclidean $O(2)$ symmetric ghost scalar model with wave function renormalization $Z = -1$ and the higher-derivative term $Y\phi\Box^2\phi$ exhibits a triple point where the symmetric phase, the symmetry-broken phase, and the phase with restored symmetry coexist in equilibrium. In general, field theory models with higher-derivative terms of alternating signs have rather rich phase structure corresponding to various periodic structures [4–7]. The existence of the triple point in ordinary $O(2)$ symmetric models with appropriate higher-derivative terms has also been shown in Ref. [8].

The phase structure of the ghost $O(2)$ model has been analyzed by us in the framework of Wegner and Houghton's (WH) renormalization group (RG) method [9] with the sharp gliding momentum cutoff k . Using WH RG framework, one is restricted to the local potential approximation (LPA), the lowest order of the gradient expansion. In the LPA, the wave function renormalization Z and couplings of the higher-derivative terms do not acquire any RG flow. Since the wave function renormalization Z is dimensionless, it can be kept constant unambiguously like $Z = +1$ for ordinary models and $Z = -1$ for ghost models. There occurs, however, an ambiguity when the couplings of higher-derivative terms are accounted for which have nonvanishing momentum dimensions. It corresponds to different approximations or RG schemes to keep either the dimensionful or the dimensionless higher-derivative couplings constant. In our previous paper [10], we argued for keeping the dimensionful coupling Y constant and showed that the model exhibits two phases:

besides the trivial symmetric phase, there occurs a phase with restored symmetry characterized by a quasi-universal dimensionful effective potential. The existence of the latter is related with the occurrence of the ghost condensate at intermediate scales k . Now we shall take another point of view and keep the *dimensionless* coupling $\tilde{Y} = Yk^{-2}$ constant during the WH RG flow. In this approximation, it shall be shown that the model has three phases and exhibits the possibility of the coexistence of all three phases in equilibrium.

One may wonder where the increased number of the phases comes from. We believe that it is related to Gibbs's phase rule $f = n_p - N_{ph}$ for one-component thermodynamical systems, where n_p is the number of parameters characterizing the thermodynamical state, and f is the number of degrees of freedom when the number N_{ph} phases are present in the system. In order to apply Gibbs's phase rule, one has to consider our model in the deep IR region. In both RG schemes, the RG flow turns out to be qualitatively independent of the internal degrees of freedom $N = 2$ in our case. This happens because the WH RG flow equations for the couplings have the same form for, say, $N = 1$ and $N = 2$, except that a few numerical constants are modified and because the tree-level RG flow (if it occurs) is independent of N . This indicates that our model can be considered as describing a one-component thermodynamical system. The number of independent dimensionless parameters included in the model is $M + 1$: that is the number M of couplings included in the potential when it is truncated at the power ϕ^{2M} and the higher-derivative coupling \tilde{Y} . In the RG scheme used in Ref. [10], the dimensionful coupling Y has been kept constant during the RG flow, so that in the IR limit, the dimensionless parameter $\tilde{Y} = Yk^2$ vanishes in a manner prescribed by the natural dimension of Y , leaving essentially the number $n_p = M$ parameters free for the determination of the IR flow. As opposed to this, we have one more dimensionless parameter characterizing the IR system, i.e., $n_p = M + 1$, when \tilde{Y} is kept constant like in the RG scheme used in the present paper. Moreover, numerics shows that only the couplings of the first two lowest-order terms of the

potential influence the phase diagram qualitatively, while the inclusion of the higher-order terms causes only quantitative changes. This means that only $M_{\text{eff}} = 2$ parameters of the potential are effective for the thermodynamic behavior of the model investigated by us, so that the number of thermodynamically interesting parameters is $n_{\text{eff}} = M_{\text{eff}} = 2$ and $n_{\text{eff}} = M_{\text{eff}} + 1 = 3$ for the RG schemes with $Y = \text{const.}$ and $\tilde{Y} = \text{const.}$, respectively. Now Gibbs's phase rule tells us that the possible number of degrees of freedom can be $f = n_{\text{eff}} - N_{ph}$. For a nonvariant system with vanishing degrees of freedom ($f = 0$), we get that the number of coexistent phases can be $N_{ph} = 2$ and $N_{ph} = 3$ for the RG schemes with $Y = \text{const.}$ and $\tilde{Y} = \text{const.}$, respectively. The arguments given above are also an indication that the RG scheme used in the present paper may yield a more reliable description of the phase structure, because the higher-derivative coupling \tilde{Y} is not turned off in the IR regime automatically.

II. WEGNER-HOUGHTON RENORMALIZATION GROUP FOR THE GHOST $O(2)$ MODEL

In this paper, we study the three-dimensional, Euclidean, $O(2)$ symmetric model for the real two-component ghost scalar field $\underline{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ using the ansatz

$$S_k[\underline{\phi}] = \frac{1}{2} \int d^3x \underline{\phi}^T \Omega(-\square) \underline{\phi} + \int d^3x U_k(\underline{\phi}^T \underline{\phi}) \quad (1)$$

for the blocked action in LPA, where $U_k(\underline{\phi}^T \underline{\phi})$ stands for the blocked potential assumed to be of the polynomial form (a Taylor expansion truncated at the order ϕ^{2M})

$$U_k(r) = \sum_{n=0}^M \frac{v_n(k)}{n!} r^n, \quad (2)$$

with $r = \frac{1}{2} \underline{\phi}^T \underline{\phi}$ and

$$\Omega(-\square) = -Z\square + Y\square^2, \quad (3)$$

with the wave function renormalization $Z = -1$ and the higher-derivative coupling $Y > 0$. The phase structure of the model has been analyzed in the framework of the WH RG method with the sharp gliding momentum cutoff k . In the LPA, the wave function renormalization Z and the higher-derivative coupling Y do not acquire RG flow. The WH RG equation for the local potential is given as [10]

$$k \partial_k U_k(r) = -\alpha k^3 [\ln s_+(k) + \ln s_-(k)], \quad (4)$$

where

$$\begin{aligned} s_+(k) &= \Omega(k^2) + U'_k(r) + 2rU''_k(r), \\ s_-(k) &= \Omega(k^2) + U'_k(r), \end{aligned} \quad (5)$$

with $U'_k(r) = \partial_r U_k(r)$, $U'' = \partial_r^2 U_k(r)$, and $\alpha = 1/(4\pi^2)$. Here r corresponds now to a constant background field Φ with $\Phi = \sqrt{2}r \geq 0$ pointing in an arbitrary direction \underline{e} in the internal space. That background field is the tool to find out the form of the potential.

In general, the WH RG equation may lose its validity. This happens when at least one of the arguments of the logarithms on the right-hand side of Eq. (4) ceases to be positive at some nonvanishing scale k_c . For scales $k \leq k_c$, the resummation of the loop expansion by means of the WH equation is no longer possible. The IR behavior can then be revealed by means of the tree-level renormalization (TLR) procedure [11] [see also its application to the $O(2)$ model in our previous paper [10]]. While in the case of the one-component ($N = 1$) real scalar field, the vanishing of $s_+(k)$ governs the singularity; in cases with $N \geq 2$, the vanishing of $s_-(k)$ does it. The critical scale k_c is given by $s_-(k_c)|_{\Phi=0} = 0$, implying $Z + \tilde{Y} + \tilde{v}_1(k_c) = 0$, just like in the case $N = 1$. (Throughout this paper, the dimensionless quantities shall be denoted by tilde, so that $\phi = k^{1/2} \tilde{\phi}$, $v_n = k^{3-n} \tilde{v}_n$, $U_k = k^3 \tilde{U}_k$, and $Y = k^{-2} \tilde{Y}$.) The spinodal instability at the singularity scale k_c reveals itself in building up an inhomogeneous field configuration $\underline{\psi}$ on the homogeneous background. The essence of TLR is to decrease the scale k by a step $\Delta k \ll k$ and to find out the inhomogeneous configuration $\underline{\psi}$ that minimizes the Euclidean action at the given scale $k < k_c$, that determines the blocked action at the lower scale $k - \Delta k$ via

$$S_{k-\Delta k}[\underline{e}\Phi] = \min_{\underline{\psi}} S_k[\underline{e}\Phi + \underline{\psi}]. \quad (6)$$

We shall restrict the function space of spinodal instabilities to those of stationary waves $\underline{\psi}$ pointing in the direction \underline{e} of the homogeneous background field in the internal space and describing sinusoidal periodicity in a given direction n_μ of the external space—i.e., to the form

$$\underline{\psi} = \underline{e} 2\rho \sin(kn_\mu x_\mu + \theta) \quad (7)$$

with the phase shift θ . Making use of the ansatz (7), the TLR blocking relation (6) reduces to the recursion relation

$$\begin{aligned} U_{k-\Delta k}(\Phi) &= \min_{\{\rho\}} \left(U_k(\Phi) + (Z + \tilde{Y})k^2\rho^2 \right. \\ &\quad \left. + \sum_{n=1}^M \frac{\rho^{2n}}{(n!)^2} \partial_\Phi^{2n} U_k(\Phi) \right). \end{aligned} \quad (8)$$

The ansatz (7) reduces the TLR of the $O(2)$ model to that of the $O(1)$ model with the same wave function renormalization Z and higher-derivative coupling \tilde{Y} . Let $\rho_k(\Phi)$ be the amplitude minimizing the expression on the right-hand side of the recursion relation (7). Clearly, a nonvanishing value of $\rho_k(\Phi)$ breaks $O(2)$ symmetry in internal space, as well as $O(3)$ and translational symmetries in the three-dimensional external space. For local potentials of the form (2) and for scales $k < k_c$, the interval $0 \leq \Phi \leq \Phi_c(k)$ [with $\Phi_c(k) = \sqrt{k}\tilde{\Phi}_c(k)$], in which the instability occurs, is determined via the relation $s_-(k) = 0$ as

$$\tilde{\Phi}_c(k) = \sqrt{-\frac{2[Z + \tilde{Y} + \tilde{v}_1(k)]}{3\tilde{v}_2(k)}}. \quad (9)$$

Our numerical procedure for the determination of the RG trajectories is just the same as in our paper [10]. The WH RG equations are rewritten as a coupled set of ordinary differential equations for the dimensionless couplings \tilde{v}_n of the local potential $\tilde{U}_k(\tilde{\Phi})$ and those solved with the truncation $M = 10$ with the fourth-order Runge-Kutta method for scales $k_c < k \leq \Lambda$. It may happen that it holds the inequality $s_-(\Lambda) < 0$ at the ultraviolet (UV) cutoff scale Λ already. Therefore, we define the singularity scale as $k_s = k_c$ for the cases with $s_-(\Lambda) > 0$, and as $k_s = \Lambda$ for cases with $s_-(\Lambda) < 0$. If there is a singularity, then TLR is applied for scales $k < k_s$ in order to determine the RG flow in the IR regime by means of the recursion relation (8) rewritten in terms of the dimensionless quantities. The scale k has then been decreased from the scale k_s by at least 2 orders of magnitude with the step size $\Delta k/k = 0.005$ and the truncation $M = 10$. The numerical precision was set to 80 digits. Generally, ~ 1000 iteration steps have been numerically performed at each value of the constant background Φ for the minimization of the blocked potential $U_k(\rho, \Phi)$ with respect to the amplitude ρ of the spinodal instability. The minimization with respect to ρ on the right-hand side of Eq. (8) and the determination of the couplings at the lower scale $k - \Delta k$ with least-square fit are performed in the interval $0 \leq \Phi \leq \bar{\Phi}$ of the background fields, which has been chosen in a similar manner as described in Ref. [10]. Namely, for ‘‘Mexican hat’’-like potential $U_{k_s}(\Phi)$, the choice $\bar{\Phi} \approx 1.5\Phi_m$ has been made, where $\pm\Phi_m$ are the positions of the local minima of the potential with $\Phi_m = \sqrt{-2v_1(k_s)/3v_2(k_s)}$. For convex potentials $U_{k_s}(\Phi)$ the choice $\bar{\Phi} \gtrsim 30$ has been made. It has been observed numerically that the blocked potential does not acquire tree-level corrections outside of the interval $0 \leq \Phi \leq \Phi_c$, with Φ_c given by Eq. (9), but the choice of the larger interval makes the minimization and fitting numerically stable.

III. PHASE STRUCTURE AND IR SCALING LAWS

A. Phase diagram

The phase structure has been investigated for RG trajectories started in the hypercube $[-1, +1] \otimes [0, 10] \otimes [0, 2]$ in the three-dimensional parameter space $(\tilde{v}_1, \tilde{v}_2, \tilde{Y})$. Various slices of the phase diagram are shown in Fig. 1. The identification of the phases is based on the concept of the so-called sensitivity matrix [12,13]. The matrix $S_{n,m}$ is built up by the derivatives of the running coupling constants with respect to the bare ones:

$$S_{n,m} = \frac{\partial g_n(k)}{\partial g_m(\Lambda)}. \quad (10)$$

We can find different phases when a singularity takes place in the IR ($k \rightarrow 0$) and the UV ($\Lambda \rightarrow \infty$) limits of the elements of the sensitivity matrix. According to this type of identification, we can find different phases in the model for which the effective potential depends on different sets of bare couplings. Using this technique, we find that there exist three phases and a triple point in all slices at constant \tilde{v}_2 . It shall be shown that there is a symmetric phase (phase I), a phase with restored symmetry (phase II), and a phase with spontaneously broken symmetry (phase III). Some remarks should be made with respect to the phase

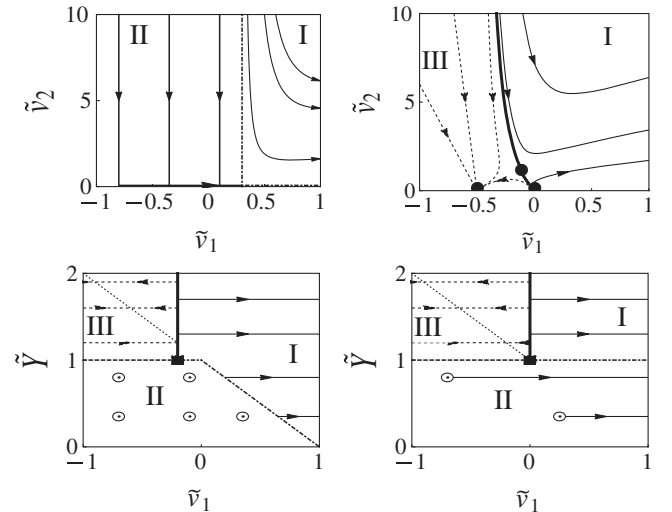


FIG. 1. Various planar slices of the phase diagram of the ghost $O(2)$ model with a few typical RG trajectories in the parameter space $(\tilde{v}_1, \tilde{v}_2, \tilde{Y})$: the slice $(\tilde{v}_1, \tilde{v}_2)$ for $\tilde{Y} = 0.7$ (at the top to the left), the slice $(\tilde{v}_1, \tilde{v}_2)$ for $\tilde{Y} = 1.5$ (at the top to the right) with the fixed points (dots), the slice (\tilde{v}_1, \tilde{Y}) for $\tilde{v}_2 > 0$ (at the bottom to the left), and the slice (\tilde{v}_1, \tilde{Y}) for $\tilde{v}_2 = 0$ (at the bottom to the right). The phase boundaries II-I, III-I, and III-II are depicted by thick dash-dotted, thick full, and dashed lines, respectively. The dotted line represents a section of the straight line $\tilde{v}_1 = 1 - \tilde{Y}$, which is the IR fixed line in the slice with $\tilde{v}_2 = 0$; the full square stands for the triple point. The dotted circles represent RG trajectories running perpendicularly to the $\tilde{v}_2 = \text{const}$ planes.

diagram. In our WHRG approach, all RG trajectories lie in one of the $\tilde{Y} = \text{const.}$ planes. The RG trajectories belonging to phase II arrive perpendicularly to the plane $\tilde{v}_2 = 0$, where they make a turn with 90° and run away to positive infinity parallel to the \tilde{v}_1 axis. This happens because the dimensionful coupling v_1 takes a nonvanishing constant value in the IR limit $k \rightarrow 0$. Therefore, the phase boundary II-I is the two-dimensional surface $(\tilde{v}_1 = 1 - \tilde{Y}, \tilde{v}_2 > 0, 0 < \tilde{Y} < 1) \cup (1 - \tilde{Y} < \tilde{v}_1 \leq 1, \tilde{v}_2 = 0, 0 < \tilde{Y} < 1)$. The Gaussian and Wilson-Fisher fixed points shown in the top-right subfigure in Fig. 1 belong to phase I and stand for fixed lines with any values of $\tilde{Y} \in [0, 2]$, but the IR fixed point (line) belongs to phase III and occurs only for $\tilde{Y} \in [1, 2]$. The positions of the fixed points are given in Table I. One can see in the top-right subfigure in Fig. 1 that both the Gaussian and Wilson-Fisher fixed points lie on the phase boundary III-I and act for the RG trajectories as crossover points. The flow of the RG trajectories in phase I is qualitatively the same, independently of the value of \tilde{Y} in the interval $0 < \tilde{Y} \leq 2$. In the slice $(\tilde{v}_1, \tilde{v}_2)$ for $1 < \tilde{Y} \leq 2$, the trajectories in phase III run into the IR fixed point (line), but their evaluation becomes numerically unstable in the close neighborhood of the fixed point. The phase boundary III-II lies in the plane $\tilde{Y} = 1$. Finally, in slices (\tilde{v}_1, \tilde{Y}) for any constant \tilde{v}_2 (subfigures at the bottom in Fig. 1), one can see all three phases and the triple point. In the three-dimensional parameter space there is a triple line, the line of intersection of the phase boundaries III-II and III-I. The detailed study of the IR scaling laws enables one to identify the symmetry properties of the various phases. This is given in the following subsections.

B. Phase I

Phase I is the symmetric phase of the model. Trajectories belonging to phase I are those along which the WHRG equation (4) does not acquire any singularity. The RG flows of the individual dimensionful couplings are qualitatively the same in phase I, regardless of the value of \tilde{Y} . They increase strictly monotonically with decreasing scale k in a rather short UV scaling region $\sim 0.3 < k \leq \Lambda = 1$, and then tend asymptotically to certain constant values $v_n(0)$ in the IR regime. Therefore, the dimensionful effective potential is convex, but very much sensitive to the bare potential. The IR limiting values of the dimensionful couplings $v_1(0)$ and $v_2(0)$ have been compared on RG trajectories started at

TABLE I. The positions of the fixed points for given values of the higher-derivative coupling \tilde{Y} .

Fixed point	\tilde{v}_1	\tilde{v}_2
Gaussian	0	0
Wilson-Fisher	$\frac{3}{13}(1 - \tilde{Y})$	$\frac{80\pi^2}{169}(1 - \tilde{Y})^2$
IR	$1 - \tilde{Y}$	0

various given “distances” $t = \tilde{v}_1(\Lambda) - \tilde{v}_u$ from the phase boundary \tilde{v}_u (I-II for $0 \leq \tilde{Y} < 1$ and I-III for $1 < \tilde{Y} \leq 2$) for given values of $v_2(\Lambda) = 0.01, 0.1$ and several values of \tilde{Y} . The linear relation

$$v_1(0) = at + b(\tilde{Y}) \quad (11)$$

has been established where the slope $a = 1 \pm .001$ is independent of \tilde{Y} , whereas the mass squared at the phase boundary ($t \rightarrow 0$),

$$b(\tilde{Y}) = (1 - \tilde{Y})b(0)\Theta(1 - \tilde{Y}), \quad (12)$$

decreases approximately linearly to zero at $\tilde{Y} = 1$ with the increasing higher-derivative coupling \tilde{Y} (see Fig. 2) and vanishes for $\tilde{Y} > 1$.

For $k \rightarrow 0$, the coupling $v_2(k)$ increases by a great amount with respect to its bare value $v_2(\Lambda)$ near the phase boundary I-II for $t \rightarrow 0$ and $0 < \tilde{Y} \ll 1$, but it accommodates very little loop corrections near the boundary I-III for $t \rightarrow 0$ and $1 < \tilde{Y} \leq 2$. In the latter case, the behavior of the coupling $v_2(k)$ is similar to its behavior in the symmetric phase of the ordinary $O(2)$ model near the boundary with the symmetry-broken phase. Far enough from the phase boundary \tilde{v}_u —i.e., at larger values of t —the loop corrections are suppressed by the large mass squared $v_1(0)$, and the coupling $v_2(k)$, as well as all higher-order couplings $v_{n>2}(k)$, keep essentially their bare values. For $t \rightarrow 0$, the IR value $v_2(0)$ shows a significant dependence on the higher-derivative coupling \tilde{Y} ; it has a minimum at $\tilde{Y} = 1$ with $v_2(0) = 0$ (Fig. 2).

C. Phase II

Phase II occurs for $0 < \tilde{Y} < 1$. As we shall argue below, phase II is a phase with restored symmetry in the IR limit; i.e., a periodic structure occurs below the singularity scale k_s which breaks $O(2)$ symmetry, but it is washed out in the limit $k \rightarrow 0$. In this phase $k_s = \Lambda$, so that the RG flow has to be followed up by the TLR procedure started at the UV scale Λ . It was found that the couplings of the dimensionful blocked potential tend to constant values in the IR limit.

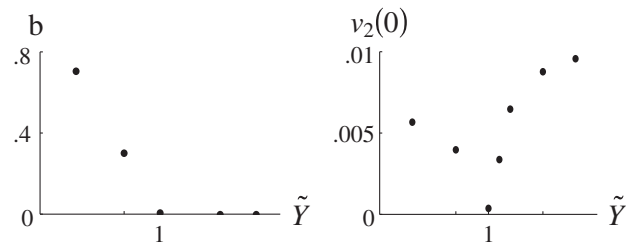


FIG. 2. The parameters b in Eq. (11) (to the left) and $v_2(0)$ (to the right) vs the higher-derivative coupling \tilde{Y} at the “distance” $t = \tilde{v}_1(\Lambda) - \tilde{v}_u = 0.001$ from the boundary of phase I.

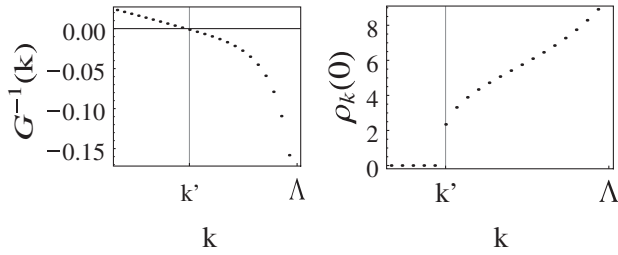


FIG. 3. The flow of the inverse propagator $G^{-1}(k)$ and that of the amplitude of the spinodal instability ρ_k at vanishing homogeneous background field $\Phi = 0$ along the RG trajectory with $\tilde{Y} = 0.7$, $\tilde{v}_1(\Lambda) = -0.1$, $\tilde{v}_2(\Lambda) = 0.01$, and the step size $\Delta k/k = 5 \times 10^{-5}$.

The typical behavior of the inverse propagator $G^{-1} = (-1 + \tilde{Y})k^2 + v_1(k)$ and that of the amplitude of the spinodal instability $\rho_k(0)$ for a vanishing homogeneous background field $\Phi = 0$ are shown in Fig. 3. It can be seen that just below the scale $k_s = \Lambda$, the inverse propagator is negative; its magnitude as well as the amplitude $\rho_k(0)$ decrease till the gliding cutoff k reaches some nonvanishing scale $k' < k_s$. It was found that $\rho_k(0)$ decreases linearly with the scale k in the interval $k' < k < \Lambda$. At the scale k' , the propagator vanishes and the amplitude of the spinodal instability jumps to zero suddenly. This means that below the scale k' , no tree-level renormalization occurs anymore. The flow of the amplitude ρ_k of the spinodal instability is qualitatively just the same as we have found it previously in our paper [10] in phase II. Namely, the periodic configuration is developed below the scale k_s , but it is washed out at some nonvanishing scale k' .

Moreover, it has been observed that for any given value of the higher-derivative coupling \tilde{Y} , the effective potential is quasi-universal in the sense that it does not depend on at which point $[v_1(\Lambda), v_2(\Lambda)]$ the RG trajectories have been started. Therefore, we have determined the mean values $\overline{v_1(0)}$ and $\overline{v_2(0)}$ of the couplings $v_1(0)$ and $v_2(0)$ with their variances via averaging them over all evaluated RG trajectories belonging to a given value of the coupling \tilde{Y} . Table II shows that the *dimensionful* mass squared decreases with increasing values of \tilde{Y} linearly as

TABLE II. Mean IR values of the dimensionful couplings of the quadratic and quartic terms of the effective potential with their errors in phase II for various values of the higher-derivative coupling \tilde{Y} .

\tilde{Y}	$\overline{v_1(0)} \pm \Delta v_1(0)$	$\overline{v_2(0)} \pm \Delta v_2(0)$
.0	.92 ± .03	-.016 ± .036
.3	.69 ± .01	-.010 ± .016
.5	.50 ± .01	-.007 ± .016
.7	.25 ± .05	.002 ± .050
1.0	.025 ± .007	-.016 ± .018

$$\overline{v_1(0)}(\tilde{Y}) = \overline{v_1(0)}|_{\tilde{Y} \rightarrow 0} (1 - \tilde{Y}) \quad (13)$$

(cf. Fig. 4), while the coupling of the quartic term vanishes. Similarly, all the higher-order couplings $\tilde{v}_{n>2}(0)$ vanish. One should recall that the theory in the limit $\tilde{Y} \rightarrow 0$ is not bounded energetically from below.

It might happen that the loop corrections would become significant for scales $k < k'$ again. Therefore, we use the values of the couplings $\tilde{v}_n(k')$ ($1 \leq n \leq 10$) obtained by the TLR procedure as initial conditions for solving the WH RG equation for $k < k'$. It has been established that the loop corrections cause less than a 0.1% change in the value of $v_1(k')$ and $\sim 30\%$ change in $v_2(k')$ on any particular RG trajectory. Since we have argued above that the nonvanishing values of $v_n(k')$ for $n \geq 2$ are due to numerical inaccuracies, we have to conclude that our TLR result obtained at the scale k' is stable against further loop corrections in the region $0 \leq k < k'$.

D. Phase III

Phase III occurs for $\tilde{Y} > 1$ and consists of two regions in the parameter plane $(\tilde{v}_1, \tilde{v}_2)$ specified by the singularity scale $k_s = \Lambda$ in the region with $-1 \leq \tilde{v}_1 \leq -1 + \tilde{Y}$ and $k_s = k_c < \Lambda$ for $-1 + \tilde{Y} < \tilde{v}_1 < \tilde{v}_u$, where \tilde{v}_u is the phase boundary III-I. It has been established numerically that phase III is characterized by spontaneous breaking of $O(2)$ symmetry and a quasi-universal *dimensionless* effective potential

$$\tilde{U}_{k \rightarrow 0}(\tilde{\Phi}) = -\frac{1}{2}(-1 + \tilde{Y})\tilde{\Phi}^2, \quad (14)$$

providing the Maxwell cut-like universal dimensionful effective potential [see Fig. 5 and the numerical value of $\tilde{v}_1(0)$ in Table III, which should be compared with its theoretical value $1 - \tilde{Y}$]. The dimensionless effective potential (14) of the form of a down-sided parabola with curvature $1 - \tilde{Y} < 0$ is the generalization of that with curvature -1 obtained in the symmetry-breaking phase of the ordinary $O(2)$ model without higher-derivative terms. The latter case is recovered as a limiting one for $\tilde{Y} = 2$. The presence of the higher-derivative coupling

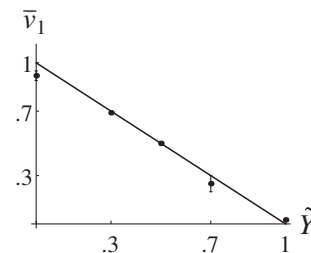


FIG. 4. The dimensionful mass squared $\overline{v_1(0)}$ vs the higher-derivative coupling \tilde{Y} in phase II.

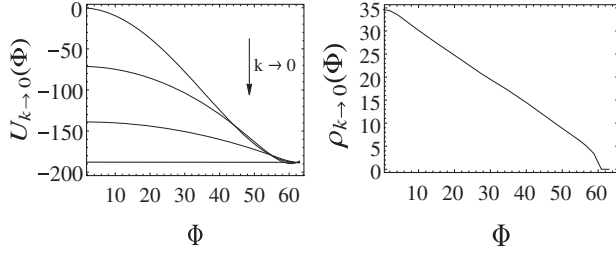


FIG. 5. The dimensionless blocked potential $\tilde{U}_k(\Phi)$ (to the left) and the amplitude $\rho_0(\Phi)$ of the spinodal instability (to the right) vs the homogeneous background field Φ for $\tilde{Y} = 1.5$ in phase III.

$\tilde{Y} > 1$ results in decreasing the magnitude of the curvature. Like in the case of the ordinary $O(2)$ model, it has been found that the amplitude of the spinodal instability survives the IR limit and depends linearly on the homogeneous background field Φ :

$$\rho_{k \rightarrow 0}(\Phi) = \beta(-\Phi + \Phi_c(0)). \quad (15)$$

The values of the coefficient β obtained numerically for various values of \tilde{Y} are compared in Table III. These values do not show any dependence on the higher-derivative coupling \tilde{Y} and yield the mean value $\bar{\beta} = -.53 \pm .01$. Based on this result and the assumption that the limits $\tilde{Y} \rightarrow 2$ were continuous, one is inclined to suggest that the exact value is $\beta = 1/2$, but our TLR procedure has some systematic error.

We also determined numerically the scaling of the dimensionless couplings in the deep IR region (see Fig. 6). There occurs a clear-cut IR scaling region in which the couplings $\tilde{v}_n(k)$ with $n \geq 2$ scale down to zero according to the power law $\tilde{v}_{n \geq 2} \sim k^{\alpha_n}$, while $\tilde{v}_1(k) - \tilde{v}_0(0) \sim k^{\alpha_1}$ remains essentially zero in the same region. The numerical values of the scaling exponents α_n turn out to be universal, as shown in Table III. This means that all the dimensionful couplings reach their constant IR values with the power law $v_n(k) - v_n(0) \sim k^2$.

E. On the phase transitions

In thermodynamics, phase transitions accompanied by a finite jump of the free energy—i.e., the presence of a latent

TABLE III. The IR limiting values of the first two couplings of the dimensionless potential, the coefficient β of the amplitude in Eq. (15), and the first few scaling exponents α_n obtained by TLR for phase III.

\tilde{Y}	$\tilde{v}_1(0)$	$\tilde{v}_2(0)$	α_1	α_2	α_3	α_4	β
1.3	-0.281	$< 10^{-5}$	0	1	2	3	.534
1.5	-0.469	$< 10^{-5}$	0	1	2	3	.531
1.8	-0.75	$< 10^{-5}$	0	1	2	3	.521
2	-0.94	$< 10^{-5}$	0	1	2	3	.531

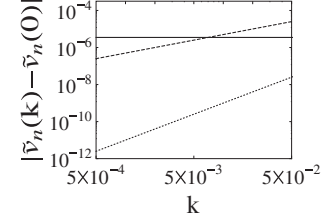


FIG. 6. Scaling of the dimensionless couplings $\tilde{v}_1(k)$ (full line), $\tilde{v}_2(k)$ (dashed line), and $\tilde{v}_3(k)$ (dotted line) for $\tilde{Y} = 1.5$ in the symmetry-broken phase III.

heat—are called transitions of the first order, while those with continuous free energy and singularities in the derivatives of the free energy are called continuous. As for the transition from phase III to phase I in our case, there is a rather straightforward way to decide that the transition III-I is continuous. Namely, one determines the behavior of the correlation length $\xi \sim 1/k_c$ approaching the boundary of phases I and III from the side of phase III. This approach is applicable only at the phase boundary III-I, because the singularity scale k_c can be detected by solving the WH RG equation (4), while this scale lies above the UV cutoff Λ for phase II; therefore we cannot make such calculations at the phase boundaries II-I or II-III. The reduced temperature is identified as $t = \tilde{v}_u - \tilde{v}_1(\Lambda)$ —i.e., the “distance” of the starting point of the RG trajectories from the phase boundary \tilde{v}_u . In order to determine the dependence of the correlation length ξ on the reduced temperature t , we have solved the WH RG equation (4) for various initial conditions $\tilde{v}_{1i}(\Lambda) = \tilde{v}_u - i \times 10^{-4}$ ($i = 1, 2, \dots, 500$) for each value of $\tilde{v}_2(\Lambda) = 0.01, 0.1$ and $\tilde{Y} = 1.2, 1.5, 2.0$. It has been established that the correlation length has a power-law behavior

$$\xi \sim 1/k_c \sim t^{-\nu} \quad (16)$$

near the phase boundary for any fixed values of the coupling \tilde{Y} (see Fig. 7). This signals that the phase transition III \rightarrow I is continuous, just like the transition in the ordinary $O(2)$ model. The critical exponent ν seems to be insensitive to the bare parameters \tilde{Y} and $\tilde{v}_2(\Lambda)$; its mean value is $\bar{\nu} = .46 \pm .03$. The ϕ^4 model can be considered as

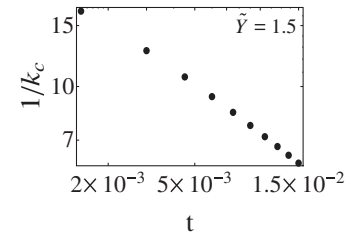


FIG. 7. Scaling of the correlation length $\xi \sim 1/k_c$ with the reduced temperature $t = \tilde{v}_u - \tilde{v}_1(\Lambda)$ (on a log-log plot) at the boundary of phases I and III for $\tilde{Y} = 1.5$ and $\tilde{v}_2(\Lambda) = 0.1$.

the textbook example of the RG technique. Therefore, it is widely investigated in various dimensions and in various levels of truncation [14–22]. Our result is close to the best value $\nu = 0.63$ obtained in the three-dimensional case.

There is, however, another way to study the continuity of the phase transition. Namely, we can determine directly the jump of the free-energy density or latent heat per unit volume—more precisely, the jump of the minimum of the effective potential going from one phase to another across the phase boundary. For that purpose, we have to determine the IR limit of the constant term $v_0^A(0)$ of the effective potential in the various phases $A = \text{I, II, III}$ at both sides of the phase boundary and compare them. For the comparison, we have to consider RG trajectories on which the bare potential has the same minimum value. Otherwise, this can be put as the correction of the IR values $v_0^A(0)$ by the minimum value of the bare potential $(U_\Lambda^A)_{\min}$; i.e., by the replacement $v_0^A(0) \rightarrow (v_0^A)_{\text{corr}} = v_0^A(0) - (U_\Lambda^A)_{\min}$. The transition from phase B to phase A is then accompanied by the jump of the potential (Euclidean action per volume) $\Delta v_0^{A \rightarrow B} = (v_0^B)_{\text{corr}} - (v_0^A)_{\text{corr}}$. The nonvanishing or vanishing value of $\Delta v_0^{A \rightarrow B}$ signals that the transition is of the first order or continuous, respectively. In our settings, $(U_\Lambda^A)_{\min}$ is nonvanishing only for RG trajectories belonging to bare potential of double-well form (those starting close to the phase boundaries III-I and III-II in phase III, and close to the phase boundary II-III in phase II). For the numerical determination of $\Delta v_0^{\text{II} \rightarrow \text{I}}$, we have chosen RG trajectories which start at the “distance” $t = 0.001$ from the phase boundary. In the case of the evaluation of $\Delta v_0^{\text{III} \rightarrow \text{I}}$, we considered RG trajectories with the values of $v_1(\Lambda)$ increased in steps $t = 0.001$ crossing the phase boundary. Finally, $\Delta v_0^{\text{III} \rightarrow \text{II}}$ has been determined from the comparison of RG trajectories for $\tilde{Y} = 1.1$ and 0.9 and various values of $\tilde{v}_1(\Lambda)$. All calculations were made for $\tilde{v}_2(\Lambda) = 0.01$. The results are shown in Fig. 8. In the plot on the left, we see that there is a jump of the free-energy density of 2 orders of magnitude larger for $0 < \tilde{Y} < 1$ than for $1 < \tilde{Y} < 2$. Together with our previous finding on the base of the study of the correlation length, this enables one to conclude

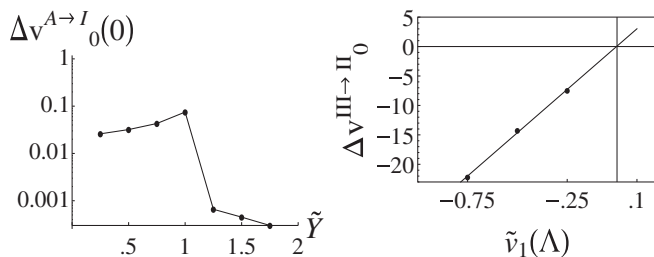


FIG. 8. The jump of the “free-energy density” $\Delta v_0^{A \rightarrow B}$ for $B = \text{I}$, $A = \text{II}$ ($0 < \tilde{Y} < 1$), and $A = \text{III}$ ($1 < \tilde{Y} < 2$) (to the left); and for $A = \text{III}$, $B = \text{II}$ (to the right).

that the phase transitions $\text{III} \rightarrow \text{I}$ and $\text{II} \rightarrow \text{I}$ are continuous and of the first order, respectively. Similarly, the plot to the right in Fig. 8 shows that the phase transition $\text{III} \rightarrow \text{II}$ is of the first order with a latent heat per unit volume decreasing to zero when the triple point is approached.

IV. CONCLUSIONS

The phase structure of the three-dimensional Euclidean $O(2)$ symmetric ghost scalar field model has been investigated in the framework of Wegner and Houghton’s (WH) renormalization group (RG), including the higher-derivative term $-\frac{1}{2} \int_x Y \phi \square \phi$ in the action and keeping the dimensionless coupling \tilde{Y} constant. The RG flow with decreasing gliding cutoff k has been determined numerically by solving the WH RG equation. When the right-hand side of the WH equation develops a singularity at some scale $k_c \neq 0$, the flow has been followed further by means of the tree-level renormalization (TLR) procedure. It has been shown that the model exhibits three phases and a triple line. The symmetric phase (phase I) is present for any values $\tilde{Y} > 0$ and shows similar features to the symmetric phase of the ordinary $O(2)$ model. Phase II is present, when $0 \leq \tilde{Y} \leq 1$. The RG flow of the trajectories belonging to phase II can only be determined by the TLR procedure on all scales below the UV cutoff Λ . The dimensionful effective potential in phase II is quasi-universal; it depends on the value of the higher-derivative coupling \tilde{Y} , but it is independent of the other bare couplings. Just below the scale Λ , there occurs a periodic spinodal instability that breaks $O(2)$ symmetry as well as rotational and translational symmetries in the external space; however, that intermediate symmetry breaking is washed out in the IR limit. Phase II has no analogue in the ordinary $O(2)$ model. It is of the same properties found in Ref. [10], and its existence is based upon the ghost-condensation mechanism available in the model with $Z < 0$ and $\tilde{Y} > 0$. Phase III occurs for $1 < \tilde{Y} \leq 2$. It separates into two regions, IIIA and IIIB, where TLR has to be used below the UV cutoff Λ and the singularity scale k_c , respectively; however, both regions IIIA and IIIB have the same deep IR behavior. In phase III, the dimensionful effective potential is universal; it exhibits the Maxwell cut which is accompanied with the nonvanishing amplitude of the periodic spinodal instability for scales $k \rightarrow 0$. Therefore, phase III is the one in which spontaneous symmetry breaking occurs, just like in the symmetry-breaking phase of the ordinary $O(2)$ model. The phase boundaries III-I and III-II intersect in a triple line.

The continuity of the various phase transitions has been studied by means of the differences of the minimum values of the effective potentials in the various phases, and it has been found that the phase transitions $\text{II} \rightarrow \text{I}$ and $\text{III} \rightarrow \text{II}$ are of the first order accompanied with a nonvanishing latent heat per unit volume, whereas the transition $\text{III} \rightarrow \text{I}$ is

continuous. The latter has been also supported by the scaling behavior of the correlation length in phase III close to the phase boundary III-I.

In the framework of the WH RG restricted to the local potential approximation (LPA), the phase structure of the model turns out to be more rich when the dimensionless higher-derivative coupling \tilde{Y} is kept constant during the RG flow than in the case in which the dimensionful coupling Y is kept constant, as we did in our previous work [10]. Therefore, it remains an open question whether the model exhibits two or three phases. The ambiguity of keeping constant either the dimensionful or the dimensionless higher-derivative coupling is an essential feature of the local potential approximation (LPA) and is unavoidable in

the WH RG approach [12]. Our work demonstrates that such an ambiguity may affect the physical results significantly when higher-derivative terms are included in the model. No similar ambiguity should occur if one goes beyond the LPA in the gradient expansion using any RG framework that is appropriate for it—e.g., the effective average action approach [23].

ACKNOWLEDGMENTS

S. Nagy acknowledges financial support from a János Bolyai Grant of the Hungarian Academy of Sciences, the Hungarian National Research Fund OTKA (Grant No. K112233).

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- [1] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2000).
 - [2] T. Chatterji, *Neutron Scattering from Magnetic Materials* (Elsevier, Amsterdam, 2006).
 - [3] Le Bellac, *Quantum and Statistical Field Theory* (Oxford University, New York, 1992).
 - [4] J. Fingberg and J. Polonyi, *Nucl. Phys.* **B486**, 315 (1997).
 - [5] V. Branchina, H. Mohrbach, and J. Polonyi, *Phys. Rev. D* **60**, 045006 (1999); **60**, 045007 (1999).
 - [6] M. D. Fournier and J. Polonyi, *Phys. Rev. D* **61**, 065008 (2000).
 - [7] G. Başar, G. V. Dunne, and M. Thies, *Phys. Rev. D* **79**, 105012 (2009).
 - [8] J. M. Carmona, S. Jimenez, J. Polonyi, and A. Tarancon, *Phys. Rev. B* **73**, 024501 (2006).
 - [9] F. J. Wegner and A. Houghton, *Phys. Rev. A* **8**, 401 (1973).
 - [10] Z. Péli, S. Nagy, and K. Sailer, *Phys. Rev. D* **94**, 065021 (2016).
 - [11] J. Alexandre, V. Branchina, and J. Polonyi, *Phys. Lett. B* **445**, 351 (1999).
 - [12] J. Polonyi, *Central Eur. J. Phys.* **1**, 1 (2003).
 - [13] S. Nagy, I. Nandori, J. Polonyi, and K. Sailer, *Phys. Rev. D* **77**, 025026 (2008).
 - [14] N. Tetradis and C. Wetterich, *Nucl. Phys.* **B422**, 541 (1994).
 - [15] S.-B. Liao, J. Polonyi, and M. Strickland, *Nucl. Phys.* **B567**, 493 (2000).
 - [16] D. F. Litim, *Phys. Rev. D* **64**, 105007 (2001).
 - [17] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, *Phys. Rev. D* **67**, 065004 (2003).
 - [18] J. Zinn-Justin, *Phys. Rep.* **344**, 159 (2001).
 - [19] T. R. Morris, *Nucl. Phys.* **B495**, 477 (1997).
 - [20] D. F. Litim and D. Zappalá, *Phys. Rev. D* **83**, 085009 (2011).
 - [21] C. Bervillier, *J. Phys. Condens. Matter* **17**, S1929 (2005).
 - [22] P. Mati, *Phys. Rev. D* **91**, 125038 (2015); arXiv:1601.00450.
 - [23] N. Tetradis and C. Wetterich, *Nucl. Phys.* **B383**, 197 (1992).