

Nonrelativistic Chern-Simons theories and three-dimensional Hořava-Lifshitz gravity

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We show that certain three-dimensional Hořava-Lifshitz gravity theories can be written as Chern-Simons gauge theories on various nonrelativistic algebras. The algebras are specific extensions of the Bargmann, Newton-Hooke and Schrödinger algebras each of which has the Galilean algebra as a subalgebra. To show this we employ the fact that Hořava-Lifshitz gravity corresponds to dynamical Newton-Cartan geometry. In particular, the extended Bargmann (Newton-Hooke) Chern-Simons theory corresponds to projectable Hořava-Lifshitz gravity with a local $U(1)$ gauge symmetry without (with) a cosmological constant. Moreover we identify an extended Schrödinger algebra containing three extra generators that are central with respect to the subalgebra of Galilean boosts, momenta and rotations, for which the Chern-Simons theory gives rise to a novel version of nonprojectable conformal Hořava-Lifshitz gravity that we refer to as Chern-Simons Schrödinger gravity. This theory has a $z = 2$ Lifshitz geometry as a vacuum solution and thus provides a new framework to study Lifshitz holography.

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I. INTRODUCTION

The local equivalence of three-dimensional (3D) Einstein gravity (with or without a cosmological constant) in terms of a Chern-Simons gauge theory [1,2] has been of crucial importance in order to gain insights into the classical and quantum properties of the theory, along with holographic dualities to two-dimensional conformal field theories (CFTs). Three-dimensional (relativistic) gravity thus plays a special role due to its simplicity, having no propagating degrees of freedom, yet being nontrivial enough to allow for black holes and numerous other interesting features.

Recently nonrelativistic geometries have gained considerable interest, in part due to their appearance in non-anti-de Sitter (AdS) holography [3–6], their relevance in condensed matter setups such as the fractional quantum Hall effect [7,8] and other fluid/field-theoretic applications [6,9–12]. Moreover these geometries lead to interesting theories of nonrelativistic gravity, beyond Newtonian gravity as embodied in the original formulation of Cartan. In particular, a novel generalization of Newton-Cartan geometry with torsion was first observed in Ref. [3] and it was subsequently shown in Ref. [13] that making this geometry dynamical leads to the known versions of Hořava-Lifshitz gravity constructed in Refs. [14–16].

Interesting supersymmetric extensions of Newton-Cartan gravity have been considered as well [17–19]. All this begs the question whether in three dimensions such nonrelativistic gravity theories are related to Chern-Simons (CS) theories, in parallel to the relativistic case.

The generalization of the CS formulation to nonrelativistic Galilean gravity was initiated in the pioneering work [20], in which the CS gauge field takes values in a Galilean algebra with two central extensions (the extended Bargmann algebra), replacing the Poincaré algebra of the relativistic setting. We will show in this paper that this vielbein formulation is equivalent to three-dimensional torsionless Newton-Cartan (NC) gravity [13], which in turn is the three-dimensional $U(1)$ -invariant projectable Hořava-Lifshitz gravity of Ref. [16].¹ By going to an extended Newton-Hooke algebra, we furthermore show that a cosmological constant can be added to the theory. Moreover, by constructing a $z = 2$ Schrödinger algebra with three extra generators, that are central with respect to the subalgebra of Galilean boosts, momenta and rotations, we obtain a novel action for conformal nonprojectable Hořava-Lifshitz gravity. The latter theory corresponds to a new version of dynamical twistless torsional Newton-Cartan geometry which we call Chern-Simons Schrödinger gravity.

The CS formulation based on the extended Bargmann algebra can be viewed as the nonrelativistic counterpart of three-dimensional Einstein gravity without a cosmological

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¹The topological nature of this theory was also discussed in Ref. [16].

constant. Adding a cosmological constant via the Newton-Hooke algebra does not have the same effect as in the relativistic case. In particular the theory is still described by projectable Hořava-Lifshitz (HL) gravity. It will be shown that the cosmological constant leads to time-dependent geometries.

In order to find the counterpart of AdS₃ gravity we need to find a CS theory that is equivalent to nonprojectable HL gravity. This is provided by considering the extended Schrödinger algebra in 2 + 1 dimensions that allows for a CS theory corresponding to twistless torsional Newton-Cartan (TTNC) gravity, or what is the same nonprojectable HL gravity, with $z = 2$ scaling symmetry. We show that this theory of Chern-Simons Schrödinger gravity admits $z = 2$ Lifshitz geometries and thus provides a new framework to study Lifshitz holography.

This paper is organized as follows. In Sec. II we discuss the basic properties of the three Lie algebras on which the CS actions are based, namely the extensions of the Bargmann, Newton-Hooke and Schrödinger algebras that admit a nondegenerate metric. In Sec. III we construct the most general CS actions compatible with these symmetries. This includes terms that are the nonrelativistic counterpart of the Lorentz CS term that can be added to the Einstein-Hilbert action in three dimensions. We continue in Sec. IV to rewrite the CS actions based on the Bargmann and Newton-Hooke algebras in the metric formulation of dynamical Newton-Cartan geometry showing that the resulting theory is a known version of projectable HL gravity. In this section we also discuss the local properties of the solutions to the flatness conditions. Finally in Sec. V we show that the CS theory based on the extended Schrödinger algebra is equivalent to a novel version of TTNC/nonprojectable HL gravity. In that section we also show that the theory admits $z = 2$ Lifshitz solutions. We conclude with a discussion and outlook in Sec. VI.

II. NONRELATIVISTIC LIE ALGEBRAS WITH NONDEGENERATE METRICS

Nonrelativistic symmetry algebras are typically non-semisimple Lie algebras, containing the Galilean algebra as a subalgebra, which consists (in 2 + 1 dimensions) of the generators J (rotation), P_a (translations, $a = 1, 2$), G_a (Galilean boosts) and H (Hamiltonian). In order to write down a Chern-Simons theory one needs a nondegenerate symmetric bilinear form (metric) on the Lie algebra that serves to define the trace in the Chern-Simons action²

$$\mathcal{L}_{\text{CS}} = \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (1)$$

²For brevity, the overall multiplicative constant $k/(4\pi)$ involving the Chern-Simons level k , appearing in this action will be omitted, as it plays no role in our discussions below.

For a nonsemisimple Lie algebra the existence of such a bilinear form is a nontrivial requirement, and in the case of the Galilean algebra with nonzero commutators

$$[J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b, \quad [H, G_a] = P_a, \quad (2)$$

it necessitates the addition of central elements. It is well known that the Galilean algebra can be centrally extended to the Bargmann algebra

$$[P_a, G_b] = N \delta_{ab}. \quad (3)$$

The mass generator N remains central even in larger algebras such as the Schrödinger and Newton-Hooke algebras.

In 2 + 1 space-time dimensions these algebras can be further extended as follows³:

$$\begin{aligned} [G_a, G_b] &= S \epsilon_{ab}, & [P_a, P_b] &= Z \epsilon_{ab}, \\ [P_a, G_b] &= N \delta_{ab} - Y \epsilon_{ab} \end{aligned} \quad (4)$$

where the generators S , Y and Z may have nontrivial commutators with the rest of the algebra, determined by Jacobi identities. In the case of the Schrödinger algebra these three extra generators are linearly independent, while for the Newton-Hooke algebra $Y = 0$ and Z is proportional to S . These extensions play an important role in obtaining nondegenerate metrics on various nonrelativistic symmetry algebras such as the Bargmann, Newton-Hooke and Schrödinger algebras. In the following, we denote by $B(x, y)$ the bilinear form where x and y are elements of the Lie algebra. Symmetry requires that $B(x, y) = B(y, x)$ and invariance under the action of the algebra corresponds to $B([z, x], y) + B(x, [z, y]) = 0$ for all z, x, y .

A. Extended Bargmann algebra

If we add the central element S in Eq. (4) (but not Y and Z) to the Bargmann algebra (2)–(3) the resulting non-semisimple Lie algebra is a semidirect sum of the normal subalgebra H, P_a, N with the Nappi-Witten algebra [23] consisting of J, G_a, S (which is a central extension of the two-dimensional Euclidean algebra). This algebra was used in the CS theory of Ref. [20] and corresponds, as shown below, to a 3D projectable Hořava-Lifshitz gravity theory. The possible nontrivial values of $B(x, y)$ for the centrally extended Bargmann algebra are given by

³Note that this is an extension of the Galilean algebra by N, S, Y, Z where in general only N and Z are central. For S to also be central one needs to remove Y from the algebra. Indeed, in the case of the Galilean algebra one cannot add Y as a central extension because there is no nontrivial cohomology associated with it [21,22] (we thank Joaquim Gomis for pointing this out to us). In the present paper we will never use Y in the context of the Galilei algebra but only in the larger Schrödinger algebra. If one wants to have Y central, one needs to remove Z from the algebra, in which case S is not central.

$$\begin{aligned}
B(H, S) &= -B(J, N) = c_1, & B(P_a, G_b) &= c_1 \epsilon_{ab}, \\
B(G_a, G_b) &= c_2 \delta_{ab}, & B(J, S) &= c_2, \\
B(J, J) &= c_3, & B(H, J) &= c_4, & B(H, H) &= c_5, \quad (5)
\end{aligned}$$

where c_i are arbitrary constants and with $c_1 \neq 0$ for the matrix to be nondegenerate. If we remove the central element S from the algebra the bilinear form becomes degenerate.

B. Extended Newton-Hooke algebra

There exists a deformation of the Bargmann algebra called the Newton-Hooke algebra. Its nonzero commutators are those of Eqs. (2)–(3) plus $[H, P_a] = -\Lambda_c G_a$. There exists an extension of this algebra involving the S generator where the central element appears in

$$\begin{aligned}
[G_a, G_b] &= S \epsilon_{ab}, & [H, P_a] &= -\Lambda_c G_a, & [P_a, P_b] &= \Lambda_c S \epsilon_{ab}. \quad (6)
\end{aligned}$$

This extended Newton-Hooke algebra, which reduces to the extended Bargmann algebra for $\Lambda_c = 0$, was studied in the context of CS theories in Ref. [24]. For $\Lambda_c \neq 0$, the parameter Λ_c can be set to one by rescaling $(H, P_a, N) \rightarrow \Lambda_c^{1/2}(H, P_a, N)$. The most general symmetric bilinear form that one can define on the algebra is given by Eq. (5) together with

$$B(H, N) = -\Lambda_c c_2, \quad B(P_a, P_b) = \Lambda_c c_2 \delta_{ab}, \quad (7)$$

and requiring $\Lambda_c \neq c_1^2/c_2^2$ ensures that the matrix is nondegenerate.

C. Extended Schrödinger algebra

The conformal extension of the Bargmann algebra is the Schrödinger algebra (with dynamical exponent $z = 2$). The Hamiltonian is extended to an $SL(2, \mathbb{R})$ algebra consisting of dilatations D with $z = 2$ and a special conformal generator K that form the subalgebra

$$[D, H] = -2H, \quad [H, K] = D, \quad [D, K] = 2K. \quad (8)$$

The Schrödinger algebra is obtained by taking this $SL(2, \mathbb{R})$ algebra and specifying how it acts on the Bargmann subalgebra (2)–(3). This action is given by

$$\begin{aligned}
[H, G_a] &= P_a, & [D, P_a] &= -P_a, \\
[D, G_a] &= G_a, & [K, P_a] &= -G_a. \quad (9)
\end{aligned}$$

The mass generator N remains central with respect to the full Schrödinger algebra.

It is possible to add dilatations to the extended Bargmann algebra of Sec. II A by taking $[D, S] = 2S$. However this

algebra has no nondegenerate metric. If we consider the full central extension (4), i.e. we add S , Y and Z to the Bargmann algebra we can add the full $SL(2, \mathbb{R})$ algebra (8) such that Eq. (9) continues to hold. The action of the $SL(2, \mathbb{R})$ subalgebra on S , Y and Z is nontrivial and fully determined by the Jacobi identities given all the other commutators.⁴ The result is that the nonzero commutators are

$$\begin{aligned}
[H, Y] &= -Z, & [H, S] &= -2Y, & [K, Y] &= S, \\
[K, Z] &= 2Y, & [D, S] &= 2S, & [D, Z] &= -2Z. \quad (10)
\end{aligned}$$

The extended Schrödinger algebra is thus given by Eqs. (2)–(4), (8), (9) and (10). The corresponding symmetric bilinear form invariant under the extended Schrödinger algebra is

$$\begin{aligned}
B(H, S) &= B(D, Y) = B(K, Z) = -B(J, N) = c_1, \\
B(P_a, G_b) &= c_1 \quad \epsilon_{ab} B(H, K) = -c_2, \\
B(D, D) &= 2c_2, & B(J, J) &= c_3, \quad (11)
\end{aligned}$$

which is nondegenerate if $c_1 \neq 0$.

III. NONRELATIVISTIC CHERN-SIMONS ACTIONS

We now turn to study the form of the CS action (1) for each of these three algebras which have the Bargmann algebra as a subalgebra and allow for a nondegenerate metric.

A. Bargmann and Newton-Hooke invariant Chern-Simons actions

The extended Bargmann algebra can be obtained by setting $\Lambda_c = 0$ in the extended Newton-Hooke algebra so we will construct the CS action using the metric (5) and (7). Expanding the gauge connection as $A = H\tau + P_a e^a + G_a \Omega^a + J\Omega + Nm + S\zeta$, the CS action becomes

$$\begin{aligned}
&\text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \\
&= 2c_1 \left[-\epsilon_{ab} R^a(G) \wedge e^b + \frac{1}{2} \epsilon_{ab} \tau \wedge \Omega^a \wedge \Omega^b - \Omega \wedge dm \right. \\
&\quad \left. + \zeta \wedge d\tau + \Lambda_c \tau \wedge e^1 \wedge e^2 \right] \\
&\quad + c_2 [\Omega^a \wedge R^a(G) + 2\zeta \wedge d\Omega + \Lambda_c e^a \wedge R^a(P) \\
&\quad - 2\Lambda_c \tau \wedge R(N) + \Lambda_c e^a \wedge \Omega^a \wedge \tau] \\
&\quad + c_3 \Omega \wedge d\Omega + 2c_4 \tau \wedge d\Omega + c_5 \tau \wedge d\tau, \quad (12)
\end{aligned}$$

⁴This is an explicit example of a more general theorem on double extensions of Lie algebras, elaborated on in Ref. [25]. We thank Jan Rosseel for useful discussions on this point.

(see also Refs. [20,24]) where the curvatures $R^a(P)$, $R^a(G)$ and $R(N)$ are given by

$$\begin{aligned} R^a(P) &= de^a - \Omega^a \wedge \tau - \epsilon^{ab}\Omega \wedge e^b, \\ R^a(G) &= d\Omega^a - \epsilon^{ab}\Omega \wedge \Omega^b, \quad R(N) = dm - \Omega^a \wedge e^a. \end{aligned} \quad (13)$$

These curvatures are defined by the expansion of the field strength

$$\begin{aligned} F &= dA + A \wedge A \\ &= HR(H) + P_a R^a(P) + G_a R^a(G) + JR(J) \\ &\quad + NR(N) + SR(S). \end{aligned} \quad (14)$$

We see that Λ_c plays the role of a cosmological constant term (in the c_1 term). The terms proportional to $c_2\Lambda_c$ are by themselves invariant under the gauge transformations $\delta A = d\Lambda + [A, \Lambda]$.

The terms with coefficients c_4 and c_5 in Eq. (12) are not interesting as they can be removed by a field redefinition of ζ . This leads to a new value for the parameter in front of the $\Omega \wedge d\Omega$ term. Hence we can always restrict ourselves to c_1 , c_2 and c_3 and set to zero $c_4 = c_5 = 0$. When $\Lambda_c = 0$ the terms proportional to c_2 and c_3 are

$$c_2(\Omega^a \wedge R^a(G) + 2\zeta \wedge d\Omega) + c_3\Omega \wedge d\Omega. \quad (15)$$

These can be thought of as the analogue of the Lorentz CS term. The term with coefficient c_2 is a novel Galilean boost invariant combination that starts as $\Omega^a \wedge d\Omega^a$ plus extra terms to make it invariant. To see the invariance explicitly we give the transformations of the connections for $\Lambda_c = 0$ appearing in Eq. (15) that read

$$\begin{aligned} \delta\Omega^a &= d\lambda^a + \epsilon^{ab}(\lambda\Omega^b - \lambda^b\Omega), \quad \delta\Omega = d\lambda, \\ \delta\zeta &= -\epsilon^{ab}\lambda^a\Omega^b. \end{aligned} \quad (16)$$

If we consider the CS theory on a manifold with a boundary they are expected to lead to Galilean boost and rotation anomalies on the boundary theory. In the simplest setting with $c_2 = c_3 = 0$ the ζ equation of motion is $d\tau = 0$. In Sec. IV A we will see that this corresponds to having no torsion in the Newton-Cartan description, or what is the same, projectable HL gravity [13].

B. Schrödinger invariant Chern-Simons action

The extended Schrödinger algebra is Eqs. (2)–(4), (8), (9) and (10). We expand the gauge field as

$$\begin{aligned} A &= H\tau + P_a e^a + G_a \omega^a + J\omega + Nm + Db + Kf \\ &\quad + S\zeta + Y\alpha + Z\beta. \end{aligned} \quad (17)$$

Using the metric on the Lie algebra (11) the Chern-Simons action can be written as

$$\begin{aligned} \mathcal{L} &= 2c_1[\tilde{R}^2(G) \wedge e^1 - \tilde{R}^1(G) \wedge e^2 + \tau \wedge \omega^1 \wedge \omega^2 \\ &\quad - m \wedge d\omega - f \wedge e^1 \wedge e^2 + \zeta \wedge (d\tau - 2b \wedge \tau) \\ &\quad + \alpha \wedge (db - f \wedge \tau) + \beta \wedge (df + 2b \wedge f)] \\ &\quad + 2c_2[b \wedge db - \tau \wedge df + 2b \wedge \tau \wedge f] + c_3\omega \wedge d\omega, \end{aligned} \quad (18)$$

where the curvature $\tilde{R}^a(G)$ is given by

$$\tilde{R}^a(G) = d\omega^a + \epsilon^{ab}\omega^b \wedge \omega - \omega^a \wedge b - f \wedge e^a. \quad (19)$$

There is no redefinition of the connections ζ , α and β that allows one to remove the term with the coefficient c_2 entirely. It transforms under the $SL(2, \mathbb{R})$ transformations inside the extended Schrödinger algebra. It would be interesting to see if it corresponds to some anomaly for a boundary theory like a Weyl-type anomaly.

The equation of motion of ζ now imposes the on-shell condition that $d\tau = 2b \wedge \tau$ which is equivalent to $\tau \wedge d\tau = 0$. In the language of Newton-Cartan geometry this corresponds to TTNC geometry [3,26] or what is the same nonprojectable HL gravity [13]. The details will be given in Sec. V A.

IV. CHERN-SIMONS ACTION FOR 3D PROJECTABLE HOŘAVA-LIFSHITZ GRAVITY

We know from Ref. [27] that gauging the Bargmann algebra leads to NC geometry. In Ref. [13] it was shown that dynamical Newton-Cartan geometry is field-redefinition equivalent to projectable Hořava-Lifshitz gravity as presented in Ref. [16]. Hence we should be able to show that the CS action given in Sec. III A is equivalent to a 3D projectable HL gravity theory.

A. Bargmann invariant projectable Hořava-Lifshitz gravity

We will now rewrite Eq. (12) with only the c_1 coefficient nonzero in a metric form using the language of NC geometry. The connections τ_μ and e_μ^a are the vielbeins of NC geometry. We define inverse vielbeins v^μ and e_a^μ via $\delta_\nu^\mu = -v^\mu\tau_\nu + e_a^\mu e_\nu^a$ so $v^\mu\tau_\mu = -1$, $e_a^\mu\tau_\mu = 0$, $v^\mu e_\mu^a = 0$ and $e_a^\mu e_\mu^b = \delta_a^b$. It can be shown that the first term in the CS action (12) can be written as

$$R^2(G) \wedge e^1 - R^1(G) \wedge e^2 = v^\mu e_a^\nu R_{\mu\nu}^a(G) \tau \wedge e^1 \wedge e^2. \quad (20)$$

With $m = -v^\mu m_\mu \tau + e_a^\mu m_\mu e^a$ it follows that the third term in Eq. (12) becomes

$$\begin{aligned} m \wedge R(J) &= \left(-\frac{1}{2} v^\mu m_\mu \mathcal{R} - e_2^\rho m_\rho v^\mu e_1^\nu R_{\mu\nu}(J) \right. \\ &\quad \left. + e_1^\rho m_\rho v^\mu e_2^\nu R_{\mu\nu}(J) \right) \tau \wedge e^1 \wedge e^2, \end{aligned} \quad (21)$$

where we used that

$$R_{ab}(J) = e_a^\mu e_b^\nu R_{\mu\nu}(J) \equiv \frac{1}{2} \epsilon_{ab} \mathcal{R}. \quad (22)$$

The action (12) is written in a first-order formalism where all the connections in A_μ are treated as independent variables. The form we are looking for treats the NC variables τ_μ , e_μ^a and m_μ as the independent variables. Hence we will integrate out the variables Ω^a , Ω and ζ . Their equations of motion are the NC curvature constraints [27] $R^a(P) = 0$, $R(N) = 0$ and $R(H) = d\tau = 0$ where the curvatures are given in Eq. (13). These are solved by expressing Ω_μ^a and Ω_μ in terms of τ_μ , e_μ^a (their inverse) and m_μ where $d\tau = 0$. The off-shell implementation of the curvature constraints makes the theory diffeomorphism invariant because the NC curvature constraints imply that the transformations of τ_μ , e_μ^a and m_μ constitute diffeomorphisms and local G^a , J , N transformations [27].

In order to rewrite the CS action it will be useful to employ the following Bianchi identity:

$$\begin{aligned} dR^a(P) - \epsilon^{ab} \Omega \wedge R^b(P) - \Omega^a \wedge d\tau \\ = -R^a(G) \wedge \tau - \epsilon^{ab} R(J) \wedge e^b. \end{aligned} \quad (23)$$

Using the curvature constraints $R^a(P) = 0$ and $d\tau = 0$ which will be implemented off shell we find $R^a(G) \wedge \tau + \epsilon^{ab} R(J) \wedge e^b = 0$. From this we conclude that

$$\begin{aligned} v^\mu e_1^\nu R_{\mu\nu}(J) &= -e_2^\mu e_a^\nu R_{\mu\nu}{}^a(G), \\ v^\mu e_2^\nu R_{\mu\nu}(J) &= e_1^\mu e_a^\nu R_{\mu\nu}{}^a(G). \end{aligned} \quad (24)$$

Using that $\Omega^1 = -v^\mu \Omega_\mu^1 \tau + e_a^\mu \Omega_\mu^1 e^a$ we conclude that Eq. (12), with $c_1 = 1$ and all other constants zero, can be written as

$$\mathcal{L} = e(2\hat{v}^\mu e_a^\nu R_{\mu\nu}{}^a(G) + (e_a^\mu e_b^\nu - e_\nu^\mu e_b^\mu) \Omega_\mu^a \Omega_\nu^b + v^\mu m_\mu \mathcal{R}), \quad (25)$$

where $e = \tau \wedge e^1 \wedge e^2$.

To massage this expression further we need a notion of a covariant derivative. This can be introduced via the vielbein postulates

$$\begin{aligned} \mathcal{D}_\mu \tau_\nu &= \partial_\mu \tau_\nu - \Gamma_{\mu\nu}^\rho \tau_\rho = 0, \\ \mathcal{D}_\mu e_\nu^a &= \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a - \Omega_\mu^a \tau_\nu - \epsilon^{ab} \Omega_\mu e_\nu^b = 0, \end{aligned} \quad (26)$$

where we take for $\Gamma_{\mu\nu}^\rho$

$$\Gamma_{\mu\nu}^\rho = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}), \quad (27)$$

in which

$$\begin{aligned} \hat{v}^\mu &= v^\mu - h^{\mu\nu} m_\nu, & \bar{h}_{\mu\nu} &= h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu, \\ h_{\mu\nu} &= \delta_{ab} e_\mu^a e_\nu^b, & h^{\mu\nu} &= \delta^{ab} e_a^\mu e_b^\nu. \end{aligned} \quad (28)$$

The connection (27) is a symmetric connection for $d\tau = 0$ that is invariant under G^a , J , and N transformations. The vielbein postulates relate $\Gamma_{\mu\nu}^\rho$ to Ω_μ^a and Ω_μ . These relations are the same as the expressions obtained by solving the curvature constraints $R_{\mu\nu}^a(P) = 0$, $R_{\mu\nu}(N) = 0$ for Ω_μ^a and Ω_μ . We denote by ∇_μ the covariant derivative containing the connection $\Gamma_{\mu\nu}^\rho$. For $d\tau = 0$ we have [13,27]

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] X_\sigma &= R_{\mu\nu\sigma}{}^\rho X_\rho, \\ R_{\mu\nu\sigma}{}^\rho &= e_a^\rho \tau_\sigma R_{\mu\nu}{}^a(G) - e_{\sigma a} e_b^\rho R_{\mu\nu}(J) \epsilon^{ab}. \end{aligned} \quad (29)$$

We now switch to employing a Lagrangian density rather than a 3-form. Using Eq. (29) and the fact that from the vielbein postulates it follows that $\Omega_\mu^a e_\nu^a = \nabla_\mu v^\nu$ we find after performing a few partial integrations and writing $v^\mu = \hat{v}^\mu + h^{\mu\nu} m_\nu$ that

$$\begin{aligned} \mathcal{L} &= e(-\nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu + \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu + v^\mu m_\mu \mathcal{R} \\ &\quad + (h^{\mu\rho} h^{\nu\sigma} - h^{\mu\sigma} h^{\nu\rho}) \nabla_\mu m_\rho \nabla_\nu m_\sigma). \end{aligned} \quad (30)$$

Finally, using partial integrations which give rise to a commutator on one of the m_μ vectors as well as properties of the Riemann tensor, it can be shown that

$$\mathcal{L} = e(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - (h^{\mu\nu} K_{\mu\nu})^2 - \tilde{\Phi} \mathcal{R}), \quad (31)$$

where

$$\tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu. \quad (32)$$

This is the same action as the action⁵ (10.10) given in Ref. [13] which in turn is based on the NC version of the results of Ref. [16]. Note that the extrinsic curvature is given by $h^{\nu\rho} K_{\mu\rho} = -\nabla_\mu \hat{v}^\nu$. One observes that the HL λ parameter which can appear between the two extrinsic curvature terms is equal to unity in Eq. (31).

If we include Λ_c appearing in the extended Newton-Hooke algebra we simply end up with the same Lagrangian to which we add $e\Lambda_c$. We note that the sign of the cosmological constant term is not fixed.

The Lagrangian (31) should be thought of as depending on the variables $\tau_\mu = \partial_\mu \tau$, $\tilde{\Phi}$ and $\bar{h}_{\mu\nu}$ and their derivatives. In projectable HL gravity τ is identified with the

⁵In the analysis of Ref. [13] a different choice was made for the connection $\Gamma_{\mu\nu}^\rho$ that was denoted by $\hat{\Gamma}_{\mu\nu}^\rho$. This other choice is related to Eq. (39) via equations (5.7) and (5.3) of Ref. [13]. It can be shown that the form of the Lagrangian is not affected by these choices.

Arnowitt-Deser-Misner (ADM) time coordinate leading to foliation preserving diffeomorphism invariance.

B. Solutions

We will solve the equations of motion of Eq. (12), $F = 0$ with F expanded as in Eq. (14), locally for the case with $c_2 = c_3 = c_4 = c_5 = 0$ but with Λ_c arbitrary. Under a gauge transformation the connection transforms as $\delta A = d\Lambda + [A, \Lambda]$. We will write Λ as $\Lambda = \xi^\mu A_\mu + \Sigma$, where $\Sigma = G_a \lambda^a + J\lambda + N\sigma + S\kappa$. In components these are the following transformations:

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu, & \delta e_\mu^a &= \mathcal{L}_\xi e_\mu^a + \lambda^a \tau_\mu + \epsilon^{ab} \lambda e_\mu^b, \\ \delta\Omega_\mu^a &= \mathcal{L}_\xi \Omega_\mu^a + \partial_\mu \lambda^a + \epsilon^{ab} (\lambda \Omega_\mu^b - \lambda^b \Omega_\mu^a), \\ \delta m_\mu &= \mathcal{L}_\xi m_\mu + \lambda^a e_\mu^a + \partial_\mu \sigma, & \delta\Omega_\mu &= \mathcal{L}_\xi \Omega_\mu + \partial_\mu \lambda, \\ \delta\zeta_\mu &= \mathcal{L}_\xi \zeta_\mu - \epsilon^{ab} \lambda^a \Omega_\mu^b + \partial_\mu \kappa. \end{aligned} \quad (33)$$

Without loss of generality we can fix the gauge redundancy by setting $\tau_\mu = \delta_\mu^i$, $e_\mu^a = \delta_\mu^i \delta_i^a$, $\Omega_\mu = 0$, $\Omega^a = -\Lambda_c \delta_i^a x^i$ and $m = \frac{1}{2} \Lambda_c x^i x^i dt + d\sigma$. The relation between NC geometry and the ADM form of the HL metric

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \quad (34)$$

uses the following identifications (see Sec. VIII of Ref. [13]):

$$\begin{aligned} \tau_t &= N, & \tau_i &= 0, & h_{ij} &= \gamma_{ij}, & h_{it} &= h_{ti} = 0, \\ m_t &= 0, & m_i &= -N^{-1} \gamma_{ij} N^j. \end{aligned} \quad (35)$$

This identification only works in special gauges of the CS theory. When written in the form (31) the HL theory is not a Lorentzian metric theory. In order to make contact with the ADM parametrization we take $\sigma = -\frac{1}{2} \Lambda_c t x^i x^i$, so that $m_t = 0$ and $m_i = \partial_i \sigma = -\Lambda_c t x^i$.

Hence the full solution for τ , e^a and m is given by $\tau = dt$, $e^a = \delta_i^a dx^i$, $m = -\Lambda_c t x^i dx^i$. This corresponds to the ADM variables $N = 1$, $N^i = \Lambda_c t x^i$, $h_{ij} = \delta_{ij}$. By making the coordinate transformation $x^i = e^{-\Lambda_c t^2/2} X^i$ this becomes

$$ds^2 = -dt^2 + e^{-\Lambda_c t^2} dX^i dX^i. \quad (36)$$

We thus find cosmological solutions for $\Lambda_c \neq 0$. Of course this is only true sufficiently locally, as there can be nontrivial identifications on a global level.

V. CHERN-SIMONS ACTIONS FOR 3D NONPROJECTABLE HOŘAVA-LIFSHITZ GRAVITY

In Ref. [26] it was shown that gauging the Schrödinger algebra leads to torsional Newton-Cartan geometry with twistless torsion $\tau \wedge d\tau = 0$. In Ref. [13] it has been shown that TTNC geometry corresponds to nonprojectable HL

gravity. We refer to Ref. [28] for an alternative derivation of the same connection between dynamical TTNC geometry and HL gravity. We now show that the CS action given in Sec. III B is equivalent to a 3D nonprojectable HL gravity theory.

A. Chern-Simons Schrödinger gravity

Our goal will be to rewrite the CS Lagrangian (18) with $c_2 = c_3 = 0$ into the metric formulation of TTNC geometry. As in the case discussed in Sec. IV A we will go from a first-order formalism to a second-order one by integrating out the connections ω^a , ω , ζ and α . The equations of motion corresponding to varying these connections are the curvature constraints $\tilde{R}^a(P) = 0$, $\tilde{R}(N) = 0$, $\tilde{R}(H) = 0$, and $\tilde{R}(D) = 0$. These curvatures can be computed by expanding the curvature of Eq. (17) as

$$\begin{aligned} F &= H\tilde{R}(H) + P_a \tilde{R}^a(P) + G_a \tilde{R}^a(G) + J\tilde{R}(J) \\ &\quad + N\tilde{R}(N) + D\tilde{R}(D) + K\tilde{R}(K) + S\tilde{R}(S) \\ &\quad + Y\tilde{R}(Y) + Z\tilde{R}(Z). \end{aligned} \quad (37)$$

Solving the constraints $\tilde{R}^a(P) = 0$, $\tilde{R}(N) = 0$, $\tilde{R}(H) = 0$ and $\tilde{R}(D) = 0$ was done in Ref. [26] and the solution can be expressed as giving ω^a , ω , b and f in terms of the vielbeins τ (obeying $\tau \wedge d\tau = 0$), e^a , m and the components $\hat{v}^\mu b_\mu$ and $\hat{v}^\mu f_\mu$. The curvature constraints also allow us to rewrite the algebra of gauge transformations acting on these fields as the algebra of diffeomorphisms and internal transformations consisting of local G^a , J , N , D and K transformations.

The expressions for ω^a and ω can also be obtained from a vielbein postulate for a specific realization of an affine connection $\tilde{\Gamma}_{\mu\nu}^\rho$ that is invariant under all the transformations except those that are diffeomorphisms. These vielbein postulates are

$$\begin{aligned} \mathcal{D}_\mu \tau_\nu &= \partial_\mu \tau_\nu - \tilde{\Gamma}_{\mu\nu}^\rho \tau_\rho - 2b_\mu \tau_\nu = 0, \\ \mathcal{D}_\mu e_\nu^a &= \partial_\mu e_\nu^a - \tilde{\Gamma}_{\mu\nu}^\rho e_\rho^a - \omega_\mu^a \tau_\nu - \epsilon^{ab} \omega_\mu e_\nu^b - b_\mu e_\nu^a = 0, \end{aligned} \quad (38)$$

where we take for $\tilde{\Gamma}_{\mu\nu}^\rho$

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}^\rho &= -\hat{v}^\rho (\partial_\mu - 2b_\mu) \tau_\nu + \frac{1}{2} h^{\rho\sigma} ((\partial_\mu - 2b_\mu) \bar{h}_{\nu\sigma} \\ &\quad + (\partial_\nu - 2b_\nu) \bar{h}_{\mu\sigma} - (\partial_\sigma - 2b_\sigma) \bar{h}_{\mu\nu}). \end{aligned} \quad (39)$$

The connection $\tilde{\Gamma}_{\mu\nu}^\rho$ is symmetric. The associated curvature is $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] X_\sigma = \tilde{R}_{\mu\nu\sigma}^\rho X_\rho$ for any vector X_ρ where [13]

$$\begin{aligned} \tilde{R}_{\mu\nu\sigma}^\rho &= -e^{\rho d} e_\sigma^c \epsilon_{cd} \tilde{R}_{\mu\nu}(J) + e_c^\rho \tau_\sigma \tilde{R}_{\mu\nu}^c(G) \\ &\quad - \delta_\mu^\rho \tau_\sigma f_\nu + \delta_\nu^\rho \tau_\sigma f_\mu + \delta_\sigma^\rho (f_\mu \tau_\nu - f_\nu \tau_\mu). \end{aligned} \quad (40)$$

The equations of motion for ζ and α are solved by

$$\begin{aligned} b_\nu &= \frac{1}{2} \hat{v}^\mu (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) - \hat{v}^\mu b_\mu \tau_\nu, \\ f_\nu &= \hat{v}^\mu (\partial_\mu b_\nu - \partial_\nu b_\mu) - \hat{v}^\mu f_\mu \tau_\nu, \end{aligned} \quad (41)$$

which is why we are left with $\hat{v}^\mu b_\mu$ and $\hat{v}^\mu f_\mu$ as independent variables on top of the usual TTNC variables τ , e^a and m . These expressions satisfy $e_a^\mu e_b^\nu R_{\mu\nu}(K) = 0$.

Using the curvature constraints the Lagrangian (18) for $c_2 = c_3 = 0$ and $c_1 = 1$ can be written as

$$\begin{aligned} \mathcal{L} &= 2(e^a \wedge \omega^a \wedge \omega - \tau \wedge \omega^1 \wedge \omega^2 + f \wedge e^1 \wedge e^2 \\ &\quad + \beta \wedge (df + 2b \wedge f)). \end{aligned} \quad (42)$$

With the help of the vielbein postulates this can be further rewritten as

$$\begin{aligned} \mathcal{L} &= -(2\epsilon^{\mu\nu\rho} m_\rho \partial_\mu \omega_\nu + \epsilon^{\mu\nu\rho} \epsilon_{\sigma\lambda\kappa} \tau_\rho v^\kappa \tilde{\nabla}_\mu v^\sigma \tilde{\nabla}_\nu v^\lambda \\ &\quad + 2\hat{v}^\mu f_\mu) \tau \wedge e^1 \wedge e^2 + 2\beta \wedge (df + 2b \wedge f). \end{aligned} \quad (43)$$

Using the above-mentioned results multiple times as well as Eq. (28) and after performing various partial integrations a lengthy calculation gives

$$\begin{aligned} \mathcal{L} &= e[(h^{\alpha\nu} h^{\beta\mu} - h^{\alpha\mu} h^{\beta\nu}) \bar{h}_{\alpha\sigma} \tilde{\nabla}_\mu \hat{v}^\sigma \bar{h}_{\beta\lambda} \tilde{\nabla}_\nu \hat{v}^\lambda - \tilde{\Phi} \tilde{\mathcal{R}} \\ &\quad - 2\hat{v}^\mu f_\mu + 2\epsilon^{\mu\nu\rho} \tau_\rho \hat{v}^\sigma \beta_\nu R_{\mu\sigma}(K)], \end{aligned} \quad (44)$$

where we defined $\tilde{R}_{ab}(J) = e_a^\mu e_b^\nu \tilde{R}_{\mu\nu}(J) \equiv \frac{1}{2} \epsilon_{ab} \tilde{\mathcal{R}}$.

The next step is to go from the connection $\tilde{\Gamma}_{\mu\nu}^\rho$ to the torsionful connection (27). The torsion comes from the fact that for TTNC we have $\tau \wedge d\tau = 0$ so that the first term in Eq. (27) is no longer symmetric. The difference between these two connections is a tensor depending on b_μ . We find

$$\begin{aligned} \mathcal{L} &= e[(h^{\alpha\nu} h^{\beta\mu} - h^{\alpha\mu} h^{\beta\nu}) K_{\alpha\mu} K_{\beta\nu} + 2\hat{v}^\mu b_\mu h^{\nu\rho} K_{\nu\rho} \\ &\quad - 2(\hat{v}^\mu b_\mu)^2 - \tilde{\Phi} \tilde{\mathcal{R}} - 2\hat{v}^\mu f_\mu + 2\epsilon^{\mu\nu\rho} \tau_\rho \hat{v}^\sigma \beta_\nu R_{\mu\sigma}(K)]. \end{aligned} \quad (45)$$

If we express the spatial curvature $\tilde{\mathcal{R}}$ in terms of the spatial curvature \mathcal{R} defined with respect to the Ω connection in Eq. (22) we find⁶ $\tilde{\mathcal{R}} = \mathcal{R} - \nabla_\mu (h^{\mu\nu} a_\nu)$. The vector a_μ is called the acceleration vector in HL gravity. In TTNC geometry it is known as the torsion vector $a_\mu = \mathcal{L}_{\hat{v}} \tau_\mu$, since all information about the torsion of Eq. (27) is contained in a_μ . The extrinsic curvatures $K_{\mu\rho}$ obey $h^{\nu\rho} K_{\mu\rho} = -\nabla_\mu \hat{v}^\nu$. We see that the DeWitt metric has

⁶Formula (12.49) of Ref. [13] contains a typo. The vector a_μ should have been b_μ . Since $h^{\mu\nu} b_\nu = \frac{1}{2} h^{\mu\nu} a_\nu$ this explains the factor of 2 difference between the expression here and formula (12.49) of Ref. [13].

$\lambda = 1$ where λ is the parameter in HL gravity that measures the relative coefficient of the two extrinsic curvature terms. The difference with Eq. (31) is that now there are couplings to $\hat{v}^\mu b_\mu$. We note that b_μ and f_μ transform as $\delta b_\mu = \partial_\mu \Lambda_D + \Lambda_K \tau_\mu$, $\delta f_\mu = \partial_\mu \Lambda_K + 2\Lambda_K b_\mu - 2\Lambda_D f_\mu$, where Λ_D and Λ_K are the local parameters of the D and K transformations. We can thus gauge fix the K transformations by setting $\hat{v}^\mu b_\mu$ to any desired value.

Finally we rewrite the last term in Eq. (45). Using that for TTNC we can always write $\tau_\mu = N \partial_\mu \tau$, it can be shown that

$$\epsilon^{\mu\nu\rho} \tau_\rho \hat{v}^\sigma \beta_\nu R_{\mu\sigma}(K) = -\frac{1}{4} \epsilon^{\mu\nu\rho} \beta_\nu \tau_\rho (\partial_\mu + 2a_\mu) I, \quad (46)$$

where I is defined as $I = B^2 - 4(\hat{v}^\mu b_\mu)^2 + 2\hat{v}^\nu \partial_\nu (B - 2\hat{v}^\mu b_\mu) - 4\hat{v}^\mu f_\mu$, in which B denotes the quantity $B = \hat{v}^\mu N^{-1} \partial_\mu N$. Our final result is thus Eq. (45) with Eq. (46). The action depends on the variables $\tau_\mu = N \partial_\mu \tau$, $\bar{h}_{\mu\nu}$, $\tilde{\Phi}$, $\hat{v}^\mu b_\mu$, $\hat{v}^\mu f_\mu$ and β_μ . The equation of motion for β_μ allows us to solve for $\hat{v}^\mu f_\mu$ on shell.

The Lagrangian (45) provides a new way of constructing conformal actions for nonprojectable HL gravity that we refer to as Chern-Simons Schrödinger gravity. The main difference with the $z = 2$ Weyl invariant construction of Refs. [13,26] is that we do not need to introduce a Stückelberg scalar, called χ in Refs. [13,26]. This Stückelberg scalar was needed in order to construct a $z = 2$ Weyl invariant combination of extrinsic curvature terms based on a DeWitt metric with λ parameter $1/2$, i.e. $(h^{\alpha\nu} h^{\beta\mu} - \frac{1}{2} h^{\alpha\mu} h^{\beta\nu}) K_{\alpha\mu}^\chi K_{\beta\nu}^\chi$ where $K_{\mu\nu}^\chi$ is the extrinsic curvature scalar with m_μ replaced by $m_\mu - \partial_\mu \chi$ (see Ref. [13] for details).

B. Lifshitz solutions

The Schrödinger invariant CS theory (18) with $c_2 = c_3 = 0$ admits $z = 2$ Lifshitz solutions. It can be readily verified that the following expressions solve the flatness conditions $F = dA + A \wedge A = 0$:

$$\tau = \frac{dt}{r^2}, \quad e^1 = \frac{dr}{r}, \quad e^2 = \frac{dx}{r}, \quad b = -\frac{dr}{r}, \quad \beta = -\frac{dx}{r}, \quad (47)$$

with all other connections equal to zero. If we use the relation to the ADM description of HL gravity expressed in Eqs. (34) and (35) we find the $z = 2$ Lifshitz metric

$$ds^2 = -\frac{dt^2}{r^4} + \frac{dr^2}{r^2} + \frac{dx^2}{r^2}. \quad (48)$$

The solution has a simpler form. If we denote $\mathbf{b} = e^{(D-P_1)\rho}$, where $r = e^{-\rho}$, then the Lifshitz solution can be written as $A = \mathbf{b}^{-1} a \mathbf{b} + \mathbf{b}^{-1} d\mathbf{b}$, where $a = Hdt + (P_2 - Z)dx$.

The 3D Lifshitz solution with $z = 2$ was also found in the context of CS theories for higher-spin theories [29,30]. However, it was pointed out in Ref. [31] that this interpretation is problematic due to a degeneracy problem: the spin connection cannot be determined from the torsion-free equation. Put another way the nonrelativistic solutions of $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ CS theory are not equivalent to metric solutions. Here we show that the solution (47) naturally emerges from a Newton-Cartan Chern-Simons theory which is not a Lorentzian metric theory.

VI. DISCUSSION

The results obtained in this paper open up a number of interesting applications and extensions. First of all, it will be interesting to examine CS actions for other nonrelativistic algebras, such as the Galilean conformal algebra, and likewise for algebras that play a role in ultrarelativistic limits, such as the Carroll algebra. In the latter case, one expects a connection to the 3D Carrollian gravity of Ref. [32].

Another worthwhile direction to pursue is to consider the CS actions of this paper in the presence of nontrivial boundaries, and consider aspects of edge physics as performed e.g. in Ref. [33] for quantum Hall states. In particular it would be interesting to study the role of the Galilean boost CS term [with the coefficient c_2 in Eq. (15)] in relation to anomalies in this context. Further one could try to find a microscopic description of the extended Bargmann CS theory, e.g. using nonrelativistic fermions with a mass gap such that the effective theory below the mass gap is described by the extended Bargmann CS theory.⁷ Moreover it is tempting to consider the CS theory with the Galilean boost and rotation CS terms (with coefficients c_2 and c_3) in Eq. (12) as the nonrelativistic analogue of topologically massive gravity [34,35]. To explore this idea further one would for example like to understand the solutions of the theory.

An important application of our findings is to use the Schrödinger invariant CS theory as a bulk holographic action for $z = 2$ Lifshitz space-times. The resulting Chern-Simons Schrödinger gravity may be regarded as a very minimal setup to do Lifshitz holography (see Ref. [36] for a review). Using HL gravity in this context was proposed in Refs. [37,38] and the CS reformulation of this paper is expected to provide new insights. In particular the CS formulation can give a proper definition of black objects (provided they exist) in these nonrelativistic gravity

theories, and therewith also give information on boundary hydrodynamics and other dynamical properties. We also stress that our results point towards Lifshitz vacua appearing naturally in nonrelativistic gravity, rather than in Lorentzian metric theories. It would thus be interesting to revisit some of the pathologies [39] and other properties (see e.g. Ref. [40]) that have been examined within the framework of Riemannian geometry.

Another relevant aspect to pursue, in close parallel with higher-spin gravity, is to employ the techniques of Refs. [41,42] to find the corresponding generalization of holographic entanglement [43] for nonrelativistic CS gravity. Moreover, a further extension of our ideas to nonrelativistic higher-spin gravity could be an interesting direction. Similar in spirit, an $SL(2, \mathbb{R}) \times U(1)$ CS theory (called lower-spin gravity) was argued to be the minimal setup to holographically describe warped CFTs [44]. In this light one could try to find a relation between the present CS theories or some close cousin thereof and two-dimensional warped CFTs [45].

All the HL gravity actions obtained via our CS formulation have the property that the HL λ parameter, which appears in the DeWitt metric contracting the extrinsic curvatures, is equal to unity. It would thus be interesting to see whether by adding appropriate scalar matter fields, i.e. considering CS matter theories, we can construct more general HL actions for which $\lambda \neq 1$.

Upon the completion of this work we were informed by Eric Bergshoeff and Jan Rosseel of Ref. [46] in which it is shown that the Bargmann invariant CS action can be obtained by a nonrelativistic limit from three-dimensional general relativity, augmented with two vector fields. This work also obtains a supersymmetric generalization, which is thus a supersymmetric extension of 3D projectable HL gravity.

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