

Modifying the pion mass in the loosely bound Skyrme modelSven Bjarke Gudnason^{1,*} and Muneto Nitta^{2,†}¹*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*²*Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan*

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We study the loosely bound Skyrme model with the addition of two different pion mass terms; this is the most general potential of polynomial form up to second order in the trace of the Skyrme field. The two pion mass terms are called the standard pion mass term and the modified pion mass term. We find that the binding energies are not reduced by the introduction of the modified pion mass, but it is analogous to the standard pion mass term with a decrease in the value of the mass parameter of the loosely bound potential (for large values of the latter parameter). We find by increasing the overall pion mass that we can reduce the classical binding energy of the 4-Skyrmion to the 2.7% level and the total binding energy including the contribution from spin-isospin quantization is reduced to the 5.8% level.

DOI: [10.1103/PhysRevD.94.065018](https://doi.org/10.1103/PhysRevD.94.065018)**I. INTRODUCTION**

The Skyrme model was made as an effective theory of pions that could describe baryons in terms of its soliton—the Skyrmion [1,2]. It was, however, not taken too serious as a model until Witten pointed out that the Skyrmion should be identified with the baryon in large- N_c quantum chromodynamics [3,4]. Although the single and charge-two Skyrmions were studied in the literature in the following years, little progress was made on finding Skyrmions with higher baryon numbers (three and above) until the idea of using rational maps was introduced [5,6]. The Skyrmion solutions were then soon found and their symmetries identified for baryon numbers up to and including $B = 22$ [7]. These Skyrmions are well described by rational maps and look like fullerenes and thus they are hollow, almost spherical shells of baryon charge B with $2B - 2$ holes in them. This approach seemed to be on the right track as it is a convenient and precise way of finding Skyrmion solutions with higher baryon numbers. For the single Skyrmion the pion mass term has little qualitative effect; naively, it seems that it just decreases the size slightly and increases the energy a little [8], nothing that refitting the parameters cannot compensate. For the Skyrmions of higher baryon numbers, however, it turned out that the pion mass has a drastic effect; the fullerene-type hollow shells are only the preferred minima of the energy when the pion mass is turned off or very small [9–11]. In fact, for a pion mass of the order of its experimentally measured value, the Skyrmions prefer to order themselves as cubes in a crystal—akin towards the alpha-particle model of nuclei [12]. However, the Skyrmions are much more complex than just point particles with interactions and

thus should not be directly compared to the alpha-particle model.

All these steps of progress towards finding Skyrmion solutions of higher baryon numbers brought us to this point and, in principle, Skyrmion solutions of any baryon number can now be constructed. The Skyrme model as was used up to this point is made of three terms, the kinetic term, the Skyrme term, and the linear pion mass term (linear in the chiral Lagrangian field U). We shall henceforth call this pion mass term the *standard* pion mass term. However, a notorious problem has been tagging along so far, namely, the binding energies of the Skyrmions with higher baryon numbers are much too large; they are about 1 order of magnitude larger than the experimentally observed values. This problem motivated several directions of improving the standard Skyrme model. One attempt at mending the problem of the large binding energies was the idea of starting from a higher-dimensional self-dual theory, performing dimensional reduction, and then identifying the Skyrme model as the leading order Lagrangian; the binding energy in this construction would go to zero if infinitely many mesons were to be integrated in [13,14]. Another direction is based on the discovery of a subsector where the model has a Bogomol’nyi bound that can actually be saturated [15,16], unlike that of the standard Skyrme model [17]. This model is constructed by squaring the baryon charge current and adding a potential and is by now called the BPS-Skyrme model. One peculiarity of this model is that it does not contain a kinetic term and not the Skyrme term either. A strength of this model is that it models a perfect fluid, which is a welcomed feature in the light of nuclear matter and neutron stars [18–21]. In a realistic model of nuclei, however, one would expect the presence of at least the kinetic term in the model. Turning on the kinetic term and the Skyrme term with order-one coefficients, however, renders the model very similar to the standard Skyrme

*bjarke@impcas.ac.cn

†nitta@phys-h.keio.ac.jp

model and the binding energies are again too large. One idea is then that the kinetic term and the Skyrme term are rather small compared to the BPS-Skyrme term [22]. This turns out to be a rather difficult technical problem; what happens here is that when only the BPS term is present in the theory (plus a potential), then the Skyrmions can take any shape. However, with the kinetic term and the Skyrme term turned on, the Skyrmions like to take their usual shapes of platonic solids; however, if the coefficient of the latter two terms is very small, then the solutions can afford very large derivatives. This fact is quite a problem for most codes for Skyrmion calculations [23]. The third direction of reducing the binding energies in the Skyrme-like models is to take the standard Skyrme model and add to it a holomorphic (quartic) potential, which is based on an energy bound that, however, can only be saturated for the single Skyrmion [23,24]. This model was called the lightly bound Skyrme model in Ref. [23]. Although the lightly bound Skyrme model, i.e., the Skyrme model with the holomorphic potential, is able to reduce the binding energies of the multi-Skyrmions, long before reaching experimentally observed values, the symmetries of the Skyrmions completely change and the platonic symmetries are lost [25]. This leads to severe problems of retaining the earlier successes of the Skyrme model; in particular, if the cubic shape of the 4-Skyrmion is lost, then the identification of the Hoyle state and the ratio of slopes of the ground state and Hoyle state rotational bands [26] should be reconsidered entirely. A related problem with the lightly bound Skyrme model is that the binding energy of the $B = 5$ Skyrmion is higher than that of the $B = 4$ Skyrmion and hence nuclear clustering [27] into n alpha particles for nucleon number $A = 4n$ is no longer possible. In Ref. [25] we have chosen to keep the cubic symmetry of the 4-Skyrmion to retain the clustering of the nuclei which, in our opinion, is a strength of the Skyrme-type models. Not only trying to keep the symmetries and hence the successes of the Skyrme model, a better potential than that of the lightly bound Skyrme model was found in Ref. [25]; we call the Skyrme model with this quadratic potential the loosely bound Skyrme model. The loosely bound Skyrme model can reach lower binding energies than the lightly bound model before the symmetries change from platonic to face-centered cubic (FCC) symmetries.¹

As pointed out several times in the literature, the pion mass can be made from infinitely many different terms, see, e.g., [25,28–31]. In Ref. [25] a class of potentials giving rise to a pion mass term was contemplated

¹Let us clarify that we use the term FCC in this paper to refer to the Skyrmions that split up into separate $B = 1$ clumps of baryon charge situated at the vertices of a cubic lattice [23]. Obviously, for finite-sized Skyrmions it then only corresponds to a part cut out from the lattice. In particular, for the 4-Skyrmion that we will study in this paper, the symmetry turns from cubic to tetrahedral; we will nevertheless call the tetrahedral state *FCC*.

$$V_{0n} = \frac{1}{n} m_{0n}^2 (1 - \sigma^n), \quad (1)$$

where $\sigma = \text{Tr}[U]/2$ and U is the Skyrme field related to the pions as $U = \sigma \mathbf{1}_2 + i\pi^a \tau^a$. For each n , the above potential gives a normalized mass term for the pions. Only the sum of these terms is measured. The loosely bound potential, on the other hand, belongs to a class of potentials that does not contribute to the pion mass

$$V_n = \frac{1}{n} m_n^2 (1 - \sigma)^n, \quad n \geq 2. \quad (2)$$

The loosely bound potential corresponds to $n = 2$ and the lightly bound potential corresponds to $n = 4$. Notice that the two classes of potentials coincide for $n = 1$.

Although the pion decay constant and the pion mass are both experimentally known quantities, a modern point of view in the Skyrme model is to consider them as renormalized (effective) constants, that should be renormalized in the baryon medium and not in the pion vacuum (i.e., at zero chemical potential and zero temperature). Therefore the pion decay constant is often taken to be around half of its measured value.² In this spirit, we will in this paper also allow for some slush in the pion mass and consider values too small and too large, in order to study the effects on the model.

In this paper, we will take the loosely bound Skyrme model, which is the Skyrme model with the potential V_2 as well as the two first terms contributing to the pion mass, i.e., $V_{01} = V_1$ and V_{02} . This is the most general potential of polynomial form up to second order in σ . Let us first contemplate what effect we could expect from switching the standard pion mass term, V_1 , with the modified pion mass term V_{02} [33,34]. Since the Skyrmion of charge B needs to wrap a 3-cycle on the target space B times, it will necessarily pass the antipodal point to the vacuum ($\sigma = -1$) B times. When the Skyrme field is near this antipodal point the standard pion mass term has its maximal contribution to the energy, whereas the modified pion mass term has none. Since the binding energy is a comparison between the 1-Skyrmion and the B -Skyrmion, we can easily see that increasing the energy more for the B -Skyrmions than for the 1-Skyrmion lowers the relative binding energy of the B -Skyrmion. Therefore, one would naively conclude that the standard pion mass term is preferred over the modified one.

This paper is thus a complete scan of the parameter space of the most general potential of the polynomial form up to second order in σ . We find in agreement with the above contemplation of the modified pion mass that it increases the binding energy of the B -Skyrmions. However, although

²Ref. [32] also argues that the mass of the delta resonance and nucleon mass can only be fitted in the standard Skyrme model if the pion mass is taken to be larger than its measure value.

the increase in binding energy is considerable when the loosely bound potential is turned off, it becomes smaller and almost insignificant when the coefficient of the loosely bound potential is turned to its maximal value, i.e., just before the cubic symmetry of the 4-Skyrmion is lost. This is related to the fact that in this region of parameter space, switching from the linear or standard pion mass term to the modified pion mass term merely results in a lower value of the mass parameter of the loosely bound potential. This can thus be compensated by increasing the value of the latter mass parameter. We study the effects of the complete parameter space on the classical binding energy, the total binding energy which takes into account the quantum contribution from the spin and isospin of the nucleon, the pion decay constant, the mass spectrum, and finally the charge radius of the proton. We are able to reduce the classical binding energy to about the 2.7% level and the total binding energy to about the 5.8% level. The conclusion is that the modified pion mass term is not advantageous, but an increase in the value of the pion mass allows for a larger value of the mass parameter of the loosely bound potential, which in turn lowers the binding energy further.

The paper is organized as follows. In Sec. II we present the loosely bound Skyrme model with the two different pion mass terms, i.e., the most general potential up to second order in σ , set the notation, and define the observables that we will study on the entire parameter space of the model. Section III explains the numerical methods used and Sec. IV presents the results. Finally Sec. V concludes with a discussion and the Appendix shows figures of numerical Skyrmion solutions at the boundary between the cubic and FCC symmetry regions in the parameter space.

II. THE MODEL AND OBSERVABLES

The model under study is the Skyrme model and the Lagrangian density in physical units reads

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathcal{L}_2 + \frac{1}{e^2} \mathcal{L}_4 - \frac{\tilde{m}_\pi^2 f_\pi^2}{4m_\pi^2} V, \quad (3)$$

where the kinetic (Dirichlet) term and Skyrme term is given by

$$\mathcal{L}_2 = \frac{1}{4} \text{Tr}(L_\mu L^\mu), \quad \mathcal{L}_4 = \frac{1}{32} \text{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]), \quad (4)$$

and $L_\mu \equiv U^\dagger \partial_\mu U$. f_π is the pion decay constant with units of energy (MeV), $e > 0$ is a real-valued dimensionless constant, \tilde{m}_π is the pion mass in MeV, and, finally, m_π is a dimensionless pion mass parameter. The indices $\mu, \nu = 0, 1, 2, 3$ are spacetime indices and U is the Skyrme field which, in terms of the pions, reads

$$U = \mathbf{1}_2 \sigma + i \tau^a \pi^a, \quad (5)$$

with $U^\dagger U = \mathbf{1}_2$ being the nonlinear sigma model constraint, which is equivalent to $\sigma^2 + \pi^a \pi^a = 1$ and τ^a are the Pauli matrices.

It will prove convenient to do a rescaling of the energy and length scales and only work with dimensionless parameters. In particular, we will make a rescaling such that $\tilde{x}^i = \mu x^i$, where both \tilde{x}^i and μ have units of length (MeV⁻¹), and similarly for the energy $\tilde{E} = \lambda E$, where \tilde{E} and λ have units of energy (MeV). In particular, we get

$$\mathcal{L} = c_2 \mathcal{L}_2 + c_4 \mathcal{L}_4 - V, \quad (6)$$

where $c_2 > 0$ and $c_4 > 0$ are positive-definite real constants and

$$\lambda = \frac{f_\pi}{2e\sqrt{c_2 c_4}}, \quad \mu = \sqrt{\frac{c_2}{c_4}} \frac{2}{ef_\pi}, \quad (7)$$

whereas the pion mass in physical units (MeV) is given by

$$\tilde{m}_\pi = \frac{\sqrt{c_4}}{2c_2} ef_\pi m_\pi. \quad (8)$$

This relation assumes that the potential will have a pion mass normalized to m_π in dimensionless units.

The main focus of this paper is to study the most general potential of polynomial form up to second order in $\text{Tr}[U]$:

$$V = V_1 + V_{02} + V_2, \quad (9)$$

and the potentials are defined as

$$\begin{aligned} V_1 &\equiv m_1^2 (1 - \sigma), \\ V_{02} &\equiv \frac{1}{2} m_{02}^2 (1 - \sigma^2), \\ V_2 &\equiv \frac{1}{2} m_2^2 (1 - \sigma)^2, \end{aligned} \quad (10)$$

where the mass parameters m_1, m_{02}, m_2 are all real and

$$\sigma = \frac{1}{2} \text{Tr}[U]. \quad (11)$$

Note that there are only 2 free parameters to second order because the constant is irrelevant for the equations of motion. However, the above basis is convenient because all mass parameters are real valued. We will nevertheless change to a simpler basis shortly.

The Lagrangian density (6) without a potential turned on enjoys $\text{SU}(2) \times \text{SU}(2)$ symmetry. This symmetry is explicitly broken down to a diagonal $\text{SU}(2)$ by the potential (9). This $\text{SU}(2)$ corresponds to isospin and we will keep it unbroken in this paper.

The target space of the Skyrme model—due to the mentioned symmetry breaking—is given by $\mathcal{M} \simeq \text{SU}(2) \simeq S^3$. The map U —the Skyrme field—is thus a map from space, i.e., $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ to the target space \mathcal{M} and is characterized by the third homotopy group

$$\pi_3(\mathcal{M}) = \mathbb{Z} \ni B, \quad (12)$$

which admits solitons called Skyrmions and the integer B is called the baryon number, which in turn, can be calculated from the baryon charge density

$$\mathcal{B}^0 = -\frac{1}{12} \epsilon^{ijk} \text{Tr}[L_i, L_j, L_k], \quad (13)$$

by integrating over space

$$B = \frac{1}{2\pi^2} \int d^3x \mathcal{B}^0. \quad (14)$$

The pion mass (squared) in the model is given by

$$m_\pi^2 = \left. \frac{\partial V}{\partial \sigma} \right|_{\sigma=1}, \quad (15)$$

and as explained in Ref. [25], the potentials V_1 and V_{02} belong to the class of potentials giving rise to a pion mass in the vacuum $\sigma = 1$, whereas V_2 gives no contribution to the pion mass. In particular, calculating the pion mass from the potential (9), we get

$$m_\pi^2 = m_1^2 + m_{02}^2, \quad (16)$$

and so we can parametrize the two mass parameters giving a pion mass contribution as

$$m_1^2 = \alpha m_\pi^2, \quad m_{02}^2 = (1 - \alpha) m_\pi^2, \quad (17)$$

where the real parameter $\alpha \in [0, 1]$ takes on a value in the interval from zero to one; $\alpha = 1$ corresponds to the traditional pion mass, whereas $\alpha = 0$ yields the modified pion mass [33,34] and any value in between is a linear interpolation between the two.

We will now switch to a simpler basis for the potential

$$\begin{aligned} V &= \alpha m_\pi^2 (1 - \sigma) + \frac{1}{2} (1 - \alpha) m_\pi^2 (1 - \sigma^2) + \frac{1}{2} m_2^2 (1 - \sigma)^2 \\ &= m_\pi^2 (1 - \sigma) + \frac{1}{2} [m_2^2 - (1 - \alpha) m_\pi^2] (1 - \sigma)^2, \end{aligned} \quad (18)$$

where we have absorbed the modified pion mass term into the loosely bound potential in the second line. If we set $\alpha = 1$ then the potential is equal to a subset of that analyzed in Ref. [25]. However, when $\alpha < 1$ the coefficient of

$(1 - \sigma)^2$ (the loosely bound potential term) is no longer positive semidefinite; i.e., the mass parameter is no longer only real valued. We will now define the parameter

$$\mathbf{m}_2^2 \equiv m_2^2 - (1 - \alpha) m_\pi^2, \quad (19)$$

which takes values in the range $[-m_\pi^2, \infty)$ and the potential is then simply

$$V = m_\pi^2 (1 - \sigma) + \frac{1}{2} \mathbf{m}_2^2 (1 - \sigma)^2. \quad (20)$$

It is easy to confirm that $\sigma = 1$ is always a (local) vacuum. The lower bound on \mathbf{m}_2^2 comes from the condition that $\sigma = -1$ should not become the global vacuum; the value of \mathbf{m}_2^2 where the two vacua $\sigma = \pm 1$ become degenerate is exactly $\mathbf{m}_2^2 = -m_\pi^2$. There is no upper bound on \mathbf{m}_2^2 ; however, when the parameter becomes too large, the platonic symmetries of the multi-Skyrmions are lost; in particular, the 4-Skyrmion loses its cubic symmetry [25].

Now we can see from Eq. (19) that when $\alpha = 1$, $\mathbf{m}_2 = m_2$ as expected and when $\alpha < 1$, the modified pion mass term is turned on, corresponding to a decrease in the effective value of the mass parameter m_2 . In order to cover the complete parameter space, however, we need to consider also the negative range of \mathbf{m}_2^2 , corresponding to the case where $m_2^2 < (1 - \alpha) m_\pi^2$, which is possible only when the modified pion mass is turned on. When $m_2 \gg m_\pi$, we can thus expect that the modified pion mass does not provide any advantage at all, since it merely reduces the effective value of m_2 and from Ref. [25] we know that the largest possible value of m_2 provides the lowest possible binding energies in that range. The negative range of $\mathbf{m}_2^2 \in [-m_\pi^2, 0)$ is, however, until now unexplored.

Now we should make a choice concerning the (dimensionless) units, i.e., fixing c_2 and c_4 . The standard Skyrme units correspond to $c_2 = c_4 = 2$ for which the energy and length are given in units of $f_\pi/(4e)$ and $2/(ef_\pi)$, respectively; see Ref. [35]. Here we will apply the same convention for units as used in Ref. [25], namely,

$$c_2 = \frac{1}{4}, \quad c_4 = 1, \quad (21)$$

and hence energies and lengths according to Eq. (7) will be given in units of f_π/e and $1/(ef_\pi)$, respectively. The pion mass in physical units (8) with the normalization convention (21) thus reads

$$\tilde{m}_\pi = 2ef_\pi m_\pi. \quad (22)$$

Because of the different normalization of the potential (9) [and of the Lagrangian density (6)] by a factor of 2, the pion mass $m = 1$, used in Ref. [12], corresponds to $m_\pi = 1/4$ and $\tilde{m}_\pi = ef_\pi/2$ in our units and normalization.

Let us define the observables that we want to compare with data for nuclei. The first, and the one of prime interest in this paper, is the classical binding energy

$$\Delta_B \equiv BE_1 - E_B, \quad (23)$$

where E_B is the total energy of the Skyrmion with baryon number B . It will, however, prove convenient to use the *relative* classical binding energy instead

$$\delta_B \equiv \frac{\Delta_B}{BE_1} = 1 - \frac{E_B}{BE_1}, \quad (24)$$

as the physical units drop out and we can use any units we like, in particular, our Skyrme units, i.e., Skyrme units in our normalization. Before we can compare honestly with the experiment, we should take into account the quantum contribution to the ground state of spin and isospin quantization, yielding

$$\delta_B^{\text{tot}} \equiv 1 - \frac{E_B + \epsilon_B}{B(E_1 + \epsilon_1)}, \quad (25)$$

where ϵ_B is the quantum contribution to the ground state of the baryon represented by the Skyrmion of baryon number B . Note, in particular, that the quantum contribution to the Skyrmion of baryon number B *decreases* the binding energy, whereas the quantum contribution to the 1-Skyrmion *increases* the binding energy. As well known the quantum contribution to the 1-Skyrmion happens to be larger than those to the higher B -Skyrmions and therefore the spin-isospin quantization has the effect of increasing the already too large binding energies of the Skyrmions. The reason why the quantum contribution to the 1-Skyrmion is larger than to the other ones is simply that the 1-Skyrmion is the smallest one and hence it has the smallest moment of inertia. Since the quantum contribution to the energy of the Skyrmion is inversely proportional to the moment of inertia the above mentioned effect on the binding energy follows.

In this paper, for practical reasons of flops economy and because of the fact that the ground state of the ${}^4\text{He}$ is a spin-0, isospin-0 state, we will focus on the $B = 1$ and $B = 4$ sectors of the model; for other baryon numbers in a subset of the model (the $\alpha = 1$ sector), see Ref. [25]. In particular, the latter fact implies that there is no quantum contribution to the $B = 4$ Skyrmion [36] and hence the total relative binding energy simplifies to

$$\delta_4^{\text{tot}} \equiv 1 - \frac{E_4}{4(E_1 + \epsilon_1)}. \quad (26)$$

Since we only need to calculate the ground state energy contribution from spin-isospin quantization of the single Skyrmion, the calculation is considerably simpler and we follow Ref. [37]. Let $U \rightarrow AUA^{-1}$, with $A = A(t)$ being an $\text{SU}(2)$ matrix which rotates the isospin of U . This gives rise to the kinetic energy

$$T = \frac{1}{2} a_i U_{ij} a_j = \Lambda \text{Tr}(\partial_0 A \partial_0 A^{-1}), \quad (27)$$

for the single ($B = 1$) Skyrmion

$$U = \cos f(r) + i \hat{x}^i \tau^i \sin f(r), \quad (28)$$

where $a_i \equiv -i \text{Tr}(\tau_i A^{-1} \dot{A})$, \hat{x}^i is the spatial unit vector, and $U_{ij} = \Lambda \delta_{ij}$, with

$$\Lambda \equiv \frac{8\pi}{3} \int dr r^2 \sin^2 f \left(c_2 + c_4 f_r^2 + \frac{c_4}{r^2} \sin^2 f \right), \quad (29)$$

where $f_r \equiv \partial_r f$. Hence we get the kinetic energy from canonical quantization

$$T = \frac{1}{8\Lambda} \ell(\ell + 2) = \frac{1}{2\Lambda} J(J + 1). \quad (30)$$

In particular, for the spin-1/2 ground state of the proton or neutron, we get

$$T_{1/2} = \frac{1}{2\Lambda} \frac{3}{4}, \quad (31)$$

where $J = \ell/2$ is the spin quantum number. Reinstating physical units and using our normalization (21), we get for the total mass

$$\tilde{E}_1 + \tilde{\epsilon}_1 = \frac{f_\pi}{e} E_1 + \frac{e^3 f_\pi}{2\Lambda} \frac{3}{4}. \quad (32)$$

When calculating the relative binding energy, the factor of f_π/e will drop out, and hence we need only calculate

$$E_1 + \epsilon_1 = \frac{e}{f_\pi} (\tilde{E}_1 + \tilde{\epsilon}_1) = E_1 + \frac{e^4}{2\Lambda} \frac{3}{4}. \quad (33)$$

We will also calculate the mass of the delta resonance. Since it is merely the spin-3/2 state of the baryon in our model, we can estimate the mass by

$$\tilde{m}_\Delta = \tilde{E}_1 + 5\tilde{\epsilon}_1. \quad (34)$$

In order to add the quantum contribution to the energy of the 1-Skyrmion to its classical contribution, we need to fit the mass and size of a selected Skyrmion to those of a corresponding nucleus. A large part of the literature used the proton and delta resonance as the two input parameters to fit f_π and e [37]; this fit suffers from the problem that the binding energies for the Skyrmions are about an order of magnitude too large compared with those of nuclei. Later a different fit was made using ${}^6\text{Li}$ [38]; the purpose here is to better match the energies of multi-Skyrmions with higher B . Other fits in the literature use ${}^{12}\text{C}$ [26] or ${}^4\text{He}$ [25]. For concreteness and simplicity, we will again use ${}^4\text{He}$ as in

Ref. [25]; this is convenient because we can use the different 4-Skyrmions to do the calibration; in fact, we will—in this paper—recalibrate each point in the parameter space of the model such that the 4-Skyrmion fits the mass and size of ${}^4\text{He}$. This will, in turn, give an accurate estimate of the effect on the quantum contribution of the different parts of the parameter space.

Another observable that we will calculate on the Skyrmion solutions is the size of the nuclei. Since we fit the size of the 4-Skyrmion, we will use that of the 1-Skyrmion as a check. We choose to define the squared radius in terms of the baryon charge density, i.e.,

$$r_B^2 = \frac{1}{2\pi^2 B} \int d^3x r^2 \mathcal{B}^0, \quad (35)$$

where \mathcal{B}^0 is the baryon charge density given in Eq. (13). Hence, the size can be estimated as $r_B \sim \sqrt{r_B^2}$.

We are now ready to perform numerical calculations on the full parameter space of the most general potential of the polynomial form up to second order in σ .

III. NUMERICAL CALCULATIONS

We will follow the approach used in Ref. [25] and use a finite difference method in conjunction with the relaxation method for the partial differential equations. Our grid sizes are typically 101^3 and we use a fourth-order stencil.

In order to save computational costs, we use the hedgehog ansatz (28) for calculating the 1-Skyrmions and solve the ordinary differential equation

$$\begin{aligned} & c_2 \left(f_{rr} + \frac{2}{r} f_r - \frac{\sin 2f}{r^2} \right) \\ & + c_4 \left(\frac{2\sin^2(f) f_{rr}}{r^2} + \frac{\sin(2f) f_r^2}{r^2} - \frac{\sin 2f \sin^2 f}{r^4} \right) \\ & = m_1^2 \sin f + \frac{1}{2} m_{02}^2 \sin 2f + m_2^2 (1 - \cos f) \sin f, \quad (36) \end{aligned}$$

to very high accuracy level, better than the 10^{-6} level. Therefore, in order to compare the $B = 4$ solutions to the $B = 1$ solutions—for the purpose of calculating the relative binding energy—we need to obtain the energy for the 4-Skyrmion very precisely. We will again utilize the trick used in Ref. [25]; namely, we relax the numerical solution down to the 10^{-3} level, locally (we denote this time τ_0), and from then on, we make an exponential fit to the energy as function of relaxation (imaginary) time. We continue the cooling process until the exponential fit is precise enough and the imaginary time where we stop the calculation is τ_2 . τ_1 is defined as the midpoint: $\tau_1 = (\tau_0 + \tau_2)/2$. After the fit has been calculated, we take the $\tau \rightarrow \infty$ limit of the energy function and the result is

$$\begin{aligned} E_B & \simeq \frac{B}{B_{\text{numerical}}} \\ & \times \frac{E_{B,\text{numerical}}(\tau_0) E_{B,\text{numerical}}(\tau_2) - E_{B,\text{numerical}}^2(\tau_1)}{E_{B,\text{numerical}}(\tau_0) - 2E_{B,\text{numerical}}(\tau_1) + E_{B,\text{numerical}}(\tau_2)}. \quad (37) \end{aligned}$$

Note that we also use another trick of compensating the total energy by a factor of $B/B_{\text{numerical}}$ as both the energy and the baryon charge is underestimated in the numerical calculation. We have checked in Ref. [25] that this reproduces the energy for the $B = 1$ sector within an accuracy of about 2.7×10^{-4} or better.

As another check on the precision of the Skyrmion solutions, we calculate the baryon charge numerically and find that all the solutions yield $B = 4$ to a precision of about 1.7×10^{-3} or better. Therefore in summary our results should be trustable down to about the per mille level.

For each data point in the parameter space, we refit the length and energy scales to the ${}^4\text{He}$ nucleus, thus determining f_π and e . After the physical units are fitted we calculate all the observables presented in the last section.

We are now ready to present the results in the next section.

IV. RESULTS

A. Classical binding energies

We start by presenting the classical binding energies, defined in Eq. (24), in Fig. 1 for various values of m_π as functions of \mathbf{m}_2^2 . The four curves correspond to $m_\pi = 0.125, 0.25, 0.375, 0.5$. The value $m_\pi = 0.25$ corresponds to the choice $m = 1$ in Refs. [12,36]. The blue dots in this figure and in the remainder of the paper correspond to Skyrmion solutions with cubic symmetry, whereas the red-dashed dots correspond to the Skyrmion breaking up into individual and weakly bound $B = 1$ clumps, situated in an FCC lattice [23]; see the Appendix.

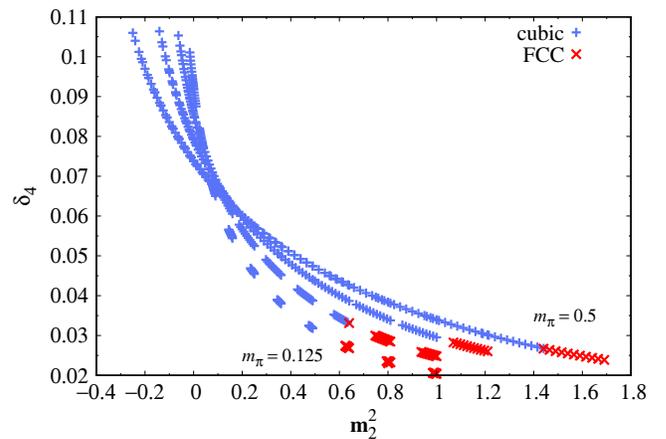


FIG. 1. Classical binding energy δ_4 as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

We clearly see that increasing the pion mass m_π , allows for larger values of \mathbf{m}_2^2 and eventually for lower classical binding energies. We note, however, that $m_\pi = 0.125$ has a smaller binding energy than $m_\pi = 0.25$ (corresponding to $m = 1$ in Refs. [12,36]) before the symmetries change from cubic to FCC. Nevertheless, larger values of the pion mass parameter decrease the binding energies; in particular, the classical binding energy is smaller for $m_\pi = 0.375$ and $m_\pi = 0.5$ than for $m_\pi = 0.125$. The largest value of \mathbf{m}_2^2 possible for cubic symmetry is reached for $m_\pi = 0.5$ and it also yields the smallest classical binding energy of about 2.7% for the 4-Skyrmion.

We can also see from Fig. 1 that the slope of the curves at large \mathbf{m}_2^2 is much smaller than for small values; hence, the difference between the standard (linear) pion mass and the modified pion mass becomes much less pronounced (recall that it corresponds merely to a negative shift in the value of \mathbf{m}_2^2).

B. Calibration

Now we will perform a calibration to the ${}^4\text{He}$ nucleus for each Skyrmion solution in the parameter space. In particular, as mentioned in Sec. II, we fit the mass and the size of the 4-Skyrmion to those of ${}^4\text{He}$; this determines f_π and e . Figures 2 and 3 show the pion decay constant and the (dimensionless) Skyrme term parameter e .

We can see from Fig. 2 that for $m_2 = 0$, the modified pion mass term—corresponding to negative values of \mathbf{m}_2^2 —increases the pion decay constant (which is good). The experimentally observed value is around 184 MeV (not shown in the figure) in the normalization used in the Skyrme model [37]. However, turning on m_2 , which corresponds to positive values of \mathbf{m}_2^2 (and decreases the binding energy), reduces the value of f_π to about a third of its experimentally measured value. We observed that larger

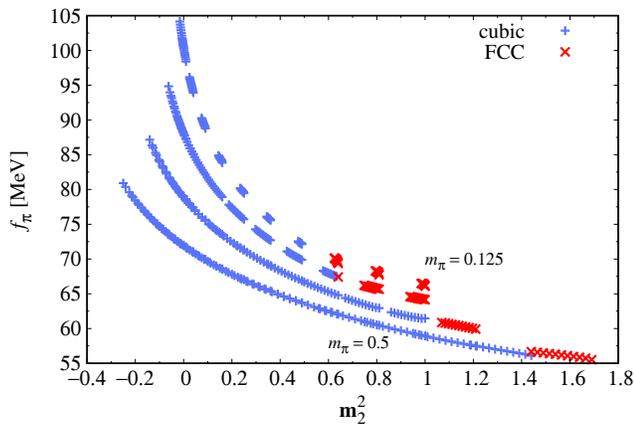


FIG. 2. Calibration of the pion decay constant, f_π , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

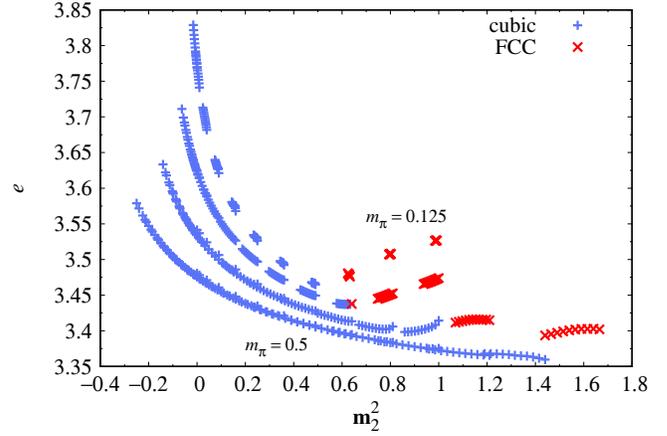


FIG. 3. Calibration of the Skyrme term coefficient, e , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

values of the pion mass directly translate into smaller values of the pion decay constant, f_π .

Since the value of the Skyrme term coefficient e is, to the best of our knowledge, not known experimentally, there is no preferred value; it is simply the result of the fit of length and energy units. Let us, however, remark that the blue points move downwards (decreasing e) for increasing \mathbf{m}_2^2 (except for the largest values of the $m_\pi = 0.375$ series before the symmetry changes from cubic to FCC), whereas when the symmetry switches to FCC the red-dashed points are moving upwards (increasing e) for increasing \mathbf{m}_2^2 (but possibly saturating at a plateau). Let us also remark that the smaller e is, the smaller the contribution from spin-isospin quantization to the 1-Skyrmion is. This means that in order to get the smallest possible total binding energy, we need an as small as possible value of e . For this, the large values of the pion mass m_π are advantageous.

Let us also remark that the curves of the binding energies shown in Fig. 1 are far smoother than those shown in Fig. 3;

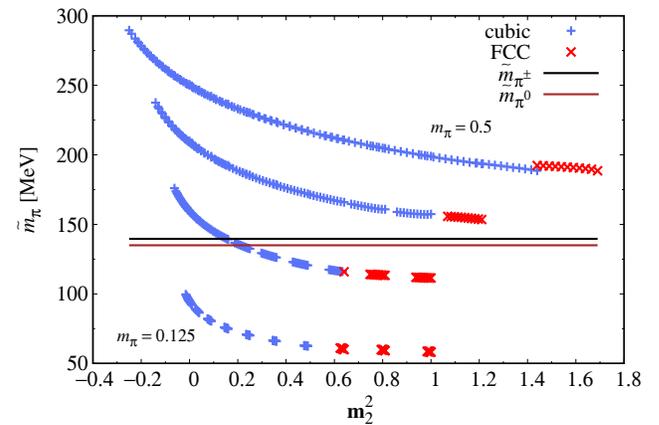


FIG. 4. Pion mass in physical unit, \tilde{m}_π , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

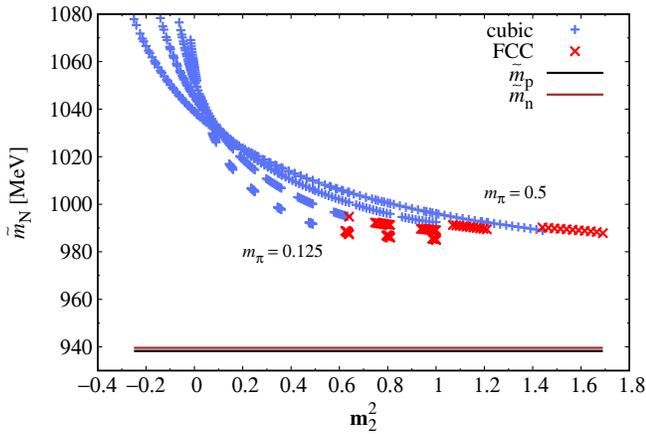


FIG. 5. Nucleon mass in physical unit, \tilde{m}_N , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

this is due to the highly precise calculations for the energies, whereas the sizes have been estimated without taking any limits of large relaxation times. This can be seen as small jumps in the curves of e in Fig. 3. The error is, however, still smaller than or about the per mille level.

C. Mass spectrum

We now turn to the mass spectrum. Since the mass and size of the 4-Skyrmion has been fitted to that of ${}^4\text{He}$, f_π and e are fixed. The masses in physical units of the pion, the nucleon and the delta resonance can thus readily be calculated and they are presented in Figs. 4–6, respectively.

Let us start with the pion mass of Fig. 4. We can see that if we want to minimize the classical relative binding energy by maximizing m_2 , then the experimentally preferred value of m_π is between 0.25 and 0.375, i.e., between the second and the third series in Fig. 4. However, as already mentioned, since the pion decay constant is almost a factor of 3 off of its experimental value and the fact that we choose

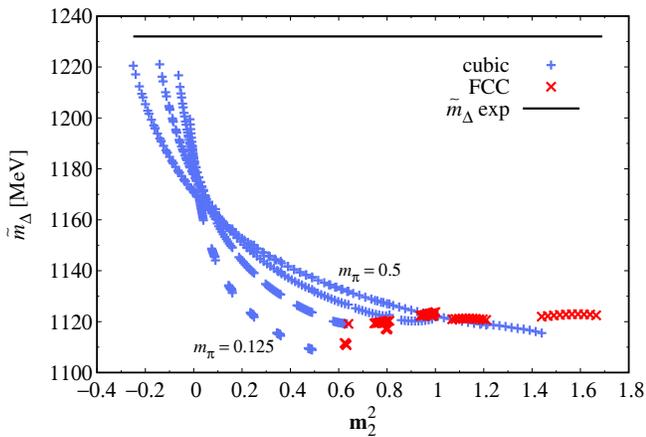


FIG. 6. Mass of the Δ resonance in physical unit, \tilde{m}_Δ , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

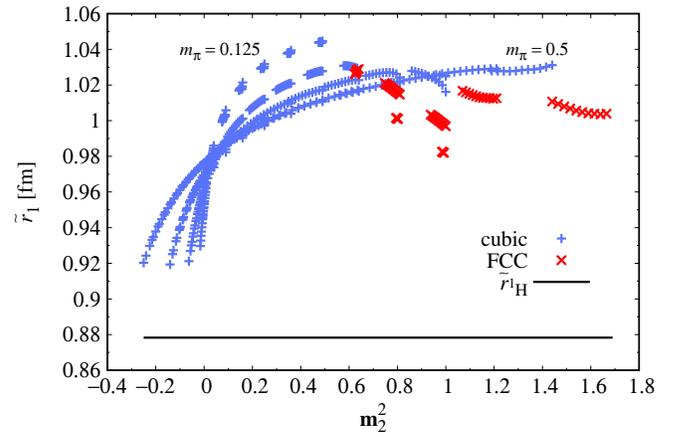


FIG. 7. Charge radius of the proton in physical unit, \tilde{r}_1 , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

to interpret f_π and m_π as renormalized constants in the baryon medium, not in the pion vacuum, then we may contemplate allowing for some slush also in the value of m_π .

Looking now at Fig. 5, interestingly, we can see that for the largest possible value of \mathbf{m}_2 , the nucleon mass is closest to the experimentally observed value for all of the pion mass values. The pion mass $m_\pi = 0.5$ gives slightly better, but nearly the same value as for $m_\pi = 0.125$ and for $m_\pi = 0.25$ the worst fit to the measured nucleon mass is found, in the limit of the largest possible value of \mathbf{m}_2^2 before the cubic symmetry is lost. For all points in the parameter space, we can conclude that the loosely bound Skyrme model overestimates the nucleon mass.

The final mass we calculate in this paper is the mass of the delta resonance. We can see from Fig. 6 that for the largest possible values of \mathbf{m}_2^2 , for each series, the model estimate is the farthest away from the measured mass of the delta resonance. For all points in the parameter space, we

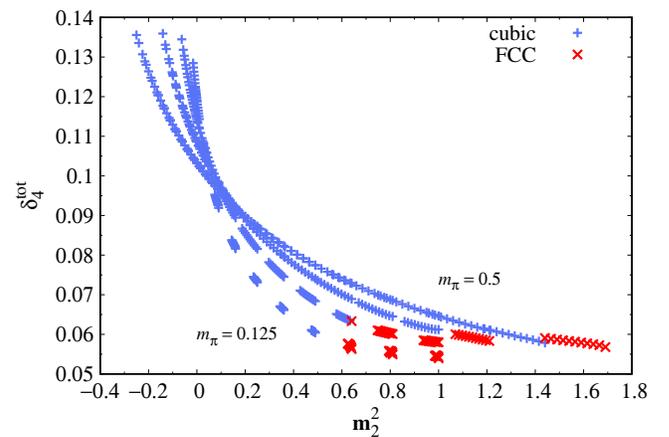


FIG. 8. Total binding energy, δ_4^{tot} , as function of \mathbf{m}_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$.

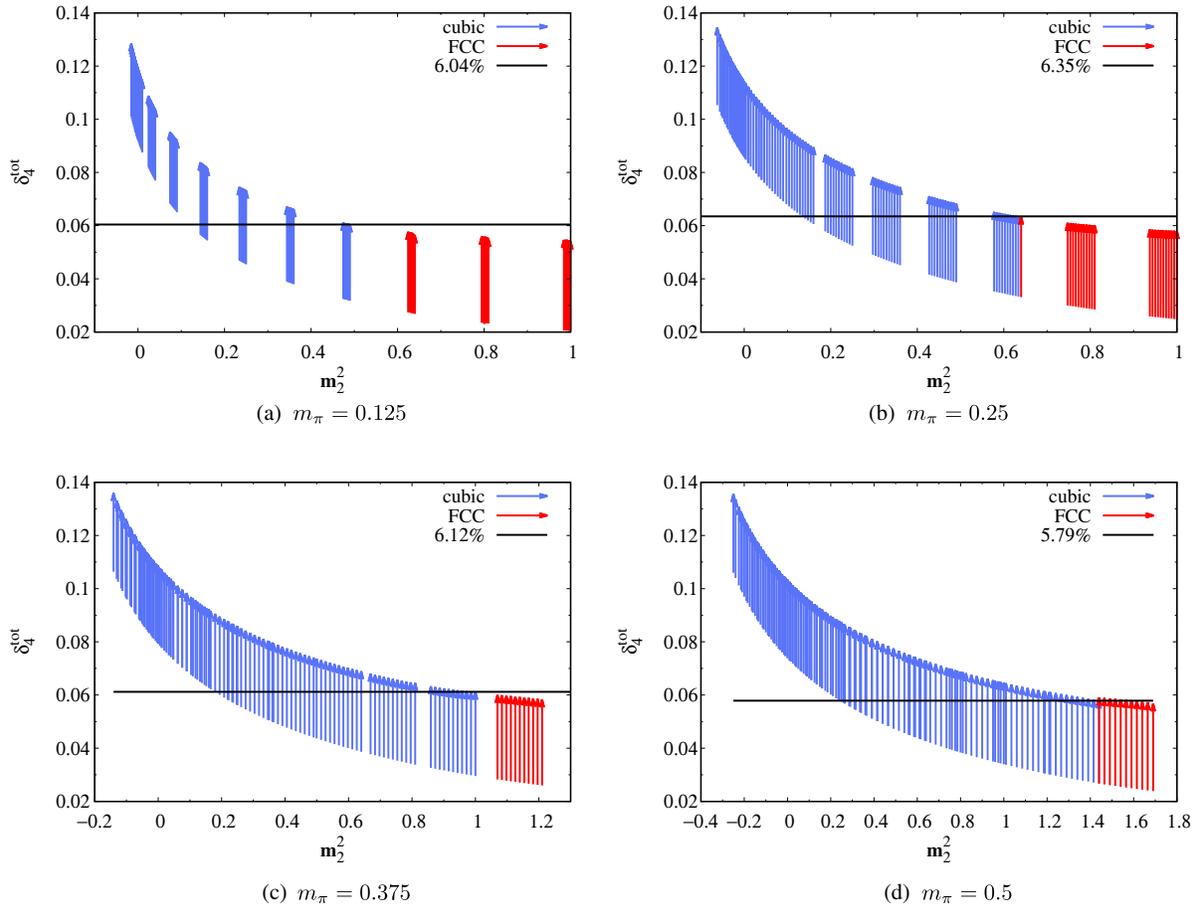


FIG. 9. Breakdown of the total binding energy, δ_4^{tot} , as function of m_2^2 ; four series of points are shown corresponding to $m_\pi = 0.125, 0.25, 0.375, 0.5$, corresponding to (a),(b),(c),(d), respectively. The bottom of the arrows corresponds to the classical contribution, whereas the arrow head includes the quantum correction.

can conclude that the loosely bound Skyrme model underestimates the mass of the delta resonance.

D. Proton charge radius

We will now turn to the proton charge radius and use it as a rough estimate of the size of the nucleon. Figure 7 shows the square root of the squared radius averaged using the baryon charge density³ of the 1-Skyrmion; see Eq. (35).

We can observe from the figure that all the proton charge radii in the entire parameter space are overestimated. This is because we fit the length scale to the size of ${}^4\text{He}$ and the 4-Skyrmion in general is too small; the addition of the loosely bound potential, $m_2 > 0$, in turn exacerbates this tendency and decreases the 4-Skyrmion even more. With this choice of fitting, this problem shows up as the charge radius for the proton being too large. We can, interestingly, observe that if the loosely bound potential is turned off ($m_2 = 0$), then the modified pion mass improves the value of the charge radius

³Reference [39] argues that the baryon charge density is a natural definition for calculating the size of a soliton; in particular in Skyrme-type models.

(corresponding to negative values of m_2^2). This is, however, in the part of the parameter space where the classical relative binding energies for the 4-Skyrmion are the largest and, hence, most at odds with experimental data.

Another effect that we can observe from Fig. 7 is that before the threshold for cubic symmetry is reached, i.e., the boundary between when the symmetry of the 4-Skyrmion is cubic or FCC, then the charge radius increases (except for $m_\pi = 0.375$). However, once the symmetry has changed to FCC, the size of the 4-Skyrmion increases quite a lot and this in turn has the effect of reducing the charge radius of the proton (because we fit the length scale to the size of the 4-Skyrmion); in fact, for increasing m_2^2 , the red-dashed points move downwards in the figure.

E. Total binding energies

The final comparison with experiment is again the relative binding energy, but now we will take the quantum contribution due to spin-isospin quantization into account. The total relative binding energy is defined in Eq. (26). Figure 8 shows the total relative binding energy and Fig. 9 displays the breakdown of the classical contribution (the

bottom of the arrows) and the quantum contribution (the length of the arrows).

As was the case for the classical relative binding energy, so is the case for the total relative binding energy; the loosely bound potential decreases the binding energy. We can see that the lowest binding energy is reached for $m_\pi = 0.5$, but the next-to-best value is for $m_\pi = 0.125$; the dependence of the total binding energy on the pion mass parameter is not linear and indeed quite nontrivial.

From Fig. 9 we can see that the quantum contribution increases slightly when the loosely bound potential is turned on, i.e. when \mathbf{m}_2^2 is large.

V. DISCUSSION AND CONCLUSION

In this paper we have found that the modified pion mass increases the binding energy of the B -Skyrmions as one would expect. We also found that the cubic symmetry is kept for slightly larger values of the coefficient of the loosely bound potential when the modified pion mass term is used, compared to when the standard pion mass term is used. This is because a given value of the modified pion mass term corresponds to the same standard pion mass albeit with a reduce value of the loosely bound mass parameter $\mathbf{m}_2^2 = m_2^2 - m_\pi^2$. We found—as pointed out many places in the literature—that the model prefers quite large values of the pion mass; this allows us to use a larger coefficient of the loosely bound potential and hence reduce the binding energy further. We are able to reduce the classical binding energy to about the 2.7% level and the total binding energy to about the 5.8% level. This corresponds, however, to a rather large pion mass at 190 MeV, a rather small pion decay constant at 56 MeV, a nucleon mass at 990 MeV, the mass of the delta resonance at 1118 MeV, and finally a charge radius of the proton at 0.97 fm.

This systematic study has only lowered the relative binding energy by about 0.6% with respect to that found in Ref. [25]. However, in this spirit of systematically surveying the parameter space of the Skyrme model, there are plenty of directions to look for improvements. One next step is to consider the BPS-Skyrme term; however, as we mentioned in the Introduction, its introduction to the model with a large coefficient has proven notoriously difficult at the technical level of numerical calculations. Naturally one can extend this systematic study to the complete potential of third order in σ . Other effects that we would like to include in the future is the breaking of the isospin symmetry and the Coulomb potential—which should be most significant for larger nuclei.

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APPENDIX: THE CUBIC TO FCC TRANSITION

In this appendix we show figures of the Skyrmions around the region in parameter space where the Skyrmion changes symmetry from cubic (platonic) symmetry to FCC symmetry. The Skyrmion thus changes from being composed of eight half-Skyrmions situated at the corners of a cube to being composed by four spheres sitting on the vertices of a tetrahedron.

Figures 10–13 show series of Skyrmion solutions near the boundary of the mentioned phase transition for $m_\pi = 0.125, 0.25, 0.375$ and 0.5 , respectively. The loosely

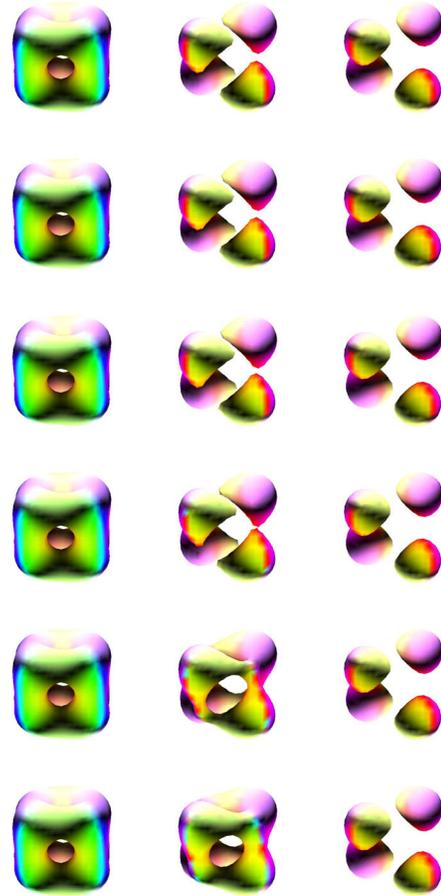


FIG. 10. The columns show $m_\pi = 0.125$ Skyrmion solutions with $m_2 = 0.7, 0.8, 0.9$ and the rows correspond to α from 0 to 1 in steps of 0.2 from top to bottom.

bound potential parameter m_2 is increased from left to right in each figure and α is varied vertically.

It is interesting to note that the Skyrmions are slightly more strongly bound and less aloof for $\alpha = 0$ (modified

pion mass) than for $\alpha = 1$ (standard pion mass). Consistently with findings in the text, we see that the larger m_π is, the larger values of m_2 are possible before the phase transition takes place.

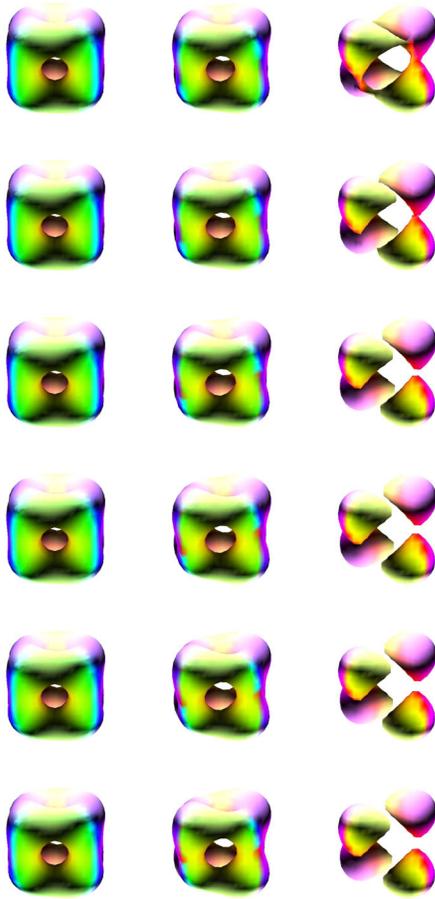


FIG. 11. The columns show $m_\pi = 0.25$ Skyrmion solutions with $m_2 = 0.7, 0.8, 0.9$ and the rows correspond to α from 0 to 1 in steps of 0.2 from top to bottom.



FIG. 12. The columns show $m_\pi = 0.375$ Skyrmion solutions with $m_2 = 0.9, 1, 1.1$ and the rows correspond to α from 0 to 1 in steps of 0.2 from top to bottom.



FIG. 13. The columns show $m_\pi = 0.5$ Skymion solutions with $m_2 = 1.1, 1.2, 1.3$ and the rows correspond to α from 0 to 1 in steps of 0.2 from top to bottom.

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