Conjugate variables in quantum field theory and a refinement of Pauli's theorem

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For the case of spin zero, we construct conjugate pairs of operators on Fock space. On states multiplied by polarization vectors, coordinate operators Q conjugate to the momentum operators P exist. In the massive case the notion of interest is derived from a geometrical quantity, the massless case is realized by taking the limit $m^2 \to 0$ on the one hand, on the other, starting with $m^2 = 0$ directly, from conformal transformations. The norm problem of the states on which the Q's act is crucial: the states determine eventually how many independent conjugate pairs exist. It is intriguing that (light-) wedge variables and, hence, the wedge-local case seem to be preferred.

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I. INTRODUCTION AND EMBEDDING

A. Preliminaries

Usually, a student of physics meets conjugate pairs first in the context of classical mechanics. Generalized coordinates $\{q_k\}_{k=1}^{k=n}$ serve together with generalized momenta $\{p_j\}_{j=1}^{j=n}$ as the constitutive elements of Poisson brackets,

$$\{F,G\} = \sum_{j,k}\left(\frac{\partial F}{\partial p^j}\frac{\partial G}{\partial q^k} - \frac{\partial G}{\partial p^j}\frac{\partial F}{\partial q^k}\right), \tag{1}$$

for F, G being functions of p, q—called observables. The p's and q's span the phase space, and the Poisson brackets define a symplectic structure. Inserting for F, G the momenta and coordinates themselves, one obtains

$$\{p_j, q_k\} = \delta_{jk}, \tag{2}$$

with δ being Kroneckers δ. The (Hamiltonian) equations of motion read

$$\frac{\partial H(p,q)}{\partial p^k} = \dot{q}_k \qquad \frac{\partial H(p,q)}{\partial q^k} = -\dot{p}_k, \tag{3}$$

with H being the Hamiltonian of the system. The equation of motion for a general observable $O = O(p,q,t)$ which may explicitly depend on time is given by the Poisson bracket

$$\frac{dO}{dt} = \frac{\partial O}{\partial t} + \{H, O\}. \tag{4}$$

The equations (3) become a case of (4) for $O = p_k$ and $O = q_k$. They are also known as *canonical* equations of motion and transformations $P{=}P(p,q), Q{=}Q(p,q)$, which leave them form invariant, are called canonical. It is one of the beautiful results of classical mechanics that the actual motion of a system in time, i.e., the solutions of (3) $p_k(t), q_j(t)$, can be understood as a canonical transformation which transports initial data $p_k(t_0), q_j(t_0)$ to the actual ones at time t.

It is to be noted that time appears rather as a kind of "external" label than as a coordinate. One may, however, incorporate it as $n+1$th coordinate and define -H as its conjugate momentum [1,2].

In relativistic point particle mechanics, time becomes part of the coordinates $x_\mu^{(j)}$ and may be reintroduced as *eigentime* $\tau^{(j)}$, serving then as an *invariant* for the labeling purpose along world lines for the jth particle.

In quantum mechanics, coordinates q and momenta p become Hermitian operators Q and P, acting on the state space of the system which is a Hilbert space. The Poisson brackets go (at least for Cartesian coordinates) over into the commutator, and the equations of motion (in the Heisenberg picture) change accordingly:

$$[P_j, Q_k] = -i\delta_{jk} \qquad i\frac{dO}{dt} = i\frac{\partial O}{\partial t} + [O, H]. \tag{5}$$

It is interesting to observe that the transformation $P \to -Q, Q \to P$ is (like in classical mechanics) a canonical transformation, which implies that if we choose as the "q-representation" square integrable functions f from, say, $\mathbb{R}^{3n} \to \mathbb{C}$ and consider their Fourier transforms (FT), their role will be interchanged by the mentioned canonical transformation. Realizing operators P_j, X_k by the prescription

$$P_j f(x) = -i\frac{\partial}{\partial x^j} f(x) \quad \text{FT} \quad P_j\tilde{f}(p) = p_j\tilde{f}(p) \tag{6}$$

$$X_k f(x) = x_k f(x) \quad \text{FT} \quad X_k\tilde{f}(p) = i\frac{\partial}{\partial p^k}\tilde{f}(p) \tag{7}$$

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(with j,k running from 1 to $3n$), the roles of the operators will change accordingly. The following *relations* stay invariant:

$$[P_j, X_k] = -i\delta_{jk}, \qquad [P_j, P_{j'}] = 0, \qquad [X_k, X_{k'}] = 0. \tag{8}$$

The operators P, Q are conjugate to each other, and the FT indeed realizes the conjugation.

The most intriguing aspect of the operator nature of observables is certainly the discovery by Heisenberg that uncertainty relations hold for observables which do not commute. Most notably, in this context, are conjugate pairs.

Here P_j generates translations in $\mathbb{R}^{3n}(x)$, whereas X_k generates translations in $\mathbb{R}^{3n}(p)$. In quantum mechanics the identification of P_j with $3n$ momentum operators and of X_k with $3n$ position operators is automatic, and the unbounded operators P_j and X_k are essentially self-adjoint. The role of the Hamiltonian and an associated time operator is, however, special: the Hamiltonian is bounded from below, whereas a time operator has to extend over the whole real line; hence, a tentative time operator cannot be self-adjoint. This is known as Pauli's theorem [3] and precludes any naive extension to the relativistic situation.

B. Embedding our approach

The literature on position and time operators in quantum mechanics, relativistic quantum mechanics, and quantum field theory (QFT) is overwhelmingly rich—for a very good reason: the respective notions are fundamental. We will not attempt to review it. Instead, we quote only a few papers with which our results may have a closer relation. We regret all omissions.

The impact of Poincaré invariance on the notion of localizability in quantum theory has been analyzed in [4]. Under plausible assumptions on the set of states associated with localization at a *point* in three-dimensional space, the authors arrive at the definition of a position operator $x^{op} = i\nabla_{\mathbf{p}} - i\mathbf{p}/(2(\mathbf{p}^2 + m^2))$ acting on one-particle solutions of the Klein-Gordon equation to mass m. Thus, spatial localization at a point is not a Lorentz-covariant concept.

In [5], the reference to a point in space has been weakened to a *finite region* in space, again quite plausible from a conceptual point of view. The group theoretic analysis leads to the theorem that all Lorentz invariant systems of $m^2 > 0$ are localizable and their position variables are unique if the systems are elementary. For $m = 0$, the only localizable elementary system has spin zero.

The next level of sophistication has been reached by *local quantum physics* in the spirit of [6]. Over finite regions in spacetime, one defines nets of algebras of observables, studies their representations, and deduces their properties. A recent review of localization based on these notions has been provided in [7]. Quantum fields may or may not be used in this context. It turns out that particle states can never be created by operators strictly localized in bounded regions of spacetime. Our findings below better be in accordance with such general statements. After this look into spatial localizability, we should have a glance at the construction of time operators.

Notable early papers are [2,8,9]. In analogy to classical mechanics, a time operator has been introduced and discussed within ordinary quantum mechanics. It has been admitted as a Hermitian but not self-adjoint operator. A wealth of further literature has been provided in [10]. On the more abstract level, time operators are understood as positive-operator-valued measures [11–14] or affiliated to C^*-algebras [15]. A very recent review within the general context of quantum spacetime, general relativity, and even cosmology has been given in [16].

For our own considerations, the reference to the role of the conformal group is quite important. In [17,18], the charges of the special conformal transformations have become candidates for a relativistic four-position operator. From a different point of view, this has also been studied in [19–21] (also subsection IV C). We will discuss [22] in detail below. Eventually, one has to consider the covering group $SU(2,2)$, which is out of reach for the time being. In [23], the simpler case of $SU(1,1)$ has been successfully treated and provides time observables with projective covariance. Presumably, it is this research where one should find the connection with our treatment of the problem.

Our intention is to understand relativistic position operators as part of a theory which otherwise has been already constructed. Since models and their dynamics which are amenable to experimental tests rely even today mainly on perturbation theory, the most important Hilbert space for particle physics is Fock space and its imbedding into systems of Green functions as off-shell continuation. Available to us are conserved currents, their associated charges, and composite operators formed as functions of the basic quantum fields. Hence, the most useful tools are invariance groups and, to some extent, geometrical quantities. Since, in flat spacetime, Poincaré invariance is relevant, the energy-momentum operator P participates in the game, and a conjugate partner Q is a natural candidate for a position operator.

If one can dispose over conjugate pairs, one may define $Q'_\mu = Q_\mu + \Theta_{\mu\lambda}P^\lambda$ and obtain

$$[Q'_\mu, Q'_\nu] = 2i\Theta_{\mu\nu} \tag{9}$$

(for commuting Q's). This relation is at the basis of some model classes realizing noncommutative coordinates. This may provide additional motivation for studying conjugate pairs in QFT. One may benefit in this context from reading [24].[1] In [25–27], we have seen that it is nontrivial to realize the commutator,

$$[P_\mu, Q_\nu] = i\eta_{\mu\nu}, \tag{10}$$

on Fock space immediately, and that it is easier to study first *preconjugate* pairs P, X which satisfy

$$[P_\mu, Q_\nu] = iN_{\mu\nu}, \tag{11}$$

where $N_{\mu\nu}$ is an operator that can (at least on states) be inverted. In fact, previously we relaxed the diagonality condition expressed in the rhs of (10), which still yields interesting results [28], but in the present paper we will study the full strength of (10) on states in Fock space and its surrounding system of Green functions.

From all preconjugate operators introduced in [28], we will consider here in detail $X(\nabla)$, $X(<)$, and $X(K)$, which are based on the mass shell belonging to four-dimensional Minkowski space, and $X(<_0)$ and $X(K)$ which are based on $(1,1)+(0,2)$-dimensional spacetime. The preconjugate $X(\omega)$ does not lead to Lorentz-covariant Q on (1,3)-dimensional spacetime and is therefore discarded. $X(p\text{-conf})$ turns out to be essentially P and, hence, does not need to be discussed.

Group theoretic considerations in Sec. III serve to recapitulate earlier work [22], to find a place for noncommutative coordinates, and, in particular—via some new interpretation on Fock space—to control our derivations there. The distinguished role played by the special conformal generators as the only preconjugate X's which are local in position space and permit a smooth transition between off shell and on shell was pointed out already in [28]. This explains why, in the group theoretic context, they have been singled out.

In Sec. IV, we discuss our results, offer some conclusions and point out open questions.

II. CONJUGATE OPERATORS IN FOCK SPACE

As mentioned already in the Introduction, we would like to construct operators Q_ν which act in a sense to be specified as conjugate to the energy-momentum operator P_μ of the system:

$$[P_\mu, Q_\nu] = i\eta_{\mu\nu}. \tag{12}$$

On Fock space, the right-hand side of (12) cannot be a multiple of the unit operator,[2] in particular, if Q is chargelike, i.e., annihilates the vacuum, since P does so by general assumptions of QFT. Since we wish to obtain the Q's also from chargelike X's, we have to understand the commutator in a weak sense, namely, applied to states—here, to states of Fock space. The definition of an appropriate Q satisfying

$$[P_\mu, Q_\nu]|\mathbf{p}_1, \dots \mathbf{p}_n\rangle = in\eta_{\mu\nu}|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle, \tag{13}$$

thus, has to be found case by case.

A. From $X(\nabla)$ to $Q(\nabla)$

In [28], we derived the operator

$$X_\nu^{(\nabla)}(a, a^\dagger) = \frac{i}{2}\int \frac{d^3p}{2\omega_p}(a^\dagger(\mathbf{p})\nabla_\nu a(\mathbf{p}) - \nabla_\nu a^\dagger(\mathbf{p})a(\mathbf{p})). \tag{14}$$

Here,

$$\nabla_\nu \equiv \frac{\partial}{\partial p^\nu} - \frac{p_\nu p^\lambda}{m^2}\frac{\partial}{\partial p^\lambda} \quad \text{with} \quad p_0 = \omega_p, \frac{\partial}{\partial p^0} = 0 \quad \text{on shell}. \tag{15}$$

The operator $X^{(\nabla)}$ is chargelike and (formally) Hermitian. It satisfies the algebraic relation

$$[P_\mu, X_\nu^{(\nabla)}] = i\int \frac{d^3p}{2\omega_p}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)a^\dagger(\mathbf{p})a(\mathbf{p}) \tag{16}$$

$$= i\eta_{\mu\nu}N - i\int \frac{d^3p}{2\omega_p}\frac{p_\mu p_\nu}{m^2}a^\dagger(\mathbf{p})a(\mathbf{p}), \tag{17}$$

where P_μ, N denote the energy-momentum and the number operator, respectively,

$$P_\mu = \int \frac{d^3p}{2\omega_p}p_\mu a^\dagger(\mathbf{p})a(\mathbf{p}),$$
$$N = \int \frac{d^3p}{2\omega_p}a^\dagger(\mathbf{p})a(\mathbf{p}). \tag{18}$$

We, therefore, qualified it as an operator *preconjugate* to P on Fock space. $X_\nu^{(\nabla)}$ transforms as a vector under Lorentz,

$$[M_{\mu\nu}, X_\rho^{(\nabla)}] = i(X_\mu^{(\nabla)}\eta_{\nu\rho} - X_\nu^{(\nabla)}\eta_{\mu\rho}), \tag{19}$$

and for the commutator of X's, we found

$$[X_\mu^{(\nabla)}, X_\nu^{(\nabla)}] = -\frac{i}{m^2}M_{\mu\nu}(a^\dagger, a). \tag{20}$$

On n-particle states, $X^{(\nabla)}$ generates

[1] We are grateful to Jochen Zahn for pointing out this reference to us.

[2] K. S. is indebted to Rainer Verch for pointing out the relevance of this fact.

$$iX_\nu^{(\nabla)}|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle = \sum_{k=1}^n\left(\nabla_\nu^{(k)} - \frac{3}{2}\frac{p_\nu^{(k)}}{m^2}\right)|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle. \quad (21)$$

The aim is now to construct an operator $Q_\nu^{(\nabla)}$ such that it satisfies

$$[P_\mu, Q_\nu^{(\nabla)}] = i\eta_{\mu\nu}N \quad (22)$$

on Fock space. Then we shall call this Q conjugate to P.

In order to proceed, we first apply (16) to the vacuum: the result is zero.

This originates from the fact that the operators involved are chargelike and implements the aforementioned projector property of the conjugation equation (12).

Applying (16) to an n-particle state yields

$$[P_\mu, X_\nu^{(\nabla)}]|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle = i\left(n\eta_{\mu\nu} - \sum_{k=1}^n\frac{p_\mu^{(k)}p_\nu^{(k)}}{m^2}\right)|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle. \quad (23)$$

This relation implies further projection content of (22): for $n = 1$, we have

$$[P_\mu, X_\nu^{(\nabla)}]|\mathbf{p}\rangle = i\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)|\mathbf{p}\rangle \quad (24)$$

and obtain zero when contracting with P^μ from the left. On states with $n > 1$, the corresponding result is nonvanishing. Furthermore, applying the commutator from the lhs of (24) to a state $X^\nu(\nabla)|\mathbf{p}\rangle$ and summing over ν, we find

$$[P_\mu, X_\nu^{(\nabla)}]X^\nu(\nabla)|\mathbf{p}\rangle = i\nabla_\mu|\mathbf{p}\rangle. \quad (25)$$

This relation can be read as $[P_\mu, X_\nu]$ being proportional to $i\eta_{\mu\nu}$ on a "nontrivial" state—a state $|\mathbf{p}\rangle$ being multiplied by a nontrivial function of p. This analysis, thus, suggests either to admit only states containing more than one particle or to consider states which are multiplied with nontrivial functions of the momenta. Let us study this latter case first.

1. Inversion on "spin" states

Since, for $n = 1$, the rhs of (24) is precisely the spin sum of a massive vector particle,

$$\sum_{l=1}^3\epsilon_\mu^{(l)}(\mathbf{p})\epsilon_\nu^{(l)}(\mathbf{p}) = -\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right), \quad (26)$$

we are led to introduce one-particle states,

$$|\mathbf{p}, l, \mu\rangle = \epsilon_\mu^{(l)}(\mathbf{p})|\mathbf{p}\rangle, \quad (27)$$

where

$$\epsilon_\rho^{(l)}(\mathbf{p}) = \begin{pmatrix}\frac{p^l}{m}\\ -\delta_\rho^l + \frac{p^l p_\rho}{m(m+\omega_p)}\end{pmatrix} \quad (28)$$

represents three $(l = 1, 2, 3)$ polarization four-vectors, the first line gives the $\rho = 0$ component, and the second line refers to their spatial components $\rho = 1, 2, 3$.[3]

They obey the orthogonality relations:

$$\epsilon_{\rho'}^{(l')}\eta^{\rho'\rho}\epsilon_\rho^{(l)} = \eta^{l'l}. \quad (29)$$

We first find

$$\sum_{l=1}^3\epsilon_\mu^{(l)}(\mathbf{p})\epsilon_\nu^{(l)}(\mathbf{p})\nabla^\nu = -\nabla_\mu, \quad (30)$$

and then

$$i[X^\nu, [i[P_\mu, X_\nu], a^\dagger]] = \nabla_\mu a^\dagger, \quad (31)$$

(recall: $p_0 = \omega_p, \partial/\partial p_0 \equiv 0$); the contribution $(3/2)p_\mu/m^2$ drops out. When applied to the vacuum state, this means that

$$[P_\mu, X_\nu]\eta^{\nu\rho}\epsilon_\rho^{(l)}|\mathbf{p}\rangle = i\epsilon_\mu^{(l)}|\mathbf{p}\rangle \quad (32)$$

i.e., the commutator operates on these states as $i\eta_{\mu\nu}$, which is the desired conjugation relation on one-particle states. [A slightly different way to derive (32) is to start from (23) for $n = 1$, to insert (26) in the rhs, to replace $|\mathbf{p}\rangle$ with $|\mathbf{p}, l, \mu\rangle$, and then to use (29)]. Due to the orthogonality relation (29), the vectors $|\mathbf{p}, l, \mu\rangle$ satisfy

$$\langle\mathbf{p}', l', \rho'|\mathbf{p}, l, \rho\rangle = 2\omega_p\delta(\mathbf{p}' - \mathbf{p})\epsilon_{\rho'}^{(l')}(\mathbf{p})\epsilon_\rho^{(l)}(\mathbf{p}), \quad (33)$$

and, thus, have positive norm if we define their scalar product with the metric $-\eta^{\rho'\rho}$.

An explicit form of operators Q can be obtained as follows. We consider

$$\eta^{\rho\sigma}[X_\rho, \epsilon_\sigma^{(l)}(\mathbf{p})a^\dagger(\mathbf{p})] = -i\eta^{\rho\sigma}\epsilon_\sigma^{(l)}\nabla_\rho a^\dagger(\mathbf{p}) \quad (34)$$

$$[X^\rho, \epsilon_\rho^{(l)}(\mathbf{p})a^\dagger(\mathbf{p})] \doteq -ie^{(l)}a^\dagger(\mathbf{p}) \quad (35)$$

$$e^{(l)} = \left(-\delta_k^l + \frac{p^l p_k}{m(m+\omega_p)}\right)\frac{\partial}{\partial p_k}. \quad (36)$$

[3] After finding (24), recalling (26), and then defining (27), the author (K. S.) understood a remark made to him earlier by Erhard Seiler, that the problem with (12) is analogous to the state space problem in QED.

These equations are all supposed to be applied to the vacuum, where, for the commutator (32) as well, the interchange of the polarization vector with the operators P, X is permitted. Then the operators

$$Q^{(l)}_{(\mathrm{eff})}|\mathbf{p}\rangle = -ie^{(l)}|\mathbf{p}\rangle \tag{37}$$

generate for $\mathbf{p}=0$, i.e. in the rest frame, precisely translations in the momentum $\mathbf{p}$: they are indeed conjugate to P. [We attached "eff" for "effective" because this equality only holds when read in the context of (32).]

For finite, i.e., nonvanishing, $\mathbf{p}$, we use the fact that the polarization vectors can be extended and then composed to form a matrix L with inverse L^{-1},

$$(L(p))^{\rho}_{\sigma} = \begin{pmatrix} \frac{\omega_p}{m} & -\frac{p^j}{m} \\ \frac{p_i}{m} & \delta^j_i - \frac{p_i p^j}{m(m+\omega_p)} \end{pmatrix} \quad \text{and}$$

$$(L^{-1}(p))_{\sigma}{}^{\rho} = \begin{pmatrix} \frac{\omega_p}{m} & \frac{p^j}{m} \\ \frac{-p_i}{m} & \delta^j_i - \frac{p_i p^j}{m(m+\omega_p)} \end{pmatrix}, \tag{38}$$

where L is the boost, mapping the 4-vector $(m,0,0,0)^T$ into the 4-vector $(\omega_p, p_1, p_2, p_3)^T$, and L^{-1} transforms the derivatives:

$$(L^{-1}(p))^{\rho}_{\sigma}\frac{\partial}{\partial p^{\rho}} = \begin{pmatrix} \frac{\omega_p}{m}\partial_0 + \frac{p^j}{m}\partial_j \\ \frac{-p_i}{m}\partial_0 + \delta^j_i\partial_j - \frac{p_i p^j}{m(m+\omega_p)}\partial_j \end{pmatrix}. \tag{39}$$

Since, in the present context, $\partial_0 \equiv 0$, we see first of all that the contraction of ϵ with ∇ results in the differential operators e in (37). We may then go a step further and use the fact that the first column of L in (38) represents a fourth four-vector $\epsilon^{(0)}_{\rho}$ (timelike) which permits the following definition:

$$Q^{\lambda}_{(\mathrm{eff})}|\mathbf{p}\rangle = X_{\nu}\eta^{\nu\rho}\epsilon^{(\lambda)}_{\rho}|\mathbf{p}\rangle \tag{40}$$

$$= X_{\nu}\eta^{\nu\rho}(-L^{\lambda}_{\rho})|\mathbf{p}\rangle \tag{41}$$

$$= -(L^{-1})^{\lambda\nu}X_{\nu}|\mathbf{p}\rangle \tag{42}$$

$$Q^{(\mathrm{eff})}_{\lambda}|\mathbf{p}\rangle = -i(L^{-1})_{\lambda}{}^{\nu}\nabla_{\nu}|\mathbf{p}\rangle. \tag{43}$$

(We have suppressed the contribution $\frac{3}{2}\frac{p_\nu}{m^2}$ within $X_\nu|\mathbf{p}\rangle$ since it does not contribute eventually in the commutator $[P,X]$.)

Comparing with (39), we see that there we only have to replace the ordinary by the tangential derivative to find the result:

$$Q^{(\mathrm{eff})}_0|\mathbf{p}\rangle = 0 \tag{44}$$

$$Q^{(\mathrm{eff})}_j|\mathbf{p}\rangle = i\left(\frac{\partial}{\partial p^j} - \frac{p_j p^l \partial_l}{m(m+\omega_p)}\right)|\mathbf{p}\rangle. \tag{45}$$

For the commutators with P_μ, this implies

$$[P_\mu, Q^{(\mathrm{eff})}_0]|\mathbf{p}\rangle = 0 \tag{46}$$

$$[P_\mu, Q^{(\mathrm{eff})}_l]|\mathbf{p}\rangle = i\epsilon^{(l)}_{\mu}|\mathbf{p}\rangle = -iL^l_\mu|\mathbf{p}\rangle. \tag{47}$$

If we define

$$P^{(\mathrm{eff})}_j = (L^{-1})^{\mu}_j P_\mu, \tag{48}$$

we obtain finally

$$[P_\mu, Q^{(\mathrm{eff})}_0] = [P^{(\mathrm{eff})}_\mu, Q^{(\mathrm{eff})}_0] = 0 \tag{49}$$

$$[P^{(\mathrm{eff})}_\mu, Q^{(\mathrm{eff})}_l]|\mathbf{p}\rangle = i\eta_{\mu l}|\mathbf{p}\rangle. \tag{50}$$

As for the interpretation, we may paraphrase the result as follows: in the rest frame, the polarization vectors are unit vectors and the X's coincide with the Q's. As can be seen from (15), at $\mathbf{p}=0 \to \nabla^0 = 0$ in accordance with geometry: at $\mathbf{p}=0$ the tangential plane is orthogonal to the p^0 axis; hence, no tangential motion into that direction can be generated by an infinitesimal change of $\mathbf{p}$. This implies $X_0 = Q_0 = 0$. At $\mathbf{p}=0$, the spatial X's are conjugate to the spatial P's. For finite $\mathbf{p}$, and with the help of polarization vectors, we may define states with "spin" and introduce Q's which evolve with the inverse of these polarization vectors such that $Q_0 = 0$ still and the commutators with the P's become polarization vectors, which can then be absorbed into new P's which are also just the evolved one's for P. In this way, the whole system remains Lorentz covariant. The obvious analogue to this (from which the idea of introducing polarization vectors has been suggested) is the quantization of a free, massive, Abelian vector field [29,30]. There, as in the present case, a structure in three-dimensional space is compatible with Lorentz covariance in four-dimensional spacetime with a correctly performed embedding: the time component is a well-determined function of the space components.

Are the states $\epsilon|\mathbf{p}\rangle$ asymptotic ones? Naively, the answer is "yes," since only on-shell momenta enter in their definition. In x space, the polarization vectors represent nonlocal differential operators, which can be seen, e.g., when acting on a scalar field. So, this may very well be an explicit realization of the general results reported in [7].

Actually, the operators $X^{(\nabla)}(a^\dagger, a)$ are already nonlocal when expressed in terms of the free scalar field, in marked contrast to the conformal case, discussed below, since there $X=K$ and the K's are *local* charges in x space.[4]

[4] $X(\nabla)$ represents the geometrical notion "tangential derivative ∇" in Hilbert space, whereas K represents the invariance of $p^2=0$ there.

We still have to check how the commutator (20) translates itself to the Q's. It turns out that, due to the presence of the polarization vectors, this commutator does not vanish. When searching for noncommutative coordinates, one may thus rely on preconjugate pairs [28], introduce Θ's like in (9), or employ the operators $Q^{(\mathrm{eff})}$ (44). We hope to come back to this question in the near future.

For $n \geq 2$, one has to construct three four-vectors which are totally symmetric in the n momenta, vanish when contracted with any one of them and are reproduced by contraction with the transverse projector in the rhs of (26). We do not pursue this construction any further, since it is essentially provided by going over to the helicity basis as used for scattering amplitudes.

2. Inversion on standard states

We now wish to invert (16) on ordinary n-particle Fock states in order to obtain effectively (12) on states.

By explicit calculation, we find

$$[P_\mu, X_\nu^{(\nabla)}]|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = i \sum_{k=1}^{n} \left(\eta_{\mu\nu} - \frac{p_\mu^{(k)} p_\nu^{(k)}}{m^2}\right)|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle, \tag{51}$$

and the question is whether the 4×4-matrix (in the indices μ, ν) is invertible. As noted above, this is not the case for $n = 1$, since P^μ projects to zero. For $n = 2$, one checks in the center-of-mass system $\mathbf{p} \equiv \mathbf{p}_1 = -\mathbf{p}_2$ that the determinant results in

$$\det(\mathrm{r.h.s.}) = -\frac{16}{m^4}\mathbf{p}^2\omega_p^2 \neq 0. \tag{52}$$

Hence, this matrix can be inverted, with the inverse applied from the right and attributed as a factor to X, which thereby becomes a Q. (The momentum $\mathbf{p} = 0$ is an unphysical point).

Since, for n larger than two, the kinematical configuration cannot become worse, we conclude that the inversion is possible for all $n \geq 2$.

Let us now discuss the case $n = 2$ in more detail. Equation (51) reads

$$[P_\mu, X_\nu^{(\nabla)}]|\mathbf{p}_1, \mathbf{p}_2\rangle = 2iN_{\mu\nu}|\mathbf{p}_1, \mathbf{p}_2\rangle \tag{53}$$

$$\text{with} \quad N_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{p_\mu^{(1)} p_\nu^{(1)}}{2m^2} - \frac{p_\mu^{(2)} p_\nu^{(2)}}{2m^2}\right). \tag{54}$$

In the center-of-mass system and after rotating to zero the y and z components of $\mathbf{p}$, the matrix $N_{\mu\nu}$ is diagonal

$$N_{\mu\nu} = -\begin{pmatrix} \frac{p_x^2}{m^2} & 0 & 0 & 0 \\ 0 & \frac{2(m^2+p_x^2)}{m^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}. \tag{55}$$

Multiplying (53) with the inverse of N,

$$(N^{-1})^{\nu\rho} = \begin{pmatrix} -\frac{m^2}{p_x^2} & 0 & 0 & 0 \\ 0 & \frac{m^2}{2(m^2+p_x^2)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\nu\rho}, \tag{56}$$

we arrive at the conjugation equation in the form

$$[P_\mu, Q_\nu]|\mathbf{p}, -\mathbf{p}\rangle = 2i\eta_{\mu\nu}|\mathbf{p}, -\mathbf{p}\rangle \tag{57}$$

$$Q_0 = -\frac{m^2}{p_x^2}X_0 \qquad Q_1 = \frac{m^2}{2(m^2 + p_x^2)}X_1 \tag{58}$$

$$Q_2 = X_2 \qquad Q_3 = X_3. \tag{59}$$

Like in the preceding subsection, we have to check now the norms of the states created by the commutator $[P_\mu, Q_\nu]$. The one generated by $[P_0, Q_0]$ is opposite to the one generated by the spatial components $[P_j, Q_j]$ ($j = 1, 2, 3$ no sum). Hence, we face the same problem as in gauge theories: the scalar component $\partial_\mu A^\mu$ of the vector field creates states with negative norm. Thus, we try to remedy it by the same means: we impose a Gupta-Bleuler condition on the allowed *states*, thereby characterizing them as physical ones. Combining the contribution from the (0,0) component with that of the (1,1) component and requiring that the sum vanishes, we find

$$\left(-\frac{2m^2}{p_x^2}\alpha_0 + \frac{m^2}{m^2 + p_x^2}\alpha_1\right)|\mathbf{p}, -\mathbf{p}\rangle = 0. \tag{60}$$

(Here the α's are real numbers.) This equation has no solution identically in $\mathbf{p}$. However, in the massless limit, such a solution exists with $\alpha_1 = 2\alpha_0$.

We conclude from this result that, in the massive case, such an inversion procedure is not consistent. Only the construction of the preceding subsection seems to be applicable. Let us have a look at the massless limit. Obviously, $Q_0 = Q_1 = 0$. This tells us that only the spatial components Q_2 and Q_3 exist and are conjugate to P_2, P_3, respectively. Effectively, the measurable quantities are these spatial ones. Hence, this solution is not *manifestly* Lorentz covariant but, nevertheless, covariant in the sense of the transition from $\{P, Q\}$ to $\{P_{\mathrm{eff}}, Q_{\mathrm{eff}}\}$ above and the case of $Q(K)$ treated below in Sec. II C.

B. From $X(<_0)$ to $Q(<_0)$

In [28], we introduced wedge variables ("<" for "wedge"):

$$\text{in } p \text{ space}\quad p_u = \frac{1}{\sqrt{2}}(p_0 - p_1) \qquad p_0 = \frac{1}{\sqrt{2}}(p_v + p_u) \tag{61}$$

$$p_v = \frac{1}{\sqrt{2}}(p_0 + p_1) \qquad p_1 = \frac{1}{\sqrt{2}}(p_v - p_u) \tag{62}$$

$$\text{in } x \text{ space}\quad u = \frac{1}{\sqrt{2}}(x^0 - x^1) \qquad x^0 = \frac{1}{\sqrt{2}}(v + u) \tag{63}$$

$$v = \frac{1}{\sqrt{2}}(x^0 + x^1) \qquad x^1 = \frac{1}{\sqrt{2}}(v - u). \tag{64}$$

Note that $p^u = p_v, p^v = p_u$. The mass shell condition is given by

$$2p_u p_v - p_a p_a = m^2 \qquad a = 2, 3 \qquad \text{summation over a.} \tag{65}$$

We then constructed differential operators $\nabla(<)_\mu$, with $\mu = u, v, 2, 3$ acting on one-particle wave functions. Here one can admit $p_0 = \pm\omega_p$ and $\omega_p \equiv \sqrt{m^2 + \mathbf{p}^2}$, and hence *both* shells of the hyperboloid $p^2 = m^2$ are covered. When aiming at operators $X(a, a^\dagger)$ for realizing these differential operators on Fock states, one has to introduce new creation and annihilation operators since the standard ones are based on $p_0 = +\omega_p$.

One can proceed as follows [31]. A scalar field satisfying the Klein-Gordon equation is being introduced as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{2p_v} e^{-i\bar{p}x} A(\mathbf{p}), \tag{66}$$

with $d^3 p \equiv dp_v dp_2 dp_3$, $\mathbf{p} = (p_v, p_a), \bar{p}_u = (m^2 + p_a p_a)/(2p_v), \bar{p}_v = p_v$, and $\bar{p}_a = p_a a = 2, 3$.

The reality of ϕ implies

$$A^\dagger(\mathbf{p}) = -A(-\mathbf{p}). \tag{67}$$

One can invert (66),

$$A(\mathbf{p}) = \frac{1}{(2\pi)^{2/3}} \int d^3 x 2p_v e^{i\bar{p}x} \phi(x). \tag{68}$$

The field is quantized by imposing

$$[A(\mathbf{p}), A(\mathbf{p}')] = 2p_v \delta^3(\mathbf{p} + \mathbf{p}') \tag{69}$$

$$A(\mathbf{p})|0\rangle = 0 \quad \text{for } p_v < 0$$

$$\langle 0|A(\mathbf{p}) = 0 \quad \text{for } p_v > 0. \tag{70}$$

Below we shall need this definition of Fock states because it will serve to clarify the relations amongst the different Q's which we study. For the purposes of the present discussion, we work, however, with the differential operators for which the respective modifications are essentially trivial.

We treat here $\nabla(<_0)$ and discuss, in terms of it, the properties of $X(<_0)$ and $Q(<_0)$). We found the following in [28]:

$$\nabla^u = \frac{1}{2}\left(\frac{\partial}{\partial p_u} - \frac{1}{p_u} p_v \frac{\partial}{\partial p_v}\right)$$

$$\nabla^v = \frac{1}{2}\left(\frac{\partial}{\partial p_v} - \frac{1}{p_v} p_u \frac{\partial}{\partial p_u}\right) \tag{71}$$

$$\nabla^2 = \frac{\partial}{\partial p_2} - \frac{p^2}{p_a p^a} p_b \frac{\partial}{\partial p_b}$$

$$\nabla^3 = \frac{\partial}{\partial p_3} - \frac{p^3}{p_a p^a} p_b \frac{\partial}{\partial p_b}. \tag{72}$$

These differential operators satisfy the algebra:

$$[\nabla^u, \nabla^v] = \frac{-1}{2p_u p_v}\left(p_u \frac{\partial}{\partial p^v} - p_v \frac{\partial}{\partial p^u}\right)$$

$$= \frac{1}{p_a p^a}\left(p_u \frac{\partial}{\partial p^v} - p_v \frac{\partial}{\partial p^u}\right) \tag{73}$$

$$[\nabla^u, \nabla^2] = [\nabla^u, \nabla^3] = [\nabla^v, \nabla^2] = [\nabla^v, \nabla^3] = 0 \tag{74}$$

$$[\nabla^2, \nabla^3] = -\frac{1}{p_a p^a}\left(p^2 \frac{\partial}{\partial p_3} - p^3 \frac{\partial}{\partial p_2}\right)$$

$$= \frac{1}{2p_u p_v}\left(p^2 \frac{\partial}{\partial p_3} - p^3 \frac{\partial}{\partial p_2}\right). \tag{75}$$

They, furthermore, obey projection properties:

$$p_u \nabla^u + p_v \nabla^v = 0 p_2 \qquad \nabla^2 + p_3 \nabla^3 = 0. \tag{76}$$

We defined accordingly the operators $X(<_0)$ acting on functions $\tilde{f}(p_u, p_v, p_2 p_3)$ as differential operators by

$$X^u(<_0) = i\nabla^u \qquad X^2(<_0) = i\nabla^2 \tag{77}$$

$$X^v(<_0) = i\nabla^v \qquad X^3(<_0) = i\nabla^3. \tag{78}$$

Their algebra is given by

$$[X^u(<_0), X^v(<_0)] = i \frac{1}{P_a P^a} M^{uv} \tag{79}$$

$$[X^2(<_0), X^3(<_0)] = i \frac{1}{2P_u P_v} M^{23} \tag{80}$$

$$[X^u(<_0), X^2(<_0)] = [X^u(<_0), X^3(<_0)] = [X^v(<_0), X^2(<_0)]$$

$$= [X^v(<_0), X^3(<_0)] = 0. \tag{81}$$

Their commutation relations with the energy-momentum operator P read

$$[P_\alpha, X_\beta(<_0)] = \frac{i}{2}\begin{pmatrix} -\frac{P_u}{P_v} & 1 \\ 1 & -\frac{P_v}{P_u} \end{pmatrix}_{\alpha\beta} \qquad \alpha, \beta = u, v \tag{82}$$

$$[P_a, X_b(<_0)] = -i\begin{pmatrix} 1+\frac{P_2P_2}{P_bp^b} & \frac{-P_2P_3}{P_bP^b} \\ \frac{-P_3P_2}{P_bP^b} & 1+\frac{P_3P_3}{P_bP^b} \end{pmatrix}_{ab} \qquad a, b = 2, 3. \tag{83}$$

The main implication of this structure is the loss of symmetry: from the original $SO(1,3)$ invariance, only $SO(1,1)\times SO(2)$ survived. The remaining generators do not exist in the limit of vanishing mass and have thus to be excluded from participation. This is to be compared with the limit $m^2 = 0$ taken at the end of the preceding subsection: there, no boost survived—the limit was effectively nonrelativistic, although Lorentz covariance was not lost.

Some more information from this limit process will be useful later on. Using the transformation equations (61), we find that

$$\text{for } p_0 = +p_1 > 0 \qquad \nabla^u \text{ does not exist} \tag{84}$$

$$\nabla^v = \frac{1}{2\sqrt{2}}(\partial_0 - \partial_1) \tag{85}$$

$$\text{for } p_0 = -p_1 > 0 \qquad \nabla^v \quad \text{does not exist} \tag{86}$$

$$\nabla^u = \frac{1}{2\sqrt{2}}(\partial_0 + \partial_1). \tag{87}$$

1. The $SO(2)$ sector

In close analogy to the massive case, we try to realize the conjugation structure on states multiplied by polarization vectors. We choose

$$\epsilon_a^{(2)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} |\mathbf{p}|\cos\alpha \\ |\mathbf{p}|\sin\alpha \end{pmatrix} \qquad \epsilon_a^{(3)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} -|\mathbf{p}|\sin\alpha \\ |\mathbf{p}|\cos\alpha \end{pmatrix}$$
$$a = 2, 3 \qquad |\mathbf{p}| \equiv \sqrt{p_2^2 + p_3^2}. \tag{88}$$

An equivalent form is

$$\epsilon_a^{(2)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} \qquad \epsilon_a^{(3)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} -p_3 \\ p_2 \end{pmatrix} \qquad a = 2, 3, \tag{89}$$

with the obvious identification $p_2 = |\mathbf{p}|\cos\alpha, p_3 = |\mathbf{p}|\sin\alpha$. They are spacelike unit vectors,

$$\epsilon_a^{(2)}\eta^{ab}\epsilon_b^{(2)} = \epsilon_a^{(3)}\eta^{ab}\epsilon_b^{(3)} = -1, \tag{90}$$

and satisfy the completeness relation:

$$\epsilon_a^{(2)}\epsilon_b^{(2)} + \epsilon_a^{(3)}\epsilon_b^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\eta_{ab}. \tag{91}$$

The right-hand side of (82) indeed is then equal to $-i\sum_c^{2,3}\epsilon_a^{(c)}\epsilon_b^{(c)}$, and we may expect that

$$X_b\eta^{bc}\epsilon_c^{(r)}|\mathbf{p}\rangle = \eta^{bc}\epsilon_c^{(r)}i\nabla_b|\mathbf{p}\rangle \tag{92}$$

gives rise to an effective conjugate:

$$Q_{\text{eff}}^{(r)}|\mathbf{p}\rangle = i\eta^{bc}\epsilon_c^{(r)}\nabla_b|\mathbf{p}\rangle. \tag{93}$$

The explicit calculation leads to

$$Q_{\text{eff}}^{(2)}|\mathbf{p}\rangle = 0 \tag{94}$$

$$Q_{\text{eff}}^{(3)}|\mathbf{p}\rangle = \frac{i}{|\mathbf{p}|}\left(-p_3\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial p_3}\right)|\mathbf{p}\rangle. \tag{95}$$

For the effective commutator with P, this implies

$$[P_2, Q_{\text{eff}}^{(3)}]|\mathbf{p}\rangle = -i\sin\alpha|\mathbf{p}\rangle \tag{96}$$

$$[P_3, Q_{\text{eff}}^{(3)}]|\mathbf{p}\rangle = i\cos\alpha|\mathbf{p}\rangle. \tag{97}$$

Therefore, the system has *one* independent conjugate pair which corresponds to the fact that the commutator matrix (82) has a vanishing determinant which in turn originates from the projector property (76).

The normalization properties (90) tell us that the states $\epsilon(r)|\mathbf{p}\rangle r = 2, 3$ have positive norm, if we introduce the metric $\eta_{rs} r, s = 2, 3$ in this state space.

2. The $SO(1,1)$ sector

Similarly to the choice of polar variables in the previous subsection, it turns out that, in the present sector, hyperbolic variables are most suitable. We introduce

$$p_u = \frac{c}{\sqrt{2}}(\cosh\phi - \sinh\phi) = \frac{c}{\sqrt{2}}e^{-\phi}$$
$$c = \sqrt{2p_up_v} = \sqrt{p_ap_a} \quad \text{sum} \quad a = 2, 3 \tag{98}$$

$$p_v = \frac{c}{\sqrt{2}}(\cosh\phi + \sinh\phi) = \frac{c}{\sqrt{2}}e^{+\phi}$$
$$\phi = -\frac{1}{2}\ln\frac{p_u}{p_v} = \frac{1}{2}\ln\frac{p_v}{p_u}. \tag{99}$$

(Note: since $p_up_v > 0$ always, the functions involved are well defined).

The commutator matrix (82) assumes the following form:

$$[P_\alpha, X_\beta(<_0)] = \frac{i}{2}\begin{pmatrix} -\frac{p_u}{p_v} & 1 \\ 1 & -\frac{p_v}{p_u} \end{pmatrix}_{\alpha\beta} = \frac{i}{2}\begin{pmatrix} -e^{-2\phi} & 1 \\ 1 & -e^{2\phi} \end{pmatrix}_{\alpha\beta}. \quad (100)$$

The tangential derivatives ∇^u, ∇^v applied to a one-particle state,

$$|\mathbf{p}\rangle = |p_u, p_v; p_2, p_3\rangle_{|p_u=\frac{p_a p_a}{2p_v}} \equiv |\cdots\rangle, \quad (101)$$

become

$$\nabla^u|\cdots\rangle = -\frac{1}{\sqrt{2}}\frac{e^{\phi}}{c}\frac{\partial}{\partial\phi}|\cdots\rangle$$
$$\nabla^v|\cdots\rangle = \frac{1}{\sqrt{2}}\frac{e^{-\phi}}{c}\frac{\partial}{\partial\phi}|\cdots\rangle. \quad (102)$$

Geometrically interpreted, this means that they generate motions on the hyperbolas $p_u = p_u(\phi)$ and $p_v = p_v(\phi)$ for fixed $c = \sqrt{p_a p_a}$. Their projection properties (76) are, of course, maintained.

We now introduce polarization vectors,

$$\epsilon_\alpha^{(u)} = \frac{N_u}{\sqrt{2}}\begin{pmatrix} -\frac{p_u}{p_v} \\ 1 \end{pmatrix} \qquad \epsilon_\alpha^{(v)} = \frac{N_v}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{p_v}{p_u} \end{pmatrix}, \quad (103)$$

where N_u, N_v are arbitrary normalization factors, then calculate their normalization,

$$\epsilon_\gamma^{(\sigma)}\eta^{\gamma\beta}\epsilon_\beta^{(\tau)} = \begin{cases} -\frac{p_u}{p_v}N_u^2 & \text{for } \sigma = u, \tau = u \\ N_u N_v & \text{for } \sigma = u, \tau = v \\ N_u N_v & \text{for } \sigma = v, \tau = u \\ -\frac{p_v}{p_u}N_v^2 & \text{for } \sigma = v, \tau = v \end{cases}, \quad (104)$$

and their completeness relation:

$$\sum_\tau \epsilon_\alpha^{(\tau)}\epsilon_\beta^{(\tau)} = \frac{N_u^2 p_u^2 + N_v^2 p_v^2}{2p_u p_v}\begin{pmatrix} \frac{p_u}{p_v} & -1 \\ -1 & \frac{p_v}{p_u} \end{pmatrix}_{\alpha\beta}. \quad (105)$$

For the commutator (100), we can therefore write

$$[P_\alpha, X(<_0)_\beta]|\cdots\rangle = i\sum_\tau \epsilon_\alpha^{(\tau)}\epsilon_\beta^{(\tau)}\frac{p_u p_v}{N_u^2 p_u^2 + N_v^2 p_v^2}|\cdots\rangle. \quad (106)$$

Applying this commutator to the state $\eta^{\beta\gamma}\epsilon_\gamma^{(\sigma)}|\cdots\rangle$, we find by explicit calculation the expected result, namely,

$$[P_\alpha, X(<_0)_\beta]\eta^{\beta\gamma}\epsilon_\gamma^{(\sigma)}|\cdots\rangle = -i\delta_\alpha^\gamma\epsilon_\gamma^{(\sigma)}|\cdots\rangle, \quad (107)$$

i.e. the lhs acts as a $[P_\alpha, Q_{\text{eff}}^\gamma]$ on these states, with

$$Q_{\text{eff}}^u = -i\frac{N_u}{\sqrt{2}}\nabla^v \qquad Q_{\text{eff}}^v = -i\frac{N_v}{\sqrt{2}}\nabla^u. \quad (108)$$

Hence, we have two pairs of conjugate operators. This is due to the fact that the singularity for vanishing p_u, p_v prohibits the transition from the upper part of the hyperboloid to the lower one (and vice versa) and that the respective reflection is not in $SO(1,1)$. If we now choose $N_u = N_v \equiv N$ which is possible (e.g. with $N = \sqrt{\frac{p_u^2+p_v^2}{2p_u p_v}}$), then the Q_{eff}^σ operate just like a rescaled X^σ (although on different states), and hence transform as a vector under $SO(1,1)$ and have a nontrivial commutator,

$$[Q_{\text{eff}}^\sigma, Q_{\text{eff}}^\tau]|\cdots\rangle = -\frac{1}{p_a p^a}\left(p^\sigma\frac{\partial}{\partial p_\tau} - p^\tau\frac{\partial}{\partial p_\sigma}\right) \equiv \frac{i}{P_a P^a}M^{\sigma\tau}|\cdots\rangle, \quad (109)$$

with M being the generator of $SO(1,1)$. From ∇^u, ∇^v, they inherit on the states $|\cdots\rangle$ the functional dependence

$$(p_u Q_{\text{eff}}^u + p_v Q_{\text{eff}}^v)|\cdots\rangle = 0. \quad (110)$$

Reading Eqs. (108) and (109) in terms of the hyperbolic variables (98) and (102), we have a perfect analogy to the purely spatial sector with its covariance under the compact group $SO(2)$.

The norms of the states $\epsilon_\alpha^{(\sigma)}|p_u, p_v; p_2, p_3\rangle_{|p_u=\frac{p_a p_a}{2p_v}}$, $\sigma \in \{u, v\}$ can be read off from (104) for $N_u = N_v \equiv N$ and are positive definite for $\sigma = u$, v, respectively,

$$\langle\mathbf{q}|\epsilon_\alpha^{(\sigma)}(\mathbf{q})\eta^{\alpha\beta}\epsilon_\beta^{(\sigma)}(\mathbf{p})|\mathbf{p}\rangle = N^2\delta^{(2)}(q-p)\delta(p_v+q_v) \times \langle q_v; q_2, q_3|p_v; p_2, p_3\rangle \times \begin{cases} \frac{p_u}{p_v} & \text{for } \sigma = u \\ \frac{p_v}{p_u} & \text{for } \sigma = v. \end{cases} \quad (111)$$

If we introduce the metric $\eta^{\alpha\beta}$, we have a positive definite norm for these states, maintaining covariance.

The extension from the one-particle situation to n particles by tensoring deserves further study: the introduction of relative momenta and separation of the center-of-mass Hamiltonian as it has been studied in light-cone quantization (see [32] for a comprehensive review) should be complemented by the analogous treatment of the q variables and could yield quite interesting results.

3. From $X(<)$ to $Q(<)$

Having discussed the massless case $<_0$ and seeing no obvious reason why the extension to the massive case should not work, we establish now the analogous structure there. For the case $<_0$, the relevant spacetime was $(1,1)\times(0,2)$, with the symmetry $SO(1,1)\times SO(2)$. In the massive case, one can also realize this symmetry manifestly, discuss the construction of Q's and state space associated with it, and thereafter implement the symmetry generators missing in the complete $SO(1,3)$.

This can be seen as follows. Again, we base our analysis on the differential operators and not on the Fock space expressions, since the difference between the two versions can safely be expected to be a contribution proportional to P_μ, hence not contributing to the commutator $[P,Q]$.

In [28] we found differential operators $\nabla(<)$ tangential to the mass shell $2p_u p_v = m^2 + p_a p_a$,

$$\nabla^u = \frac{\partial}{\partial p_u} - \frac{p_v}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda}$$

$$\nabla^2 = \frac{\partial}{\partial p_2} + \frac{p^2}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda} \tag{112}$$

$$\nabla^v = \frac{\partial}{\partial p_v} - \frac{p_u}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda}$$

$$\nabla^3 = \frac{\partial}{\partial p_3} + \frac{p^3}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda}, \tag{113}$$

which gave rise to operators

$$X(<) = i\nabla(<), \tag{114}$$

with commutator

$$[P_\mu, X_\nu^{(<)}]f(p) = i\left(\bar{\eta}_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)f(p). \tag{115}$$

Here the indices μ, ν run over the ranges $\{u, v, 2, 3\}$, and the metric $\bar{\eta}$ reads

$$\bar{\eta}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}. \tag{116}$$

The functions f stand for eigenfuctions of the energy-momentum operator in terms of the wedge variables p, thus permitting the transition to the mass shell accordingly.

It is now crucial to observe that, in the massive case, a partial rest system with $p_2 = p_3 = 0$ exists in which the $\{2,3\}$ sector is diagonal, whereas the $\{u, v\}$ sector assumes the form

$$[P_\alpha, X_\beta^{(<)}]f(p) = -i\begin{pmatrix} \frac{p_u^2}{m^2} & -1 + \frac{p_u p_v}{m^2} \\ -1 \frac{p_v p_u}{m^2} & \frac{p_v^2}{m^2} \end{pmatrix}_{\alpha\beta} f(p)$$

$$= \frac{i}{2}\begin{pmatrix} -\frac{p_u}{p_v} & 1 \\ 1 & -\frac{p_v}{p_u} \end{pmatrix}_{\alpha\beta} f(p). \tag{117}$$

The second part of the equation follows by use of the mass shell condition at $p_a p_a = 0$, but this is precisely Eq. (100). Hence, with $c = m$, (98), we have precisely the same solution. Using the polarization vectors of that case, we conclude that there exist two conjugate pairs in the $\{u, v\}$ sector.

In the $\{2,3\}$ sector, which is already diagonal, we may also choose the same polarization vectors as before and thus have one conjugate pair. The symmetry $SO(1,1) \times SO(2)$ is manifest. However, with the mass now nonzero, we may apply the boosts M_{02}, M_{03} and the rotations M_{12}, M_{13} and realize the complete $SO(1,3)$ of the four-dimensional Minkowski momentum space. After any one of these transformations we have to identify the physical states as the ones obtained from the previously chosen states together with their transformed polarization vectors, but this is a covariant procedure. The massless limit cannot, of course, be performed and requires the transition to a $(1,1) \times (0,2)$ spacetime as shown above in the discussion of the case $X(<_0)$ to $Q(<_0)$.

C. From $X = K$ to $Q(K)$

In [28], we constructed Hermitian operators K_μ as charges on Fock space which form together with translations, Lorentz transformations, and dilatations the conformal algebra. In covariant normalization of the annihilation and creation operators, they read

$$K_0 = \int \frac{d^3p}{2\omega_p} \omega_p a^\dagger(\mathbf{p}) \partial^l \partial_l a(\mathbf{p}) \tag{118}$$

$$K_j = \int \frac{d^3p}{2\omega_p} a^\dagger(\mathbf{p})(p_j \partial^l \partial_l - 2p^l \partial_l \partial_j - 2\partial_j) a(\mathbf{p}). \tag{119}$$

In the present subsection, we inquire which operators Q_μ one can find such that (13) is satisfied. As states, we use one-particle states with vanishing mass. The operators K give rise to the following variations of the creation operator:

$$[K_0, a^\dagger(\mathbf{p})] = \omega_p \partial^l \partial_l a^\dagger(\mathbf{p}) \tag{120}$$

$$[K_j, a^\dagger(\mathbf{p})] = (p_j \partial^l \partial_l - 2p^l \partial_l \partial_j - 2\partial_j) a^\dagger(\mathbf{p}). \tag{121}$$

It will turn out that two cases have to be distinguished: in the first one, the complete group $SO(2,4)$ is realized [as fitting to a spacetime (1,3)]; in the second, the rotations M_{12}, M_{13} and the boosts M_{02}, M_{03} are not realized; we have at our disposal only the group $SO(1,1) \times SO(2)$ (as fitting to a conformal group over a $(1,1) + (0,2)$ spacetime). We use the group names as labels for the two cases.

1. The $SO(2,4)$ case

We start from (121),

$$K_j|\mathbf{p}\rangle = (p_j \partial^l \partial_l - 2p^l \partial_l \partial_j - 2\partial_j)|\mathbf{p}\rangle$$
$$j = 1, 2, 3. \tag{122}$$

We form

$$K_r P^r P_j|\mathbf{p}\rangle = p_j p^r(p_r\partial^l\partial_l - 2p^l\partial_l\partial_r - 2\partial_r)|\mathbf{p}\rangle \quad r = 1,2,3. \tag{123}$$

Rewriting (122) with the use of (123), we get

$$-2(p^l\partial_l + 1)\partial_j|\mathbf{p}\rangle = \left(K_j + \frac{1}{\omega_p^2}K_r P^r P_j + \frac{2}{\omega_p^2}p_j p^l\partial_l p^r\partial_r\right)|\mathbf{p}\rangle. \tag{124}$$

With the identifications

$$Q_j|\mathbf{p}>= i\partial_j|\mathbf{p}\rangle \qquad j = 1,2,3 \tag{125}$$

$$D = i(1 + p^l\partial_l)|\mathbf{p}\rangle, \tag{126}$$

we arrive at

$$Q_j D|\mathbf{p}\rangle = \frac{1}{2}\left(K_j + K^r\frac{P_r P_j}{P_0^2} + 2(D-i)^2\frac{P_j}{P_0^2}\right)|\mathbf{p}\rangle \tag{127}$$

$$Q_j|\mathbf{p}\rangle = \frac{1}{2}\left(K_j + K^r\frac{P_r P_j}{P_0^2} + 2(D-i)^2\frac{P_j}{P_0^2}\right)D^{-1}|\mathbf{p}\rangle. \tag{128}$$

An equivalent form is

$$Q_j|\mathbf{p}\rangle = \frac{1}{2}\left(K_j - K^r\frac{P^r P_j}{P^l P_l} - 2(D-i)^2\frac{P_j}{P^l P_l}\right)D^{-1}|\mathbf{p}\rangle, \tag{129}$$

which refers to spatial components of four-vectors only and is manifestly covariant with respect to spatial rotations.

The identification (125) implies that we have conjugate pairs for the three spatial components. It also implies, however, that

$$[P_0, Q_j]|\mathbf{p}\rangle = -i\frac{p_j}{\omega_p}|\mathbf{p}\rangle. \tag{130}$$

Lorentz covariance is definitely not manifest, and the conjugation commutator is not diagonal. The rhs of (130) would project to zero on states carrying the projector $\eta_{jk} - P_j P_k/(P^l P_l)$. This will require further study to follow shortly.

In the next step, when searching for a Q_0, we may proceed in a completely analogous manner. We start from

$$K_0|\mathbf{p}\rangle = (\omega_p\partial^l\partial_l)|\mathbf{p}\rangle, \tag{131}$$

form

$$\left(K_0 + \frac{p^r}{\omega_p}K_r\right)|\mathbf{p}\rangle = \frac{2}{\omega_p}p^r\partial_r(-p^l\partial_l)|\mathbf{p}\rangle, \tag{132}$$

and end up with

$$\left(K_0 + \frac{p^r}{\omega_p}K_r\right)|\mathbf{p}\rangle = -2Q_0 D|\mathbf{p}\rangle \tag{133}$$

once we identify D as usual and

$$Q_0 = \frac{i}{\omega_p}p^r\partial_r. \tag{134}$$

This Q_0, however, is not Hermitian, and its Hermitian part commutes with P.

We might, of course, accept a non-Hermitian Q_0 and pursue the respective analysis (we shall take up this discussion below), but for the time being we prefer to choose $Q_0 = 0$ and to go along with this choice. The choice is suggested by two observations to be presented below in Sec. III A 2 and corresponds, in the analogy to the quantization of a massless vector field, to using the Coulomb gauge: in that context, one works with a vanishing zeroth component of the vector field, $A_0 = 0$, thus giving up manifest Lorentz covariance and showing afterwards that covariance is nevertheless maintained for physical quantities. With these considerations in mind, we first collect the commutation relations of P_μ with Q_ν,

$$[P_\mu, Q_\nu]|\mathbf{p}\rangle \equiv iC_{\mu\nu}|\mathbf{p}\rangle = i\begin{pmatrix} 0 & -p_k/\omega_p \\ 0 & \\ 0 & \eta_{jk} + \frac{p_j p_k}{\omega_p^2} \\ 0 & \end{pmatrix}_{\mu\nu}|\mathbf{p}\rangle, \tag{135}$$

and then define polarization vectors [33]: in the given Lorentz frame, we choose two unit vectors $\epsilon^{(\lambda)}(\mathbf{p})$, $(\lambda = 2,3)$ with time component zero, orthogonal to each other and to the unit vector $\mathbf{p}/\omega_p$ with the orientation $\mathbf{p}/\omega_p = \epsilon^{(2)} \times \epsilon^{(3)}$. In addition, we introduce a timelike unit vector $\eta = (1,0,0,0)^T$ (T for transposed) with the help of which a third independent spacelike unit polarization vector $\hat{p}$ with vanishing time component can be defined:

$$\epsilon_\mu^{(\lambda)}\eta^{\mu\nu}\epsilon_\nu^{(\lambda)} = -1 \qquad \lambda = 2,3 \qquad \epsilon_\mu^{(2)}\eta^{\mu\nu}\epsilon_\nu^{(3)} = 0 \tag{136}$$

$$\frac{p_\mu}{\omega_p}\eta^{\mu\nu}\epsilon_\nu^{(\lambda)} = 0 \qquad \lambda = 2,3$$

$$\hat{p}_\mu = \frac{p_\mu - (p\eta)\eta_\mu}{\sqrt{(p\eta)^2 - p^2}} \tag{137}$$

$$\hat{p}_\mu\eta^{\mu\nu}\hat{p}_\nu = -1 \qquad \hat{p}_\mu\eta^{\mu\nu}\epsilon_\nu^{(\lambda)} = 0 \qquad \lambda = 2,3 \qquad \eta_\mu\eta^{\mu\nu}\eta_\nu = 1 \tag{138}$$

$$\eta_\mu\eta^{\mu\nu}\hat{p}_\nu = 0 \qquad \eta_\mu\eta^{\mu\nu}\epsilon^{(\lambda)}\nu = 0 \qquad \lambda = 2,3. \tag{139}$$

These polarization vectors satisfy the completeness relation:

$$-\eta_{\mu\nu} = \sum_{\lambda=2}^{\lambda=3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)} + \hat{p}_\mu \hat{p}_\nu - \eta_\mu \eta_\nu. \tag{140}$$

It expresses the fact that the four-vectors $\epsilon^{(\lambda)}$ with $\lambda = 2$, 3 and $\epsilon_\mu^{(0)} \equiv \eta_\mu, \epsilon_\mu^{(1)} \equiv \hat{p}_\mu$ span a four-dimensional space.

In analogy to (32), we calculate now the action of the commutator (135) on the states $\epsilon_\rho^{(\lambda)}|\mathbf{p}\rangle$ for $\lambda = 0, \ldots, 3$:

$$iC_{\mu\nu}\eta^{\nu\rho}\epsilon_\rho^{(\lambda)}|\mathbf{p}\rangle = i\begin{Bmatrix} 0 & \text{for } \lambda = 0 \\ \epsilon_\mu^{(0)} & \text{for } \lambda = 1 \\ \epsilon_\mu^{(2)} & \text{for } \lambda = 2 \\ \epsilon_\mu^{(3)} & \text{for } \lambda = 3 \end{Bmatrix}|\mathbf{p}\rangle. \tag{141}$$

The "scalar" state $\lambda = 0$ is mapped to zero; the "longitudinal" state $\lambda = 1$ is mapped onto the scalar state; the "transverse" states $\lambda = 2$, 3 are diagonally mapped onto themselves. Using $\eta_{\lambda\lambda'}$ as a metric in the transverse sector, those states have positive definite norm. On the quotient space $\{\lambda = 0, 1, 2, 3\}/\{\lambda = 0, 1\}$, we have two conjugate pairs for the spatial directions two and three.

The completeness relation (140) contains information on the Lorentz covariance of the setting presented here. Since spacelike vectors remain spacelike and timelike vectors remain timelike, it is obvious that the whole state space changes under a Lorentz transformation, but the divisor also changes and just removes the offending pieces which could introduce an indefinite metric in the transverse states. Effectively, the quotient space is Lorentz covariant.

This result also sheds light on the "Lorentz gauge": if we were to use a non-Hermitian Q_0, we could introduce manifestly Lorentz-covariant polarization vectors, but due to the non-Hermitian nature of Q_0, we would also have to form a quotient space which would then be just equivalent to the Coulomb gauge case. As to locality, a similar comment applies as in the case of $Q(\nabla)$. Although K is local in x space, Q has to be generated from it by "dividing" through D. And this is certainly a nonlocal operation (Eq. (103) in [27]).

The solution for general n-particle states has to be constructed via symmetrized tensor products. We do not go into the details of this problem.

2. The $SO(1,1) + S(0,2)$ case

The conformal algebra can also be represented in a form which is closely related to the symmetry which governed the $<_0$ case: $SO(1,1) \times SO(2)$. Here, two boosts and two rotations are trivially represented. One may interpret this type of model as being fully realized on four-dimensional spacetime with a standard representation of the Lorentz group for all quantities but the ("would-be") observables X and Q, respectively. Alternatively, one can interpret the underlying spacetime to be $(1,1) \times (0,2)$ and the full algebra of it to be implemented. In any of the two interpretations, we have to restrict the generators and relabel the states accordingly if we wish to realize this algebra correctly on suitable one-particle Fock states. For the states, we shall write

$$|\mathbf{p}\rangle = |p_1\rangle|p_a\rangle \equiv |p_1; p_a\rangle \qquad a = 2, 3. \tag{142}$$

For the algebra, we introduce

$$\begin{aligned} P_0|\rangle &= \omega_p|\rangle \quad P_2|\rangle = p_2|\rangle \\ P_1|\rangle &= p_1|\rangle \quad P_3|\rangle = p_3|\rangle \\ M_{10}|\rangle &= i\omega_p\partial_1|\rangle \quad M_{23}|\rangle = -i(p_2\partial_3 - p_3\partial_2)|\rangle \\ M_{01}|\rangle &= -M_{10}|\rangle \quad M_{32}|\rangle = -M_{23}|\rangle \\ D^{(1,1)}|\rangle &= ip^1\partial_1|\rangle \quad D^{(0,2)}|\rangle = i(1 + p^2\partial_2 + p^3\partial_3)|\rangle \\ K_0|\rangle &= \omega_p\partial^1\partial_1|\rangle \quad K_2|\rangle = (p_2\partial^b\partial_b - 2(p^b\partial_b + 1)\partial_2)|\rangle \\ K_1|\rangle &= -p_1\partial^1\partial_1|\rangle \quad K_3|\rangle = (p_3\partial^b\partial_b - 2(p^b\partial_b + 1)\partial_3)|\rangle \\ &\left(\omega_p \equiv \sqrt{-p^1p_1}\right) \quad |\rangle \equiv |p_1; p_a\rangle. \end{aligned}$$

Hence, on the factors $|p_1\rangle$ and $|p_2, p_3\rangle$, respectively, the conformal algebras for spacetimes with one time + one space dimension (1,1) and zero time + two space dimensions (0,2), respectively, are realized.

The boosts M_{20}, M_{30} and the rotations M_{12}, M_{13} of the ambient spacetime (1,3) with conformal group $SO(2,4)$ are not realized; they correspond to those Lorentz transformations whose massless limit did not exist and had to be discarded there.

Turning our attention now to the construction of Q, we first observe that on the purely spatial part (0,2), we have identical formulas as compared with the previous case (1,3), with the range of the indices being restricted to $a = 2$, 3. Hence, we have identical results. The operators $Q_a, a = 2$, 3 are given by

$$Q_a|p_1; p_a\rangle = \left[K_a - (D - i)^2 \frac{P_a}{P^bP_b}\right] D^{-1}|p_1; p_a\rangle, \tag{143}$$

where the range of b (summation) is also 2,3 and $D \equiv D^{(0,2)}$. They have the canonical form

$$Q_a|p_1; p_a\rangle = i\partial_a|p_1; p_a\rangle \qquad a = 2, 3. \tag{144}$$

Again, we have to have a look to the fate of the commutator $[P_0, Q_j]$. That it is indeed vanishing in the present situation can be checked when using the full expression (143), e.g., on the state $P^bP_bD|p_1; p_a\rangle$.

In the (1,1) part, we note that the $D^{(1,1)}$ and the M_{10} as well as the K_0 and K_1 transformations differ, at most, by a sign from each other. This implies, on the one hand, that the M contribution in the commutator $[P, K]$ is simply related to the D contribution and, on the other hand, that we can avoid using the projector P/PP contracted with K. Indeed,

$$[P_\alpha, K_\beta]|p_1; p_a\rangle = 2\begin{pmatrix} -p^1\partial_1 & \omega_p\partial_1 \\ -\omega_p\partial_1 & p^1\partial_1 \end{pmatrix}_{\alpha\beta} |p_1; p_a\rangle \qquad \alpha, \beta = 0, 1. \tag{145}$$

Hence, on $\frac{1}{2}D^{-1}|p_1; p_a\rangle$ (note: D commutes with $[P, K]$),

$$[P_\alpha, K_\beta]\frac{1}{2}D^{-1}|p_1; p_a\rangle = i\begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & -1 \end{pmatrix}|p_1; p_a\rangle_{\alpha\beta} \qquad \alpha, \beta = 0, 1; \varepsilon = \frac{p_1}{\omega_p} = \pm 1. \tag{146}$$

In order to diagonalize the system, we introduce

$$P^{(\pm)} = \frac{1}{2}(P_1 \pm P_0) \qquad K^{(\pm)} = \frac{1}{2}(K_0 \pm K_1) \tag{147}$$

and then find

$$[P^{(+)}, K^{(-)}]\frac{1}{2}D^{-1}|p_1; p_a\rangle = +i|p_1; p_a\rangle \qquad \varepsilon = +1 \tag{148}$$

$$[P^{(-)}, K^{(+)}]\frac{1}{2}D^{-1}|p_1; p_a\rangle = -i|p_1; p_a\rangle \qquad \varepsilon = -1, \tag{149}$$

whereas the other commutator entries vanish. In matrix form, this reads

$$[P^{(\pm)}, K^{(\mp)}]\frac{1}{2}D^{-1}|p_1; p_a\rangle = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}|p_1; p_a\rangle = i(\eta)_{\alpha\beta}|p_1; p_a\rangle \qquad \alpha, \beta = +, -. \tag{150}$$

It is, thus, legitimate to interpret the operator on the lhs as the commutator of a conjugate pair P, Q. The rhs tells one that the norms of the states generated by this pair are opposite in sign; hence, the best one can do is to prescribe a kind of Gupta-Bleuler condition by requiring that the physical states must always contain an equal number of $[P^{(+)}, K^{(-)}]$ factors. The relation is covariant under application of Lorentz boosts in the (0,1) plane, i.e. the boost belonging to the little group of $SO(1,3)$, since P_α, K_β are vector operators with respect to $SO(1,1)$ and $D^{(1,1)}$ commutes with $M_{\gamma,\delta}$ ($\alpha, \beta, \delta, \gamma = 0$, 1). The $SO(2)$ factor is not touched by these transformations and is itself covariant under the $SO(2)$ transformations.

Again, for general n, one has to construct tensor products.

III. GROUP THEORETIC APPROACH

The construction of conjugate pairs of operators in relativistic QFT has, in particular, been pursued by using group theoretic methods. In [22], it has been based on the algebra of the conformal group $SO(2,4)$ interpreted as acting on four-dimensional Minkowski spacetime. In the first subsection, we review this work to some extent and, thereafter, put it into the perspective of our present paper.

A. Representation of the conformal group including Q

In [22], a representation of the conformal algebra has been established by going over to the enveloping algebra, where the standard generators $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$, translations, Lorentz transformations, dilatations, and special conformal transformations, respectively, have been replaced by $\{P_\mu, S_{\mu\nu}, Y, Q_\mu\}$ such that P and Q form a conjugate pair and operate on a Hilbert space H_Q, S satisfies commutation relations with itself like M, represented on a Hilbert space H_S, and the single Y generates an irreducible, hence, one-dimensional, representation on a Hilbert space H_Y. Assuming that these three Hilbert spaces are different, the representation is based on the tensor product $H_Q \otimes H_S \otimes H_Y$. In our notations and conventions, one starts with some Hilbert space of functions of one variable and defines on it differential operators P, Q which satisfy

$$[P_\mu, Q_\nu] = i\eta_{\mu\nu} \quad [P_\mu, P_\nu] = 0 \quad [Q_\mu, Q_\nu] = 0. \tag{151}$$

Next, one introduces operators,

$$M_{\mu\nu} = Q_\mu P_\nu - Q_\nu P_\mu + S_{\mu\nu} \tag{152}$$

$$D = \frac{1}{2}(PQ + QP) + Y = QP + 2i + Y \equiv QP + Y' \tag{153}$$

$$K_\mu = 2\left(-Q_\mu QP + \frac{1}{2}Q^2 P_\mu + Q_\mu D + Q^\lambda M_{\mu\lambda}\right), \tag{154}$$

with $S_{\mu\nu} = -S_{\nu\mu}$. One can convince oneself that the new set of operators $\{P, Q, S, Y\}$ closes once one assumes that Y commutes with P, Q, S. The aim is now to express Q, Y, S in terms of the original operators $\{P, M, D, K\}$. In [22], it has been shown that Y, S can be expressed in terms of the Casimir operators of the conformal group. This information has then been used, first, for giving an interpretation of these Casimir operators as *"conformal spin"* for S and *"fundamental length"* for Y' and, second, for discussing the irreducibility of this new representation of the conformal group. Of particular importance is the inversion for Q. It is performed via combination:

$$\frac{1}{2}K^{\lambda}\left(\eta_{\lambda\mu}-\frac{P_{\lambda}P_{\mu}}{P^2}\right)=Q^{\lambda}Y'\left(\eta_{\mu\lambda}-\frac{P_{\mu}P_{\lambda}}{P^2}\right)+Q^{\lambda}M_{\rho\lambda}\left(\eta^{\rho}_{\mu}-\frac{P^{\rho}P_{\mu}}{P^2}\right). \quad (155)$$

Here the expression for D and the commutator (151) have been used, and Y has been replaced by Y'. Clearly, this formula makes sense only if P^2 does not vanish. In [22], it has been argued by counting the number of unknowns and the number of equations that one can solve for Q. We note, however, and discuss in more detail below that K and Q are contracted with the transverse projection operator $\eta_{\mu\nu}-P^{\mu}P^{\nu}/P^2$; hence, their relation might be determined only up to a longitudinal term, proportional to P/P^2.

In the case $S=0$, one inserts the expression for M in terms of P, Q, uses D as a function of P, Q, and arrives at

$$Q_{\mu}=\left[\frac{1}{2}K^{\lambda}\left(\eta_{\mu\lambda}-\frac{P_{\mu}P_{\lambda}}{P^2}\right)+D(D-2Y-4i)\frac{P_{\mu}}{P^2}\right]D^{-1}; \quad (156)$$

i.e., in this case, a suitable longitudinal term showed up and the solution is unique. Returning to the general case $S\neq 0$, one notes that in a representation of (151) on a Hilbert space H_Q, the latter must contain at least square integrable functions $f(p)$, with the scalar product being given by $(f,g)=\int_{V_+}d^4p f^*(p)g(p)$ with V_+ denoting the forward cone of $p^2>0$. On this domain, P_{μ} is self-adjoint and the Q_{μ}'s are given by $i\partial/\partial p_{\mu}$ which is Hermitian but not self-adjoint. Their domain of Hermiticity is the dense set of differentiable functions of p_{μ} which vanish on the boundary of V_+. The operators K are self-adjoint on H_Q: an irreducible representation for the conformal group has been found, and the Casimir invariants are multiples of the identity.

Other, equivalent representations are given by functions which have support either for spacelike p_{μ}, i.e. $p^2<0$, or lightlike p_{μ}, i.e. $p^2=0$, or the negative cone $V^-=\{p\in\mathbb{R}^4|p^2>0,p_0<0\}$. But due to the fact that a self-adjoint Q_{μ} has its spectrum on the entire line, the decomposition into several irreducible representations does only yield Hermitian Q_{μ}.

1. *Noncommutative coordinates*

With (156) at hand, having an operator Q which forms, together with P, a conjugate pair, we can realize a *noncommutative coordinate operator* via

$$Q^{\mathrm{nc}}_{\mu}=Q_{\mu}+\Theta_{\mu\nu}P^{\nu}, \quad (157)$$

with Θ real and antisymmetric. Q^{nc} clearly satisfies

$$[Q^{\mathrm{nc}}_{\mu},Q^{\mathrm{nc}}_{\nu}]=2i\Theta_{\mu\nu} \quad (158)$$

[(9)].

The definition of Q^{nc} and the commutation relation (158) hold on the function space described before for Q and P, and likewise they have the same domain of Hermiticity. In which sense these properties indeed qualify Q^{nc} as a "true" noncommutative coordinate operator remains an open question.

We note, however, that restricting the functions f, on which Q^{nc} acts to obey equations of motion, i.e. to go on shell, one will encounter the intricacies which have been presented for $Q=Q(K)$ in subsection II C. These will be discussed now.

2. *Consistency of off-shell and on-shell treatment for S = 0*

The above considerations hold on a Hilbert space of functions $f(p)$ which do not necessarily satisfy any differential equation. In the parlance of QFT, one could understand them as off-shell one-particle Green functions. The considerations of Sec. II refer to one-particle states, i.e., wave functions solving the respective Klein-Gordon equation. It is then natural to inquire how the results of the preceding subsection are related to them. As a first topic, we show how our variations of one-particle states with respect to K (120) and (121) come out from (152). K has been defined as

$$K_{\mu}=2\left(-Q_{\mu}QP+\frac{1}{2}Q^2P_{\mu}+Q_{\mu}D+Q^{\lambda}M_{\mu\lambda}\right). \quad (159)$$

We interpret now the operators as differential operators $\delta^A(A=P,M,D)$ acting on some eigenfunction of P and, hence, obtain in the first step

$$\delta^K_{\mu}=2\left(-p^{\nu}\delta^Q_{\nu}\delta^Q_{\mu}+\frac{1}{2}p_{\mu}\delta^Q_{\lambda}\delta^{\lambda}_Q+\delta^D\delta^Q_{\mu}+\delta^M_{\mu\nu}\delta^{\nu}_Q\right). \quad (160)$$

Eventually, we wish to realize Q_j by $i\partial/\partial p^j$ and, therefore, use as δ's for $A=D$, M our standard variations and find in the second step

$$\delta^K_0|\mathbf{p}\rangle=\left(2i\delta^Q_0+\omega_p\frac{\partial^2}{\partial p^l\partial p_l}\right)|\mathbf{p}\rangle \quad (161)$$

$$\delta^K_j|\mathbf{p}\rangle=\left(-2i\omega_p\delta^Q_0\frac{\partial}{\partial p^j}+2i\omega_p\frac{\partial}{\partial p^j}\delta^Q_0+p_j\frac{\partial^2}{\partial p^l\partial p_l}-2p^l\frac{\partial^2}{\partial p^l\partial p^j}-2\frac{\partial}{\partial p^j}\right)|\mathbf{p}\rangle. \quad (162)$$

This result tells us that, for $\delta^Q_0=\partial/\partial p^0$, the construction within [22] provides a relation between all variations δ^K and all variations δ^Q which, as we know from the $S=0$ case, one is able to invert. For $\delta^Q_0=0$ in the relation for δ^K_0, i.e., no independent variation with respect to direction 0, i.e. $\partial/\partial p^0 \equiv 0$, we obtain precisely our on-shell variations δ^K. Hence, we conclude that the two approaches match.

As the second topic, we discuss—for the [22] case $S = 0$ and a representation with $P^2 = 0$—what we shall call the "gauge" problem.

We use the solution (156) and apply it to a one-particle state $DP^2|\mathbf{p}\rangle$:

$$(Q_0 DP^2)|\mathbf{p}\rangle = \left(\frac{1}{2}\times 0 - \frac{1}{2}\omega_p p^\lambda \delta^K_\lambda + \omega_p (i(1+p^l\partial_l) - 4i - 2y) i(1+p^l\partial_l)\right)|\mathbf{p}\rangle \qquad (163)$$

$$= 0 \quad \text{for } y = -i. \qquad (164)$$

The "direct" term K_0 is annihilated by $p^2 = 0$ (on-shell-ness); however, the projector contribution $K^\lambda P_\lambda P_0$ is *nontrivially* cancelled by the contribution coming from the D terms. For $\mu = j$, however, no cancellation takes place, and we arrive at a contradiction: the lhs vanishes, and the rhs does not. Hence, like in the quantization of (massless) gauge fields, we have to give up at least one of the fundamental properties which we would have liked to see realized. In Sec. II C 1, we gave up manifest Lorentz covariance, used $Q_0 = 0$ (Coulomb gauge), and were able to realize two conjugate pairs on states with a definite metric. If we had stuck to manifest covariance, we would have had to give up Hermiticity for Q_0.

We shall see in the next subsection that a similar phenomenon happens in the massive case.

B. Representation of Poincaré and dilatations

For $n = 1$, the relation (16) can be rewritten as

$$[P_\mu, X^\nabla_\nu] = i\left(\eta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right). \qquad (165)$$

It is then suggestive to introduce an operator $X^{(\mathrm{com})}$ ("com" for "composite"),

$$X^{(\mathrm{com})}_\mu = M_{\mu\lambda}\frac{P^\lambda}{P^2}, \qquad (166)$$

which is a Lorentz vector,

$$[M_{\mu\nu}, X^{(\mathrm{com})}_\rho] = -i(\eta_{\mu\rho}X^{(\mathrm{com})}_\nu - \eta_{\nu\rho}X^{(\mathrm{com})}_\mu), \qquad (167)$$

that fulfils

$$[X^{(\mathrm{com})}_\mu, X^{(\mathrm{com})}_\nu] = iM_{\mu\nu}\frac{1}{P^2}, \qquad (168)$$

i.e., the analogue of (20), and reproduces (165).

We now choose eigenfunctions of P as representation space, interpret the operators involved accordingly as differential operators, and apply $X^{(\mathrm{com})}$ to an eigenfunction $\phi(p)$:

$$X^{(\mathrm{com})}_\mu \phi(p) = i\left(\frac{\partial}{\partial p^\mu} - \frac{p_\mu}{p^2}p^\lambda\partial_\lambda\right)\phi(p). \qquad (169)$$

The first derivative term points to an operator Q which indeed is realized once we add a term $(D - \hat{Y})(P_\mu/P^2)$ with $D = i(1 + p^\lambda\partial p^\lambda), \hat{Y} = i$ on $\phi(p)$. Hence,

$$Q_\mu = X^{(\mathrm{com})}_\mu + (D - \hat{Y})\frac{P_\mu}{P^2} \qquad (170)$$

yields

$$Q_\mu\phi(p) = i\frac{\partial}{\partial p^\mu}\phi(p). \qquad (171)$$

We note, first of all, that adding $(D - \hat{Y})P_\mu/P^2$ to $X^{(\mathrm{com})}$ generates an *Abelian* operator Q_μ (four components) and, second, that Q_μ obviously satisfies the conjugation relation $[P_\mu, Q_\nu] = i\eta_{\mu\nu}$ on the eigenfunctions $\phi(p)$.

We, therefore, succeed in finding an operator (in the enveloping algebra of Poincaré + dilatations) which realizes $Q_\mu = i\partial/\partial p^\mu$. It is also noteworthy that the differential operator on the rhs of (169) is just an off-shell continuation of ∇_μ.

1. Noncommutative coordinates

In perfect analogy to the conformal case, we are also in the present context able to define a differential operator which qualifies—at least formally—as a *noncommutative coordinate operator*:

$$Q^{nc}_\mu = Q_\mu + \Theta_{\mu\nu}P^\nu = X^{(\mathrm{com})}_\mu + (D - \hat{Y})\frac{P_\mu}{P^2} + \Theta_{\mu\nu}P^\nu. \qquad (172)$$

(Again, Θ is real and antisymmetric.) It operates on functions $\phi(p)$, with P, M, D accordingly interpreted as differential operators. It is to be noted that the mass can be either nonvanishing or (for off-shell ϕ) vanishing.

2. Consistency of off-shell and on-shell treatment for S = 0

Let us now choose Fock space as representation space. Then formulas exactly analogous to the above ones hold on one-particle states with range of indices λ restricted to $\{1, 2, 3\}$:

$$X^{(\mathrm{com})}_\mu|\mathbf{p}\rangle = i\left(\frac{\partial}{\partial p^\mu} - \frac{p_\mu}{m^2}p^l\partial_l\right)|\mathbf{p}\rangle \qquad (173)$$

$$Q_\mu = X_\mu^{(\mathrm{com})} + (D - \hat{Y})\frac{P_\mu}{P^2} \tag{174}$$

$$Q_\mu|\mathbf{p}\rangle = i\frac{\partial}{\partial p_\mu}|\mathbf{p}\rangle. \tag{175}$$

We obtain $Q_0 = 0$ once we put the derivative $\partial/\partial p^0 \equiv 0$. Thus, these considerations confirm, on the one hand, that one can invert off shell and, on the other hand, that our on-shell arguments on the vanishing of Q_0 in subsubsection II A 1 are correct.

Obviously, the above formulas are very close to those of [22] for an operator Q_μ derived from the conformal generators K_μ. The precise derivation proceeds as follows. We use the definitions of (152) for M and D and obtain

$$M_{\mu\lambda}P^\lambda = Q_\mu P^2 - QPP_\mu + S_{\mu\lambda}P^\lambda \tag{176}$$

$$D = QP + 2i + Y \tag{177}$$

$$Q_\mu = \frac{M_{\mu\lambda}P^\lambda}{P^2} + ((D - (Y + 2i))\eta_{\mu\lambda} - S_{\mu\lambda})\frac{P^\lambda}{P^2} \tag{178}$$

$$= X_\mu^{(\mathrm{com})} + ((D - (Y + 2i))\eta_{\mu\lambda} - S_{\mu\lambda})\frac{P^\lambda}{P^2}. \tag{179}$$

For $S = 0, \hat{Y} = Y + 2i$, this is precisely our expression (174). The only difference is that, in our *ad hoc* approach, the Abelian character of Q comes out as a result, whereas here, going along the lines of [22], it has been assumed from the start. But clearly, the main content is the same.

In the vein of the present section, these considerations can be interpreted as the fact, that the operators $\{P, Q, Y\}$ generate the same representation of the group Poincaré × dilatation as the set of generators $\{P, M, D\}$ via the identification (152) with $S = 0,\ Y = i$.

Finding one and the same Q on Fock space starting from different expressions in different algebras is just analogous to the well-known fact in QFT, à la Lehmann-Symanzik-Zimmermann, that different interpolating fields may represent one and the same particle on shell.

IV. DISCUSSION, CONCLUSIONS, OPEN QUESTIONS

A. Universality

We first summarize our findings schematically in Table I and then describe them in detail.

In the massive case we started from $X(\nabla)$, s. (14), which has geometrical meaning, and then derived the on-shell quantities $Q(\nabla)$, s. (44). Here it is crucial to rely on the presence of polarization vectors. The fact that $Q_0^{(\mathrm{eff})} = 0$ can however be seen already when looking at the off-shell quantities [22]-type Q_μ, s. (170), which originate from group theoretic considerations. Going on shell there confirms the vanishing of $Q(\nabla)_0$. Universality clearly means "equality on Fock space" which obviously has been achieved. Three (spatial) conjugate pairs exist. Due to the polarization vectors they operate on states with positive definite norm. Lorentz covariance is nonmanifestly realized.

In the limit of vanishing mass, this structure of physical state space can be maintained, but Q_1 vanishes; hence, only two spatial pairs survive.

The generically massless case has been based on the preconjugate $X_\mu = K_\mu$ (118), with K generating the special conformal transformations. The version relevant for this universality sector is based on the spacetime with dimension (1,3). Here also $Q(K)_0 = 0$, confirmed via off-shell reasoning, (163), and—in order to diagonalize the conjugation commutator—one has to mode out one spatial component. Two spatial conjugate pairs exist. Quite natural, however, seems to be a truncation of the algebra to $SO(1,1) \times SO(2)$ and the spacetime to $(1,1) + (0,2)$. On the state space (142), we found two conjugate pairs for the spatial part (0,2); those over the (1,1) part have to be moded out for norm reasons. A class of special interest is formed by $X(<_0)$ with its associated operator $Q(<_0)$. In the massless limit (of $X(<)$ to $X(<_0)$) the symmetry shrinks to $SO(1,1) \times SO(2)$ and accordingly also the spacetime to $(1,1) + (0,2)$. Since however in momentum space a

TABLE I. Cases of preconjugate and conjugate variables.

	Preconjugate		Conjugate	Symmetry of spacetime	State space type state space symm.	Number of conj. pairs
$m^2 \neq 0$	$X(\nabla)$	$\to$	$Q(\nabla)$	$SO(1,3)$	Standard $SO(1,3)$	3 spatial
$m^2 \to 0$		$\to$	$Q(\nabla)_{\vert m^2=0}$	$SO(1,3)$	Standard $SO(1,3)$	2 spatial
$m^2 = 0$	$X(K)$	$\to$	$Q(K)$ $Q(K)$	$SO(2,4)$ $SO(1,1) \times SO(2)$	Quotient $SO(1,3)$ Quotient $SO(1,1) \times SO(2)$	2 spatial 2 spatial
$m^2 = 0$	$X(<_0)$	$\to$	$Q(<_0)$	$SO(1,1) \times SO(2)$	Standard $SO(1,1) \times SO(2)$	$2 + 1$
$m^2 \neq 0$	$X(<)$	$\to$	$Q(<)$	$SO(1,3)$	Standard $SO(1,3)$	$2 + 1$

double cone is realized as opposed to the *single* (forward) cone in the previous examples $(Q(\nabla), Q(K))$ the resulting outcome for $Q(<_0)$ and the state space differs from the analogous conformal case: on the (0,2) part of spacetime one independent conjugate pair is realized on two states with polarization vectors $\epsilon(r)$, $r = 2, 3$, s. (96). In the (1,1) part of spacetime which appears however as (u, v) and as (p_u, p_v) on momentum space we have two conjugate pairs operating on two states with positive definite norm. Due to the nondiagonal form of the metric, the operators $Q^u_{\rm eff}, Q^v_{\rm eff}$ have a nonvanishing commutator, (109).

Once this structure has been found one can establish exactly the same one also for nonvanishing mass, $X(<) \to Q(<)$ s. (117), and—just due to the nonzero mass—one can extend it to the full Lorentz group. The number and type of conjugate pairs coincides with the massless case and thus reaches the maximal number obtainable: two in the $\{u, v\}$-sector, one in the $\{2, 3\}$ sector. The relevant state space is the standard Fock space augmented by the polarization vectors.

An intriguing result of our analysis may therefore be that wedge-local quantum field theories just provide by definition the right balance between position and momentum variables on the quantum field theoretic level to form respective operators which come as conjugate pairs on shell. Time does not play a preferred role any more.

In order to find a direct relation between $Q(<_0)$, on the one hand, and the massless limit of $Q(\nabla)$ and $Q(K)(1, 3)$ on the other, we first recall that $Q(\nabla)_0 = Q(\nabla)_1 = 0$ in the massless limit (57), and that $Q(K)_0$ and $Q(K)_1$ are moded out in the relevant state space (subsection II C 1). Let us consider the *quadruple* $\{Q(<_0)_{u,v}, \epsilon_\alpha^{(u,v)}, A(\mathbf{p})|0\rangle, \langle 0|A(\mathbf{p})\}$ and compare it with the corresponding quadruples $\{Q(K)_{0,1}, \epsilon_\alpha^{(0,1)}, a^\dagger(\mathbf{p})|0\rangle, \langle 0|a(\mathbf{p})\}$, $\{Q(\nabla)_{0,1}, \epsilon_\alpha^{(0,1)}, a^\dagger(\mathbf{p})|0\rangle, \langle 0|a(\mathbf{p})\}$. (The writing should indicate that due to $A^\dagger(\mathbf{p}) = -A(\mathbf{p})$, (67), as opposed to $(a^\dagger(\mathbf{p})|0\rangle)^\dagger = \langle 0|a(\mathbf{p})$, the $Q(<_0)$ lives in a bigger space than the other two Q's.) Now it becomes clear that the latter two are effectively the projection to zero of the first one (referring to the Q's). The reason for the nontriviality of $Q(<_0)_{u,v}$ is the presence of the *double* cone: $Q(<_0)$ operates on a bigger space as compared with $Q(K)$ and $Q(\nabla)$ (massless limit). Those have only *one* cone as their area of definition. This corresponds precisely to the nonexistence of ∇^u, *resp.* ∇^v as expressed in equations (84) which prohibits a $1 \leftrightarrow 1$ relation.

B. The gauge problem

In the course of our investigations, it has become clear that the postulate $[P_\mu, Q_\nu] = i\eta_{\mu\nu}$ has, first of all, to be understood in a weak sense: as applied to spaces of functions or states. It further became clear that the rhs of the commutator equation may be interpreted like in gauge theories: the "pure" $\eta_{\mu\nu}$ form corresponds to Lorentz gauge and is naturally realized off shell: in the *ad hoc* version as a Fourier transform (no realization of Q as function of other operators of the theory), in the [22] version $Q = Q(K)$, and in the [22]-type construction in subsection III B. On shell, i.e., on Fock states, we met the Landau gauge in $X(\nabla) \to Q(\nabla)$, massive version; the Coulomb gauge in $X(K) \to Q(K)$, $(1, 3)$ spacetime; and the light cone gauge in $X(<_0) \to Q(<_0)$. In hindsight, the explanation is simple: the desired $\eta_{\mu\nu}$ can be expanded into a sum over polarization vectors $-\eta_{\mu\nu} = \sum_{\lambda=0}^{\lambda=3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)}$, where the polarization vectors provide a basis for the space spanned by $\eta_{\mu\nu}$, leading one to define new states $\epsilon_\mu^{(\lambda)}|\mathbf{p}\rangle$. It is then nontrivial, but true, that on these states the inversion from a preconjugate X to a conjugate Q is possible. The different signs within $\eta_{\mu\nu}$ determine the norm of the eventual state. The solution $Q_\mu = i\partial/\partial p^\mu$ on these states leads to $Q_0 = 0$, since, on shell, no independent motion in direction zero, driven by $\partial/\partial p^0$, is generated. Q_0 is, however, a tentative time operator. Pauli's theorem is refined in a very bold sense: Q_0 is not only not self-adjoint—it vanishes. This must not be understood as a surprise, after all. On-shell states are constructed within the limit of ±infinite time and, hence, do not move in the flow of time. They cannot serve as direct instruments to measure time.

In the context of the case $Q(<_0)$, the gauge nature of the definition of conjugate pairs points to a possible relation with the construction of gauge theories in noncommutative field theories, notably [34]. This aspect remains to be explored.

C. General fields, more general states

Obviously, fields and states carrying spin should be studied along the lines presented in this paper. The reader finds the respective discussion in the "Discussion and conclusions" section of [28], so we do not duplicate it here, but instead concentrate on the aspects of the problems related to *conjugate* pairs.

For the construction of conjugate pairs, we introduced polarization vectors multiplying ordinary Fock states. They solved the gauge, i.e., the norm problem associated with conjugate pairs. Hence, these polarization vectors should be considered as a new, essential attribute for constructing the observables Q. They may be interpreted as tensoring the state space with some factor. But this factor is in our derivation, not arbitrary. This might be in contrast with [16].

The quadruples $\{Q(<_0), \epsilon, A(\mathbf{p})|0\rangle, \langle 0|A(\mathbf{p})\}$, $\{Q(\nabla), \epsilon, a^\dagger(\mathbf{p})|0\rangle, \langle 0|a(\mathbf{p})\}$, and $\{Q(K), \epsilon, a^\dagger(\mathbf{p})|0\rangle, \langle 0|a(\mathbf{p})\}$ serve as "detectors" in the one-particle states of Fock space for determining the value of Q.

On a formal level, these "dressed" states are asymptotic with respect to their spacetime variables; a deeper understanding of them, however, would be desirable. The inherent nonlocality in x space when deriving the Q's from the X's and taking into account the effect of the polarization vectors seems to be in accordance with [7].

Even off shell, one could probably introduce analogous quantities and discuss in these terms the domain questions of the operators Q which would then be related to norm properties as well.

A last point of discussion concerns the relevance of the state space, here chosen to be the Fock space for reasons of practical importance. As far as mathematics is concerned, this choice fits well with the Poincaré group. Coherent states have been employed in [19] for the group $SU(2,2)$ which is the universal covering group for $SO(2,4)$. (The general machinery for finding coherent states relative to symmetry groups has later been provided in [35].) Group theoretic methods lead to conjugate pairs, for the conformal case carefully studied in [19–21]. The respective norm problem has, however, not been addressed. Rather, the authors discarded the component Q_0 and did not discuss the ensuing problem of covariance with respect to Lorentz transformations.

In the present context, it would be most interesting to extend our considerations to thermal states (s. [36]) and thermal quantum fields ([37]) because this would provide a first step to widen the view to the far more general problem that, obviously, quite a few notions of time exist.

One of them is associated with irreversible processes giving rise to an arrow in time. Realizing something like this in relativistic systems requires generalization of entropy and other thermodynamic quantities and the introduction of respective state spaces. In the general relativistic context, this might provide even more insight and explain phenomena not understood today.

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