Non-Abelian vortex in four dimensions as a critical string on a conifold

P. Koroteev,[1,2] M. Shifman,[3] and A. Yung[3,4,5]

[1]Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L2Y5, Canada
[2]Department of Mathematics, University of California, Davis, California 95616, USA
[3]William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA
[4]National Research Center "Kurchatov Institute," Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia
[5]St. Petersburg State University, Universitetskaya naberejnaya, St. Petersburg 199034, Russia

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Non-Abelian vortex strings supported in a certain four-dimensional $\mathcal{N} = 2$ Yang-Mills theory with fundamental matter were shown [1] to become critical superstrings. In addition to translational moduli, the non-Abelian strings under consideration carry orientational and size moduli. Their dynamics is described by the two-dimensional sigma model whose target space is a tautological bundle over the complex projective space. For the $\mathcal{N} = 2$ theory with the $U(2)$ gauge group and four fundamental hypermultiplets, there are six orientational and size moduli. After combining with four translational moduli, they form a ten-dimensional target space, which is required for a superstring to be critical. For the theory in question, the target space of the sigma model is $\mathbb{C}^2 \times Y_6$, where Y_6 is a conifold. We study closed string states which emerge in four dimensions (4D) and identify them with hadrons of the 4D bulk $\mathcal{N} = 2$ theory. It turns out that most of the states arising from the ten-dimensional graviton spectrum are nondynamical in 4D. We find a single dynamical massless hypermultiplet associated with the deformation of the complex structure of the conifold. We interpret this degree of freedom as a monopole-monopole baryon of the 4D theory (at strong coupling).

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I. INTRODUCTION

This paper builds on the previous discovery of the non-Abelian solitonic vortex string in a certain four-dimensional (4D) Yang-Mills theory shown to be critical in the strong coupling limit [1]. The results reported below are summarized in Ref. [2]. The particular 4D theory where the non-Abelian vortex is critical is $\mathcal{N} = 2$ supersymmetric QCD with a U(2) gauge group and $N_f = 4$ quark flavors. The target space of the two-dimensional (2D) theory on the vortex string is $\mathbb{C}^2 \times Y_6$, where Y_6 is a conifold. Analyzing the closed string spectrum, we find one massless hypermultiplet associated with the deformation of the complex structure of the conifold. Then we interpret this hypermultiplet in terms of the four-dimensional Yang-Mills theory at strong coupling.

In quantum chromodynamics Regge trajectories show almost perfect linear J behavior (J stands for spin). However, in all controllable examples at weak coupling, a solitonic confining string exhibits linear behavior for the Regge trajectories only at asymptotically large spins [3,4]. The reason for this is that at $J \sim 1$ the physical "string" becomes short and thick and cannot yield linear Regge behavior. Linear Regge trajectories at $J \sim 1$ have a chance to emerge only if the string at hand satisfies the thin-string condition [1],

$$T \ll m^2, \tag{1.1}$$

where T is the string tension and m is a typical mass scale of the bulk fields forming the string. The former parameter determines the string length, while the latter determines the string width. At weak coupling $g^2 \ll 1$, where g^2 is the bulk coupling constant, we have $m \sim g\sqrt{T}$. The thin-string condition (1.1) is therefore badly broken.

For most solitonic strings in four dimensions, like the Abrikosov-Nielsen-Olesen (ANO) vortices [5], the low-energy two-dimensional effective Nambu-Goto theory on the string world sheet is not ultraviolet (UV) complete. To make the world-sheet theory sensible to the dimension of the target space, one has to take into account higher derivative corrections [6]. Higher derivative terms run in inverse powers of m and blow up at weak coupling, making the string world-sheet "crumpled" [7]. The blowup of higher derivative terms in the world-sheet theory corresponds to the occurrence of thick and short "strings."

The question of whether one can find an example of a solitonic string which might produce linear Regge trajectories at $J \sim 1$ was addressed and answered in Ref. [1]. Such a string should satisfy the thin-string condition (1.1). This condition means that higher derivative corrections are parametrically small and can be ignored. If so, the low-energy world-sheet theory should be UV complete. This implies the following necessary conditions:

(i) The low-energy world-sheet theory must be conformally invariant.

(ii) The theory must have the critical value of the Virasoro central charge.

These conditions are satisfied by the fundamental string. In Ref. [1] it was shown that (i) and (ii) above are met by non-Abelian vortex strings [8–11] supported in four-dimensional $\mathcal{N}=2$ supersymmetric QCD with the $\mathrm{U}(N)$ gauge group, $N_f=2N$ matter hypermultiplets and the Fayet-Iliopoulos (FI) parameter ξ. The non-Abelian part of the gauge group has vanishing β function.

The non-Abelian vortex string is $1/2$ Bogomolny-Prasad-Sommerfeld (BPS) saturated and, therefore, has $\mathcal{N}=(2,2)$ supersymmetry on its world sheet. In addition to translational moduli characteristics of the ANO strings, the non-Abelian string carries orientational moduli, as well as size moduli if $N_f>N$ [8–11]; see Refs. [12–15] for reviews. Their dynamics is described by a two-dimensional sigma model with the target space

$$\mathcal{O}(-1)^{\oplus(N_f-N)}_{\mathbb{CP}^1}, \tag{1.2}$$

which we will refer to as $\mathrm{WCP}(N,N_f-N)$ model. It has a natural description in terms of the gauged linear sigma model (GLSM) [16] containing N positive and N_f-N negative $U(1)$ charged chiral multiplets. For $N_f=2N$ the model becomes conformal, and condition (i) above is satisfied. Moreover, for $N=2$ the dimension of orientational/size moduli space is six, and they can be combined with four translational moduli to form a ten-dimensional space required for critical superstrings.[1] Thus, the second condition is also satisfied [1].

Given that the necessary conditions are met, a hypothesis was put forward [1] that this non-Abelian vortex string does satisfy the thin-string condition (1.1) at the strong coupling regime in the vicinity of a critical value of $g_c^2\sim 1$. This implies that $m(g^2)\to\infty$ at $g^2\to g_c^2$.

Moreover, a version of the string-gauge duality for the four-dimensional bulk Yang-Mills theory was proposed: At weak coupling this theory is in the Higgs phase and can be described in terms of (s)quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as string states formed on the non-Abelian vortex string. In this paper we further explore this hypothesis by studying string theory for the critical non-Abelian vortex. This analysis allows us to confirm and enhance the construction [1].

Vortices in $\mathrm{U}(N)$ theories are topologically stable and can be realized as either closed or open strings. Open strings need to end on some object, e.g., branes. However, there are no such objects in $\mathcal{N}=2$ SQCD.[2] Therefore, we focus on the closed strings emerging from four dimensions, and we will be able to identify closed string states with hadrons of the four-dimensional bulk theory.

It is worth mentioning at this point that our solitonic vortex describes only nonperturbative states. Perturbative states, in particular, massless states associated with the Higgs branch of the four-dimensional theory (see Sec. II), are present at all values of gauge couplings and are not captured by the vortex string dynamics.

The onset of the thin-string regime (1.1) is determined by the ratio T/m^2. While the string tension is exactly determined by FI parameter ξ,

$$T=2\pi\xi, \tag{1.3}$$

there is no exact formula known for mass m. The latter is a (common) mass parameter for the (s)quarks and Higgsed gauge bosons, which form long non-BPS multiplets. Their masses receive quantum corrections (see Ref. [14] and Sec. II below). Thus, condition (1.1) can be argued for, but it is problematic to rigorously prove it since we are in the strong coupling regime. We can test it, however. The effective hadron four-dimensional theory that emerges from quantization of the non-Abelian string should respect general properties of the original $\mathcal{N}=2$ theory.

We perform the following four major tests of our proposal:

(a) $\mathcal{N}=2$ *space-time supersymmetry in 4D.*—From the string side it emerges due to $\mathcal{N}=(2,2)$ world-sheet supersymmetry and the fact that we only have closed string states in our theory. In fact, we show that our non-Abelian vortex is a Type-IIA superstring.

(b) *Absence of 4D massless graviton.*—Our original bulk theory is $\mathcal{N}=2$ QCD without gravity. Thus, we expect that 4D massless string modes do not include graviton.

(c) *Absence of unwanted massless vector multiplets.*

(d) *The 4D massless monopole-monopole baryon.*—It exists only at strong coupling and cannot be continued to the weak coupling, where its presence would contradict previous semiclassical analysis.

Note that if the Calabi-Yau manifold Y_6 is compact, then there certainly is a massless 4D graviton in the spectrum.[3] However, since the conifold is noncompact, we do not expect any massless spin-2 states to appear after the reduction to 4D, nor do they exist in the bulk 4D $\mathcal{N}=2$ theory. We explicitly demonstrate that the 4D graviton is absent due to non-normalizability of its wave function.

Moreover, we show that 4D massless vector multiplets associated with the Killing vectors on the conifold are also absent due to non-normalizability of their wave functions over the internal six-dimensional space. Massless vector

[1] It corresponds to $\hat{c}=\frac{c}{3}=3$.

[2] There is a possibility for a string to end on BPS monopoles in $\mathcal{N}=1$ theory, which is a deformation of the $\mathcal{N}=2$ SQCD by a superpotential.

[3] An alternative—a massless 4D spin-2 state with no interpretation in terms of 4D gravity—is ruled out by the Weinberg-Witten theorem [17].

multiplets have a natural interpretation as gauge bosons. If they were present at strong coupling at g^2 close to g_c^2, they would remain massless at arbitrary g^2, in particular, at weak coupling.[4] However, we know that there are no massless gauge multiplets at weak coupling in the bulk $\mathcal{N}=2$ Yang-Mills theory—all gauge fields are Higgsed. In particular, we show that the 4D vector multiplet associated with deformation of the Kähler structure of the conifold Y_6 in Type-IIA string theory is nondynamical.

We address the physical meaning of the above non-normalizability. For certain non-normalizable modes we see that their background values should be considered as coupling constants in 4D Yang-Mills theory [18]. For others, non-normalizability is related to instability due to the presence of the Higgs branch in the bulk (and associated massless states).

The paper is organized as follows. In Sec. II we review physics of the $\mathcal{N}=2$ SQCD, non-Abelian vortices and introduce a string description for these vortices. In Sec. III we discuss $\mathcal{N}=2$ supersymmetry on the world sheet and show that we deal with Type-IIA strings. In Sec. IV we briefly review the general framework to obtain 4D states from 10D massless close string states like graviton and discuss the normalizability of these states. In Sec. V we consider the massless vector multiplet and hypermultiplet associated with deformations of Kähler and complex structures of the conifold, respectively. In Sec. VI we give a physical interpretation of the hypermultiplet associated with deformation of the complex structure of the conifold as a monopole-monopole baryon. We summarize our conclusions in Sec. VII. The Appendix contains explicit expressions for the metric of resolved and deformed conifolds.

II. NON-ABELIAN VORTEX AS A CRITICAL SUPERSTRING

In this section we briefly review our bulk $\mathcal{N}=2$ Yang-Mills theory, the non-Abelian strings that it supports, and the corresponding world-sheet model.

A. $\mathcal{N}=2$ supersymmetric Yang-Mills theory in 4D

The basic bulk model we start from is $\mathcal{N}=2$ SQCD with the $\mathrm{U}(N)$ gauge group and N_f massless matter hypermultiplets. It is described in detail in Ref. [10]; see also the review [14]. The field content is as follows.

The $\mathcal{N}=2$ vector multiplet consists of the U(1) gauge field A_μ and $\mathrm{SU}(N)$ gauge fields A_μ^a, where $a=1,\ldots,N^2-1$, as well as their Weyl fermion superpartners plus complex scalar fields a and a^a and their Weyl superpartners, respectively.

[4] One could avoid this conclusion if gauge fields were Higgsed at weak coupling. However, this would require an appropriate amount of massless charged matter multiplets.

The matter sector of the $\mathrm{U}(N)$ theory contains N_f (s) quark hypermultiplets each consisting of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners—all in the fundamental representation of the $\mathrm{SU}(N)$ gauge group. Here $k=1,\ldots,N$ is the color index while A is the flavor index, $A=1,\ldots,N_f$. In this paper we assume that the matter mass parameters vanish.

In addition, we introduce the FI parameter ξ in the U(1) factor of the gauge group. It does not break $\mathcal{N}=2$ supersymmetry.

We consider the bulk theory with $N_f=2N$. In this case the $\mathrm{SU}(N)$ gauge coupling does not run since the corresponding β function vanishes. Note, however, that the conformal invariance of the bulk theory is explicitly broken by the FI parameter.

Let us review the vacuum structure and the excitation spectrum of the bulk theory assuming weak coupling, $g^2\ll 1$, where g^2 is the $\mathrm{SU}(N)$ gauge coupling. The FI term triggers the squark condensation. The squark vacuum expectation values (VEVs) are

$$\langle q^{kA}\rangle=\sqrt{\xi}\begin{pmatrix}1 & \ldots & 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & 1 & 0 & \ldots & 0\end{pmatrix},\qquad \langle\bar{\tilde{q}}^{kA}\rangle=0,$$
$$k=1,\ldots,N,\qquad A=1,\ldots,N_f, \tag{2.1}$$

where we present the squark fields as matrices in the color (k) and flavor (A) indices.

The squark condensate (2.1) results in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global $\mathrm{SU}(N)$ combining the gauge $\mathrm{SU}(N)$ and an $\mathrm{SU}(N)$ subgroup of the flavor $\mathrm{SU}(N_f)$ group survives, however. This is the well-known phenomenon of color-flavor locking.

Thus, the unbroken global symmetry of the bulk is

$$\mathrm{SU}(N)_{C+F}\times\mathrm{SU}(\tilde{N})\times\mathrm{U}(1), \tag{2.2}$$

where

$$\tilde{N}=N_f-N.$$

Here $\mathrm{SU}(N)_{C+F}$ represents a global unbroken color-flavor rotation, which involves the first N flavors, while the $\mathrm{SU}(\tilde{N})$ factor stands for the flavor rotation of the remaining $\tilde{N}$ quarks.

Now, let us briefly discuss the perturbative excitation spectrum. Since both U(1) and $\mathrm{SU}(N)$ gauge groups are broken by the squark condensation, all gauge bosons become massive. In particular, the mass of the $\mathrm{SU}(N)$ gauge bosons is given by

$$m\approx g\sqrt{\xi} \tag{2.3}$$

at weak coupling.

As was already mentioned, $\mathcal{N}=2$ supersymmetry remains unbroken. In fact, with the nonvanishing ξ, both the squarks and adjoint scalars combine with the gauge bosons to form long $\mathcal{N}=2$ supermultiplets with eight real bosonic components [19]. All states appear in the representations of the unbroken global group (2.2), namely, in the singlet and adjoint representations of $\mathrm{SU}(N)_{C+F}$,

$$(1,1,0), \qquad (\mathbf{Adj},1,0), \tag{2.4}$$

and in the bifundamental representations of $\mathrm{SU(N)}_{C+F}\times \mathrm{SU}(\tilde{N})$,

$$\left(\bar{\mathbf{N}},\tilde{\mathbf{N}},\frac{N_f}{2\tilde{N}}\right), \qquad \left(\mathbf{N},\bar{\tilde{\mathbf{N}}},-\frac{N_f}{2\tilde{N}}\right). \tag{2.5}$$

The representations in Eqs. (2.4) and (2.5) are labeled according to three factors in Eq. (2.2). The singlet and adjoint fields are the gauge bosons and the first N flavors of squarks q^{kP} $(P=1,\ldots,N)$, together with their fermion superpartners. In particular, the mass of adjoint fields is given by Eq. (2.3).

The physical reason behind the fact that the (s)quarks transform in the adjoint or bifundamental representations of the global group is that their color charges are screened by the condensate (2.1); therefore, they can be considered as mesons.

The bifundamental fields (2.5) represent the (s)quarks of the type q^{kK} with $K=N+1,\ldots,N_f$. They belong to short BPS multiplets with four real bosonic components. These fields are massless, provided that the matter mass terms vanish. In fact, in this case the vacuum (2.1) in which only N first squark flavors develop VEVs is not an isolated vacuum. Rather, it is a root of a Higgs branch on which other flavors can also develop VEVs. This Higgs branch forms a cotangent bundle to the complex Grassmannian

$$\mathcal{H}=T^*\mathrm{Gr}^{\mathbb{C}}_{N_f,N}, \tag{2.6}$$

whose real dimension is [20,21]

$$\dim\mathcal{H}=4N\tilde{N}. \tag{2.7}$$

The above Higgs branch is noncompact and is hyper-Kähler [20,22]; therefore, its metric cannot be modified by quantum corrections [20]. In particular, once the Higgs branch is present at weak coupling, we can continue it all the way into strong coupling. In principle, it can intersect with other branches if present, but it cannot disappear in the theory with vanishing matter mass parameters. We see below that the presence of the Higgs branch and associated massless bifundamental quarks has a deep impact on non-Abelian vortex dynamics.

The Higgs branch (2.6) has a compact base defined by the condition

$$\bar{\tilde{q}}^{kA}=0. \tag{2.8}$$

This is the complex Grassmannian of real dimension $2N\tilde{N}$. The BPS vortex solutions exist only on the base of the Higgs branch. Therefore, we limit ourselves to the vacua that belong to the base manifold.

Let us comment on the U(1) charges in Eqs. (2.4) and (2.5). The global unbroken U(1) factor in Eq. (2.2) acts as follows. Let us make a $\mathrm{U}(1)_g$ gauge transformation on quarks q^{kA} [we define the U(1) quark charge as 1/2]. To preserve the vacuum (2.1) we compensate it by the action of the generator

$$\left(-\frac{1}{2},\ldots,-\frac{1}{2};\frac{N}{2\tilde{N}},\ldots,\frac{N}{2\tilde{N}}\right), \tag{2.9}$$

which belongs to flavor $\mathrm{SU}(N_f)$. Here we separated the first N and the last $\tilde{N}$ entries. As a result, the quarks q^{kP} do not transform [hence the vacuum (2.1) is invariant], while the quarks q^{kK} acquire charges $\frac{N_f}{2\tilde{N}}$, where $P=1,\ldots,N$ and $K=N+1,\ldots,N_f$. This is reflected in Eqs. (2.4) and (2.5).

What is usually referred to as the baryonic U(1) symmetry is part of the $\mathrm{U}(N)$ gauge group in our 4D Yang-Mills theory. Still, we can identify the unbroken U(1) factor in Eq. (2.2) as a "baryonic" $\mathrm{U}(1)_B$ symmetry. The reason is clear: The baryonic operators constructed as a product of two bifundamental quarks,

$$\begin{aligned}\mathcal{B}&=\varepsilon_{KK'}\varepsilon_{ll'}q^{lK}q^{l'K'},\\ \tilde{\mathcal{B}}&=\varepsilon^{KK'}\varepsilon^{ll'}\tilde{q}_{Kl}\tilde{q}_{K'l'},\\ K,K'&=N+1,\ldots,N_f,\end{aligned} \tag{2.10}$$

have the $\mathrm{U}(1)_B$ baryonic charges

$$\begin{aligned}Q_B(\mathcal{B})&=\frac{N_f}{\tilde{N}}=2,\\ Q_B(\tilde{\mathcal{B}})&=-\frac{N_f}{\tilde{N}}=-2,\end{aligned} \tag{2.11}$$

where, in what follows, we indicated the numerical values for the case we are interested in, $N=\tilde{N}=2$.

Certainly, the physical meaning of the baryonic charge above is not the same as, say, in actual QCD. As we saw above, in our theory bifundamental quarks (which can be viewed as mesons upon Higgs screening) also carry baryonic charges. Therefore, baryons can decay into bifundamental mesons. We will see an example of such a behavior below.

The above analysis of the Higgs phase assumes weak coupling. What happens if we increase the coupling constant g^2? In fact, the bulk theory at zero ξ is invariant under S-duality, which interchanges the strong and weak coupling regimes [20,23],

$$\tau \to \tau_D = -\frac{1}{\tau}, \qquad \tau = \frac{4\pi i}{g^2} + \frac{\theta_{4D}}{2\pi}, \tag{2.12}$$

where θ_{4D} is the θ angle. Therefore, even at nonzero ξ the region of $g^2 \gg 1$ can be described in terms of the dual weakly coupled gauge theory.

B. Non-Abelian vortex strings

The presence of the global $\mathrm{SU}(N)_{C+F}$ symmetry is the reason for the formation of non-Abelian flux tubes (vortex strings) [8–11]. The most important feature of these vortices is the presence of orientational and size zero modes. In $\mathcal{N} = 2$ bulk theory these strings are $1/2$ BPS saturated; hence, their tension is determined exactly by the FI parameter; see Eq. (1.3).

Non-Abelian vortices confine BPS monopoles of the four-dimensional theory. However, as was already mentioned, the monopoles cannot be attached to the string ends. In fact, in the $\mathrm{U}(N)$ theories confined elementary monopoles are junctions of two "neighboring" non-Abelian strings; see Ref. [14] and Sec. VI for a more detailed discussion.

Let us have a closer look at the effective world-sheet theory for a non-Abelian vortex. The dynamics of the translational modes (which are also present for the conventional ANO string) in the Polyakov formulation [24] is described by the action

$$S_{\mathrm{tr}} = \frac{T}{2}\int d^2\sigma\sqrt{h}h^{kl}\partial_k x^\mu \partial_l x_\mu, \tag{2.13}$$

where σ^k ($k = 1,\ 2$) are the world-sheet coordinates, x^μ ($\mu = 1, \ldots, 4$) are 4D coordinates, and $h = \det(h_{kl})$, where h_{kl} is the world-sheet metric which is understood as an independent variable.

If one chooses $N_f = N$, the dynamics of the orientational zero modes on the non-Abelian vortex (they become orientational moduli fields on the world sheet) would be described by a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric $\mathbb{CP}^{N-1}$ model which is compact [8–11]; see Refs. [12–14] for reviews. Size moduli do not appear in this case. If one adds extra quark flavors, non-Abelian vortices become semilocal. They acquire size moduli (see the review paper [25] devoted to Abelian semilocal vortices).

Non-Abelian semilocal vortices in $\mathcal{N} = 2$ Yang-Mills theory with $N_f > N$ were studied in Refs. [8,11,26–28]. The world-sheet theory for the orientational (size) moduli of the semilocal vortex is given by the sigma model on the tautological bundle over the same projective space $\mathcal{O}(-1)^{\oplus \tilde{N}}_{\mathbb{CP}^{N-1}}$, where $\tilde{N} = (N_f - N)$, which we call $WCP(N, \tilde{N})$. Its GLSM formulation is as follows [16]. One introduces two types of complex fields $n^P, P = 1, \ldots, N$ and $\rho^K, K = N+1, \ldots, N_f$, which have $U(1)$ charges $+1$ and -1, respectively. The orientational moduli are described by the N-plets n^P, while the size moduli are parametrized by the $\tilde{N}$-plet ρ^K.

The effective two-dimensional theory on the world sheet has the action

$$S_{\mathrm{or}} = \int d^2\sigma\sqrt{h}\bigg\{h^{kl}(\tilde{\nabla}_k\bar{n}_P\nabla_l n^P + \nabla_k\bar{\rho}_K\tilde{\nabla}_l\rho^K) + \frac{e^2}{2}(|n^P|^2 - |\rho^K|^2 - \beta)^2\bigg\} + \text{fermions}. \tag{2.14}$$

Since fields n^P and ρ^K have charges $+1$ and -1 with respect to the gauge U(1), we have

$$\nabla_k = \partial_k - iA_k, \qquad \tilde{\nabla}_k = \partial_k + iA_k.$$

The limit $e^2 \to \infty$ is implied.[5]

The coupling constant β in Eq. (2.14) is related to the bulk coupling via

$$\beta \approx \frac{4\pi}{g^2}. \tag{2.15}$$

This formula was derived at the weak coupling regime in the bulk theory [9,10] and is quasiclassical. It is modified at strong coupling.

Note that the first (and the only) coefficient of the β function $\beta_1 = N - \tilde{N}$ is the same for the bulk and world-sheet theories. It vanishes provided $N = \tilde{N}$.

The bosonic part of the total string action for the non-Abelian vortex under consideration is the sum of Eqs. (2.13) and (2.14),

$$S = S_{\mathrm{tr}} + S_{\mathrm{or}}. \tag{2.16}$$

As was already mentioned, the two necessary conditions for a thin string regime are met for the non-Abelian semilocal vortex supported in four-dimensional $\mathcal{N} = 2$ Yang-Mills theory, provided the gauge group is $U(N = 2)$ and the number of quark hypermultiplets is $N_f = 4$ [1]. Indeed, in the conformal gauge the translational part of the action is a free theory and therefore conformal, while the β function of the orientational (size) part is proportional to $\beta_1 = N - \tilde{N}$. Thus, the condition of conformality $\beta_1 = 0$ implies

$$N = \tilde{N} \quad \text{or} \quad N_f = 2N. \tag{2.17}$$

[5] A remark in passing: In fact, the world-sheet theory on the semilocal non-Abelian string is not exactly the $WCP(N, \tilde{N})$ model [28]. Both orientational and size moduli have logarithmically divergent norms [26]. After an appropriate infrared regularization, logarithmically divergent norms can be absorbed into the definition of two-dimensional fields. The actual theory is called the zn model. Nevertheless, it has the same infrared physics as the GLSM in question [29].

Moreover, the number of orientational (and size) degrees of freedom in Eq. (2.14) is

$$2(N+\tilde{N}-1)=2(2N-1), \tag{2.18}$$

where we subtracted 2 because of the D-term condition [see the last line in Eq. (2.14) and $U(1)$. Requiring that this number is equal to 6 gives the solution[6] $N=\tilde{N}=2$, $N_f=4$. For these values of N and $\tilde{N}$ the target space of the sigma model (2.14) is resolved conifold [16]

$$Y_6=\mathcal{O}(-1)_{\mathbb{CP}^1}\oplus\mathcal{O}(-1)_{\mathbb{CP}^1}. \tag{2.19}$$

The global symmetry of our world-sheet sigma model (2.14),

$$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1), \tag{2.20}$$

is the same as the unbroken global group of the bulk theory (2.2) for $N=\tilde{N}=2$. The fields n and ρ transform in the following representations:

$$n\colon\ (\mathbf{2},0,0), \qquad \rho\colon\ (0,\mathbf{2},1). \tag{2.21}$$

C. Bulk duality vs world-sheet duality

If $\tilde{N}<N$ the bulk $\mathcal{N}=2$ Yang-Mills theory is asymptotically free. Its coupling constant g^2 is frozen at the scale $\sqrt{\xi}$. The theory is in the weak coupling regime if $\sqrt{\xi}\gg\Lambda$, where Λ is the dynamical scale. If we make $\sqrt{\xi}\ll\Lambda$ the physics can be described by weakly coupled infrared-free $\mathcal{N}=2$ SQCD with the gauge group $\mathrm{U}(\tilde{N})\times\mathrm{U}(1)^{N-\tilde{N}}$ and N_f flavors of *dual* quarks [30]; see also Ref. [31] for a review. This bulk duality is reflected in the world-sheet duality for the sigma model on the non-Abelian vortex. Namely, the coupling constant β is reflected, $\beta\to-\beta$, and the roles of N-orientational moduli n^P and $\tilde{N}$ size moduli ρ^K are interchanged [30].

In the theory at hand, $\tilde{N}=N=2$ and the SU(2) gauge coupling constant does not run. However, as was already mentioned, our bulk theory has weak-strong self-duality [Eq. (2.12)]. This duality should be reflected in the world-sheet model as well. Indeed, the world-sheet model (2.14) is obviously self-dual under the reflection of the coupling constant β,

$$\beta\to\beta_D=-\beta. \tag{2.22}$$

Under this duality the orientational and size moduli n^P and ρ^K interchange. Note that the 4D self-dual point $g^2=4\pi$ is mapped onto the 2D self-dual point $\beta=0$. The 2D coupling constant β can be naturally complexified if we include the θ term in the action of the $\mathbb{CP}^{N-1}$ model,

$$\beta\to\beta+i\frac{\theta_{2D}}{2\pi}.$$

Given the complexification of β, we expect to get a generalization of Eq. (2.22) to complex values of the coupling which has the same fix point $\beta=0$.

It was conjectured in Ref. [1] that the thin-string condition (1.1) is in fact satisfied in this theory at the strong coupling limit $g_c^2\sim1$. The conjecture is equivalent to the assumption that the mass of quarks and gauge bosons m has a singularity as a function of g^2. If we assume, for simplicity, that there is only one singular point, then by symmetry, a natural choice is the self-dual point $\tau_c=i$ or $g_c^2=4\pi$. This gives

$$m^2\to\xi\times\begin{cases} g^2 & g^2\ll1\\ \infty & g^2\to4\pi\\ 16\pi^2/g^2 & g^2\gg1, \end{cases} \tag{2.23}$$

where the dependence of m at small and large g^2 follows from the tree-level formula (2.3) and duality (2.12).

Thus, we expect that the singularity of mass m lies at $\beta=0$. This is the point where the non-Abelian string becomes infinitely thin, higher derivative terms can be neglected, and the theory of the non-Abelian string reduces to Eq. (2.16). The point $\beta=0$ is a natural choice because at this point we have a regime change in the 2D sigma model *per se*. This is the point where the resolved conifold defined by the D term in Eq. (2.14) develops a conical singularity [32].

The term "thin string" should be understood with care. As was mentioned previously, the target space of our sigma model is *noncompact*; see Eq. (2.19). Since the noncompact string moduli ρ^K have the string-size interpretation, one might think that at large $|\rho|$ our string is not thin. Note that by the thin-string condition (1.1), we mean that the string core is thin, and higher-derivative corrections run in powers of ∂^2/m^2 and are negligible.

Note that there are massless states in the bulk theory, namely, bifundamental quarks (2.5) which give rise to the continuous spectrum. Most of these light modes are *not* localized on the string and do not participate in the string dynamics. The only zero modes which are localized (in addition to the translational modes) are the size and the orientational modes [26] indicated in Eq. (2.14). They have logarithmically divergent norms, while other light modes are power non-normalizable in the infrared. All other localized modes are massive, with mass $\sim m$. Integrating

[6]See Ref. [1] for details of the calculation of the Virasoro central charge for our sigma model. Technically, there are two other pairs of N and $\tilde{N}$ which formally fit our construction (the vanishing beta function and vanishing Virasoro central charge): $N=1$, $\tilde{N}=3$ and $N=3$, $\tilde{N}=1$, with the ratio of the $U(1)$ charges for n^A and for ρ^K fields being equal to -3 and $-1/3$, respectively. Although it is straightforward to generalize GLSM [Eq. (2.14)], we cannot proceed further since the derivation of such GLSMs as theories of dynamical vortices in $\mathcal{N}=2$ SQCD along the lines of Ref. [26] is not available at the moment.

out these massive modes leads to higher-derivative corrections running in powers of ∂^2/m^2. They are negligible if m is large; see Eq. (1.1). We do *not* integrate out zero modes.

III. TYPE-IIA DESCRIPTION

A. Vortex string and bulk supersymmetry

In this section we discuss the space-time supersymmetry of the non-Abelian vortex superstring (2.16). Let us first describe the fermionic content of the world-sheet theory. The action of the translational sector of the string in the static gauge $\sigma_1 = x_0$, $\sigma_2 = x_3$ can be written as a free theory,

$$S_{\rm tr} = \frac{T}{2}\int d^2x\{\partial_k x^i \partial_k x^i + \bar{\zeta}_L \partial_R \zeta_L + \bar{\zeta}_R \partial_R \zeta_R\}, \tag{3.1}$$

where the world-sheet integral in the static gauge is taken over x_0 and x_3, $k = 0, 3$, while x^i are transversal translational moduli, $i = 1, 2$. There are 4 real degrees of freedom associated with complex free fermions ζ_L and ζ_R in the translational sector.

Note that we use the static gauge because the effective world-sheet theory for the string was derived in the static gauge from the solitonic vortex solution of the bulk theory [9,10].

The bosonic part of the world-sheet action for orientational-size moduli (of the GLSM) is given by Eq. (2.14). The fermionic superpartners of n^P and ρ^K are fermionic fields $\xi^P_{L,R}$ and $\chi^K_{L,R}$ made of left- and right-moving modes. They are subject to the constraint

$$\bar{n}_P \xi^P_{L,R} - \bar{\rho}_K \chi^K_{L,R} = 0. \tag{3.2}$$

These fermions are related to n^P and ρ^K via $\mathcal{N} = (2,2)$ world-sheet supersymmetry.

The total number of real degrees of freedom in the fermionic orientational-size sector is $4(N + \tilde{N} - 1) = 12$ for $N = \tilde{N} = 2$. Thus, altogether we have 16 fermions in the world-sheet theory in the static gauge. This corresponds to 32 fermions in the reparametrization invariant description (which reduces to 16 fermions upon fixing a physical gauge like light-cone or static gauge). These fermions are interpreted as θ variables in 10D space for a closed string. The number of θ variables corresponds to the number of supercharges. This number is reduced to eight upon considering the string on a six-dimensional Calabi-Yau manifold with SU(3) holonomy [33]. Eight supercharges are required in order to have $\mathcal{N} = 2$ supersymmetry in 4D space. The rest of the 10D supersymmetry is broken by the Calabi-Yau background.

As was mentioned in the Introduction, this is one of the successful tests of our picture: The 4D $\mathcal{N} = 2$ supersymmetry which we get on the string side matches with $\mathcal{N} = 2$ supersymmetry present in the bulk QCD from the very beginning. Imagine that we had open vortices in our bulk QCD. Open strings would break 4D supersymmetry down to $\mathcal{N} = 1$ on the string side. This would contradict $\mathcal{N} = 2$ supersymmetry of our initial theory. Fortunately, we do not have open vortex strings.

B. Type-IIA superstring

Given the $\mathcal{N} = 2$ supersymmetry in 4D, the next question to address is whether our vortex is described by Type-IIA or type-IIB superstring theory. To answer this question we consider 10D parity transformation. As it is well known, type-IIB string is a chiral theory, and it breaks parity; Type-IIA string theory is left-right symmetric, and it conserves parity [33].

The parity transformation acts on 4D fermions as

$$\psi^\alpha \to \bar{\tilde{\psi}}_{\dot{\alpha}} \tag{3.3}$$

(for notations see Ref. [26] or [14]). Explicit expressions presented in Ref. [34] (in the static gauge) for profile functions of the fermion zero modes show that the U(1) supertranslational and SU(2) superorientational modes are proportional to

$$\begin{aligned} &\bar{\psi}_{\dot{2}} \sim (x_1 + ix_2)\zeta_L, \qquad \bar{\psi}_{\dot{2}Pk} \sim n_P \bar{\xi}_{Lk}. \\ &\bar{\tilde{\psi}}_{\dot{1}} \sim (x_1 - ix_2)\zeta_R, \qquad \bar{\tilde{\psi}}^{kP}_{\dot{1}} \sim -\xi^k_R \bar{n}^P. \end{aligned} \tag{3.4}$$

Since $x_{1,2,3} \to -x_{1,2,3}$ and $n \to -n$, $\rho \to -\rho$ under parity transformation, we have

$$\begin{aligned} &\zeta_L \to -\bar{\zeta}_R, \qquad \zeta_R \to -\bar{\zeta}_L, \\ &\xi^P_R \to -\xi^P_L, \qquad \chi^K_R \to -\chi^K_L. \end{aligned} \tag{3.5}$$

Our 2D world-sheet theory is invariant under this transformation (3.1); thus, we conclude that the string theory of the vortex string (2.16) is of type IIA.

Certainly, this result matches our expectations because we started with $\mathcal{N} = 2$ supersymmetric Yang-Mills theory preserving 4D parity (it is a vectorlike theory). Therefore, we expect that the closed string spectrum in this theory should respect 4D parity.

IV. FOUR-DIMENSIONAL REDUCTION

In this section we discuss massless states in four dimensions which are predicted by our string theory.

A. Generalities

Now let us consider a Type-IIA string propagating in 10D space with a nonflat metric,

$$\mathbb{C}^2 \times Y_6, \tag{4.1}$$

where Y_6 is the noncompact target space of sigma model (2.14), which is a resolved Calabi-Yau conifold [32]. As was argued above, we expect that the non-Abelian vortex becomes parametrically thin and can be described by the string action (2.16) at strong coupling near the self-dual point $\beta = 0$. Therefore, below we assume that β is small, $|\beta| \ll 1$.

Strictly speaking, at small β quantum corrections in the world-sheet sigma model blow up. In other words, we can say that at small β the gravity approximation does not work. However, if we are interested in the massless states, we can perform the supergravity computations at large β and then extrapolate the results to strong coupling. The massless states in the sigma model language correspond to chiral primary operators. They are protected by $\mathcal{N} = (2,2)$ world-sheet supersymmetry. Their masses are not lifted by quantum corrections. However, kinetic terms (the Kähler potentials) can acquire corrections.

The massless 10D bosonic fields of Type-IIA string theory in flat ten dimensions are the graviton, dilaton, and antisymmetric tensor B_{MN}, in the NS-NS sector. In the R-R sector Type-IIA strings give one-form and three-form [35]. Here, $M, N = 1, \ldots, 10$ are 10D indices. We start with the massless 10D graviton and examine what states it can produce in four dimensions. In fact, the states coming from other massless 10D fields listed above can be recovered from $\mathcal{N} = 2$ supersymmetry in 4D; see, for example, Ref. [36]. We follow the standard string theory method, which is well developed for compact Calabi-Yau spaces [33]. Our only novel aspect is that for each 4D state, we have to check normalizability of its wave function over the noncompact Y_6.

The massless 10D graviton is a fluctuation of the metric

$$\delta G_{MN} = G_{MN} - G^{(0)}_{MN}$$

where $G^{(0)}_{MN}$ is the metric on Eq. (4.1) which has a block form: the flat metric for $\mathbb{R}^4$ and the Calabi-Yau metric for the conifold (see the next sections and the Appendix for an explicit expression for this metric).

The graviton should satisfy the Lichnerowicz equation

$$D_A D^A \delta G_{MN} + 2R_{MANB}\delta G^{AB} = 0, \tag{4.2}$$

where D^A and R_{MANB} are the covariant derivative and the Riemann tensor, respectively, calculated in the background $G^{(0)}_{MN}$. Here, the gauge

$$D_A \delta G^A_N - \frac{1}{2} D_N \delta G^A_A = 0$$

is imposed. For the block form of the metric $G^{(0)}_{MN}$, only the six-dimensional part R_{ijkl} of R_{MANB} is nonvanishing, while the operator $D_A D^A$ is given by

$$D_A D^A = \partial_\mu \partial^\mu + D_i D^i$$

where the indices $\mu, \nu = 1, \ldots, 4$ and $i, j = 1, \ldots, 6$ belong to flat 4D space and Y_6, respectively, and we use the 4D metric with diagonal entries $(-1, 1, 1, 1)$.

Following a standard string theory method [33] we look for solutions of Eq. (4.2) assuming the factorized form of δG_{MN},

$$\begin{aligned} \delta G_{\mu\nu} &= \delta g_{\mu\nu}(x)\phi_6(y), \\ \delta G_{\mu i} &= B_\mu(x) V_i(y), \\ \delta G_{ij} &= \phi_4(x)\delta g_{ij}(y), \end{aligned} \tag{4.3}$$

where x_μ and y_i are coordinates in $\mathbb{R}^4$ and Y_6, respectively. Moreover, $\delta g_{\mu\nu}(x)$, $B_\mu(x)$ and $\phi_4(x)$ are graviton, vector and scalar fields in 4D, while $\phi_6(y)$, $V_i(y)$ and $\delta g_{ij}(y)$ are fields on Y_6.

In order for the fields $\delta g_{\mu\nu}(x)$, $B_\mu(x)$ and $\phi_4(x)$ to be dynamical in 4D, the fields $\phi_6(y)$, $V_i(y)$ and $\delta g_{ij}(y)$ should have finite norms when integrated over the six-dimensional internal space Y_6. Otherwise, the 4D fields come with infinite kinetic energy and are not dynamical [18]. They just decouple, and this is very important.

Symbolically, the Lichnerowicz equation (4.2) can be written as

$$(\partial_\mu \partial^\mu + \Delta_6) g_4(x) g_6(y) = 0, \tag{4.4}$$

where Δ_6 is the two-derivative operator from Eq. (4.2) reduced to Y_6, while $g_4(x)g_6(y)$ symbolically denotes the factorization form (4.3). If we expand g_6 in eigenfunctions,

$$-\Delta_6 g_6(y) = \lambda_6 g_6(y), \tag{4.5}$$

the eigenvalues λ_6 will play the role of the mass squared of the 4D states.

Since our conifold is asymptotically flat, g_6 for $\lambda_6 > 0$ behaves as a plane wave at large y_i^2 and is non-normalizable. Thus, we are looking for massless 4D states with $\lambda_6 = 0$,

$$-\Delta_6 g_6(y) = 0. \tag{4.6}$$

Solutions of this equation for Calabi-Yau manifolds are given by elements of Dolbeault cohomology $H^{(p,q)}(Y_6)$, where (p, q) denotes numbers of holomorphic and anti-holomorphic indices in the form. The dimensions of these spaces $h^{(p,q)}$ are called Hodge numbers for a given Y_6.

B. 4D graviton

For the 4D graviton $g_{\mu\nu}(x)$ in Eq. (4.3), Eq. (4.6) takes the form

$$-D_i D^i \phi_6 = -D_i \partial^i \phi_6 = 0. \tag{4.7}$$

It has only one solution,

$$\phi_6(y) = \text{const.} \tag{4.8}$$

For a compact Calabi-Yau space this is expressed as $h^{(0,0)} = 1$ and leads to the presence of a single graviton in 4D. For the conifold under consideration the solution (4.8) has an infinite norm on Y_6, so there is no 4D graviton in our theory.

This result is expected and most welcome. As was already mentioned, the original $\mathcal{N}=2$ Yang-Mills theory in four dimensions had no gravity and, therefore, we do not expect a 4D graviton to appear as a closed string state. The result above is a nontrivial check of our approach and, in particular, of the validity of the main conjecture of the thin-string regime for vortex strings (1.1).

The non-normalizability of wave function (4.8), besides the graviton, also rules out other 4D states of the $\mathcal{N}=2$ gravitational and tensor multiplets: the vector field, the dilaton, the antisymmetric tensor and the two scalars coming from the 10D three-form.

Note also that even if we "forgot" about the GSO projection, the tachyon would be absent in 4D anyway due to the non-normalizability of Eq. (4.8).

C. Killing vectors

Consider now the second option in Eq. (4.3): The 10D graviton $\delta G_{\mu i}$ gives rise to a vector field in 4D. This possibility is related to the presence of continuous symmetries on Y_6. Our conifold Y_6 has a global symmetry, so we expect to have seven Killing vectors associated with the generators of Eq. (2.20).

The Killing vectors obey the following equation:

$$D_i V_j^m + D_j V_i^m = 0, \qquad m = 1, \ldots, 7. \tag{4.9}$$

For the Calabi-Yau manifold it then follows that V_i should satisfy Eq. (4.6), which reads

$$D_j D^j V_i^m = 0. \tag{4.10}$$

Being integrated by parts over compact Calabi-Yau spaces, this equation implies that V_i is a covariantly constant vector $D_j V_i = 0$. Such vectors are incompatible with the SU(3) holonomy. This leads to the conclusion that there are no global continuous symmetries on compact Calabi-Yau manifolds [33].

For noncompact Y_6 this conclusion can be avoided, and we expect the presence of seven Killing vectors associated with the symmetry (2.20). However, it is easy to see that V_i^m, produced by rotations of coordinates y_i by the generators of Eq. (2.20), do not fall off at large y_i^2 (where the Y_6 metric tends to be flat). Thus, they are non-normalizable, and the associated 4D vector fields $B_\mu(x)$ are absent.

This result also matches our expectations. Vector fields $B_\mu(x)$ naturally have the interpretation of gauge fields. Their presence would mean that we have a low energy gauge group (2.20) in 4D. However, as we explained in Sec. II A, symmetry (2.20) is a global unbroken group of our bulk $\mathcal{N}=2$ QCD. It is not gauged. Therefore, the presence of gauge fields $B_\mu(x)$ would lead to inconsistency of our picture. Happily, they are absent.

Moreover, as was noted in Sec. I, massless gauge fields, if present at strong coupling, could be continued all the way to the weak coupling domain. Then their presence would contradict the quasiclassical analysis of Sec. II A, where it is shown that we do not have massless gauge multiplets at weak coupling.

D. Physical nature of non-normalizable modes

If we were studying the fundamental string on a non-compact Calabi-Yau space, we would conclude that the string propagates in the full 10D space and that its 4D subspace has no special role. However, our string is a solitonic vortex in 4D gauge theory. Clearly, we have to interpret string states as states living in this 4D theory. Most of the string states are not localized near the 4D subspace, and from the 4D perspective they represent non-normalizable states. What is the physical nature of these non-normalizable modes, in particular, those we found above?

One option is that non-normalizable modes, being non-dynamical, correspond to the coupling constants of 4D theory [18]. One example of this is the 4D graviton considered above. It comes with the infinite kinetic term; hence, the 4D metric cannot fluctuate. It is fixed to be flat and can be viewed as a fixed background rather than a dynamical field. In other words, the 4D "Planck mass" is infinite in our theory.

Another example is the 4D gauge fields $B_\mu(x)$ associated with the Killing vectors. As was noted above, they correspond to gauging of the global bulk symmetry (2.2) which, if present, would contradict the consistency of our picture. However, these gauge fields also come with the infinite kinetic terms, which means that the gauge coupling constants of these fields are in fact zero. This confirms that the symmetry (2.2) is global rather than local.

The most straightforward example of this situation will be discussed in Sec. V. We will see that the coupling constant β is a non-normalizable modulus of 4D theory.

There are also non-normalizable massive 4D states associated with the continuous spectrum of Eq. (4.5). We interpret these modes as follows. For these modes the associated integrals over Y_6 are divergent at large y^i. Large y^i means large n^P and ρ^K; see Eq. (2.14). In particular, ρ^K have a size moduli interpretation; they represent long-range tails of the non-Abelian vortex in the directions orthogonal to the string axis. The very presence of these long-range tails (and logarithmic divergence of orientational and size zero modes [26]) is related to the presence of the Higgs branch (2.7) and associated massless bifundamental quarks (2.5).

We see that the wave functions of non-normalizable states are saturated at large distances from the vortex string axis in four dimensions. Therefore, these states are *not* localized on the string. The infinite norm of these states is interpreted as an instability. These states are massive and therefore unstable. Namely, they decay into massless bifundamental quarks.

As we already mentioned in the Introduction, the vortex string of Ref. [1] is conceptually different in comparison with fundamental string theory. In the theory of fundamental strings, *all* states present in four dimensions are string states. The string theory for vortex strings of Ref. [1] is slightly different. The string states should describe only nonperturbative physics at strong coupling, such as mesons and baryons. The perturbative states seen at weak coupling are not described by this theory. In particular, the Higgs branch (and associated massless bifundamental quarks) found at weak coupling can be continued to the strong coupling. It can intersect other branches but cannot disappear (for quarks with the vanishing mass terms) [20].

V. DEFORMATIONS OF THE CONIFOLD METRIC

In this section we consider the last option in Eq. (4.3), namely, 4D scalar fields associated with deformations of the conifold metric $\delta g_{ij}(y)$. Equation (4.6), in this case, reduces to the Lichnerowicz equation on Y_6, namely,

$$D_k D^k \delta g_{ij} + 2R_{\mathrm{ikjl}} \delta g^{kl} = 0. \tag{5.1}$$

Solutions of this equation for the Calabi-Yau spaces reduce to deformations of the Kähler form or deformations of complex structure [18,32]. For a generic Calabi-Yau manifold the numbers of these deformations are given by $h^{(1,1)}$ and $h^{(1,2)}$, respectively. Before describing these deformations we briefly review conifold geometry.

A. Conifold

The target space of the sigma model (2.14) is defined by the D-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \tag{5.2}$$

and the U(1) phase is gauged away. We can construct the U(1) gauge invariant variables to be referred to as "mesonic,"

$$w^{PK} = n^P \rho^K. \tag{5.3}$$

In terms of these variables the condition (5.2) can be written as

$$\det w^{PK} = 0, \tag{5.4}$$

or, alternatively,

$$\sum_{\alpha=1}^{4} w_\alpha^2 = 0, \tag{5.5}$$

where

$$w^{PK} = \sigma_\alpha^{PK} w_\alpha$$

and σ matrices are chosen $(1, -i\tau^a)$, $a = 1, 2, 3$. Equation (5.5) defines the conifold, which is a cone whose section is $S_2 \times S_3$.

At $\beta = 0$ this conifold develops a conical singularity, and both S_2 and S_3 shrink to zero. It has the Kähler Ricci-flat metric and represents a noncompact Calabi-Yau manifold [16,32,37]. The explicit form of this metric is [37]

$$ds^2 = dr^2 + \frac{r^2}{6}(ds_1^2 + ds_2^2) + \frac{r^2}{9} ds_3^2, \tag{5.6}$$

where

$$ds_1^2 = d\theta_1^2 + (\sin\theta_1)^2 d\varphi_1^2, \tag{5.7}$$

$$ds_2^2 = d\theta_2^2 + (\sin\theta_2)^2 d\varphi_2^2, \tag{5.8}$$

$$ds_3^2 = (d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2)^2. \tag{5.9}$$

Here, r is the radial coordinate on the cone, while the angles above are defined at $0 \le \theta_{1,2} < \pi$, $0 \le \varphi_{1,2} < 2\pi$, $0 \le \psi < 4\pi$.

The volume integral associated with this metric is

$$(\mathrm{Vol})_{Y_6} = \frac{1}{108} \int r^5 dr d\psi d\theta_1 d\varphi_1 d\theta_2 d\varphi_2 \sin\theta_1 \sin\theta_2. \tag{5.10}$$

We can introduce another radial coordinate,

$$\tilde{r}^2 = \sum_{\alpha=1}^{4} |w_\alpha|^2.$$

It is related to r in (5.6) via [37]

$$r^2 = \frac{3}{2} \tilde{r}^{4/3}. \tag{5.11}$$

The conifold singularity can be smoothed in two different ways: by deformation of the Kähler form or deformation of the complex structure. The first option is called a "resolved conifold" and amounts to introducing nonzero β in Eq. (5.2). This resolution preserves Kähler structure and Ricci flatness of the metric. If we put $\rho^K = 0$ in Eq. (5.2), we get the $\mathbb{CP}^1$ model with target space S^2 of radius $\sqrt{\beta}$. The explicit metric for the resolved conifold can be found in Refs. [37–39]; see also the Appendix.

If $\beta = 0$ there is another option—deformation of the complex structure. It also preserves the Kähler property and Ricci flatness of the metric of the conifold. This is called a "deformed conifold." It is defined by deformation of Eq. (5.5), namely,

$$\sum_{\alpha=1}^{4} w_\alpha^2 = b, \tag{5.12}$$

where b is a complex number. Now, if we take the radial coordinate $\tilde{r} = 0$, the S_3 does not shrink to zero; its size is determined by b. The explicit metric on the deformed conifold is presented in Refs. [37,40,41]; see the Appendix.

B. Kähler structure deformations

Consider the 4D scalar field $\beta(x)$ associated with deformation of the Kähler form of the conifold β; see Eq. (5.2). The effective action for this field is

$$S(\beta) = T \int d^4x h_\beta (\partial_\mu \beta)^2, \tag{5.13}$$

where the metric $h_\beta(\beta)$ is given by the normalization integral over the conifold Y_6,

$$h_\beta = \int d^6 y \sqrt{g} g^{li} \left(\frac{\partial}{\partial \beta} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \beta} g_{kl} \right). \tag{5.14}$$

Here, $g_{ij}(\beta)$ is the resolved conifold metric, while g is its determinant. Using the explicit expression for the resolved conifold metric (A1), we find

$$g^{li} \left(\frac{\partial}{\partial \beta} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \beta} g_{kl} \right) = \frac{90}{r^4} \tag{5.15}$$

to the leading approximation at small β. Taking into account the volume integral (5.10), we arrive at the following β normalization integral:

$$h_\beta = (4\pi)^3 \frac{5}{6} \int dr r = \infty. \tag{5.16}$$

It is seen that the β normalization integral is quadratically divergent in the infrared. Thus, the scalar 4D $\beta(x)$ decouples in the bulk QCD; it is not represented by a localized state.

As was already mentioned, β can be naturally complexified; see Sec. II C. On the string theory side the imaginary part of β comes from the 10D antisymmetric tensor. Moreover, in Type-IIA superstring the complex scalar β is a part of the $\mathcal{N} = 2$ massless vector multiplet which also includes a 4D vector field coming from the 10D three-form (see Ref. [36] for a review). All fields of this 4D massless vector multiplet are nondynamical because of their infinite norm on Y_6.

Much in the same way as in the case of massless vector multiplets associated with the Killing vectors, the absence of the vector β multiplet matches our expectations. Indeed, massless gauge fields, if present at strong coupling, could be continued all the way up to the weak coupling domain where their presence would contradict the quasiclassical analysis of Sec. II A.

As was explained in Sec. IV D, non-normalizable modes can be interpreted as (frozen) coupling constants in 4D bulk theory. The β field is the most straightforward example of this since the 2D coupling β is known to be related to the 4D coupling.

C. Complex structure deformations

Now let us focus on the singular point $\beta = 0$. At this self-dual value of the coupling constant, there is different deformation of the conifold metric satisfying Eq. (5.1). Namely, the deformation of the complex structure (5.12) induced by the complex modulus b. The effective action for this field is

$$S(b) = T \int d^4x h_b |\partial_\mu b|^2, \tag{5.17}$$

where the metric $h_b(b)$ is given by the normalization integral over the conifold Y_6,

$$h_b = \int d^6 y \sqrt{g} g^{li} \left(\frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \bar{b}} g_{kl} \right). \tag{5.18}$$

Here, $g_{ij}(b)$ is the deformed conifold metric.

We calculate h_b below using two distinct methods. The first one follows the general framework developed in Ref. [18].[7]

Using the constraint (5.12) we can nominate, say, w_2, w_3 and w_4 as independent variables. Then the volume form of the Y_6 conifold can be written as

$$(\mathrm{Vol})_{Y_6} \sim \int \left| \frac{dw_2 dw_3 dw_4}{w_1} \right|^2. \tag{5.19}$$

The metric (5.18) can be expressed as

$$h_b \sim \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{b}} \int \left| \frac{dw_2 dw_3 dw_4}{w_1} \right|^2, \tag{5.20}$$

(see Eq. [42]). Calculating the derivatives under the constraint (5.12), we arrive at

$$h_b \sim \int \frac{d\tilde{r}}{\tilde{r}} \sim \log \frac{\tilde{r}^2_{\max}}{|b|}, \tag{5.21}$$

where the logarithmic integral at small distances is cut off by the minimal size of S_3, which is equal to $|b|$.

Now let us verify this result by explicit calculations. Starting from the explicit expression for the deformed conifold metric (A3), we obtain (to the leading order in b)

$$g^{li} \left(\frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \bar{b}} g_{kl} \right) = \frac{(\sin \psi)^2}{\tilde{r}^4}, \tag{5.22}$$

[7] We are very grateful to Cumrun Vafa for illuminating communications and for bringing our attention to this paper.

where $\tilde{r}$ is given by Eq. (5.11). Substituting this into the volume integral (5.10) and using the relation (5.11), we finally get

$$h_b = (4\pi)^3 \frac{4}{3} \log \frac{\tilde{r}_{\max}^2}{|b|}. \tag{5.23}$$

It is seen that the norm of the field $b(x)$ is logarithmically divergent in the infrared. The modes with logarithmically divergent norms are on the borderline between normalizable and non-normalizable modes. Usually such states are considered as "localized" on the string. We follow this rule. In our framework (vortex string vs string theory) we can relate this logarithmic behavior with the marginal stability of the b state; see Sec. VI. In fact, this mode is localized on the string in the same sense as the orientational and size zero modes are localized on the vortex solution in the bulk theory: They also have logarithmically divergent norms in the infrared in 4D space [26].

The upper bound in Eq. (5.21) can be related to the (infinite) size L of $\mathbb{R}^4$. Noting;[8] that $\tilde{r}_{\max} \sim |n_{\max}\rho_{\max}| \sim \xi L^2$, we finally get

$$h_b = (4\pi)^3 \frac{4}{3} \log \frac{\xi^2 L^4}{|b|}. \tag{5.24}$$

In Type-IIA superstring the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters, in fact, as a 4D hypermultiplet. Thus, our 4D scalar b is part of a hypermultiplet. Another complex scalar $\tilde{b}$ comes from the 10D three-form (see Ref. [36] for a review). Together they form the bosonic component of the 4D $\mathcal{N} = 2$ hypermultiplet. Thus, we expect that the bosonic part of the full effective action for the b hypermultiplet takes the $\mathrm{SU}(2)_R$ invariant form,

$$S(b) = T \int d^4x \{|\partial_\mu b|^2 + |\partial_\mu \tilde{b}|^2\} \log \frac{T^4 L^8}{|b|^2 + |\tilde{b}|^2}, \tag{5.25}$$

where we absorb the constant in front of the logarithm term in Eq. (5.24) into field normalization. The fields b and $\tilde{b}$, being massless, can develop VEVs. Thus, we have a new Higgs branch with the metric determined by the logarithmic factor in Eq. (5.25). This branch develops only at the self-dual value of the coupling constant $g^2 = 4\pi$. Due to the nonrenormalization theorem of Ref. [20], the logarithmic Higgs branch metric (5.25) does not depend on the 4D coupling constant g^2.

To conclude this section we would like to stress that the presence of the new "nonperturbative" Higgs branch at a single point $g^2 = 4\pi$ at strong coupling is more successful evidence for the validity of our picture. Indeed, a hypermultiplet is a BPS state. Were it present in some interval of τ at strong coupling, it could be continued all the way up to weak coupling where its presence would contradict[9]; the quasiclassical analysis; see Sec. II A.

[8]See Sec. IV D for a more detailed explanation.

VI. PHYSICAL INTERPRETATION OF STRING STATES

In this section we reveal a physical interpretation of the b state as a monopole-monopole baryon.

A. String states at weak coupling

Consider first the weak coupling region $g^2 \ll 1$ in $\mathcal{N} = 2$ SQCD. Since squarks develop condensates (2.1), non-Abelian vortices confine monopoles. As was already mentioned, confined elementary monopoles are in fact junctions of two distinct elementary non-Abelian strings [10,11,43]. As a result, in the bulk SQCD we have monopole-antimonopole mesons in which the monopole and antimonopole are connected by two confining strings; see Fig. 1(a). In the $\mathrm{U}(N)$ gauge theory we can have baryons appearing as a closed necklace configuration [14]. For the U(2) gauge group this necklace configuration consists of two monopoles; see Fig. 1(b).

Moreover, monopoles acquire quantum numbers with respect to the global symmetry group (2.2). To see this note that in the world-sheet theory on the vortex string, the confined monopole is seen as a kink interpolating between two different vacua (which are distinct elementary non-Abelian strings) of the corresponding 2D sigma model [10,11,43]. On the other hand, we know that the sigma model kinks at strong coupling are described by n^P and ρ^K fields [44,45] [for the sigma model described by Eq. (2.14), it was shown in Ref. [46]] and therefore transform in the fundamental representations[10] of non-Abelian factors in Eq. (2.2).

As a result, monopole-antimonopole mesons and baryons in our case can be singlets or triplets of both SU(2) global groups in Eq. (2.2), as well as in the bifundamental representations. With respect to baryonic $\mathrm{U}(1)_B$ symmetry in Eq. (2.2), the mesons at hand have charges $Q_B(\text{meson}) = 0,\ 1$, while baryons can have charges

$$Q_B(\text{baryon}) = 0, 1, 2 \tag{6.1}$$

[see (2.21)]. All these nonperturbative stringy states are heavy, with mass of the order of $\sqrt{\xi}$, and therefore can decay into screened quarks which are lighter and, eventually, into massless bifundamental screened quarks [Eq. (2.5)].

[9]In principle, one can avoid this conclusion if other massless BPS states are present. Together they can combine into a massive non-BPS multiplet.

[10]Strictly speaking, to make both bulk monopoles and world-sheet kinks well defined as localized objects, we should introduce an infrared regularization, say, a small quark mass term. When we take the limit of the zero quark masses, the kinks become massless and smeared all over the closed string. However, their global quantum numbers stay intact.

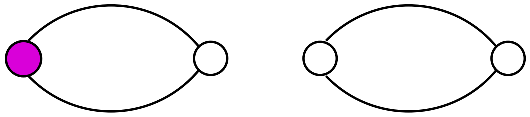

FIG. 1. (a) Monopole-antimonopole stringy meson. (b) Monopole-monopole stringy baryon. Open and closed circles denote the monopole and antimonopole, respectively.

B. Monopole-monopole baryon

Now we pass to the self-dual point $\beta = 0$ in the strong coupling region. We show that the b state of the string associated with the deformation of the complex structure of the deformed conifold can be interpreted as a baryon constructed from two monopoles; see Fig. 1(b). From Eq. (5.12) we see that the complex parameter b (which is promoted to a 4D scalar field) is a singlet with respect to two SU(2) factors of the global world-sheet group (2.20). What about its baryonic charge? Since

$$w_\alpha = \frac{1}{2}\mathrm{Tr}[(\bar{\sigma}_\alpha)_{KP} n^P \rho^K] \tag{6.2}$$

we see that the b state transforms as

$$(1, 1, 2), \tag{6.3}$$

where we used Eqs. (2.5) and (5.12). In particular, it has baryon charge $Q_B(b) = 2$.

Since the world-sheet and the bulk global symmetries are isomorphic, we are led to the conclusion that the massless b hypermultiplet is a monopole-monopole baryon with the quantum numbers (6.3) under symmetry (2.20).

We have observed that at infinite coupling of the two-dimensional theory ($\beta = 0$), a new "exotic" Higgs branch opens up, which is parametrized by the VEV of the hypermultiplet of the effective string compactification. This branch emanates only from that locus and does not exist at nonzero β. Being massless, this state is marginally stable at $\beta = 0$ and can decay into a pair of massless bifundamental quarks in the singlet channel with the same baryon charge $Q_B = 2$; see Eq. (2.10). The b hypermultiplet does not exist at nonzero β. One way to interpret this fact in terms of bulk SQCD is as follows. The b hypermultiplet may have a "wall of marginal stability" in the complex β plane—a closed loop shrunk to a single point $\beta = 0$. Outside this point the b hypermultiplet does not exist as a stable state, but at this point it is marginally stable.

This interpretation is supported by the logarithmic divergence of the norm of the b state kinetic term (5.25), which in turn suggests that the b state is only marginally stable. Detailed studies of how this can happen and how the b hypermultiplet interacts with massless bifundamental quarks is left for future work.

VII. CONCLUSIONS

In this paper we studied the massless spectrum produced by closed non-Abelian vortex strings in $\mathcal{N} = 2$ QCD with the U(2) gauge group and $N_f = 4$ flavors of quark multiplets. We interpreted 4D closed string states as a hadrons of the bulk QCD. Most of the string states turn out to be nondynamical due to the noncompactness of the six-dimensional internal Calabi-Yau space Y_6. In particular, we showed the absence of the 4D graviton and unwanted vector fields in full accordance with the expected properties of $\mathcal{N} = 2$ bulk QCD. We found one massless 4D hypermultiplet associated with deformations of the complex structure of the conifold Y_6. This state is present only at the self-dual point $g^2 = 4\pi$. We interpreted it as a baryon constructed from two monopoles connected by confining strings; see Fig. 1(b).

We expect that this massless hypermultiplet is the lowest state of the whole Regge trajectory of states with higher spins in 4D. Since 4D space is flat, we expect this Regge trajectory to be linear with respect to spin J. The explicit construction of this Regge trajectory is left for a future work.

Let us make some comments to connect our results with other developments in string theory. Non-Abelian vortices appear as D2-branes extended along the finite interval between NS5-branes and D3-branes. The length of this interval is proportional to the FI parameter, which gives the string tension [8,11]. In some other examples within the AdS/CFT framework, the solitonic vortices turn out to be D-branes or D-strings wrapping some compact cycles [39,47,48]. Yet, to the best of our knowledge, in the current literature so far, solitonic strings have not been treated as fundamental superstrings.

In the present paper (and in Ref. [1]) we have neither assumed the presence of the ten-dimensional space-time, fundamental strings or D-branes, nor used any holographic duality. Instead, our starting point is a four-dimensional $\mathcal{N} = 2$ supersymmetric QCD. Certainly, this theory can be realized as a low-energy limit of the fundamental string theory with D-branes or via geometric engineering. However, we do not assume this construction from the beginning since our starting basic bulk theory *per se* is well defined.

Next, we explored the case $N_f = 2N$ in $\mathcal{N} = 2$ SQCD and found that it supports 1/2 BPS non-Abelian vortex strings. If $N = 2$ the world-sheet theory on this vortex has ten real moduli which can be interpreted as coordinates on the target space $\mathbb{R}^4 \times Y_6$ of the two-dimensional sigma model. This supersymmetric sigma model describes critical superstring.

Our theory predicts nonperturbative hadronic states of the original SQCD at strong coupling (at $\beta = 0$). The tension of the vortex string is fixed by the 4D Fayet-Iliopoulos term ξ, which is a scale for "strong interactions," not the bona fide Planck scale. In a sense, we returned to the early days of string theory and tried to obtain (supersymmetric) hadrons as closed string excitations of a solitonic SQCD string. It turns out that in a proper setup it is possible.

Within our approach we certainly should not think of the solitonic vortex string [1] as from a D-brane since we do not have any supergravity or D-branes to begin with.[11] However, it would be stimulating to find a possible connection between the results reported in Refs. [1,2] and the literature on solitonic strings engineered in string theory. Presumably, one can see the spectrum of light states which we described in this work by applying some string dualities.

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[11] We are deeply indebted to Igor Klebanov for raising this issue, bringing our attention to Refs. [39,47,48], and suggesting that there might be a string theory S duality which relates D strings and fundamental strings.

APPENDIX: METRICS OF RESOLVED AND DEFORMED CONIFOLDS

The Kähler, Ricci flat metric on the resolved conifold has the form [37–39]

$$ds^2 = \kappa(r)^{-1}dr^2 + \frac{r^2}{6}ds_1^2 + \frac{1}{6}(r^2+6\beta)ds_2^2 + \kappa(r)\frac{r^2}{9}ds_3^2, \tag{A1}$$

where the angle differentials are defined in Eq. (5.7), while the function $\kappa(r)$ is equal to

$$\kappa(r) = \frac{r^2+9\beta}{r^2+6\beta}. \tag{A2}$$

Consider now the metric on the deformed conifold. The deformation (5.12) preserves Kähler structure and Ricci flatness of the conifold metric. The metric of the deformed conifold has the form [37,40,41]

$$ds^2 = |b|^{2/3}K(u)\left\{\frac{(\sinh u)^3}{3(\sinh 2u - 2u)}(du^2+ds_3^2) + \frac{\cosh u}{4}(ds_1^2+ds_2^2) + \frac{1}{2}ds_4^2\right\}, \tag{A3}$$

where angle differentials are defined in Eq. (5.7), while

$$ds_4^2 = \sin\psi(\sin\theta_1 d\theta_2 d\varphi_1 + \sin\theta_2 d\theta_1 d\varphi_2) + \cos\psi(d\theta_1 d\theta_2 - \sin\theta_1\sin\theta_2 d\varphi_1 d\varphi_2). \tag{A4}$$

Here

$$K(u) = \frac{(\sinh 2u - 2u)^{1/3}}{2^{1/3}\sinh u}, \tag{A5}$$

and the radial variable u is defined as

$$\tilde{r}^2 = |b|\cosh u. \tag{A6}$$

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