# Novel nonlinear kinetic terms for gravitons

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A set of novel derivative terms for spin-2 fields are proposed. They are the wedge products of curvature two-forms and vielbeins. In this work, we investigate the properties of novel two-derivative terms in the context of bigravity. Based on a minisuperspace analysis, we identify a large class of bigravity models where the Boulware-Deser ghost could be absent. We give a new perspective that Weyl gravity and new massive gravity belong to this class of bigravity models involving novel derivative terms. This is related to the fact that this class of models contains spin-2 ghosts. In addition, we discuss the UV cutoff scales, dynamical symmetric conditions, and novel higher-derivative terms.

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## I. INTRODUCTION

In the pursuit of going beyond Einstein, extensive efforts have been devoted to constructing gravitational theories that are different from general relativity. It is believed that the Einstein-Hilbert action with a cosmological constant is the only consistent nonlinear action for a single massless spin-2 field, so additional ingredients are necessary: Fierz-Pauli theory gives the graviton a mass term [1], Brans-Dicke theory introduces a scalar degree of freedom [2], and Lovelock theory involves higher spacetime dimensions and higher-derivative terms [3].

Motivated by the unexpected accelerating expansion of the present Universe [4], more models were constructed in recent attempts, such as high-derivative scalar theories [5] and nonlinear massive gravity [6]. One of the guiding principles is that a consistent model should be free of Ostrogradsky's ghost arising from higher-order equations of motion.

Antisymmetrization is a universal element in these new models. Based on this ingredient, a general framework was developed for ghost-free,[1] Lorentz-invariant, Lagrangian field theories [7,8]. In this framework, a set of novel kinetic terms for spin-2 fields were proposed:

$$\mathcal{L}_{\mathrm{kin}} = R(E^{(1)}) \wedge E^{(2)} \wedge \cdots \wedge E^{(d-1)}, \tag{1}$$

where $d$ is the number of spacetime dimensions. They are the wedge products of geometric differential forms: a curvature two-form and several vielbeins. The vielbeins $E^{(k)}$ can be the same or different.[2] Geometric intuition was used to construct these nonlinear terms.

Along the lines of Lovelock theory, we can build novel higher-derivative terms for spin-2 fields by introducing more curvature two-forms into the wedge products,

$$R(E^{(1)}) \wedge \cdots \wedge R(E^{(1)}) \wedge E^{(2)} \wedge \cdots \wedge E^{(n)}, \tag{2}$$

which is possible when spacetime has more than four dimensions. Lovelock terms correspond to the cases where all the vielbeins are the same.

If the wedge products do not involve derivative terms, they are nonlinear potential terms for spin-2 fields,

$$\mathcal{L}_{\mathrm{pot}} = E^{(1)} \wedge \cdots \wedge E^{(d)}, \tag{3}$$

which include the cosmological constant term and other interacting potentials for spin-2 fields [6,9–11] in the vielbein formulation [10].[3] These potential terms are usually free of the Boulware-Deser (BD) ghost.[4] For simplicity, they are denoted by de Rham–Gabadadze–Tolley (dRGT) terms.[5]

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[1]Let us clarify that "ghost-free" in this framework means the building blocks are potentially free of Ostrogradsky's scalar ghost. In other words, the corresponding equations of motion for the scalar modes could be at most of second order. The scalar modes may come from the Helmholtz-Hodge decomposition of tensor fields.

[2]If all the vielbeins coincide, we have the standard Einstein-Hilbert kinetic term.

[3]In 1970, Wess and Zumino proposed using vielbeins, rather than metrics, as the building blocks of the low energy effective potentials for spin-2 fields [12]. However, it is still not fully clear why the vielbein formulation is more efficient in eliminating the Boulware-Deser ghost.

[4]In a concrete model, the BD ghost may be present even if the building blocks themselves are free of it. For example, the BD ghost is excited if loop-type interactions are introduced in the metric formulation [10]. Analogously, we find constraints on novel derivative terms.

[5]We refer to [13] for reviews of this subject.

 

The search for new derivative interactions was initiated in [14], where a new BD-ghost-free term in 4d was discovered and it reads

$$h_{\mu}{}^{[\mu} h_{\nu}{}^{\nu} \partial_{\rho} \partial^{\rho} h_{\sigma}{}^{\sigma]}. \tag{4}$$

This cubic term can be thought of as a generalization of the perturbative Lovelock and dRGT terms and there are more possible terms in higher dimensions [15]. It was also conjectured that they should admit nonlinear completions in parallel to the dRGT terms in the context of massive gravity [15].

Some of the novel kinetic terms in (1) are nonlinear, multigravity completions of this cubic term in terms of differential forms. If we consider two spin-2 fields, impose the symmetric condition [16], and fix the second metric to Minkowski, the novel kinetic terms reduce to the two-derivative terms proposed in [17], which are nonlinear derivative terms for a massive graviton around a Minkowski background. They can also be obtained by dimensionally deconstructing the Gauss-Bonnet term [18].

In the discussions of [17,18], only one metric is dynamical and the Boulware-Deser ghost was shown to be present. In this work, we will not make the single dynamical metric assumption, and the conclusion is different concerning the fate of the Boulware-Deser ghost.

In [18], new kinetic interactions in 4d with a dynamical metric and a fixed Minkowski metric were investigated in detail. In the minisuperspace approximation, problematic $N^{-2}$ terms were found in the Lagrangians and the corresponding Hamiltonians are nonlinear in the lapse function $N$. This indicates the secondary constraint from the time derivative of $\pi_N = 0$ is an equation for $N$.[6] Then the dangerous, sixth degree of freedom remains dynamical. In 4d massive gravity, the sixth degree of freedom is ghostlike. It plagues a generic nonlinear completion of Fierz-Pauli theory and is known as the Boulware-Deser ghost [19].[7] This ghostlike degree of freedom is eliminated in nonlinear massive gravity [6] thanks to the existence of a secondary constraint and an associated tertiary constraint [20]. Besides the minisuperspace discussion, an impressive no-go theorem was established in [18] on new kinetic interactions for single dynamical metric models that are Lorentz invariant and free of the BD ghost.

Inspired by the successful extensions of massive gravity [6] to bigravity [9] and multigravity [10], we would like to examine the bigravity models involving the novel derivative terms (1). Given that pathologies were found in single dynamical metric models, it is very likely that the bigravity theories are sick as well. In fact, more recently, bigravity models with new kinetic interactions were studied in the first-order formulation [21], where negative results were presented again.[8] Other obstructions were discussed in [22] as well.

Contrary to the single dynamical metric models, our analysis in Sec. III shows that the sixth degree of freedom could be absent in some bigravity models constructed from (1). But the price to pay is that one of the linearized kinetic terms has a wrong sign or at least one of them vanishes. In the former case, we encounter spin-2 ghosts, which can lead to tree-level nonunitarity upon quantization. In the latter case, the bigravity theories are strongly coupled due to the absence of kinetic terms.

The presence of a spin-2 ghost is a well-known feature of a generic model of higher-derivative gravity. In this work, we propose a new viewpoint that, when the couplings to matter are not introduced, many types of bigravity models are equivalent to higher-derivative gravity without Ostrogradsky's scalar ghost. They include Weyl gravity, 3d new massive gravity, and some of their generalizations.

At the classical level, a spin-0 ghost is more dangerous than a spin-2 ghost. Usually, the Hamiltonian of a scalar ghost is unbounded from below, while that of a massless spin-2 field may simply vanish. In this sense, it is more crucial to eliminate the Boulware-Deser ghost. Furthermore, the Boulware-Deser ghost should be removed if a massive spin-2 field contains a correct number of dynamical degrees of freedom, which is at most five in four dimensions.

Upon quantization, a spin-2 ghost will lead to tree-level nonunitarity when coupled to matter. Let us remind that a notorious problem in quantizing gravity is that the Einstein-Hilbert action is nonrenormalizable [23], which is very different from the other fundamental forces. By introducing higher-order curvature terms (thus unitarity is sacrificed), one can improve the short-distance behavior of the propagators and obtain a perturbative renormalizable theory for gravity [24]. Roughly speaking, the improved high energy behavior is due to the relatively negative contributions from the ghostlike modes.

From a different perspective, we can consider a metric as an effective description of some microscopic physics. In an effective field theory (EFT) of gravity, higher-order curvature terms are expected in a low energy expansion, because they are compatible with the symmetries [25]. Even if gravity itself is not quantized, they can be generated by quantum corrections from the matter. Therefore, some bigravity models with novel kinetic terms belong to a

[6]The absence of an additional constraint was already found in [17], which should be present after a change of variables.

[7]By an abuse of terminology, the sixth degree of freedom of a massless spin-2 field in 4d is also denoted as the Boulware-Deser ghost.

[8]To avoid confusion, let us emphasize that we consider second-order formulation in this work, so spin connections are not independent. In other words, we assume the torsion-free condition.

special class of effective field theories of gravity where Ostrogradsky's scalar ghost is removed.

The cutoff scale of an EFT of gravity is usually associated with the Planck mass. However, if some of the high-order curvature terms have unnaturally large coefficients, then the cutoff scale will be lowered. By eliminating the BD ghost, we could increase the cutoff scale set by large high-order curvature corrections.

This paper is organized as follows. In Sec. II, we give the precise formulation of the novel kinetic terms for 4d bigravity models. In Sec. III, we carry out a minisuperspace analysis to identify the candidate theories that are free of the dangerous, sixth degree of freedom. In Sec. IV, we perform a field redefinition to obtain more explicit expressions of the novel kinetic terms. In Sec. V, we focus on the novel kinetic term $\mathcal{L}_2{}^{\rm kin}$ and discuss its relation to higher-curvature gravity. In Sec. VI, we linearize the nonlinear models and diagonalize the quadratic actions. In Sec. VII, the issue of the spin-2 ghost is discussed. In Sec. VIII, we examine the cutoff scales of the effective field theories of gravity involving novel kinetic terms. In Sec. IX, we explain how to obtain the symmetric condition from the equations of motion. In Sec. X, higher-derivative generalizations are discussed and a general argument for the absence of the BD ghost is presented. In Sec. XI, we summarize our results and discuss their implications.

## II. NOVEL KINETIC TERMS FOR BIGRAVITY

To be more specific, we mainly consider four-dimensional spacetime and models with two vielbeins/metrics. There are six possible nonlinear kinetic terms:

$$\mathcal{L}_1{}^{\rm kin} = R(E) \wedge E \wedge E, \tag{5}$$

$$\mathcal{L}_2{}^{\rm kin} = R(E) \wedge E \wedge F, \tag{6}$$

$$\mathcal{L}_3{}^{\rm kin} = R(E) \wedge F \wedge F, \tag{7}$$

$$\mathcal{L}_4{}^{\rm kin} = R(F) \wedge F \wedge F, \tag{8}$$

$$\mathcal{L}_5{}^{\rm kin} = R(F) \wedge F \wedge E, \tag{9}$$

$$\mathcal{L}_6{}^{\rm kin} = R(F) \wedge E \wedge E, \tag{10}$$

where $R(E)$, $R(F)$ are curvature two-forms,

$$R(E) = d\omega^E + \omega^E \wedge \omega^E, \tag{11}$$

$$R(F) = d\omega^F + \omega^F \wedge \omega^F. \tag{12}$$

Both $E$ and $F$ are dynamical vielbeins. $\omega^E$ and $\omega^F$ are the spin connections compatible with $E$ and $F$, respectively:

$$D^E E = dE + \omega^E \wedge E = 0, \tag{13}$$

$$D^F F = dF + \omega^F \wedge F = 0. \tag{14}$$

Notice that one of the vielbeins in $\mathcal{L}_2{}^{\rm kin}$ and $\mathcal{L}_4{}^{\rm kin}$ is a Lagrange multiplier. Another interesting observation is that $\mathcal{L}_3{}^{\rm kin}$ and $\mathcal{L}_6{}^{\rm kin}$ can be thought of as the Palatini formulation of the Einstein-Hilbert term, where the spin connections are expressed in terms of the associated vielbeins.[9]

We also have five potential terms:

$$\mathcal{L}_1{}^{\rm pot} = E \wedge E \wedge E \wedge E, \tag{15}$$

$$\mathcal{L}_2{}^{\rm pot} = E \wedge E \wedge E \wedge F, \tag{16}$$

$$\mathcal{L}_3{}^{\rm pot} = E \wedge E \wedge F \wedge F, \tag{17}$$

$$\mathcal{L}_4{}^{\rm pot} = E \wedge F \wedge F \wedge F, \tag{18}$$

$$\mathcal{L}_5{}^{\rm pot} = F \wedge F \wedge F \wedge F. \tag{19}$$

When we discuss other dimensions, the subscripts $n$ in $\mathcal{L}_n$ means that the number of $F$ vielbeins in a wedge product is $(n-1)$. The two vielbeins $E_\mu{}^A$ and $F_\mu{}^A$ are related to two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$:

$$g_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB}, \qquad f_{\mu\nu} = F_\mu{}^A F_\nu{}^B \eta_{AB}. \tag{20}$$

The nonlinear kinetic terms $\mathcal{L}_1$ and $\mathcal{L}_4$ are the standard Einstein-Hilbert kinetic terms, while $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_5$, $\mathcal{L}_6$ are novel kinetic terms for spin-2 fields.

To simplify our notation, the Levi-Civitá symbol $\epsilon_{ABCD}$ is not written explicitly in a wedge product. In terms of the components, the bigravity kinetic terms (5)–(10) and the bigravity potential terms (15)–(19) are

$$\mathcal{L}_1{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(E)_{\mu\nu}{}^{AB} E_\rho{}^C E_\sigma{}^D d^4x, \tag{21}$$

$$\mathcal{L}_2{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(E)_{\mu\nu}{}^{AB} E_\rho{}^C F_\sigma{}^D d^4x, \tag{22}$$

$$\mathcal{L}_3{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(E)_{\mu\nu}{}^{AB} F_\rho{}^C F_\sigma{}^D d^4x, \tag{23}$$

$$\mathcal{L}_4{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(F)_{\mu\nu}{}^{AB} F_\rho{}^C F_\sigma{}^D d^4x, \tag{24}$$

$$\mathcal{L}_5{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(F)_{\mu\nu}{}^{AB} F_\rho{}^C E_\sigma{}^D d^4x, \tag{25}$$

$$\mathcal{L}_6{}^{\rm kin} = \delta^{\mu\nu\rho\sigma}_{ABCD} R(F)_{\mu\nu}{}^{AB} E_\rho{}^C E_\sigma{}^D d^4x, \tag{26}$$

and

[9]The difference is that varying the action with respect to the vielbein in the curvature two-form will give rise to second-order equations. "Torsion-free condition" is not the only solution, so $E$ and $F$ are not necessarily proportional to each other.

$$\mathcal{L}_1{}^{\rm pot} = \delta^{\mu\nu\rho\sigma}_{ABCD} E_\mu{}^A E_\nu{}^B E_\rho{}^C E_\sigma{}^D d^4x, \tag{27}$$

$$\mathcal{L}_2{}^{\rm pot} = \delta^{\mu\nu\rho\sigma}_{ABCD} E_\mu{}^A E_\nu{}^B E_\rho{}^C F_\sigma{}^D d^4x, \tag{28}$$

$$\mathcal{L}_3{}^{\rm pot} = \delta^{\mu\nu\rho\sigma}_{ABCD} E_\mu{}^A E_\nu{}^B F_\rho{}^C F_\sigma{}^D d^4x, \tag{29}$$

$$\mathcal{L}_4{}^{\rm pot} = \delta^{\mu\nu\rho\sigma}_{ABCD} E_\mu{}^A F_\nu{}^B F_\rho{}^C F_\sigma{}^D d^4x, \tag{30}$$

$$\mathcal{L}_5{}^{\rm pot} = \delta^{\mu\nu\rho\sigma}_{ABCD} F_\mu{}^A F_\nu{}^B F_\rho{}^C F_\sigma{}^D d^4x, \tag{31}$$

where $R_{\mu\nu}{}^{AB}$ are the components of the curvature two-forms. The antisymmetric Kronecker delta or the generalized Kronecker delta is an antisymmetric product of the Kronecker deltas,

$$\delta^{\mu\nu\rho\sigma}_{ABCD} = \delta_A{}^{[\mu}\delta_B{}^\nu\delta_C{}^\rho\delta_D{}^{\sigma]}, \tag{32}$$

where the antisymmetrization $[\ldots]$ is not normalized. The Planck mass is not written explicitly. Greek letters $\mu, \nu, \rho, \sigma, \ldots$ indicate external spacetime indices and capital latin letters $A, B, C, D, \ldots$ denote internal Lorentz indices.

To minimalize the numbers of dynamical degrees of freedom, we impose the symmetric condition or the Deser–van Nieuwenhuizen condition [16]. In Sec. IX, we discuss how the symmetric condition originates in equations of motion.

## III. MINISUPERSPACE ANALYSIS

In this section, we study the minisuperspace approximation of the bigravity models constructed from (5)–(10). The minisuperspace analysis is a simple test of the ghost-free condition before investigating the fully nonlinear structure. Although it is not sufficient to prove healthiness, it is very efficient in detecting pathologies and gives the necessary conditions for the absence of ghostlike degrees of freedom. For example, it was used in [26] to show that loop-type interactions of multigravity in the metric formulation can excite the BD ghost.

In [18], new kinetic interactions with a single dynamical metric were ruled out, because their Hamiltonians are not linear in the lapse function in the minisuperspace approximation, which indicates the presence of the BD ghost. Despite the failure of single dynamical metric models, a large class of bigravity theories does satisfy the criterion that the minisuperspace Hamiltonians are linear in lapse functions, as we discuss below.

Now we start the minisuperspace analysis. The two metrics in the minisuperspace approximation are diagonal:

$$ds_1^2 = g^E_{\mu\nu} dx^\mu dx^\nu = -(N_1)^2 dt^2 + e^{2A} dx^2, \tag{33}$$

$$ds_2^2 = g^F_{\mu\nu} dx^\mu dx^\nu = -(N_2)^2 dt^2 + e^{2B} dx^2, \tag{34}$$

where the metric components are functions of time,

$$N_1 = N_1(t), \qquad N_2 = N_2(t), \\ A = A(t), \qquad B = B(t). \tag{35}$$

The corresponding symmetric vielbeins are

$$E_\mu{}^A = \begin{pmatrix} N_1 & 0 \\ 0 & e^A \delta_i{}^j \end{pmatrix}, \\ F_\mu{}^A = \begin{pmatrix} N_2 & 0 \\ 0 & e^B \delta_i{}^j \end{pmatrix}. \tag{36}$$

Let us consider a linear combination of $\mathcal{L}_1{}^{\rm kin}, \ldots, \mathcal{L}_6{}^{\rm kin}$, so the Lagrangian reads

$$\mathcal{L} = a_1 \mathcal{L}_1{}^{\rm kin} + a_2 \mathcal{L}_2{}^{\rm kin} + a_3 \mathcal{L}_3{}^{\rm kin} \\ + b_1 \mathcal{L}_4{}^{\rm kin} + b_2 \mathcal{L}_5{}^{\rm kin} + b_3 \mathcal{L}_6{}^{\rm kin}. \tag{37}$$

In the minisuperspace approximation, it becomes

$$\begin{aligned}\mathcal{L}_{\rm mini} = {} & a_1 \frac{1}{N_1} (\dot{A})^2 e^{3A} + b_1 \frac{1}{N_2} (\dot{B})^2 e^{3B} \\ & + a_2 \frac{1}{N_1} \left( \dot{A}^2 + 2\dot{A}\,\dot{B} - \frac{N_2}{N_1} \dot{A}^2 e^{A-B} \right) e^{2A+B} \\ & + a_3 \frac{1}{N_1} \left( 2\dot{A}\,\dot{B} - \frac{N_2}{N_1} \dot{A}^2 e^{A-B} \right) e^{A+2B} \\ & + b_2 \frac{1}{N_2} \left( \dot{B}^2 + 2\dot{A}\,\dot{B} - \frac{N_1}{N_2} \dot{B}^2 e^{B-A} \right) e^{A+2B} \\ & + b_3 \frac{1}{N_2} \left( 2\dot{A}\,\dot{B} - \frac{N_1}{N_2} \dot{B}^2 e^{B-A} \right) e^{2A+B},\end{aligned} \tag{38}$$

where some normalization factors are inserted to simplify the expression of $\mathcal{L}_{\rm mini}$. Time derivatives are denoted by dots:

$$\frac{d}{dt} A = \dot{A}, \qquad \frac{d}{dt} B = \dot{B}. \tag{39}$$

Integration by parts is performed in order to eliminate

$$\dot{N}_1, \quad \dot{N}_2, \quad \ddot{A}, \quad \ddot{B}. \tag{40}$$

The conjugate momenta can be derived from the minisuperspace Lagrangian:

$$P_A = \frac{\partial \mathcal{L}_{\rm mini}}{\partial \dot{A}}, \qquad P_B = \frac{\partial \mathcal{L}_{\rm mini}}{\partial \dot{B}}, \tag{41}$$

$$P_{N_1} = P_{N_2} = 0, \tag{42}$$

where the Eq. (42) are primary constraints. Then we derive the minisuperspace Hamiltonian by the Legendre transform

$$\mathcal{H}_{\text{mini}} = \dot{A}P_A + \dot{B}P_B - \mathcal{L}_{\text{mini}}. \tag{43}$$

Using the relations between momenta and velocities, one can express the Hamiltonian $\mathcal{H}_{\text{mini}}$ in terms of

$$N_1,\ N_2,\ A,\ B,\ P_A,\ P_B. \tag{44}$$

The explicit expression of $\mathcal{H}_{\text{mini}}$ is a fraction,

$$\mathcal{H}_{\text{mini}} = \frac{\mathcal{H}_{\text{n}}}{4\mathcal{H}_{\text{d}}}, \tag{45}$$

where the numerator $H_{\text{n}}$ and the denominator $H_{\text{d}}$ are

$$\begin{aligned}\mathcal{H}_{\text{n}} = &(b_3e^A + b_2e^B)e^{-2A}P_A^2(N_1)^3\\ &+ (a_2e^A + a_3e^B)e^{-2B}P_B^2(N_2)^3\\ &+ 2(b_3e^A + b_2e^B)e^{-(A+B)}P_AP_B(N_1)^2N_2\\ &+ 2(a_2e^A + a_3e^B)e^{-(A+B)}P_AP_BN_1(N_2)^2\\ &- (b_2e^A + b_1e^B)e^{-2A}P_A^2(N_1)^2N_2\\ &- (a_1e^A + a_2e^B)e^{-2B}P_B^2N_1(N_2)^2,\end{aligned} \tag{46}$$

$$\begin{aligned}\mathcal{H}_d = &(b_3e^A + b_2e^B)[(a_1 + b_3)e^A + (a_2 + b_2)e^B](N_1)^2\\ &+ (a_2e^A + a_3e^B)[(a_2 + b_2)e^A + (a_3 + b_1)e^B](N_2)^2\\ &- [(a_1b_2 - a_2b_3)e^{2A} + (a_2b_1 - a_3b_2)e^{2B}\\ &+ (a_1b_1 - a_3b_3)e^{A+B}]N_1N_2.\end{aligned} \tag{47}$$

We require $\mathcal{H}_{\text{mini}}$ to be linear in $N_1$ and $N_2$, so $N_1$ and $N_2$ are Lagrange multipliers. Then the secondary constraints

$$\dot{P}_{N_1} = \{P_{N_1}, \mathcal{H}_{\text{mini}}\} \approx 0, \tag{48}$$

$$\dot{P}_{N_2} = \{P_{N_2}, \mathcal{H}_{\text{mini}}\} \approx 0 \tag{49}$$

are equations for the canonical variables and could remove the scalar modes related to the BD ghost.[10] The Poisson bracket is defined as

$$\{\alpha, \beta\} = \sum_{q=A,B,N_1,N_2} \left(\frac{\partial\alpha}{\partial q}\frac{\partial\beta}{\partial P_q} - \frac{\partial\alpha}{\partial P_q}\frac{\partial\beta}{\partial q}\right). \tag{50}$$

The numerator $H_{\text{n}}$ and the denominator $H_{\text{d}}$ are polynomials of degree 3 and 2 in $N_1$ and $N_2$. To satisfy the requirement that lapse functions are Lagrange multipliers, $H_{\text{d}}$ should be a factor of $H_{\text{n}}$. This is true when only one of the three monomials in $H_d$ has a nonzero coefficient, which indicates two classes of bigravity models:

[10]Strictly speaking, we should use the total Hamiltonian that contains the primary constraints to compute the time derivative.

(i) The first class is

$$a_2 = a_3 = b_2 = b_3 = 0, \tag{51}$$

where the Lagrangian contains two Einstein-Hilbert terms and the minisuperspace Hamiltonian reads

$$\mathcal{H}^{\text{I}}_{\text{mini}} = N_1\left(\frac{e^{-3A}}{4a_1}P_A^2\right) + N_2\left(\frac{e^{-3B}}{4b_1}P_B^2\right). \tag{52}$$

One can introduce the cosmological constant terms (15) and (19). The interactions between the two metrics are through the nonlinear potential terms (16)–(18). The minisuperspace Hamiltonian is still linear in the lapse functions.

Since we have two Planck masses in front of two Einstein-Hilbert terms, one can take the limit where one of them goes to infinity. In this decoupling limit, a bigravity model reduces to that of a single dynamical metric with a fixed metric.

(ii) The second class is

$$a_1 = a_2 = a_3 = 0 \tag{53}$$

or

$$b_1 = b_2 = b_3 = 0. \tag{54}$$

The bigravity models in the second class contain at most one Einstein-Hilbert (EH) kinetic term (no EH term if $a_1 = b_1 = 0$), so one cannot take the decoupling limit that fixes one of the metrics.[11,12,13] Therefore, they are not ruled out by the no-go theorems in [18]. We focus on the case of (54) in the discussions below.

The minisuperspace Hamiltonian with (54) reads

$$\begin{aligned}\mathcal{H}^{\text{II}}_{\text{mini}} = &N_1\left[\frac{e^{-(A+B)}}{2(a_2e^A + a_3e^B)}P_AP_B\right.\\ &\left.- \frac{(a_1e^A + a_2e^B)e^{-2B}}{4(a_2e^A + a_3e^B)^2}P_B^2\right]\\ &+ N_2\left[\frac{e^{-2B}}{4(a_2e^A + a_3e^B)}P_B^2\right],\end{aligned} \tag{56}$$

[11]There exists another single metric limit

$$E - F = H/\lambda, \qquad \lambda \to \infty, \tag{55}$$

where the novel kinetic terms reduce to the Einstein-Hilbert term.

[12]The failure of obtaining nonlinear partially massless gravity from a consistent truncation of conformal gravity [27] is related to the absence of this decoupling limit.

[13]The fact that only certain combinations of kinetic terms are allowed is analogous to the scalar-tensor theories discussed in [28], where the degeneracy conditions can break down for some combinations of degenerate Lagrangians.

where we assume $a_2$ and $a_3$ are not zero at the same time.

We can introduce the potential terms $\mathcal{L}_1{}^{\rm pot}, \ldots, \mathcal{L}_5{}^{\rm pot}$. The lapse functions will still be Lagrange multipliers in the Hamiltonians.

From the holographic point of view, the diagonal diffeomorphism invariance is fundamental in bigravity and multigravity theories [29]. In the context of the AdS/CFT correspondence, a conformal field theory with conserved stress tensor is dual to a diffeomorphism invariant theory. The massive gravitons (or more generally the spin-2 fields without gauge invariance) correspond to spin-2 operators that are not conserved. From this perspective, a massive gravity theory should always admit enhancement to a diffeomorphism invariant theory by turning on the conserved stress tensor in the boundary field theory. However, the converse is less justified. A bigravity or a multigravity theory may not have a decoupling limit that breaks the diagonal diffeomorphism symmetry, which indicates that the conserved stress tensor decouples.

According to the minisuperspace Hamiltonians, there are two classes of bigravity models that are potentially free of the Boulware-Deser ghost. The first class of bigravity theories was proposed in [9] by promoting the fixed metric in consistent nonlinear massive gravity [6]. Only the standard Einstein-Hilbert kinetic terms are used. The absence of the BD ghost was proved in [20].

The second class of bigravity models (54) is in a different region of the parameter space, where novel kinetic terms are used. Let us investigate them in more detail. We can compute the Poisson bracket of the two constraints associated with $N_1$ and $N_2$:

$$\left\{\frac{\partial\mathcal{H}^{\rm II}_{\rm mini}}{\partial N_1},\frac{\partial\mathcal{H}^{\rm II}_{\rm mini}}{\partial N_2}\right\} = \frac{a_3 e^{-2B}P_B^2}{8(a_2e^A+a_3e^B)^4}[a_1e^{A-B}P_B - a_3e^{-(A-B)}P_A - a_2(P_A-P_B)]. \tag{57}$$

If $a_2\neq 0$ and $a_3=0$, the Poisson bracket (57) vanishes. It seems that the two constraints are first-class constraints, which could be associated with two sets of gauge symmetries. If $a_2\neq 0$ and $a_3\neq 0$, an independent constraint is generated by the stability of secondary constraints, which involves cubic momentum terms according to (57). This constraint eliminates one more dynamical variable, which signals the complete absence of the sixth degree of freedom.

Note that it is not justified to take the minisuperspace approximation before computing Poisson brackets. But this simple computation does capture the essential features of the Hamiltonian structure:

1. when $a_3=0$, the Poisson brackets of secondary constraints vanish on the constraint surface;
2. when $a_3\neq 0$, an independent tertiary constraint is generated, rendering absent the sixth degree of freedom.

At this point, it is not clear whether the minisuperspace results can be extended to the full theories. To verify this, we need to examine the Hamiltonian structure of the full theories, which is highly technical.

In the first class of bigravity models (51), the kinetic terms are the standard Einstein-Hilbert terms. The secondary constraints are those in general relativity supplemented by the contribution from the potential terms (16)–(18), which do not involve momenta and spatial derivatives. To compute the Poisson brackets of the constraints, one can use Dirac's hypersurface deformation algebra.

However, in the second class of bigravity models (54), the kinetic parts of the Lagrangians are modified by the novel kinetic terms. To obtain the Hamiltonians already requires some work. The constraints have more involved dependence on momenta and spatial derivative terms. Dirac's algebra is not applicable. The computations of constraint brackets are considerably more challenging. We leave the technical analysis of the Hamiltonian structure to a separate work [30] in which we verify that the sixth degree of freedom is indeed eliminated, and the case $a_2\neq 0$, $a_3=0$ does describe two interacting massless spin-2 fields.

## IV. FIELD REDEFINITIONS

Let us derive the explicit expressions of the novel kinetic terms. It is difficult to achieve this step directly because the components of a curvature two-form are complicated functions of the associated vielbein. To circumvent this difficulty, we make use of a mathematical identity for tensors in $d$ dimensions[14]

$$T^{[\mu_1\ldots\mu_d]}_{\cdots} = \det(E)T^{\nu_1\ldots\nu_d}_{\cdots}(E^{-1})_{\nu_1}{}^{[\mu_1}\ldots(E^{-1})_{\nu_d}{}^{\mu_d]}, \tag{58}$$

where the antisymmetrized product of $E^{-1}$ gives $\det(E^{-1})$ and cancels $\det(E)$ out. For example, in two dimensions we have

$$T^{01}-T^{10} = \det(E)T^{01}[(E^{-1})_0{}^0(E^{-1})_1{}^1-(E^{-1})_0{}^1(E^{-1})_1{}^0] + \det(E)T^{10}[(E^{-1})_1{}^0(E^{-1})_0{}^1-(E^{-1})_1{}^1(E^{-1})_0{}^0].$$

We notice the minisuperspace Lagrangian (38) contains the ratio of two lapse functions. From the mathematical identity (58), it is natural to introduce a new tensor field as the "ratio" of two vielbeins

[14]By an abuse of notation, the local Lorentz indices are denoted by Greek letters as well.

$$e_\mu{}^\nu = F_\mu{}^A (E^{-1})^\nu{}_A, \tag{59}$$

where $\mu$ and $\nu$ are external spacetime indices and $A$ is an internal Lorentz index.

Using the identity (58), the novel kinetic terms become

$$\mathcal{L}_2{}^{\rm kin} = d^4x\sqrt{-g}R(g)_{ab}{}^{\mu\nu}\delta_\mu{}^{[a}\delta_\nu{}^b\delta_\rho{}^\rho e_\sigma{}^{\sigma]},$$
$$\mathcal{L}_3{}^{\rm kin} = d^4x\sqrt{-g}R(g)_{ab}{}^{\mu\nu}\delta_\mu{}^{[a}\delta_\nu{}^b e_\rho{}^\rho e_\sigma{}^{\sigma]}. \tag{60}$$

After simple manipulations, we have

$$\mathcal{L}_2{}^{\rm kin} = (-)\frac{1}{4}\sqrt{-g}R(g)_{\mu\nu}{}^{[\mu\nu}e_\rho{}^{\rho]}d^4x, \tag{61}$$

$$\mathcal{L}_3{}^{\rm kin} = \frac{1}{4}\sqrt{-g}R(g)_{\mu\nu}{}^{[\mu\nu}e_\rho{}^\rho e_\sigma{}^{\sigma]}d^4x, \tag{62}$$

where the antisymmetrization $[\ldots]$ is not normalized. The normalization factors in (61) and (62) are made precise. These choices simplify the expressions below. More explicitly, $\mathcal{L}_2{}^{\rm kin}$ and $\mathcal{L}_3{}^{\rm kin}$ are functions of the metric $g_{\mu\nu}$ and the new tensor field $e_\mu{}^\nu$:

$$\mathcal{L}_2{}^{\rm kin} = \sqrt{-g}\left(R_\mu{}^\nu - \frac{1}{2}R\delta_\mu{}^\nu\right)e_\nu{}^\mu d^4x, \tag{63}$$

$$\mathcal{L}_3{}^{\rm kin} = \sqrt{-g}[R_{\mu\nu}{}^{\rho\sigma}e_\rho{}^\mu e_\sigma{}^\nu - 2R_\mu{}^\nu(e_\nu{}^\mu e_\rho{}^\rho - e_\nu{}^\rho e_\rho{}^\mu) + \frac{1}{2}R(e_\mu{}^\mu e_\nu{}^\nu - e_\mu{}^\nu e_\nu{}^\mu)]d^4x. \tag{64}$$

In terms of the new tensor field $e_{\mu\nu}$, the potential terms (15)–(19)are

$$\mathcal{L}_1{}^{\rm pot} = \sqrt{-g}, \tag{65}$$

$$\mathcal{L}_2{}^{\rm pot} = \sqrt{-g}e_\mu{}^\mu, \tag{66}$$

$$\mathcal{L}_3{}^{\rm pot} = \sqrt{-g}e_\mu{}^{[\mu}e_\nu{}^{\nu]}, \tag{67}$$

$$\mathcal{L}_4{}^{\rm pot} = \sqrt{-g}e_\mu{}^{[\mu}e_\nu{}^\nu e_\rho{}^{\rho]}, \tag{68}$$

$$\mathcal{L}_5{}^{\rm pot} = \sqrt{-g}e_\mu{}^{[\mu}e_\nu{}^\nu e_\rho{}^\rho e_\sigma{}^{\sigma]}. \tag{69}$$

We can lower and raise the indices of $e_\mu{}^\nu$ by the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. For example, we have

$$e_{\mu\nu} = e_\mu{}^\rho g_{\rho\nu} = F_\mu{}^A E_\nu{}^B \eta_{AB}. \tag{70}$$

The symmetric condition then translates into

$$e_{\mu\nu} = F_\mu{}^A E_\nu{}^B \eta_{AB} = F_\nu{}^A E_\mu{}^B \eta_{AB} = e_{\nu\mu} \tag{71}$$

and we have

$$e^\mu{}_\rho e^\rho{}_\nu = g^{\mu\alpha}e_{\alpha\beta}g^{\beta\rho}e_{\rho\nu} = g^{\mu\alpha}f_{\alpha\nu}. \tag{72}$$

When $e_{\mu\nu}$ is symmetric, $e^\mu{}_\nu$ is the square root of $g^{-1}f$ with $g_{\mu\nu}, f_{\mu\nu}$ defined in (20).

## V. $\mathcal{L}_2{}^{\rm kin}$ AND QUADRATIC CURVATURE GRAVITY

In this section, we focus on the novel kinetic term $\mathcal{L}_2{}^{\rm kin}$ and explain its connections to the models of quadratic curvature gravity.

### A. Gauge symmetries of $\mathcal{L}_2{}^{\rm kin}$

As anticipated in the minisuperspace analysis, the case of $a_3 = 0$, $a_2 \neq 0$ seems to be related to bigravity models with two sets of gauge symmetries. Let us emphasize that the coefficient $a_1$ of the Einstein-Hilbert term is not fixed.

From the explicit form of $\mathcal{L}_2{}^{\rm kin}$, we can see that the linear combination of $\mathcal{L}_1{}^{\rm kin}$ and $\mathcal{L}_2{}^{\rm kin}$ is invariant under several gauge transformations:

(i) Standard diffeomorphism invariance:

$$\delta g_{\mu\nu} = \pounds_\xi g_{\mu\nu}, \qquad \delta e_{\mu\nu} = \pounds_\xi e_{\mu\nu}, \tag{73}$$

where $\xi^\mu$ is a four-vector. This symmetry is expected in a covariant bigravity theory.

(ii) Additional diffeomorphismlike invariance:

$$\delta e_{\mu\nu} = \pounds_{\xi'} g_{\mu\nu} = \nabla_\mu \xi'_\nu + \nabla_\nu \xi'_\mu, \tag{74}$$

where $\xi^\mu$ is a four-vector. The Lagrangians are invariant because $\mathcal{L}_2{}^{\rm kin}$ is the product of the Einstein tensor and the new tensor $e_{\mu\nu}$. After integrating by parts, the covariant derivative acts on the Einstein tensor and the changes in the Lagrangians vanish due to the second Bianchi identity.

(iii) Local Lorentz invariance:

$$E_\mu{}^A \to \Lambda_B{}^A E_\mu{}^B, \qquad F_\mu{}^A \to \Lambda_B{}^A F_\mu{}^B. \tag{75}$$

For an infinitesimal transformation, we have

$$\delta E_\mu{}^A = \omega_B{}^A E_\mu{}^B, \qquad \delta F_\mu{}^A = \omega_B{}^A F_\mu{}^B, \tag{76}$$

where

$$\Lambda_B{}^A = \delta_B{}^A + \omega_B{}^A + \mathcal{O}(\omega^2). \tag{77}$$

From the definition (59), we know that $e_{\mu\nu}$ is invariant under a diagonal local Lorentz transformation, so the Lagrangians are invariant as well.

(iv) Additional local-Lorentz-like invariance:

$$\delta e_{\mu\nu} = t_{\mu\nu}, \qquad t_{\mu\nu} = -t_{\nu\mu} \tag{78}$$

or in the infinitesimal form

$$\delta F_\mu{}^A = \omega'_B{}^A E_\mu{}^B, \qquad \omega'_{AB} = -\omega'_{BA}. \tag{79}$$

The antisymmetric part of $e_{\mu\nu}$ is projected out by the symmetric Einstein tensor, so the Lagrangians are invariant under a change in the antisymmetric part of $e_{\mu\nu}$.

Since the antisymmetric part drops out when $a_3 = 0$, we can identify the symmetric part of $e^\mu{}_\nu$ with the square root of a metric product $g^{\mu\rho} f'_{\rho\nu}$.[15]

These gauge symmetries persist even if we turn on the potential terms $\mathcal{L}_1{}^{\text{pot}}$ and $\mathcal{L}_2{}^{\text{pot}}$, which are related to the cosmological constant. However, the "additional" gauge symmetries will be broken when $\mathcal{L}_3{}^{\text{kin}}$, $\mathcal{L}_3{}^{\text{pot}}$, $\mathcal{L}_4{}^{\text{pot}}$, and $\mathcal{L}_5{}^{\text{pot}}$ are introduced.

If we substitute $\delta e_{\mu\nu}$ on the left-hand side of (74) with $\delta g_{\mu\nu}$ and $\delta F_\mu{}^A$ on the left-hand side of (79) with $\delta E_\mu{}^A$, they are the off-diagonal gauge transformations.

### B. New massive gravity

One may suspect that the bigravity models with $a_3 = 0$ can be transformed into two Einstein-Hilbert terms after some field redefinitions. It is not clear what field redefinition can make this connection. Nevertheless, it was shown in [31] that $\mathcal{L}_2{}^{\text{kin}}$ in (63) can be obtained by taking a scaling limit of two Einstein-Hilbert terms,[16]

$$\lambda[\sqrt{-f}R(f) - \sqrt{-g}R(g)] \to \sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)e_{\mu\nu} = \mathcal{L}_2{}^{\text{kin}}, \tag{80}$$

where $e_{\mu\nu}$ is defined as

$$f_{\mu\nu} = g_{\mu\nu} + e_{\mu\nu}/\lambda \tag{81}$$

with

$$\lambda \to \infty. \tag{82}$$

In fact, $\mathcal{L}_2{}^{\text{kin}}$ in the form of (63) already appeared in the auxiliary field representation of 3d new massive gravity [32,33], which is a theory of quadratic curvature gravity.

In the language of vielbeins [34], the Lagrangian of 3d new massive gravity reads[17]

$$\mathcal{L}_{\text{NMG}} = \sigma\mathcal{L}_{\text{EH}} + \mathcal{L}_2{}^{\text{kin}} + c_1\mathcal{L}_1{}^{\text{pot}} + c_3\mathcal{L}_3{}^{\text{pot}}, \tag{83}$$

$$\mathcal{L}_{\text{EH}} = R(E) \wedge E, \tag{84}$$

$$\mathcal{L}_2{}^{\text{kin}} = R(E) \wedge F, \tag{85}$$

$$\mathcal{L}_1{}^{\text{pot}} = E \wedge E \wedge E, \tag{86}$$

$$\mathcal{L}_3{}^{\text{pot}} = E \wedge F \wedge F, \tag{87}$$

where $\sigma = \pm 1$ is the sign of the Einstein-Hilbert term, $c_1$ is proportional to the cosmological constant, and $c_3$ corresponds to the mass squared $m^2$. The usual auxiliary field is identified with the second tensor field

$$e_{\mu\nu} = e_\mu{}^\rho g_{\rho\nu} = F_\mu{}^A E_\nu{}^B \eta_{AB}. \tag{88}$$

Note that the symmetric condition

$$e_{\mu\nu} = e_{\nu\mu} \tag{89}$$

is imposed dynamically by the equations of motion.[18]

Therefore, new massive gravity is an example of 3d bigravity models in the second class. It is known that 3d new massive gravity does not contain the Boulware-Deser ghost [35], which furnishes evidence that the second class of bigravity models is free of the BD ghost.

A straightforward generalization of 3d new massive gravity is to introduce other potential terms $\mathcal{L}_2{}^{\text{pot}}$ and $\mathcal{L}_4{}^{\text{pot}}$ [31]:

$$\mathcal{L}_2{}^{\text{pot}} = E \wedge E \wedge F, \tag{90}$$

$$\mathcal{L}_4{}^{\text{pot}} = F \wedge F \wedge F. \tag{91}$$

Note that, if $\mathcal{L}_4{}^{\text{pot}}$ is considered, the Lagrangian does not reduce to that of quadratic curvature gravity when $e_{\mu\nu}$ is integrated out. Instead, it contains infinitely many higher-order curvature corrections.

In 3d new massive gravity, the Einstein-Hilbert term is always present. One of the reasons may be that the second spin-2 field $e_{\mu\nu}$ is usually considered to be an auxiliary field, which seems to have no dynamics. We want to emphasize that $\mathcal{L}_2{}^{\text{kin}}$ is a kinetic term as well, and the use of the Einstein-Hilbert term is not necessary.[19] Therefore, the

[15]Note that the metric $f'_{\mu\nu}$ is different from $f_{\mu\nu}$ defined in (20) and they coincide only when the antisymmetric part of $e_{\mu\nu}$ vanishes.

[16]This scaling limit can also be achieved in the vielbein formulation [11].

[17]In this representation, we can see that one of the BD-ghost-free potentials has already appeared in the context of new massive gravity.

[18]In Sec. IX, we discuss how to generalize this example of a dynamical symmetric condition.

[19]To eliminate the second-order time-derivative terms due to the curvature tensor in $\mathcal{L}_2{}^{\text{kin}}$, we need to supplement the action by boundary terms analogous to the York-Gibbons-Hawking term. Then $\mathcal{L}_2{}^{\text{kin}}$ generates a time-derivative term $\partial_t e_{\mu\nu}$ in the Lagrangian, so both $g_{\mu\nu}$ and $e_{\mu\nu}$ have dynamical degrees of freedom. Furthermore, if we expand the Lagrangian around a Minkowski background and diagonalize the quadratic kinetic terms, $\mathcal{L}_2{}^{\text{kin}}$ will give rise to two linearized Einstein-Hilbert terms.

Einstein-Hilbert term could be absent, and then the 3d bigravity Lagrangian reads

$$\mathcal{L} = a_2 \mathcal{L}_2{}^{\rm kin} + \sum_{i=1}^{4} c_i \mathcal{L}_i{}^{\rm pot}, \qquad a_2 \neq 0. \tag{92}$$

The number of dynamical degrees of freedom should be the same as that of new massive gravity and the Boulware-Deser ghost should not be propagating. The cases without the Einstein-Hilbert term are related to the generalized new massive gravity in [31] by a field redefinition

$$e_{\mu\nu} \to e_{\mu\nu} + c g_{\mu\nu}, \tag{93}$$

where $c$ depends on the coefficients of $\mathcal{L}_{\rm EH}$ and $\mathcal{L}_2{}^{\rm kin}$.

In 3d, $\mathcal{L}_2{}^{\rm kin}$ is the only novel kinetic term from (1) due to the limited number of spacetime indices. Since a massless graviton in 3d has no dynamical degree of freedom, we can choose $a_2$ such that the kinetic term of the massive graviton has a correct sign. Then (92) is a unitary theory of 3d massive gravity. The special case of $c_1 = c_2 = c_4 = 0$ was discussed in [36] and that of $c_1 \neq 0$, $c_2 = c_4 = 0$ in [37].

### C. Critical gravity

There exists a continuous family of critical points [38] in the parameter space of 3d new massive gravity (83) and its higher-dimensional generalization [39],

$$\begin{aligned}\mathcal{L} = {}& R(E) \wedge E \wedge \cdots \wedge (E + F) \\ & + E \wedge \cdots \wedge E \wedge (\Lambda E \wedge E + m^2 F \wedge F),\end{aligned} \tag{94}$$

which are known as critical gravity. At these critical points, the cosmological constant $\Lambda$ is proportional to the mass squared $m^2$ with dimension-dependent coefficients. Integrating out the auxiliary field, the linearized fourth-order equations of motion contain two massless spin-2 modes.[20]

Here we want to point out that

$$\mathcal{L}_2{}^{\rm kin} = R(E) \wedge F \tag{95}$$

and its higher-dimensional version

$$\mathcal{L}_2{}^{\rm kin} = R(E) \wedge E \wedge \cdots \wedge E \wedge F \tag{96}$$

can be thought of as a special limit of critical gravity in the bigravity formulation.[21] Note that in this limit, $e_{\mu\nu}$ cannot be integrated out because it is a Lagrange multiplier.

It is shown in the Hamiltonian analysis of [30] that $\mathcal{L}_2{}^{\rm kin}$ in 4d has two sets of first-class constraints, corresponding to two sets of gauge symmetries. More generally, the Lagrangian

$$\mathcal{L} = a_1 \mathcal{L}_1{}^{\rm kin} + a_2 \mathcal{L}_2{}^{\rm kin} + c_1 \mathcal{L}_1{}^{\rm pot} + c_2 \mathcal{L}_2{}^{\rm pot}, \tag{97}$$

with

$$a_2 \neq 0, \tag{98}$$

is a theory of two interacting, massless gravitons in various dimensions ($d > 2$), where $c_1$ and $c_2$ are related to the cosmological constant.[22]

The same gauge symmetries are also realized in higher-derivative counterparts of the two-derivative term $\mathcal{L}_2{}^{\rm kin}$

$$R(E) \wedge \cdots \wedge R(E) \wedge E \wedge \cdots \wedge E \wedge F, \tag{99}$$

where one of the $E$-vielbeins in the Lovelock terms [3] is replaced by an F-vielbein. The additional symmetries are due to the fact that Lovelock tensors are both symmetric and divergence free.

Along the lines of the second-class bigravity theories (54), we propose a general Lagrangian describing two interacting, massless, gauge-invariant gravitons, which is a linear combination of Lovelock terms, the novel derivative terms (96), (99) and two potential terms

$$\begin{aligned}\mathcal{L}_1{}^{\rm pot} &= E \wedge \ldots \wedge E, \\ \mathcal{L}_2{}^{\rm pot} &= E \wedge \ldots \wedge E \wedge F,\end{aligned} \tag{100}$$

where at least one of the novel derivative terms is present.[23]

[20] However, the total number of dynamical degrees of freedom should be the same as that of one massless and one massive graviton. The second massless graviton seems to be an artifact of the linearized equation of motion at the critical points, as there is no symmetry enhancement. For example, logarithmic modes are allowed if we do not assume the Brown-Henneaux boundary conditions [40]. They were claimed to be dual to logarithmic conformal field theories [37,41–43].

[21] The Einstein-Hilbert term is absorbed into $\mathcal{L}_2{}^{\rm kin}$ by redefining $F$.

[22] There is a no-go theorem for interacting theories of massless, gauge-invariant, spin-2 fields if the Lagrangian has at most two derivatives [44]. This is not in contradiction to the present work, because one of the linearized kinetic terms has a wrong sign, which violates one of the assumptions in the no-go theorem. The details of the linearized actions are discussed in Sec. VI. A recent construction of color-decorated gravity [45] evades this no-go theorem by including extra fields.

[23] Deforming these massless models by other potential terms with two vielbeins, one obtains the generalizations of new massive gravity proposed in [31].

### D. Weyl gravity

Weyl gravity is a well-known theory of quadratic curvature gravity in 4d, which is both diffeomorphism and conformal invariant. Interestingly, the Lagrangian of Weyl gravity can be reformulated as[24]

$$\mathcal{L}_{\text{Weyl}} = R(E) \wedge E \wedge F + E \wedge E \wedge F \wedge F, \tag{101}$$

where $F$ has dimension 2.

Then the absence of Ostrogradsky's scalar ghost is translated into the absence of the BD ghost. Ostrogradsky's spin-2 ghost in the four-derivative formulation now becomes a basic spin-2 ghost due to a wrong sign kinetic term.

In this representation, Weyl gravity is built from a novel kinetic term and a dRGT potential term. Despite the presence of a spin-2 ghost, Weyl gravity is the first example of nonlinear completions of Fierz-Pauli massive gravity that are free of the BD ghost, where the nonlinear theory was proposed 20 years before the linear one.

Furthermore, Weyl gravity is a special bigravity model in the second class (54) with an emergent gauge symmetry (conformal symmetry). This gauge symmetry is a nonlinear completion of the additional gauge symmetry of a massive spin-2 field around a de Sitter background at the partially massless point [27]. To make this connection more clear, we linearize (101) around the de Sitter background and diagonalize the quadratic Lagrangian in the next section.

## VI. LINEARIZED LAGRANGIANS

In the previous section, we show that some of the bigravity models with novel kinetic terms are equivalent to higher-derivative gravity models. It is well known that higher-derivative gravity models usually contain spin-2 ghosts, which could lead to the problem of nonunitarity. In this section, we derive the quadratic actions of the novel kinetic terms and examine whether this is a general feature of the bigravity theories in the second class (54).

### A. Minkowski background

Consider a bigravity model in 4d whose Lagrangian reads

$$\mathcal{L} = a_1 \mathcal{L}_{\text{EH}} + a_2 \mathcal{L}_2{}^{\text{kin}} + a_3 \mathcal{L}_3{}^{\text{kin}}, \tag{102}$$

where $\mathcal{L}_{\text{EH}}$ is the Einstein-Hilbert kinetic term

$$\mathcal{L}_{\text{EH}} = \sqrt{-g} R(g), \tag{103}$$

[24]By redefining $F = F' + E$, one has the auxiliary field reformulation of Weyl gravity with an Einstein-Hilbert term and a cosmological constant term, where the second spin-2 field $e_{\mu\nu}$ can be thought of as a matter field that couples to the Einstein tensor.

and $\mathcal{L}_2{}^{\text{kin}}$ and $\mathcal{L}_3{}^{\text{kin}}$ are the novel nonlinear kinetic terms defined in (61)–(62).

Let us expand the metric field $g_{\mu\nu}$ and the symmetric tensor field $e_{\mu\nu}$ around the Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \tag{104}$$

$$e_{\mu\nu} = \eta_{\mu\nu} + \delta e_{\mu\nu}, \tag{105}$$

where we assume there is no numerical factors in front of $\eta_{\mu\nu}$. These factors can always be absorbed into $a_1$, $a_2$, $a_3$ by redefining $g_{\mu\nu}$ and $e_{\mu\nu}$.

Note that $e_{\mu\nu}$ simply vanishes in a different kind of background solutions. The two kinds of background solutions are related by a redefinition of $F_\mu{}^A$:

$$F' = F + E. \tag{106}$$

To the quadratic order, the linearized Lagrangian reads

$$\bar{\mathcal{L}} = c_1 \delta g_\mu{}^{[\mu} \partial_\nu \partial^\nu \delta g_\rho{}^{\rho]} + c_2 \delta g_\mu{}^{[\mu} \partial_\nu \partial^\nu \delta e_\rho{}^{\rho]}, \tag{107}$$

where the first term is the linearized Einstein-Hilbert term and the coefficients are

$$c_1 = \frac{1}{4}(-a_1 + a_2 - a_3), \qquad c_2 = \frac{1}{2} a_2 - a_3. \tag{108}$$

Now we can diagonalize the quadratic Lagrangian

$$\bar{\mathcal{L}} = c_1 (h_\mu{}^{[\mu} \partial_\nu \partial^\nu h_\rho{}^{\rho]} - H_\mu{}^{[\mu} \partial_\nu \partial^\nu H_\rho{}^{\rho]}), \tag{109}$$

where we assume $c_1 \neq 0$ and the diagonalized spin-2 fields are

$$h_{\mu\nu} = \delta g_{\mu\nu} + \frac{c_2}{2c_1} \delta e_{\mu\nu}, \qquad H_{\mu\nu} = \frac{c_2}{2c_1} \delta e_{\mu\nu}. \tag{110}$$

If $c_1 = 0$ and $c_2 \neq 0$, then the first term in (107) vanishes and the diagonalized Lagrangian is

$$\bar{\mathcal{L}} = c_2 (h_\mu{}^{[\mu} \partial_\nu \partial^\nu h_\rho{}^{\rho]} - H_\mu{}^{[\mu} \partial_\nu \partial^\nu H_\rho{}^{\rho]}), \tag{111}$$

and the diagonalized fields are

$$h_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} + \delta e_{\mu\nu}), \qquad H_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} - \delta e_{\mu\nu}). \tag{112}$$

The linearized Lagrangian after diagonalization is a linear combination of two linearized Einstein-Hilbert terms[25] except in some special cases. The diagonalized kinetic terms always have opposite signs due to the absence of the quadratic term of $e_{\mu\nu}$, which can be traced back to the absence of $R(F)$.

The special cases are

$$c_2 = 0, \quad \text{or} \quad \frac{1}{2}a_2 = a_3; \tag{114}$$

then $H_{\mu\nu} = 0$ and the second diagonalized kinetic term vanishes. The first kinetic term has a right sign when

$$c_1 = \frac{1}{4}(-a_1 + a_3) < 0. \tag{115}$$

A more extreme case is

$$c_1 = c_2 = 0, \quad \text{or} \quad a_1 = \frac{1}{2}a_2 = a_3; \tag{116}$$

then the linearized Lagrangian is empty. In these special cases, the bigravity models are strongly coupled due to the absence of some linearized kinetic terms.

Note that the Lagrangians of these special cases can be schematically written as

$$\mathcal{L} = (a_1 - a_3)R \wedge E \wedge E + a_3 R \wedge (F - E) \wedge (F - E). \tag{117}$$

If we consider the other kind of background solutions where $\bar{e}_{\mu\nu}$ vanishes, both of the linearized kinetic terms are present and there is no issue of strong coupling. So the problem of strong coupling depends on the choice of background solutions.[26] The existence of these strongly coupled backgrounds is related to the presence of spin-2 ghosts.

The potential terms (65)–(69) can generate linear terms around a Minkowski background, which signals a wrong choice of vacuum. In the next subsection, we discuss the linearized actions around general maximally symmetric backgrounds.

### B. Constant curvature background

Let us introduce nonlinear potential terms to the 4d Lagrangian,

$$\begin{aligned}\mathcal{L} = {} & a_1 \mathcal{L}_{\rm EH} + a_2 \mathcal{L}_2{}^{\rm kin} + a_3 \mathcal{L}_3{}^{\rm kin} \\ & + b_1 \mathcal{L}_1{}^{\rm pot} + b_2 \mathcal{L}_2{}^{\rm pot} + b_3 \mathcal{L}_3{}^{\rm pot} \\ & + b_4 \mathcal{L}_4{}^{\rm pot} + b_5 \mathcal{L}_5{}^{\rm pot},\end{aligned} \tag{118}$$

where the potential terms are defined in (65)–(69).

The spin-2 fields $g_{\mu\nu}$ and $e_{\mu\nu}$ are expanded around a cosmological background $\bar{g}_{\mu\nu}$,

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \\ e_{\mu\nu} &= \delta e_{\mu\nu},\end{aligned} \tag{119}$$

where the background spacetime has constant curvature

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}). \tag{120}$$

We assume the background value of $e_{\mu\nu}$ vanishes, which is not necessary. However, if the background solution of $e_{\mu\nu}$ is proportional to the background metric $\bar{g}_{\mu\nu}$, then we can always set it to zero by a shift in $F_\mu{}^A$.[27]

To avoid the strong coupling problem, we require

$$a_2 \neq 0; \tag{121}$$

otherwise we should consider a different background solution.

Around a background solution, the linear terms in the perturbative Lagrangian should vanish, which indicates

$$b_1 = -2a_1\Lambda, \qquad b_2 = a_2\Lambda. \tag{122}$$

Before the shift in $F$, the two equations correspond to the solution of the background metric $\bar{g}_{\mu\nu}$ and the ratio between two background spin-2 fields.

The linearized Lagrangian is

$$\begin{aligned}\bar{\mathcal{L}} = {} & \sqrt{-\bar{g}}c_1(\delta g_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu\delta g_\rho{}^{\rho]} + \delta g_\mu{}^\nu[\bar{\nabla}_\rho, \bar{\nabla}^\mu]\delta g_\nu{}^\rho) \\ & + \sqrt{-\bar{g}}c_2(\delta e_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu\delta g_\rho{}^{\rho]} + \delta e_\mu{}^\nu[\bar{\nabla}_\rho, \bar{\nabla}^\mu]\delta g_\nu{}^\rho) \\ & + \sqrt{-\bar{g}}\Bigg[2a_1\Lambda - a_1\Lambda\left(\frac{1}{4}\delta g_\mu{}^\mu\delta g_\nu{}^\nu - \frac{1}{2}\delta g_\mu{}^\nu\delta g_\nu{}^\mu\right) \\ & + a_2\Lambda\left(\frac{1}{2}\delta g_\mu{}^\mu\delta e_\nu{}^\nu - \delta g_\mu{}^\nu\delta e_\nu{}^\mu\right) \\ & + b_3(\delta e_\mu{}^\mu\delta e_\nu{}^\nu - \delta e_\mu{}^\nu\delta e_\nu{}^\mu)\Bigg],\end{aligned} \tag{123}$$

where the coefficients

[25]The diagonalized form is invariant under a continuous family of field redefinitions

$$\begin{aligned} h_{\mu\nu} &= \cosh(\theta)h'_{\mu\nu} + \sinh(\theta)H'_{\mu\nu}, \\ H_{\mu\nu} &= \sinh(\theta)h'_{\mu\nu} + \cosh(\theta)H'_{\mu\nu}.\end{aligned} \tag{113}$$

[26]Around a background where $e_{\mu\nu}$ vanishes, $\mathcal{L}_3{}^{\rm kin}$ does not contribute to the linearized Lagrangian. We will encounter the strong coupling problem when $a_2 = 0$.

[27]The definitions of fluctuating fields are modified accordingly.

$$c_1 = -\frac{1}{4}a_1, \qquad c_2 = \frac{1}{2}a_2 \tag{124}$$

are simplified due to a shift in $F$. Total derivative terms are neglected.

If $a_1 \neq 0$, the diagonalized fields are

$$h_{\mu\nu} = \delta g_{\mu\nu} - \frac{a_2}{a_1}\delta e_{\mu\nu}, \quad H_{\mu\nu} = \frac{a_2}{a_1}\delta e_{\mu\nu}, \tag{125}$$

and the linearized Lagrangian becomes

$$\begin{aligned}\bar{\mathcal{L}} = &-\frac{1}{4}a_1\sqrt{-\bar{g}}(h_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu h_\rho{}^{\rho]} + h_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]h_\nu{}^\rho)\\ &+\frac{1}{4}a_1\sqrt{-\bar{g}}(H_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu H_\rho{}^{\rho]} + H_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]H_\nu{}^\rho)\\ &+\sqrt{-\bar{g}}\left[-a_1\Lambda\left(\frac{1}{4}h_\mu{}^\mu h_\nu{}^\nu - \frac{1}{2}h_\mu{}^\nu h_\nu{}^\mu\right)\right.\\ &+a_1\Lambda\left(\frac{1}{4}H_\mu{}^\mu H_\nu{}^\nu - \frac{1}{2}H_\mu{}^\nu H_\nu{}^\mu\right)\\ &\left.+b_3\left(\frac{a_1}{a_2}\right)^2(H_\mu{}^\mu H_\nu{}^\nu - H_\mu{}^\nu H_\nu{}^\mu)\right],\end{aligned} \tag{126}$$

where the constant term is neglected. The first four lines are the linearized Einstein-Hilbert terms with cosmological constant $\Lambda$ around the background solutions. The last line is the Fierz-Pauli mass term for $H_{\mu\nu}$. The coefficient of the massless spin-2 field $h_{\mu\nu}$ is $a_1$, while that of $H_{\mu\nu}$ is $-a_1$. One of them is a spin-2 ghost. The mass squared of $H_{\mu\nu}$ is determined by $b_3$.

If $a_1 = 0$, the diagonalized fields are

$$h_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} + \delta e_{\mu\nu}), \quad H_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} - \delta e_{\mu\nu}). \tag{127}$$

In addition, $b_3$ should vanish in order to be consistent with our choice of background solution $\bar{e}_{\mu\nu} = 0$, so the mass terms vanish. The linearized Lagrangian is

$$\begin{aligned}\bar{\mathcal{L}} = &\frac{1}{2}a_2\sqrt{-\bar{g}}(h_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu h_\rho{}^{\rho]} + h_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]h_\nu{}^\rho)\\ &-\frac{1}{2}a_2\sqrt{-\bar{g}}(H_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu H_\rho{}^{\rho]} + H_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]H_\nu{}^\rho)\\ &+\sqrt{-\bar{g}}\left[2a_2\Lambda\left(\frac{1}{4}h_\mu{}^\mu h_\nu{}^\nu - \frac{1}{2}h_\mu{}^\nu h_\nu{}^\mu\right)\right.\\ &\left.-2a_2\Lambda\left(\frac{1}{4}H_\mu{}^\mu H_\nu{}^\nu - \frac{1}{2}H_\mu{}^\nu H_\nu{}^\mu\right)\right],\end{aligned} \tag{128}$$

which corresponds to two interacting massless gravitons.

### C. Linearized Weyl gravity

In this subsection, we would like to discuss the linearized action of Weyl gravity (101) around the de Sitter background. As shown in [27], the conformal transformation in Weyl gravity can be recast into a nonlinear partially massless transformation for a spin-2 matter field after some field redefinitions. So we expect that after diagonalization the massive spin-2 field has linear partially massless gauge symmetry.

The explicit expression of (101) is

$$\begin{aligned}\mathcal{L}_{\mathrm{Weyl}} &= \mathcal{L}_2{}^{\mathrm{kin}} + \mathcal{L}_3{}^{\mathrm{pot}}\\ &= \sqrt{-g}\left[\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)e_{\mu\nu} + e_\mu{}^\mu e_\nu{}^\nu - e_{\mu\nu}e^{\mu\nu}\right],\end{aligned} \tag{129}$$

where the equations of motion for $e_{\mu\nu}$ are

$$e_{\mu\nu} = \frac{1}{2}R_{\mu\nu} - \frac{1}{12}Rg_{\mu\nu}. \tag{130}$$

The nonlinear gauge symmetry transformations are

(i) conformal invariance,

$$\begin{aligned}g_{\mu\nu} &\to (1+2\phi)g_{\mu\nu},\\ e_{\mu\nu} &\to e_{\mu\nu} - \nabla_\mu\partial_\nu\phi,\end{aligned} \tag{131}$$

(ii) and diffeomorphism invariance,

$$g_{\mu\nu} \to g_{\mu\nu} + \pounds_\xi g_{\mu\nu}, \qquad e_{\mu\nu} \to e_{\mu\nu} + \pounds_\xi e_{\mu\nu}. \tag{132}$$

Let us consider the de Sitter background solution

$$\bar{g}_{\mu\nu} = g^{\mathrm{dS}}_{\mu\nu}, \qquad \bar{e}_{\mu\nu} = \frac{\Lambda}{6}g^{\mathrm{dS}}_{\mu\nu}, \tag{133}$$

where we keep the nonzero background value of $e_{\mu\nu}$.

The fluctuations around the de Sitter vacuum are

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}, \qquad \delta e_{\mu\nu} = e_{\mu\nu} - \bar{e}_{\mu\nu}. \tag{134}$$

Then we can expand the full action to the quadratic order

$$\begin{aligned}\bar{\mathcal{L}}_{\mathrm{Weyl}} = &\sqrt{-\bar{g}}\left[-\frac{\Lambda^2}{3} + (\delta e^2 - \delta e_{\mu\nu}\delta e^{\mu\nu})\right.\\ &\left.+\frac{\Lambda}{6}(\delta e\delta g - 4\delta e_{\mu\nu}\delta g^{\mu\nu}) - \frac{\Lambda^2}{72}(\delta g^2 - 4\delta g_{\mu\nu}\delta g^{\mu\nu})\right]\\ &+\frac{1}{2}\sqrt{-\bar{g}}(\delta e_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu\delta g_\rho{}^{\rho]} + \delta e_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]\delta g_\nu{}^\rho)\\ &-\frac{\Lambda}{24}\sqrt{-\bar{g}}(\delta g_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu\delta g_\rho{}^{\rho]} + \delta g_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]\delta g_\nu{}^\rho),\end{aligned} \tag{135}$$

where total derivative terms are neglected, the covariant derivative $\bar{\nabla}$ is compatible with the background metric $\bar{g}_{\mu\nu}$, and

$$\delta g = \delta g_\mu{}^\mu, \qquad \delta e = \delta e_\mu{}^\mu. \tag{136}$$

The diagonalized fields are

$$h_{\mu\nu} = \frac{6}{\Lambda}\delta e_{\mu\nu},$$
$$H_{\mu\nu} = \delta g_{\mu\nu} - \frac{6}{\Lambda}\delta e_{\mu\nu}. \tag{137}$$

The quadratic Lagrangian in terms of $h_{\mu\nu}$, $H_{\mu\nu}$ reads

$$\begin{aligned}\bar{\mathcal{L}}_{\rm Weyl} = &\frac{\Lambda}{24}\sqrt{-\bar{g}}(h_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu h_\rho{}^{\rho]} + h_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]h_\nu{}^\rho)\\ &-\frac{\Lambda^2}{48}\sqrt{-\bar{g}}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right)\\ &-\frac{\Lambda}{24}\sqrt{-\bar{g}}(H_\mu{}^{[\mu}\bar{\nabla}_\nu\bar{\nabla}^\nu H_\rho{}^{\rho]} + H_\mu{}^\nu[\bar{\nabla}_\rho,\bar{\nabla}^\mu]H_\nu{}^\rho)\\ &+\frac{\Lambda^2}{18}\sqrt{-\bar{g}}\left(H_{\mu\nu}H^{\mu\nu} - \frac{1}{4}H^2\right),\end{aligned} \tag{138}$$

where we neglect the constant term. We can see that $h$ is a massless spin-2 field with a negative Planck mass, while $H$ is a massive spin-2 field with a positive Planck mass. The signs of the kinetic terms are opposite.

We can further examine the gauge symmetries at the linearized level:

(i) linearized conformal symmetry,

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} + 2\phi\bar{g}_{\mu\nu}, \tag{139}$$

$$\delta e_{\mu\nu} \to \delta e_{\mu\nu} - \bar{\nabla}_\mu\partial_\nu\phi, \tag{140}$$

and

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{6}{\Lambda}\bar{\nabla}_\mu\partial_\nu\phi, \tag{141}$$

$$H_{\mu\nu} \to H_{\mu\nu} + \left(\bar{\nabla}_\mu\partial_\nu + \frac{\Lambda}{3}\bar{g}_{\mu\nu}\right)\left(\frac{6}{\Lambda}\phi\right). \tag{142}$$

(ii) and linearized diffeomorphism symmetry,

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu, \tag{143}$$

$$\delta e_{\mu\nu} \to \delta e_{\mu\nu} + \frac{\Lambda}{6}(\bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu), \tag{144}$$

and

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu, \tag{145}$$

$$H_{\mu\nu} \to H_{\mu\nu}. \tag{146}$$

Therefore, the linear partially massless gauge transformation

$$H_{\mu\nu} \to H_{\mu\nu} + \left(\bar{\nabla}_\mu\partial_\nu + \frac{\Lambda}{3}\bar{g}_{\mu\nu}\right)\alpha \tag{147}$$

is a combination of linearized conformal and diffeomorphism transformations.

Interestingly, only the massless spin-2 field $h$ transforms under a change of coordinate. The massive spin-2 field cannot transform because the Lagrangian of the massive mode is not invariant.

In this way, we provide a different perspective of Weyl gravity by using a novel kinetic term and a dRGT term. In this representation, one may understand better why unitary partially massless gravity in 4d is not found [46].[28] Along the lines of dRGT massive gravity, there have been many recent investigations on nonlinear partially massless gravity [47–49]. In 4d, a promising candidate was identified in dRGT massive gravity [47], which makes use of precisely the same potential term

$$E \wedge E \wedge F \wedge F, \tag{148}$$

but the kinetic term is assumed to be the Einstein-Hilbert term and the $F$ vielbein is fixed to be de Sitter. Partially massless gauge symmetry is only an artifact of the perturbative Lagrangian at low orders. We also confirm the suspicion in [48] that a new kinetic term is required in order to extend the partially massless symmetry to the nonlinear level, which becomes trivial from our bigravity reformulation of Weyl gravity.

The fact that the novel kinetic terms have no nontrivial single dynamical metric limit indicates that we cannot truncate Weyl gravity in a consistent manner to obtain a nonlinear theory of partially massless gravity with a single dynamical metric and a fixed de Sitter metric.

In addition, partially massless gauge symmetry in 4d can be thought of as an emergent gauge symmetry of the 4d bigravity models constructed from novel kinetic terms and dRGT potential terms. It is tempting to consider the case of three dimensions, where a massless spin-2 field has no dynamical degree of freedom and we do not need to worry about the sign of its kinetic term. However, there is only one new kinetic term in 3d, and these bigravity models were well investigated along the lines of 3d new massive gravity. In particular, the 3d version of Weyl gravity proposed in [36] is an example of the conflict between diffeomorphism and conformal invariances in three

[28] To be more precise, we require the off-shell action to be gauge invariant.

dimensions. Partially massless symmetry has no nonlinear extension in 3d to date.

## VII. SPIN-2 GHOST

In the previous section, we show that, generically, the two linearized kinetic terms have opposite signs. A kinetic term with a wrong sign usually results in an unbounded Hamiltonian and nonunitarity. If we associate the Hamiltonian with the energy, then we could extract infinite energy from a system whose Hamiltonian is unbounded from below, which leads to classical instability. Nonunitarity in a quantum theory means unphysical negative probability. Therefore, when a kinetic term has a wrong sign, the corresponding degree of freedom is considered to be an unwanted "ghost."

From the classical point of view, it is not clear which is the correct sign for a massless spin-2 kinetic term, as the Hamiltonian simply vanishes on shell. Naively, the Einstein-Hilbert term seems to have a wrong sign in the minisuperspace approximation (33),

$$\sqrt{-g}R(g) \to -\frac{12}{N}(\dot{A})^2 e^{3A}, \qquad N > 0, \tag{149}$$

but the Hamiltonian is still bounded because it vanishes. If we modify the sign of the Einstein-Hilbert term, Newton's constant will become negative and gravity will be a repulsive force. Certainly, this contradicts our physical world, but this is not ruled out as a theoretical possibility. We know the Coulomb force is repulsive for like charges.

The Hamiltonian of novel kinetic terms vanishes on shell as well [30], which is expected in covariant theories, so a bounded Hamiltonian is not strong evidence for classical stability.[29] One might need to examine other definitions of energy. As the local definition of gravitational energy is controversial, it may be more sensible to consider global energies (masses) according to the isometries of asymptotic spacetime. They are the conserved charges associated with the global symmetries of the vacuum where the infinite-dimensional diffeomorphism group is spontaneously broken to a finite-dimensional global symmetry group. In critical gravity models, the Abbott-Deser-Tekin mass [50] of black hole solutions was shown to be zero [39]. As discussed before, the novel kinetic term $\mathcal{L}_2{}^{\text{kin}}$ can be considered as a special limit of critical gravity, and we expect it to have the same property.

From the quantum perspective, the correct sign for spin-2 kinetic terms has a more definite answer. Unitarity requires particle poles to have positive residues. A propagator can be derived from the quadratic action, so the correct signs of the kinetic term and the mass term are determined.

The residue of a spin-2 propagator depends on the coupling to matter. It vanishes if the spin-2 field is not coupled to the energy-momentum tensor. So one can avoid negative residues by identifying the physical metric with the spin-2 field whose kinetic term has a correct sign. Using the effective metric [51], we can escape the problem of tree-level nonunitarity.

Furthermore, to obtain the solutions of a model we need boundary conditions. When the Lagrangian allows for ghostlike excitations, they can still be avoided by proper boundary conditions. In this way, we could eliminate the ghostlike modes whose kinetic terms have wrong signs; then the bigravity models should reduce to healthy vector-tensor theories when the mass squared has a correct sign.

## VIII. CUTOFF SCALE

Ghosts are ubiquitous in the framework of effective field theories. Their presence does not disqualify the models from describing nature. They just tell us when the theories stop providing consistent descriptions and microscopic details become important.

As an effective field theory, Einstein's gravity has a cutoff scale set by the Planck mass. Higher-curvature terms are also compatible with diffeomorphism invariance, so they should be present. The natural values of their coefficients are of order unity in terms of the Planck mass, and the cutoff scale remains the same. Ostrogradsky's ghosts due to higher-derivative equations of motion are not excited below the Planck scale because their masses are around the Planck value. The corrections due to higher-curvature terms are negligible at a low energy scale.

However, if for some unknown reasons, the coefficients of some correction terms are considerably larger than their natural values, then we need to worry about Ostrogradsky's instability even below the Planck scale. The cutoff scale of an effective field theory of gravity is lowered by the ghost modes.

Let us consider an example in 4d that admits a bigravity reformulation

$$\mathcal{L} = M_p^2\sqrt{-g}[R + R(e + ee + \cdots) + \Lambda(1 + e) + m^2(ee + eee + \cdots)], \tag{150}$$

where the tensor structures are not written explicitly, and the two spin-2 fields $g_{\mu\nu}$, $e_{\mu\nu}$ are dimensionless. After linearization and diagonalization, $m^2$ corresponds to the

[29]Since the on-shell Hamiltonian is zero, one of the spin-2 fields has negative energy if the other spin-2 field has positive energy. The spin-2 field with negative energy seems as problematic as the BD ghost. But the presence of negative energy modes is a general feature of covariant theories. For example, consider Einstein's gravity coupled to healthy matter: if matter has positive energies, the massless graviton will have negative energy even if its kinetic term has a correct sign. In this sense, the gravitational sector, which we identify with the spin-2 fields, with negative energy may still be acceptable. In contrast, the BD ghost is an additional degree of freedom that does not belong to spin-2 fields, so it behaves as ghostlike matter and is more troublesome.

mass of the massive spin-2 field, which is ghostlike.[30] In principle, higher-curvature terms are also allowed.

Integrating out the auxiliary field $e_{\mu\nu}$, we have a Lagrangian of higher-curvature gravity, which schematically reads

$$\mathcal{L} = \sqrt{-g}\left(M_p^2\Lambda + M_p^2 R + \frac{M_p^2}{m^2}RR + \cdots\right). \tag{151}$$

We can see that a small mass in the bigravity formulation translates into large higher-curvature terms.

To be more precise, there are two kinds of Ostrogradsky's ghosts in a model of higher-curvature gravity, which can be rephased more transparently in the bigravity formulation. In the bigravity representation, the massive modes contain two kinds of ghostlike degrees of freedom. The first one is the ghostlike spin-2 mode due to a wrong sign kinetic term.[31] The second one is Ostrogradsky's scalar ghost or the Boulware-Deser ghost in a generic theory of massive spin-2 field.

For simplicity, let us assume Minkowski spacetime is the background solution. We also assume, after linearization and diagonalization, the kinetic terms of (150) are given by the linearized Einstein-Hilbert terms to avoid more ghosts. These assumptions already constrain the possible terms in (150). The scale of the quadratic potential terms is set by the mass squared $m^2$. Then both the spin-2 ghost and the scalar ghost are excited and interact with the healthy modes at a low scale,

$$\Lambda = m. \tag{152}$$

By requiring that the quadratic term of the linearized Lagrangian take the form of the Fierz-Pauli mass term, the scalar ghost is absent in the quadratic action. But it can still appear in the interaction terms, which is known as the BD ghost. If we assume the effective Lagrangian is given by the bigravity models in the second class (54), this scalar ghost can be eliminated completely. Then we can focus on the problem of spin-2 ghosts.

To increase the cutoff scale, let us first examine the quadratic action in detail. The massive spin-2 field is denoted by $H_{\mu\nu}$. According to Sec. VI, its linearized action is given by the Fierz-Pauli theory:

$$\bar{\mathcal{L}} = M_p^2(H_\mu{}^{[\mu}\partial_\nu\partial^\nu H_\rho{}^{\rho]} + \alpha m^2 H_\mu{}^{[\mu}H_\nu{}^{\nu]}), \tag{153}$$

where the kinetic term has a wrong sign and $\alpha$ is a model-dependent numerical factor.

[30]It is assumed that the coefficients of the potential terms are of the same order, which is not a necessary assumption. When their magnitude are different, the smallest one is the most important.

[31]The kinetic term of the helicity-1 mode is from the mass terms, so they are ghosts when the spin-2 ghost is also a tachyon. Let us assume the mass squared is positive.

Let us decompose $H_{\mu\nu}$ *à la* Helmholtz,

$$H_{\mu\nu} = \frac{1}{M_p}H^T_{\mu\nu} + \frac{1}{M_p m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{M_p m^2}\partial_\mu\partial_\nu\phi, \tag{154}$$

$$\partial^\mu H^T_{\mu\nu} = 0, \qquad \partial^\mu A_\mu = 0, \tag{155}$$

where the dimensions of $H^T_{\mu\nu}$, $A_\mu$, and $\phi$ are 1 and the use of $M_p$, $m$ is to canonically normalize the kinetic terms of the decomposed fields.[32]

In terms of the decomposed modes, the Fierz-Pauli Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {}& \partial_\mu(H^T)_{\nu\rho}\partial^\mu(H^T)^{\nu\rho} - \partial_\mu(H^T)_\nu{}^\nu\partial^\mu(H^T)_\rho{}^\rho \\ &+ \alpha m^2[(H^T)_\mu{}^\mu(H^T)_\nu{}^\nu - (H^T)_{\mu\nu}(H^T)^{\mu\nu}] \\ &- 2\alpha\partial_\mu A_\nu\partial^\mu A^\nu + 2\alpha(H^T)_\mu{}^\mu(\Box\Phi),\end{aligned} \tag{158}$$

where the total derivative terms are neglected. The last term is a cross term, so we introduce

$$\bar{H}_{\mu\nu} = H^T_{\mu\nu} + \frac{\alpha}{3}\phi\eta_{\mu\nu} \tag{159}$$

to diagonalize the kinetic terms. After diagonalization, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {}& \partial_\mu\bar{H}_{\nu\rho}\partial^\mu\bar{H}^{\nu\rho} - \partial_\mu\bar{H}_\rho{}^\rho\partial^\mu\bar{H}_\nu{}^\nu \\ &+ \alpha m^2(\bar{H}_\mu{}^\mu\bar{H}_\nu{}^\nu - \bar{H}_{\mu\nu}\bar{H}^{\mu\nu}) + \frac{4}{3}\alpha^2\partial_\mu\phi\partial^\mu\phi \\ &+ 2\alpha^2 m^2\bar{H}_\mu{}^\mu\phi + \frac{4}{3}\alpha^3 m^2\phi^2 - 2\alpha\partial_\mu A_\nu\partial^\mu A^\nu.\end{aligned} \tag{160}$$

The kinetic terms of $\bar{H}_{\mu\nu}$ and $\phi$ have wrong signs. If $\alpha > 0$, then the helicity-1 modes $A_\mu$ are healthy modes. Although the ghostlike modes can be excited at scale $\Lambda = m$, they are harmless before healthy degrees of freedom are coupled to them. The cutoff scales are then determined by the lowest scale of the interaction terms that involve both the ghosts and the healthy modes.

By considering nonlinear redefinitions, we can always make the massless spin-2 field transverse:

[32]Another natural decomposition is with respect to the covariant derivative of the massless spin-2 field $h_{\mu\nu}$:

$$H_{\mu\nu} = \frac{1}{M_p}H^T_{\mu\nu} + \frac{1}{M_p m}(\nabla^{(h)}_\mu A_\nu + \nabla^{(h)}_\nu A_\mu) + \frac{1}{M_p m^2}\nabla^{(h)}_\mu\partial_\nu\phi \tag{156}$$

$$\nabla^{(h)\mu}H^T_{\mu\nu} = 0, \qquad \nabla^{(h)\mu}A_\mu = 0. \tag{157}$$

$$h_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_p} h^T_{\mu\nu}. \tag{161}$$

Then we perform a general power counting of the possible perturbative terms without using the specific structures of the nonlinear terms. From the two derivative terms, we have

$$M_p^{2-i-2j-k-l} m^{-2j-2k} \partial^2 (H^T)^i (\partial A)^{2j} (\partial\partial\phi)^k (h^T)^l. \tag{162}$$

Assuming the coefficients of the potential terms in (150) are of the same order, we have

$$M_p^{2-i-2j-k-l} m^{2-2j-2k} (H^T)^i (\partial A)^{2j} (\partial\partial\phi)^k (h^T)^l. \tag{163}$$

The interaction terms start from the cubic order

$$i + 2j + k + l = 3, 4, 5, \ldots, \tag{164}$$

and a perturbative term

$$M_p^{-a} m^{-b} \partial^m (H^T)^i (A)^{2j} (\phi)^k (h^T)^l \tag{165}$$

becomes important at the energy scale

$$\Lambda = (M_p^a m^b)^{1/(a+b)}. \tag{166}$$

The lowest scales of the interaction terms involving both the ghostlike modes $(H^T, \phi)$ and the healthy modes $(h^T, A)$ can be found[33]:

(i) The lowest scale of the two-derivative terms is

$$\Lambda_5 = (M_p m^4)^{1/5}, \tag{167}$$

where the cubic terms

$$j = k = 1, \qquad i = l = 0, \tag{168}$$

and

$$k = 2, \quad l = 1, \quad i = j = 0 \tag{169}$$

become important.

(ii) The lowest scale of the potential terms is

$$\Lambda_3 = (M_p m^2)^{1/3}, \tag{170}$$

where infinitely many terms

$$j = 1, \qquad k = 1 + n, \qquad i = l = 0 \tag{171}$$

and

$$k = 2 + n, \qquad l = 1, \qquad i = j = 0 \tag{172}$$

$$n = 0, 1, 2, \ldots \tag{173}$$

become important.

The cutoff scale of the two-derivative terms is lower than that of the potential terms, so a generic bigravity model in the second class (54) is a consistent effective field theory at least up to $\Lambda_5$.[34]

We can further improve this by turning off $\mathcal{L}_3^{\mathrm{kin}}$; then the kinetic terms have more gauge symmetries. The kinetic terms contain only helicity-2 modes whose interaction scale is set by the Planck mass. In this way, we are able to increase the cutoff scale to $\Lambda_3$ or some higher energy[35]

$$\Lambda \to \Lambda_3 \equiv (M_p m^2)^{1/3}. \tag{174}$$

It should be noted that we need to make sure the spin-2 ghost does not couple to the matter below the cutoff scale. This indicates that we should consider an effective metric [51].[36]

In a different region of parameter space, it is possible that the massless spin-2 field has a wrong sign, while the massive one is healthy. By eliminating the BD ghost and using the gauge-invariant kinetic terms, the cutoff scale is set by the interaction terms $k = 2 + n$, $l = 1$, $i = j = 0$ from the potentials, which is $\Lambda_3$ again. When the mass squared has a correct sign, these effective field theories contain a ghost-free massive graviton and a decoupled, ghostlike, massless spin-2 field below the cutoff scale.

Here we want to give one possible reason for "naturally" large coefficients of the higher-curvature terms. In the bigravity formulation, the gauge symmetries are enhanced when

$$m^2 = 0, \qquad a_3 = 0, \tag{175}$$

so small values of $m^2$ and $a_3$ are technically natural, which is analogous to the mass of the electron. Quantum corrections to these parameters should be multiplicative, rather than additive.

[33] If $\alpha < 0$, then the helicity-1 mode $A_\mu$ becomes a ghost due to a wrong sign kinetic term. If $a = 0$, the helicity-1 mode is strongly coupled. In addition, $A_\mu$ could be Ostrogradsky's vector ghost if the equations of motion involve higher-order time-derivative terms of $A_\mu$. But from the constraint analysis [30], we can count the numbers of dynamical degrees of freedom, and we know the novel kinetic terms do not contain Ostrogradsky's vector ghost.

[34] If the potential terms are modified, then Ostrogradsky's scalar ghost is eliminated only at the linear level. If the massive spin-2 field has a correct sign kinetic term, then the cutoff scale set by the Boulware-Deser ghost is $\Lambda_5$ as well.

[35] In [52], it was shown that the $\Lambda_3$ interaction terms vanish in some cases that are equivalent to quadratic curvature gravity. This indicates that the cutoff scale may be higher than $\Lambda_3$

[36] But the BD ghost will be revived at some scale above $\Lambda_3$ due to the use of an effective metric.

## IX. SYMMETRIC CONDITION

In the above sections, we impose the symmetric condition or the Deser–van Nieuwenhuizen condition [16] to minimalize the numbers of dynamical degrees of freedom. In this section, we present a general way to derive the symmetric condition from the equations of motion.

Let us decompose the rank-2 tensor $e_{\mu\nu}$ into two parts,

$$e_{\mu\nu} = F_\mu{}^A E_\nu{}^B \eta_{AB} = e^{\mathrm{s}}_{\mu\nu} + e^{\mathrm{as}}_{\mu\nu}, \tag{176}$$

where $e^{\mathrm{s}}_{\mu\nu}$ is the symmetric part and $e^{\mathrm{as}}_{\mu\nu}$ is the antisymmetric part:

$$e^{\mathrm{s}}_{\mu\nu} = e^{\mathrm{s}}_{\nu\mu}, \qquad e^{\mathrm{as}}_{\mu\nu} = -e^{\mathrm{as}}_{\nu\mu}. \tag{177}$$

In the Lagrangians, an antisymmetric product vanishes if it contains an odd number of $e^{\mathrm{as}}_{\mu\nu}$. In 4d bigravity models, the antisymmetric part of $e_{\mu\nu}$ only appears in the terms below:

$$R_{\mu\nu}{}^{[\mu\nu}(e^{\mathrm{as}})_\rho{}^\rho(e^{\mathrm{as}})_\sigma{}^{\sigma]}, \tag{178}$$

$$(e^{\mathrm{as}})_\mu{}^{[\mu}(e^{\mathrm{as}})_\nu{}^{\nu]},$$
$$(e^{\mathrm{as}})_\mu{}^{[\mu}(e^{\mathrm{as}})_\nu{}^\nu(e^{\mathrm{s}})_\rho{}^{\rho]}, \tag{179}$$

$$(e^{\mathrm{as}})_\mu{}^{[\mu}(e^{\mathrm{as}})_\nu{}^\nu(e^{\mathrm{s}})_\rho{}^\rho(e^{\mathrm{s}})_\sigma{}^{\sigma]}, \tag{180}$$

$$(e^{\mathrm{as}})_\mu{}^{[\mu}(e^{\mathrm{as}})_\nu{}^\nu(e^{\mathrm{as}})_\rho{}^\rho(e^{\mathrm{as}})_\sigma{}^{\sigma]}. \tag{181}$$

We argue that the equations of motion for $e^{\mathrm{as}}_{\mu\nu}$ generally lead to the symmetric condition

$$\frac{\delta}{\delta e^{\mathrm{as}}_{\mu\nu}}\int \mathcal{L} = 0 \Rightarrow e^{\mathrm{as}}_{\mu\nu} = 0, \tag{182}$$

because the equations for $e^{\mathrm{as}}_{\mu\nu}$ can be written in a matrix form,

$$\mathcal{A}_{\mu\nu,\rho\sigma}(e^{\mathrm{as}})^{\rho\sigma} = 0, \tag{183}$$

which gives the symmetric condition if $\mathcal{A}$ is invertible. An important point is that the Lagrangians do not contain linear terms, so the equations of motion for $e^{\mathrm{as}}$ are homogeneous.[37]

The argument is clear if (181) is not considered. When the Lagrangian contains (181), the equations of motion contain cubic terms of $e^{\mathrm{as}}_{\mu\nu}$. Then we can write the cubic terms as products of quadratic terms and linear terms, and think of the quadratic terms as part of the matrix $\mathcal{A}$.[38]

We do not have a proof that the above argument works in general, but we check several examples and always find that

$$\det \mathcal{A} \neq 0. \tag{184}$$

The spirit is close to [10], where the symmetric condition is derived from a local Lorentz transformation. In addition, we do not rule out the possibility that $\mathcal{A}$ could be degenerate at some singular points of the phase space.

## X. HIGHER-DERIVATIVE GENERALIZATIONS

Along the lines of Lovelock terms, the novel kinetic terms can be generalized to novel higher-derivative terms [7],[39]

$$R(E)\wedge\cdots\wedge R(E)\wedge E\wedge\cdots\wedge E\wedge F\wedge\cdots\wedge F, \tag{185}$$

which might be inconsistent with terms involving $R(F)$.

In Sec. V, we discuss the special cases with only one $F$ vielbein,

$$R(E)\wedge\cdots\wedge R(E)\wedge E\wedge\cdots\wedge E\wedge F, \tag{186}$$

which have the same gauge symmetries as $R(E)\wedge E\wedge F$. They describe the derivative interactions between two massless, gauge-invariant spin-2 fields, where the Boulware-Deser ghost is absent. We expect that other higher-derivative terms in (185) do not contain the BD ghost as well.

For the extension to multigravity, we have

$$R(E^{(1)})\wedge\cdots\wedge R(E^{(1)})\wedge E^{(2)}\wedge\cdots\wedge E^{(d-n)}, \tag{187}$$

where $n$ is the number of curvature two-forms and $E^{(k)}$ vielbeins can be the same or different. Lovelock terms and dRGT terms are unified in (187).

In the end, we would like to connect with some results in the literature. The bigravity models with (186) in the metric formulation were already proposed in [31] as generalizations of new massive gravity. The BD ghost was argued to be absent by counting the degrees of freedom using symmetries [31,53]. The antisymmetric structure guarantees that the equations of motion for the decomposed fields

$$e_{\mu\nu} = \tilde{e}_{\mu\nu} + \nabla_\mu A_\nu + \nabla_\nu A_\mu + \nabla_\mu\partial_\nu\phi \tag{188}$$

[37]We assume that the matter does not couple to the antisymmetric part of $e_{\mu\nu}$ linearly. For example, if the physical vielbein is a linear combination of $E_\mu{}^A$ and $F_\mu{}^B$, the corresponding physical metric will contain a quadratic term $e^{\mathrm{as}}_{\mu\rho}e^{\mathrm{as}\,\rho}_\nu$, so the equations of motion for $e^{\mathrm{as}}_{\mu\nu}$ are homogeneous.

[38]Note that $\mathcal{A}$ can be degenerate for special values of $e^{\mathrm{as}}_{\mu\nu}$ if they are not the solutions of the equations of motion at the same time.

[39]If we impose the symmetric condition and fix the second metric to be Minkowski, they are equivalent to the higher-derivative interactions proposed in [17].

are of second order, so additional degrees of freedom are avoided.[40] Then one can count the dynamical degrees of freedom in the bigravity models and show that the total number is at most $(d^2 - 2d - 1)$, which is that of a massless and a massive spin-2 field. Therefore, the Boulware-Deser ghost is absent.[41]

From this argument, we can see why the curvature tensors should be associated with the same spin-2 field $g_{\mu\nu}$. If a curvature tensor contains the second spin-2 field $e_{\mu\nu}$, then the equations of motion for the decomposed modes will usually be of higher order, because they are not gauge modes in a curvature tensor and no apparent antisymmetric structure is protecting them.[42]

The decomposed field argument concerning (186) is based on the fact that Lovelock tensors are divergence free. For the other novel derivative terms in (187), we can generalize this argument by using the second Bianchi identity,

$$\nabla_{[\mu} R_{\nu\rho]}{}^{\alpha\beta} = 0. \tag{189}$$

The covariant derivatives in front of the decomposed fields will not act on the Riemann tensor after integrating by parts, so the equations of motion for the decomposed fields will not contain fourth-order derivative terms of the metric. The variation of a Riemann tensor $R_{\mu\nu}{}^{\rho\sigma}$ contains some second covariant derivative terms of $\delta g_\mu{}^\nu$ that are antisymmetrized, so the equations of motion for the metric will not contain fourth-order derivative terms of $\phi$.

In the vielbein formulation, the second Bianchi identity stems from a basic identity of the exterior derivative,

$$d^2 = 0, \tag{190}$$

which is the key element of the unifying framework [7,8].

## XI. CONCLUSION

In summary, we present evidence that

$$R(E^{(1)}) \wedge \cdots \wedge R(E^{(1)}) \wedge E^{(2)} \wedge \cdots \wedge E^{(n)} \tag{191}$$

are basic building blocks for the actions of interacting spin-2 fields that are free of the Boulware-Deser ghost.[43] Models that can be constructed from these building blocks include Einstein gravity, Weyl gravity, Lovelock gravity, new massive gravity, dRGT massive gravity, and some of their generalizations. The parameter space is further extended by novel derivative terms.

Curiously, the building blocks (191) can be obtained from Lovelock terms by replacing some of the vielbeins in the wedge products with other vielbeins.

The novel two-derivative terms in 4d are studied in detail:

(i) Based on a minisuperspace analysis, a large class of bigravity models (54) is identified, which are potentially free of the Boulware-Deser ghost.

(ii) The bigravity models in this class (54) do not have the usual single dynamical metric limit with a fixed metric, which is in accordance with the no-go theorem for a new kinetic interaction for the single dynamical metric in [18].

(iii) We reformulate some well-understood models of higher-curvature gravity as bigravity models in this class (54), Their spectra are known to contain one massless and one massive spin-2 field without the BD ghost.

(iv) The argument that new massive gravity is free of the BD ghost is extended to other bigravity models in this class (54), which applies to novel higher-derivative terms as well.[44]

This class of bigravity models is interesting despite the issue of spin-2 ghosts. Firstly, as toy models of quantum gravity, they have a better chance to be perturbatively renormalizable and there are fewer negative norm states because the BD ghost is absent. Secondly, as effective field theories of gravity, they can increase the cutoff scale set by higher-derivative terms with large coefficients.[45]

In general, we can avoid the ghost modes by reducing the number of dynamical degrees of freedom. A useful strategy of eliminating ghost modes is to impose specific boundary conditions. Another method to remove the spin-2 ghosts is done simply by setting the decomposed helicity-2 modes $\tilde{e}_{\mu\nu}$ in (188) to zero. They may give rise to healthy vector-tensor theories.[46]

[40] This argument is dangerous. The equation of motion for the decomposed field $\phi$ contains third-order derivative terms of the metric, in the form of covariant derivatives of curvature tensors. In addition, if one varies the action with respect to the metric after the substitution, the equations of motion will contain third-order derivative terms of $\phi$ due to the variations of covariant derivatives. But it is possible that the third-order time-derivative terms can be removed by the time derivatives of some second-order equations [54]; then the counting of the degrees of freedom is correct.

[41] When there are additional gauge symmetries, the number of dynamical degrees of freedom is reduced.

[42] In [55], an Einstein-Hilbert term for $e_{\mu\nu}$ was introduced to obtain unitary models. However, we suspect that the absence of ghostlike degrees of freedom is an artifact of linearization. For example, in the minisuperspace approximation, the Hamiltonians are not linear in the lapse functions.

[43] Parity is assumed to be preserved; otherwise there are more possible terms. For example, in 3d, one could introduce a gravitational Chern-Simons term that violates parity [56]. The critical points of higher-derivative gravity theories were first investigated in this context [57].

[44] The absence of the BD ghost in 4d novel kinetic terms is proved by the constraint analyses in [30].

[45] An important question is whether the special structure of BD-ghost-free building blocks is detuned by quantum corrections.

[46] We refer to [58] for recent developments on vector-tensor models.

In light of the AdS/CFT correspondence [59], the large class of nonunitary bigravity models may be dual to nonunitary conformal field theories. It is interesting to explore nonunitary holography in the extended parameter space.

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*Note Added.*—When the first version of this work appeared in arXiv, we were informed that similar nonlinear completions were studied before by K. Hinterbichler and R. A. Rosen.

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