

# Earliest stages of the nonequilibrium in axially symmetric, self-gravitating, dissipative fluids

L. Herrera<sup>\*</sup>

*Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, 1050 Caracas, Venezuela  
and Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca,  
37007 Salamanca, Spain*

A. Di Prisco<sup>†</sup>

*Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, 1050 Caracas, Venezuela*

J. Ospino<sup>‡</sup>

*Departamento de Matemática Aplicada and Instituto Universitario de Física Fundamental y Matemáticas,  
Universidad de Salamanca, 37007 Salamanca, Spain*

J. Carot<sup>§</sup>

*Departament de Física, Universitat Illes Balears, E-07122 Palma de Mallorca, Spain*

(Received 23 March 2016; published 26 September 2016)

We report a study on axially and reflection symmetric dissipative fluids, just after its departure from hydrostatic and thermal equilibrium, at the smallest time scale at which the first signs of dynamic evolution appear. Such a time scale is smaller than the thermal relaxation time, the thermal adjustment time, and the hydrostatic time. It is obtained that the onset of nonequilibrium will critically depend on a single function directly related to the time derivative of the vorticity. Among all fluid variables (at the time scale under consideration), only the tetrad component of the anisotropic tensor in the subspace orthogonal to the four-velocity and the Killing vector of axial symmetry, shows signs of dynamic evolution. Also, the first step toward a dissipative regime begins with a nonvanishing time derivative of the heat flux component along the meridional direction. The magnetic part of the Weyl tensor vanishes (not so its time derivative), indicating that the emission of gravitational radiation will occur at later times. Finally, the decreasing of the effective inertial mass density, associated to thermal effects, is clearly illustrated.

DOI: [10.1103/PhysRevD.94.064072](https://doi.org/10.1103/PhysRevD.94.064072)

## I. INTRODUCTION

Many issues related with the structure of self-gravitating fluids may be addressed within the static regime. In this case, the spacetime admits a timelike, hypersurface orthogonal, Killing vector. Thus, a coordinate system can always be chosen, such that all metric and physical variables are independent on the timelike coordinate. The static case, for axially and reflection symmetric spacetimes, was studied in [1]. In such a case the fluid is in equilibrium, implying that the hydrostatic equilibrium equations (Eqs. (21,22) in [1]) are satisfied.

If, instead, the system evolves with time, we have to consider the full dynamic case where the system is out of equilibrium (thermal and dynamic), the general formalism to analyze this situation, for axially and reflection symmetric spacetimes was developed in [2] using a framework based on the 1 + 3 formalism [3–6].

However, some part of the life of stars (at any stage of evolution), may be described on the basis of the quasistatic approximation (slowly evolving regime). This is so, because many relevant processes in star interiors take place on time scales that are usually, much larger than the hydrostatic time scale [7,8]. In this case, the system is assumed to evolve, although slowly enough, so that the hydrostatic equilibrium equations (Eqs. (21,22) in [1]) are assumed to be satisfied, all along the evolution.

This regime has been recently described in detail, within the context of the 1 + 3 formalism [9].

Nevertheless, during their evolution, self-gravitating objects may pass through phases of intense dynamical activity for which the quasistatic approximation is clearly not reliable (e.g., the quick collapse phase preceding neutron star formation).

It is worth mentioning that both regimes (“quick” and “quasistatic”), may be present, at different phases of the collapse of massive stars. Indeed, after the core bounce, leading to a supernova, the hydrostatic equilibrium is reached within few milliseconds, while the subsequent, Kelvin–Helmholtz phase, lasts for about 20 seconds, during

<sup>\*</sup>lherrera@usal.es

<sup>†</sup>alicia.diprisco@ciens.ucv.ve

<sup>‡</sup>j.ospino@usal.es

<sup>§</sup>jcarot@uib.cat

which the system is in the quasistatic regime, thereby satisfying the hydrostatic equilibrium equations [10]. We recall, that the hydrostatic time for a neutron star is of the order of  $10^{-3}$  seconds, while the order of magnitude of the relaxation time for neutron star matter range from  $10^{-3}$  to  $10^{-1}$  seconds.

All these phases of star evolution (“slow” and “quick”) are generally accompanied by intense dissipative processes, usually described in the diffusion approximation.

This assumption, in its turn, is justified by the fact that frequently, the mean free path of particles responsible for the propagation of energy in stellar interiors is very small as compared with the typical length of the star.

Here we shall focus on the “quick” phase, with the inclusion of all the dissipative processes.

However, instead of following the evolution of the system for a long time after its departure from equilibrium, we shall analyze its behavior immediately after such departure.

In this work “immediately” means at the smallest time scale, at which we can observe the first signs of dynamical evolution. Such a time scale is assumed to be smaller than the thermal relaxation time, the hydrostatic time, and the thermal adjustment time.

Doing so we shall be able to extract important conclusions about the very early stages of nonequilibrium, avoiding the introduction of numerical procedures which might lead to model dependent conclusions.

The price to pay for such a simplification, is that we shall describe only the very early stages of the evolution. The reward is that we shall be able to answer to the following questions:

- (1) what are the first signs of nonequilibrium?
- (2) what physical variables do exhibit such signs?
- (3) what does control the onset of the dynamic regime, from an equilibrium initial configuration?

Our approach may be summarized as follows: We observe a system which is initially static, and leaves the equilibrium for unknown causes which are not relevant for the discussion. At this moment we put the clock to work, and watch the system until the first signs of nonequilibrium appear. At this very moment, we stop the clock. It is during this time scale that we describe the behavior of the system.

As we shall see, a specific function related with the time derivative of the vorticity vector, appears as the fundamental variable, controlling the departure from equilibrium and the ensuing evolution. By analogy (in its physical meaning) with the Bondi’s news function [11], we shall refer to this quantity as the fluid news function.

From the analysis of the transport equation we shall see that the time derivative of one of the heat flux components (“radial”) vanishes at the time scale under consideration, whereas the time derivative of the other (“meridional”) component, is controlled by the fluid news function.

Also we shall see that, at the time scale under consideration, the only fluid variable which exhibits deviation from the equilibrium is the tetrad component of the anisotropic tensor in the subspace spanned by the two spacelike vectors orthogonal to the four-velocity and the Killing vector of axial symmetry.

At this same time scale, the magnetic part of the Weyl tensor vanishes, implying that no emission of gravitational radiation is produced at this stage of evolution. However, the time derivative of the magnetic part of the Weyl tensor does not vanish and depends upon the fluid news function, in such a way, that the vanishing of the latter imply the vanishing of the former. In other words the emission of gravitational process occurs at a time scale larger than the one considered here, and is tightly related to the fluid news function.

Finally, by using the transport equations together with the “conservation” laws, we put in evidence the decreasing of the effective inertial mass density, associated with thermal effects.

In this work we shall heavily rely on the formalism developed in [2], thus in order to avoid the rewriting of some of the equations we shall frequently refer to [2], however we warn the reader of some important changes in the notation.

## II. BASIC DEFINITIONS AND NOTATION

In this section we shall deploy all the variables required for our study, some details of the calculations are given in [2], and therefore we shall omit them here.

### A. The metric, the source, and the kinematical variables

We shall consider, axially (and reflection) symmetric sources. For such a system the line element may be written in “Weyl spherical coordinates” as:

$$ds^2 = -A^2 dt^2 + B^2(dr^2 + r^2 d\theta^2) + C^2 d\phi^2 + 2Gd\theta dt, \quad (1)$$

where  $A, B, C, G$  are positive functions of  $t, r$ , and  $\theta$ . We number the coordinates  $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$ .

We shall assume that our source is filled with an anisotropic and dissipative fluid. We are concerned with either bounded or unbounded configurations. In the former case we should further assume that the fluid is bounded by a timelike surface  $S$ , and junction (Darmois) conditions should be imposed there.

The energy momentum tensor may be written in the “canonical” form, as

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + P g_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \quad (2)$$

The above is the canonical, algebraic decomposition of a second order symmetric tensor with respect to unit timelike

vector, which has the standard physical meaning when  $T_{\alpha\beta}$  is the energy-momentum tensor describing some energy distribution, and  $V^\mu$  the four-velocity assigned by certain observer.

With the above definitions it is clear that  $\mu$  is the energy density (the eigenvalue of  $T_{\alpha\beta}$  for eigenvector  $V^\alpha$ ),  $q_\alpha$  is the heat flux, whereas  $P$  is the isotropic pressure, and  $\Pi_{\alpha\beta}$  is the anisotropic tensor. We emphasize that we are considering an Eckart frame where fluid elements are at rest.

Since we choose the fluid to be comoving in our coordinates, then

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right); \quad V_\alpha = \left(-A, 0, \frac{G}{A}, 0\right). \quad (3)$$

We shall next define a canonical orthonormal tetrad (say  $e_\alpha^{(a)}$ ), by adding to the four-velocity vector  $e_\alpha^{(0)} = V_\alpha$ , three spacelike unitary vectors (these correspond to the vectors **K**, **L**, **S** in [2])

$$e_\alpha^{(1)} = (0, B, 0, 0); \quad e_\alpha^{(2)} = \left(0, 0, \frac{\sqrt{A^2 B^2 r^2 + G^2}}{A}, 0\right), \quad (4)$$

$$e_\alpha^{(3)}(0, 0, 0, C), \quad (5)$$

with  $a = 0, 1, 2, 3$  (latin indices labeling different vectors of the tetrad).

The dual vector tetrad  $e_{(a)}^\alpha$  is easily computed from the condition

$$\eta_{(a)(b)} = g_{\alpha\beta} e_{(a)}^\alpha e_{(b)}^\beta, \quad e_{(a)}^\alpha e_\alpha^{(b)} = \delta_{(a)}^{(b)}, \quad (6)$$

where  $\eta_{(a)(b)}$  denotes the Minkowski metric.

In the above, the tetrad vector  $e_{(3)}^\alpha = (1/C)\delta_\phi^\alpha$  is parallel to the only admitted Killing vector (it is the unit tangent to the orbits of the group of 1-dimensional rotations that defines axial symmetry). The other two basis vectors  $e_{(1)}^\alpha$ ,  $e_{(2)}^\alpha$  define the two *unique* directions that are orthogonal to the 4-velocity and to the Killing vector.

For the energy density and the isotropic pressure, we have

$$\mu = T_{\alpha\beta} e_{(0)}^\alpha e_{(0)}^\beta, \quad P = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}, \quad (7)$$

where

$$h_\beta^\alpha = \delta_\beta^\alpha + V^\alpha V_\beta, \quad (8)$$

whereas the anisotropic tensor may be expressed through three scalar functions defined as (see [2], but notice the change of notation):

$$\Pi_{(2)(1)} = e_{(2)}^\alpha e_{(1)}^\beta T_{\alpha\beta}, \quad (9)$$

$$\Pi_{(1)(1)} = \frac{1}{3} \left( 2e_{(1)}^\alpha e_{(1)}^\beta - e_{(2)}^\alpha e_{(2)}^\beta - e_{(3)}^\alpha e_{(3)}^\beta \right) T_{\alpha\beta}, \quad (10)$$

$$\Pi_{(2)(2)} = \frac{1}{3} \left( 2e_{(2)}^\alpha e_{(2)}^\beta - e_{(3)}^\alpha e_{(3)}^\beta - e_{(1)}^\alpha e_{(1)}^\beta \right) T_{\alpha\beta}. \quad (11)$$

This specific choice of these scalars is justified by the fact, that the relevant equations used to carry out this study, become more compact and easier to handle, when expressed in terms of them.

Finally, we may write the heat flux vector in terms of the two tetrad components  $q_{(1)}$  and  $q_{(2)}$ :

$$q_\mu = q_{(1)} e_\mu^{(1)} + q_{(2)} e_\mu^{(2)} \quad (12)$$

or, in coordinate components (see [2])

$$q^\mu = \left( \frac{q_{(2)} G}{A \sqrt{A^2 B^2 r^2 + G^2}}, \frac{q_{(1)}}{B}, \frac{A q_{(2)}}{\sqrt{A^2 B^2 r^2 + G^2}}, 0 \right), \quad (13)$$

$$q_\mu = \left( 0, B q_{(1)}, \frac{\sqrt{A^2 B^2 r^2 + G^2} q_{(2)}}{A}, 0 \right). \quad (14)$$

Of course, all the above quantities depend, in general, on  $t$ ,  $r$ ,  $\theta$ .

The expressions for the kinematical variables are (see [2]).

For the four acceleration we have

$$a_\alpha = V^\beta V_{\alpha\beta} = a_{(1)} e_\mu^{(1)} + a_{(2)} e_\mu^{(2)}, \quad (15)$$

with

$$a_{(1)} = \frac{A'}{AB}; \quad a_{(2)} = \frac{A}{\sqrt{A^2 B^2 r^2 + G^2}} \left[ \frac{A_{,\theta}}{A} + \frac{G}{A^2} \left( \frac{\dot{G}}{G} - \frac{\dot{A}}{A} \right) \right], \quad (16)$$

where the dot and the prime denote derivatives with respect to  $t$  and  $r$  respectively.

For the expansion scalar

$$\Theta = V^\alpha_{;\alpha} = \frac{1}{A} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{G^2}{A(A^2 B^2 r^2 + G^2)} \left( -\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G} \right). \quad (17)$$

Next, the shear tensor

$$\sigma_{\alpha\beta} = \sigma_{(a)(b)} e_\alpha^{(a)} e_\beta^{(b)} = V_{(\alpha\beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta}, \quad (18)$$

may be defined through two independent tetrad components (scalars)  $\sigma_{(1)(1)}$  and  $\sigma_{(2)(2)}$ , which may be written in terms of the metric functions and their derivatives as (see [2]):

$$\sigma_{(1)(1)} = \frac{1}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) + \frac{G^2}{3A(A^2B^2r^2 + G^2)} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{G}}{G} \right), \quad (19)$$

$$\sigma_{(2)(2)} = \frac{1}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) + \frac{2G^2}{3A(A^2B^2r^2 + G^2)} \left( -\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G} \right). \quad (20)$$

It is worth noticing that the shear tensor has no projection in the subspace  $e_\alpha^{(1)} e_\beta^{(2)}$ .

Finally, for the vorticity tensor

$$\Omega_{\beta\mu} = \Omega_{(a)(b)} e_\beta^{(a)} e_\mu^{(b)}, \quad (21)$$

we find that it is determined by a single basis component:

$$\Omega_{(1)(2)} = -\Omega_{(2)(1)} = -\Omega, \quad (22)$$

where the scalar function  $\Omega$  is given by

$$\Omega = \frac{G(\frac{G'}{G} - \frac{2A'}{A})}{2B\sqrt{A^2B^2r^2 + G^2}}. \quad (23)$$

Now, from the regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry, and in particular at the center (see [12–14]), we should require that as  $r \approx 0$

$$\Omega = \sum_{n \geq 1} \Omega^{(n)}(t, \theta) r^n, \quad (24)$$

implying, because of (23) that in the neighborhood of the center

$$G = \sum_{n \geq 3} G^{(n)}(t, \theta) r^n. \quad (25)$$

Beside the kinematical variables defined above, it would be convenient for our discussion to introduce the “specific velocities”, defined in [9] (with the change of notation already mentioned):

$$V_{(1)(1)} = e_\alpha^{(1)} e_\beta^{(1)} \left( \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \Omega_{\alpha\beta} \right), \quad (26)$$

$$V_{(2)(2)} = e_\alpha^{(2)} e_\beta^{(2)} \left( \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \Omega_{\alpha\beta} \right), \quad (27)$$

$$V_{(3)(3)} = e_\alpha^{(3)} e_\beta^{(3)} \left( \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \Omega_{\alpha\beta} \right), \quad (28)$$

$$V_{(1)(2)} = e_\alpha^{(1)} e_\beta^{(2)} \left( \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \Omega_{\alpha\beta} \right), \quad (29)$$

which become, using (17), (19), (20), and (22)

$$V_{(1)(1)} = \frac{1}{3} (3\sigma_{(1)(1)} + \Theta), \quad V_{(2)(2)} = \frac{1}{3} (3\sigma_{(2)(2)} + \Theta), \quad (30)$$

$$V_{(3)(3)} = \frac{1}{3} (\Theta - 3\sigma_{(1)(1)} - 3\sigma_{(2)(2)}), \quad V_{(1)(2)} = -\Omega, \quad (31)$$

satisfying

$$V_{(1)(1)} + V_{(2)(2)} + V_{(3)(3)} = \Theta. \quad (32)$$

The physical meaning of the above expressions becomes intelligible when we recall that the tensor  $\sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \Omega_{\alpha\beta}$  defines the proper time variation of the infinitesimal distance  $\delta l$  between two neighboring points on the three-dimensional hypersurface (say  $\Sigma$ ), orthogonal to the four velocity, divided by  $\delta l$  (see [9] for details).

## B. The electric and magnetic part of the Weyl tensor and the super-Poynting vector

Let us now introduce the electric ( $E_{\alpha\beta}$ ) and magnetic ( $H_{\alpha\beta}$ ) parts of the Weyl tensor ( $C_{\alpha\beta\gamma\delta}$ ), defined as usual by

$$E_{\alpha\beta} = C_{\alpha\nu\beta\delta} V^\nu V^\delta, \quad H_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\nu\epsilon\rho} C_{\beta\delta}^{\epsilon\rho} V^\nu V^\delta. \quad (33)$$

The electric part of the Weyl tensor has only three independent nonvanishing components, whereas only two components define the magnetic part. Thus we may write these two tensors, in terms of five tetrad components ( $\mathcal{E}_{(1)(1)}$ ,  $\mathcal{E}_{(2)(2)}$ ,  $\mathcal{E}_{(1)(2)}$ ,  $H_{(1)(3)}$ ,  $H_{(3)(2)}$ ), respectively as:

$$E_{\alpha\beta} = \left[ (2\mathcal{E}_{(1)(1)} + \mathcal{E}_{(2)(2)}) \left( e_\alpha^{(1)} e_\beta^{(1)} - \frac{1}{3} h_{\alpha\beta} \right) \right] + \left[ (2\mathcal{E}_{(2)(2)} + \mathcal{E}_{(1)(1)}) \left( e_\alpha^{(2)} e_\beta^{(2)} - \frac{1}{3} h_{\alpha\beta} \right) \right] + \mathcal{E}_{(2)(1)} (e_\alpha^{(1)} e_\beta^{(2)} + e_\beta^{(1)} e_\alpha^{(2)}), \quad (34)$$

and

$$H_{\alpha\beta} = H_{(1)(3)}(e_{\beta}^{(1)}e_{\alpha}^{(3)} + e_{\alpha}^{(1)}e_{\beta}^{(3)}) + H_{(2)(3)}(e_{\beta}^{(3)}e_{\alpha}^{(2)} + e_{\alpha}^{(2)}e_{\beta}^{(3)}). \quad (35)$$

Also, from the Riemann tensor we may define three tensors  $Y_{\alpha\beta}$ ,  $X_{\alpha\beta}$ , and  $Z_{\alpha\beta}$  as

$$Y_{\alpha\beta} = R_{\alpha\nu\beta\delta}V^{\nu}V^{\delta}, \quad (36)$$

$$X_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu}{}^{\epsilon\rho}R_{\epsilon\rho\beta\delta}^*V^{\nu}V^{\delta}, \quad (37)$$

and

$$Z_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\epsilon\rho}R_{\delta\beta}{}^{\epsilon\rho}V^{\delta}, \quad (38)$$

where  $R_{\alpha\beta\nu\delta}^* = \frac{1}{2}\eta_{\epsilon\rho\nu\delta}R_{\alpha\beta}{}^{\epsilon\rho}$  and  $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho}V^{\nu}$ .

The above tensors in turn, may be decomposed, so that each of them is described through four scalar functions known as structure scalars [15]. These are (see [2] for details)

$$\begin{aligned} Y_T &= 4\pi(\mu + 3P), & X_T &= 8\pi\mu, \\ Y_I &= 3\mathcal{E}_{(1)(1)} - 12\pi\Pi_{(1)(1)}, & X_I &= -3\mathcal{E}_{(1)(1)} - 12\pi\Pi_{(1)(1)} \\ Y_{II} &= 3\mathcal{E}_{(2)(2)} - 12\pi\Pi_{(2)(2)}, & X_{II} &= -3\mathcal{E}_{(2)(2)} - 12\pi\Pi_{(2)(2)}, \\ Y_{III} &= \mathcal{E}_{(2)(1)} - 4\pi\Pi_{(2)(1)}, & X_{III} &= -\mathcal{E}_{(2)(1)} - 4\pi\Pi_{(2)(1)}, \end{aligned} \quad (39)$$

and

$$\begin{aligned} Z_I &= (H_{(1)(3)} - 4\pi q_{(2)}); & Z_{II} &= (H_{(1)(3)} + 4\pi q_{(2)}); \\ Z_{III} &= (H_{(2)(3)} - 4\pi q_{(1)}); & Z_{IV} &= (H_{(2)(3)} + 4\pi q_{(1)}). \end{aligned} \quad (40)$$

From the above tensors, we may define the super-Poynting vector by

$$P_{\alpha} = \epsilon_{\alpha\beta\gamma}(Y_{\delta}^{\gamma}Z^{\beta\delta} - X_{\delta}^{\gamma}Z^{\delta\beta}), \quad (41)$$

where  $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho}V^{\nu}$ .

In our case, we may write:

$$P_{\alpha} = P_{(1)}e_{\alpha}^{(1)} + P_{(2)}e_{\alpha}^{(2)}, \quad (42)$$

with

$$\begin{aligned} P_{(1)} &= 2H_{(2)(3)}(2\mathcal{E}_{(2)(2)} + \mathcal{E}_{(1)(1)}) + 2H_{(1)(3)}\mathcal{E}_{(2)(1)} \\ &\quad + 32\pi^2q_{(1)}[(\mu + P) + \Pi_{(1)(1)}] + 32\pi^2q_{(2)}\Pi_{(2)(1)}, \\ P_{(2)} &= -2H_{(1)(3)}(2\mathcal{E}_{(1)(1)} + \mathcal{E}_{(2)(2)}) - 2H_{(2)(3)}\mathcal{E}_{(2)(1)} \\ &\quad + 32\pi^2q_{(2)}[(\mu + P) + \Pi_{(2)(2)}] + 32\pi^2q_{(1)}\Pi_{(2)(1)}. \end{aligned} \quad (43)$$

In the theory of the super-Poynting vector, a state of gravitational radiation is associated to a nonvanishing component of the latter (see [16–18]). This is in agreement with the established link between the super-Poynting vector and the news functions [19], in the context of the Bondi–Sachs approach [11,20].

We can identify two different contributions in (43). On the one hand we have contributions from the heat transport process. These are in principle independent of the magnetic part of the Weyl tensor, which explains why they remain in the spherically symmetric limit. Next we have contributions related to the gravitational radiation. These require, both, the electric and the magnetic part of the Weyl tensor to be different from zero.

### III. THE HEAT TRANSPORT EQUATION

In order to avoid the drawbacks generated by the standard (Landau–Eckart) irreversible thermodynamics [21,22], (see [23–26] and references therein) we shall need a transport equation derived from a causal dissipative theory [27–32]. In this work we shall resort to Müller-Israel-Stewart second order phenomenological theory for dissipative fluids [27–30]. However, as we shall see, the main conclusions generated by our study are not dependent on the transport equation chosen, as far as it is a causal one, i.e., that it leads to a Cattaneo type [33] equation, leading thereby to a hyperbolic equation for the propagation of thermal perturbations.

Thus, the transport equation for the heat flux reads [24,28–30],

$$\tau h_{\nu}^{\mu}q_{;\beta}^{\nu}V^{\beta} + q^{\mu} = -\kappa h^{\mu\nu}(T_{;\nu} + Ta_{\nu}) - \frac{1}{2}\kappa T^2\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}q^{\mu}, \quad (44)$$

where  $\tau$ ,  $\kappa$ ,  $T$  denote the relaxation time, the thermal conductivity, and the temperature, respectively.

Contracting (44) with  $e_{\mu}^{(2)}$  we obtain

$$\begin{aligned} &\frac{\tau}{A}(\dot{q}_{(2)} + Aq_{(1)}\Omega) + q_{(2)} \\ &= -\frac{\kappa}{A}\left(\frac{G\dot{T} + A^2T_{;\theta}}{\sqrt{A^2B^2r^2 + G^2}} + ATa_{(2)}\right) - \frac{\kappa T^2q_{(2)}}{2}\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}, \end{aligned} \quad (45)$$

where (23), has been used.

On the other hand, contracting (44) with  $e_\mu^{(1)}$ , we find

$$\frac{\tau}{A}(\dot{q}_{(1)} - Aq_{(2)}\Omega) + q_{(1)} = -\frac{\kappa}{B}(T' + BTa_{(1)}) - \frac{\kappa T^2 q_{(1)}}{2} \left( \frac{\tau V^\alpha}{\kappa T^2} \right)_{;\alpha}. \quad (46)$$

It is worth noticing that the two equations above are coupled through the vorticity.

#### IV. LEAVING THE EQUILIBRIUM

We shall now take a snapshot of the system, just after it has abandoned the equilibrium. As mentioned before, by “just after” we mean on the smallest time scale, at which we can detect the first signs of dynamical evolution.

The general “philosophy” of our approach consists of considering a fluid distribution which is in equilibrium (in the sense exposed in the Introduction), and assume that, for a reason which is not relevant for the discussion, at some initial time (say  $t_0$ ) the system abandons such a state. Thus, at  $t_0$  the clock is put to measure time, and we stop it as soon as we detect the first sign of dynamic evolution. The scale time under consideration is defined by the time interval measured by our clock. This is, so to speak, the “philosophy” of the approach.

However, in practice we shall proceed slightly differently. Indeed, we are going to choose a given time scale, which we shall specify below. Two possible results may then appear:

- (i) No signs of dynamic evolution are observed within the chosen time scale.
- (ii) Such signs do appear, at such time scale.

Of course in the case of the first result, we should have to enlarge our time scale.

Now, in the study of dissipative fluids, there are three fundamental time scales, each of which are endowed with a distinct physical meaning, namely: the hydrostatic time (sometimes also called the hydrodynamic time), the thermal relaxation time, and the thermal adjustment time (see [7,8] for details).

The hydrostatic time is the typical time in which a fluid element reacts on a slight perturbation of hydrostatic equilibrium, it is basically of the order of magnitude of the time taken by a sound wave to propagate through the whole fluid distribution.

The thermal relaxation time is the time taken by the system to return to the steady state in the heat flux (whether of thermodynamic equilibrium or not), after it has been removed from it.

Finally, the thermal adjustment time is the time it takes a fluid element to adjust thermally to its surroundings. It is, essentially, of the order of magnitude of the time required for a significant change in the temperature gradients. From

the above it is evident that the thermal adjustment time is, generally, larger than the thermal relaxation time.

We shall evaluate the system at a time scale which is smaller than the three time scales described above. It should be emphasized that such a time scale is chosen heuristically. Thus, as mentioned before, if no sign of evolution could be detected within this time scale, it should be enlarged until these signs appear. However, as we shall see below, such signs do appear within the time scale under consideration.

The above comments imply that:

- (i) At the time scale at which we are observing the system, which is smaller than the hydrostatic time scale, the kinematical quantities  $\Omega(G)$ ,  $\Theta$ ,  $\sigma_{(1)(1)}$ ,  $\sigma_{(2)(2)}$  as well as the “velocities”  $V_{(1)(1)}$ ,  $V_{(2)(2)}$ ,  $V_{(3)(3)}$ ,  $V_{(1)(2)}$  keep the same values they have in equilibrium, i.e., they are neglected (of course not so their time derivatives which are assumed to be small, say of order  $O(\epsilon)$ , where  $\epsilon \ll 1$ ), but nonvanishing.
- (ii) From (A5) (A6) (Eqs. B6, B7 in [2]), it follows at once that the heat flux vector should also be neglected (once again, not so its time derivative). The vanishing of the flux vector also follows at once from the fact that the time scale under consideration is smaller than the relaxation time.
- (iii) From the above conditions it follows at once that first order time derivatives of the metric variables  $A$ ,  $B$ ,  $C$  can be neglected.

Then, we have for the four acceleration

$$a_{(1)} = \frac{A'}{AB}; \quad a_{(2)} = \frac{1}{Br} \left( \frac{A_{,\theta}}{A} + \frac{\dot{G}}{A^2} \right). \quad (47)$$

Also, from the conditions above and (17), (19), (20), (23), (30), (31), it follows that

$$\dot{\Theta} = \frac{1}{A} \left( \frac{2\ddot{B}}{B} + \frac{\ddot{C}}{C} \right),$$

$$\dot{\sigma}_{(1)(1)} = \dot{\sigma}_{(2)(2)} \equiv \dot{\sigma} = \frac{1}{3A} \left( \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} \right), \quad (48)$$

$$\dot{\Omega} = \frac{1}{AB^2 r} \left( \frac{\dot{G}'}{2} - \frac{\dot{G}A'}{A} \right), \quad (49)$$

and

$$V_{(1)(1)} = V_{(2)(2)} \equiv V, \quad \dot{V} = \frac{\ddot{B}}{AB}, \quad \dot{V}_{(3)(3)} = \frac{\ddot{C}}{AC}. \quad (50)$$

Now, at thermal equilibrium, when the heat flux vanishes, the Tolman conditions for thermal equilibrium [34]

$$(TA)' = (TA)_{,\theta} = 0, \quad (51)$$

are valid.

Therefore just after the system leaves the equilibrium, at a time scale which is smaller than the thermal adjustment time and the thermal relaxation time, the Eqs. (51) are still valid, even though the system starts to leave the thermal equilibrium. This is so because of the fact that our time scale is smaller than the relaxation time, and therefore the temperature gradients have the same values they had in equilibrium. However, the fulfillment of (51) is not enough to ensure the vanishing of  $\dot{q}_{(2)}$ , due to the appearance of a  $\dot{G}$  term in (45) (through  $a_{(2)}$ ), which eventually would lead to the breaking of the thermal equilibrium in the meridional direction (at later time).

Thus, the evaluation of (46) and (45) just after leaving the equilibrium, produces respectively

$$\dot{q}_{(1)} = 0, \quad (52)$$

and

$$\tau\dot{q}_{(2)} = -\frac{\kappa AT_{,\theta}}{\text{Br}} - \kappa AT a_{(2)}, \quad (53)$$

or, using (51)

$$\tau\dot{q}_{(2)} = -\frac{\kappa T \dot{G}}{A \text{Br}}. \quad (54)$$

Therefore, at the very beginning of the evolution, the dissipative process starts with contributions along the  $e_{\mu}^{(2)}$  (meridional) direction.

We shall now turn to fluid variables ( $\mu, P, \Pi_{(1)(1)}, \Pi_{(2)(2)}, \Pi_{(2)(1)}$ ). Using MAPLE we shall calculate the components of the Einstein tensor  $G_{\alpha\beta}$  and evaluate them just after the system leaves the equilibrium. At this time scale, this tensor have three types of terms: On the one hand, terms with first time derivatives of the metric functions  $A, B, C$ , which are set to zero, next, there are terms that neither contain  $G$ , nor first time derivatives of  $A, B, C$ , these correspond to the expression in equilibrium, finally, there are terms with first time derivatives of  $G$  and/or second time derivatives of  $A, B, C$ , which of course are not neglected. Then using (7), (9), (10), (11) and the Einstein equations,

$$G_{\alpha\beta} = -8\pi T_{\alpha\beta}, \quad (55)$$

we obtain

$$8\pi\mu = 8\pi\mu_{(eq)}, \quad (56)$$

$$8\pi P = 8\pi P_{(eq)} - \frac{2}{3A}\dot{\Theta} + \frac{2}{3A^2 B^2 r^2} \left( \dot{G}_{,\theta} + \dot{G} \frac{C_{,\theta}}{C} \right), \quad (57)$$

$$8\pi\Pi_{(1)(1)} = 8\pi\Pi_{(1)(1)(eq)} + \frac{\dot{\bar{\sigma}}}{A} + \frac{1}{3A^2 B^2 r^2} \left[ \dot{G}_{,\theta} - \dot{G} \left( \frac{3B_{,\theta}}{B} - \frac{C_{,\theta}}{C} \right) \right], \quad (58)$$

$$8\pi\Pi_{(2)(2)} = 8\pi\Pi_{(2)(2)(eq)} + \frac{\dot{\bar{\sigma}}}{A} + \frac{1}{3A^2 B^2 r^2} \left[ -2\dot{G}_{,\theta} + \dot{G} \left( \frac{3B_{,\theta}}{B} + \frac{C_{,\theta}}{C} \right) \right], \quad (59)$$

$$8\pi\Pi_{(2)(1)} = 8\pi\Pi_{(1)(1)(eq)} - \frac{\dot{\Omega}}{A} + \frac{\dot{G}}{A^2 B^2 r} \left[ \frac{(Br)'}{\text{Br}} - \frac{A'}{A} \right], \quad (60)$$

where  $eq$  stands for the value of the quantity at equilibrium.

Now, from (56) it follows at once that the energy density, after leaving the equilibrium, at the time scale considered here, has the same value it had in equilibrium. Then since there should be a generic equation of state relating the energy density with the isotropic pressure, it is reasonable to assume that at the time scale under consideration we have  $P = P_{(eq)}$ , and following this line of arguments it would be also reasonable to assume  $\Pi_{(1)(1)} = \Pi_{(1)(1)(eq)}, \Pi_{(2)(2)} = \Pi_{(2)(2)(eq)}$ .

Once again, it is important to remark that such assumptions are purely heuristic. Therefore if it would happen that as a consequence of their imposition, we detect no signs of evolution (at the time scale under consideration), we should relax them and enlarge our time scale, until these signs become observable. However this is not the case. Indeed, from these latter conditions and (23), (57), (58), (59), it follows at once that:

$$\dot{G} = B^2 f(t, r), \quad (61)$$

$$\dot{\bar{\sigma}} = -\frac{f(t, r)}{3Ar^2} \left( \frac{C_{,\theta}}{C} - \frac{B_{,\theta}}{B} \right), \quad (62)$$

$$\dot{\Theta} = \frac{f(t, r)}{Ar^2} \left( \frac{2B_{,\theta}}{B} + \frac{C_{,\theta}}{C} \right), \quad (63)$$

$$\dot{\Omega} = \frac{f(t, r)}{Ar} \left( \ln \frac{B\sqrt{f}}{A} \right)', \quad (64)$$

where  $f(t, r)$  is an arbitrary function of its arguments.

Two comments are in order at this point:

- (i) Because of (25) it is obvious that  $f = \sum_{n \geq 3} f^{(n)}(t) r^n$  in the neighborhood of the center.
- (ii) Observe that  $f$  controls the evolution of  $G(\Omega), \Theta$  and  $\bar{\sigma}$ .

The situation is quite different for the scalar  $\Pi_{(2)(1)}$ . In fact, as we shall see, we cannot assume that  $\Pi_{(2)(1)} = \Pi_{(2)(1)(eq)}$ .

Indeed, because of (60), to assume that  $\Pi_{(2)(1)} = \Pi_{(2)(1)(eq)}$ , amounts to impose the condition

$$\frac{\dot{\Omega}}{A} = \frac{\dot{G}}{A^2 B^2 r} \left[ \frac{(Br)'}{Br} - \frac{A'}{A} \right], \quad (65)$$

which together with (23) produces

$$\dot{G} = B^2 r^2 g(t, \theta), \quad (66)$$

where  $g$  is an arbitrary function of its arguments. But, (66) clearly violates the regularity condition (25), close to the center. Accordingly, at the time scale under consideration we have  $\Pi_{(2)(1)} \neq \Pi_{(2)(1)(eq)}$ , more precisely

$$8\pi\Pi_{(2)(1)} = 8\pi\Pi_{(2)(1)(eq)} + \frac{f(t, r)}{2A^2 r} \left( \ln \frac{r^2}{f} \right)'. \quad (67)$$

Thus we see that, after leaving the equilibrium, at the time scale under consideration, the energy density, the isotropic pressure and the (1)(1) and (2)(2) tetrad components of the anisotropic tensor may be assumed to keep the values they have in equilibrium. However for the transverse tension  $\Pi_{(2)(1)}$  the situation is different, and the first signs of the dynamic regime are already present in this tetrad component of the anisotropic tensor, at our time scale.

Using MAPLE we can also easily calculate the scalars defining the electric part of the Weyl tensor, after the system leaves the equilibrium, we obtain:

$$\begin{aligned} \mathcal{E}_{(1)(1)} &= \mathcal{E}_{(1)(1)(eq)} - \frac{\dot{\sigma}}{2A} \\ &\quad - \frac{1}{6A^2 B^2 r^2} \left[ \dot{G}_{,\theta} - \dot{G} \left( \frac{3B_{,\theta}}{B} - \frac{C_{,\theta}}{C} \right) \right], \end{aligned} \quad (68)$$

$$\begin{aligned} \mathcal{E}_{(2)(2)} &= \mathcal{E}_{(2)(2)(eq)} - \frac{\dot{\sigma}}{2A} \\ &\quad + \frac{1}{6A^2 B^2 r^2} \left[ 2\dot{G}_{,\theta} - \dot{G} \left( \frac{3B_{,\theta}}{B} + \frac{C_{,\theta}}{C} \right) \right] \end{aligned} \quad (69)$$

$$\mathcal{E}_{(2)(1)} = \mathcal{E}_{(2)(1)(eq)} + \frac{\dot{\Omega}}{2A} - \frac{\dot{G}}{2A^2 B^2 r} \left[ \frac{(Br)'}{Br} - \frac{A'}{A} \right]. \quad (70)$$

Using (60), (61), and (63) in (68), (69), and (70), it follows at once that

$$\begin{aligned} \mathcal{E}_{(1)(1)} &= \mathcal{E}_{(1)(1)(eq)}, & \mathcal{E}_{(2)(2)} &= \mathcal{E}_{(2)(2)(eq)}, \\ \mathcal{E}_{(2)(1)}^{oeq} &= -4\pi\Pi_{(2)(1)}^{oeq}, \end{aligned} \quad (71)$$

which imply, because of (39)

$$X_I = X_{I(eq)}, \quad X_{II} = X_{II(eq)}, \quad X_{III} = X_{III(eq)}, \quad (72)$$

and

$$Y_I = Y_{I(eq)}, \quad Y_{II} = Y_{II(eq)}, \quad Y_{III}^{oeq} = -8\pi\Pi_{(2)(1)}^{oeq}, \quad (73)$$

where  $oeq$  stands for the value of the quantity ‘‘out of equilibrium,’’ and as it follows at once from (67)

$$8\pi\Pi_{(2)(1)}^{oeq} = \frac{f(t, r)}{2A^2 r} \left( \ln \frac{r^2}{f} \right)'. \quad (74)$$

Let us now analyze the ‘‘generalized Euler equations’’ (A2) (Eq. A7 in [2]), derived from the ‘‘conservation laws’’ ( $T_{;\nu}^{\mu\nu} = 0$ ). Evaluated within the time scale under consideration, these are the Eqs. (A3) and (A4) in the Appendix:

Observe that these two equations have the ‘‘Newtonian’’ form

$$\text{Massdensity} \times \text{Acceleration} = \text{Force}, \quad (75)$$

and where we can clearly identify the ‘‘effective inertial mass density’’ as the factors multiplying  $\dot{V}$  and  $\dot{V}_{(3)(3)}$ . Also, it is worth noticing that the first term in the right-hand side of (A3), and the first term in the right-hand side of (A4), represent the ‘‘gravitational force.’’ This is in agreement with the equivalence principle, according to which, the ‘‘effective inertial mass density’’ equals the ‘‘passive gravitational mass density’’ [the factor multiplying the square brackets in (A3) and (A4)].

We observe that, according to (A3) and (A4) there are two different ‘‘effective inertial mass densities’’, depending on the anisotropy of the fluid. This is a clear reminiscence of the situation appearing in relativistic dynamics, where a moving particle offers different inertial resistances to the same force, according to whether it is subjected to that force longitudinally or transversely.

Finally, replacing  $\dot{q}_{(2)}$ , by its expression from (53), into (A4) we obtain

$$\begin{aligned} &(\mu + P + \Pi_{(2)(2)}) \left[ 1 - \frac{\kappa T}{\tau(\mu + P + \Pi_{(2)(2)})} \right] \dot{V}_{(3)(3)} \\ &= -(\mu + P + \Pi_{(2)(2)}) \left[ 1 - \frac{\kappa T}{\tau(\mu + P + \Pi_{(2)(2)})} \right] \\ &\quad \times \left[ 4\pi A(P + 2\Pi_{(1)(1)} + 2\Pi_{(2)(2)}) - \frac{AC' a(1)}{BC} \right] \\ &\quad + \text{force and dissipative terms.} \end{aligned} \quad (76)$$

This last equation illustrates the well known decreasing of the inertial mass density (and consequently, of the passive gravitational mass density) associated to thermal effects, which was discovered in [35], and that has been

shown to appear in a great variety of scenarios (see [36–48] and references therein).

Next, observe that by evaluating the physical variables out of equilibrium, we may obtain

$$\dot{V} = \frac{B_{,\theta}\dot{G}}{AB^3r^2} = \frac{B_{,\theta}f}{Ar^2B}, \quad (77)$$

$$\dot{V}_{(3)(3)} = \frac{C_{,\theta}\dot{G}}{ACB^2r^2} = \frac{C_{,\theta}f}{Ar^2C}, \quad (78)$$

from where it is apparent that  $f$  controls the evolution of the different “velocities.”

We can now turn to the equations (B1, B3) and (B5) in [2]. They describe the evolution of  $\Theta$ ,  $\bar{\sigma}$ , and  $\Omega$ , and using (62), (63), (64) they become identities. On the other hand (B4) becomes an identity when using (61), (67), (71).

Next we have the Eqs. (A5), (A6) (B6, B7 in [2]), which from the all results obtained above become identities, whereas the Eqs. (B8) and (B9) imply

$$H_{(1)(3)} = H_{(3)(2)} = 0, \quad (79)$$

of course their time derivatives do not vanish, as we shall see below.

Equations (B10–B13) in [2] describe the evolution of the structure scalars  $X_I, X_{II}, X_{III}$ . It is a simple matter to check that within the time scale considered here  $\dot{X}_I = \dot{X}_{II} = \dot{X}_{III} = 0$ . Also, it is a simple matter to see that Eqs. (B14–B16) in [2] do not provide any additional information.

Finally, the Eqs. (A7) and (A8) (B17, B18 in [2]) describe the evolution of the magnetic part of the Weyl tensor in terms of the function  $f(t, r)$ , more specifically, these equations become:

$$\begin{aligned} \dot{H}_{(1)(3)} = & \frac{f}{4ABr} \left[ \frac{f'}{rf} - \frac{f''}{f} - \left( \frac{2}{r} - \frac{f'}{f} \right) \left( \frac{A'}{A} - \frac{2B'}{B} + \frac{C'}{C} \right) \right] \\ & + \frac{fB(2\mathcal{E}_{(1)(1)} + \mathcal{E}_{(2)(2)})}{Ar}, \end{aligned} \quad (80)$$

$$\begin{aligned} \dot{H}_{(3)(2)} = & \frac{f}{4ABr^2} \left( \frac{2}{r} - \frac{f'}{f} \right) \left( \frac{A_{,\theta}}{A} - \frac{2B_{,\theta}}{B} + \frac{C_{,\theta}}{C} \right) \\ & + \frac{fB\mathcal{E}_{(2)(1)(eq)}}{Ar}, \end{aligned} \quad (81)$$

from which it is evident that the evolution of the magnetic part of the Weyl tensor is fully controlled by the function  $f$ .

## V. CONCLUSIONS

We have carried out an exhaustive analysis of axially symmetric fluid distributions, just after its departure from equilibrium, at the smallest time scale at which we can detect signs of dynamical evolution.

As our main result, we have found that the evolution of all variables is controlled by a single function  $f$ , which we call the fluid news function, in analogy with the Bondi’s news function. Indeed, if anything happens at all at the source leading to changes in the field, it can only do so through the function  $f$ , and viceversa, exactly as it appears from the analysis of the spacetime outside the source (Bondi). However, an important difference between these two functions must be emphasized, namely: our function  $f$  controls the evolution only within the time scale considered here, a limitation which does not apply to the Bondi’s news function (see below for a deeper discussion on this point).

Among all the physical variables, there are two which play a significant role in the departure from equilibrium. On the one hand, it is the heat flow along the  $e_{(2)}^\mu$  direction, the one which shall appear first. On the other hand, it is also remarkable that it is the tetrad component of the anisotropic tensor, in the subspace spanned by the tensor  $e_{(2)}^\mu e_{(1)}^\nu$ , the one which shows the first indications of the departure from equilibrium.

It is worth mentioning, that at the time scale used here, there is not gravitational radiation, as it follows at once from (43). Thus, the emission of gravitational waves is an event which occurs at later times. This fact becomes intelligible in the light of the following comments.

For a second order phenomenological theory for dissipative fluids we obtain from Gibbs equation and conservation equations (see [24,43] for details):

$$\begin{aligned} TS_{;\alpha}^\alpha = & -q^\alpha \left[ h_{\alpha}^\mu (\ln T)_{;\mu} + V_{\alpha;\mu} V^\mu + \beta_1 q_{\alpha;\mu} V^\mu \right. \\ & \left. + \frac{T}{2} \left( \frac{\beta_1}{T} V^\mu \right)_{;\mu} q_\alpha \right], \end{aligned} \quad (82)$$

where  $S^\alpha$  is the entropy four-current, and  $\beta_1 = \frac{\tau}{\kappa T}$ .

From which it becomes evident that at the time scale under consideration  $S_{;\alpha}^\alpha = 0$ .

We recall that in the above expression, terms involving couplings of heat flux to the vorticity, vanish at the time scale under consideration. Also, we have excluded shear and bulk viscosity contributions in (82). The fact is that these absent terms are proportional to the shear tensor, the expansion scalar, terms quadratic in the bulk viscosity pressure, terms proportional to the bulk viscosity pressure multiplied by its time derivative, and terms proportional to the anisotropic stress tensor associated to the shear viscosity multiplied by itself, or by its time derivative (see Eq. (2.20) in [24]), (we recall that the anisotropic stress tensor may, but does not need to, be related to viscosity effects, since it may be sourced by many other physical phenomena. Thus, for example it may be different from zero for a static configuration). Of course, within the time scale used here, all these terms vanish. However, it should be clear that in the study of any specific astrophysical

scenario, these dissipative phenomena may be present and might play an important role in the detailed description of the structure and evolution of the object (at a time scale larger than the one considered here).

Thus, within our time scale, our observers do not detect a real (entropy producing) dissipative process. But as it was already pointed out in the seminal Bondi's paper on gravitational radiation (see Sec. VI in [11]), in the absence of dissipation, the system is not expected to radiate (gravitationally) due to the reversibility of the equation of state, at variance with the fact that radiation is an irreversible process (see also [49] for a further discussion on this point).

Therefore, it is obvious that, in the presence of gravitational radiation, an entropy generator factor should also be present in the description of the source. But as we have just seen, such a factor does not appear within the time scale under consideration. Accordingly it is reasonable not to detect gravitational radiation at that same time scale.

The reversibility of the evolution, at the time scale under consideration, implied by the above comments, could also be inferred from a simple inspection of (54), (61), (62), (63), (64), (77), (78), (80), (81).

Indeed, it results at once from these equations, that if the function  $f$  is different from zero until some time, and vanishes afterwards (always within the time scale under consideration), the system will turn back to equilibrium, without "remembering" to have been out of it previously.

In other words, the fluid news function, unlike the Bondi's news function, is the precursor of, (appears before), the dissipative process related to the emission of gravitational radiation, and should be different from zero until such emission starts.

In relation with the point above, another comment is in order: in [19] the link between radiation and vorticity was put in evidence (see also [50]), more specifically it was explicitly assumed that such a link was a causal one (the title of [19] is: "Why does gravitational radiation produce vorticity?"), i.e., it was assumed that radiation precedes the appearance of vorticity. However as we have just shown, both the magnetic part of the Weyl tensor, and  $\Omega$  vanish at the time scale under consideration, whereas their first time derivatives do not vanish at that same time scale, suggesting that both phenomena (radiation and vorticity) occur essentially simultaneously.

An interesting particular case is represented by the situation appearing if we impose that the system was initially spherically symmetric (besides being in equilibrium), and assume that it remains spherically symmetric afterwards. In such a case, it is obvious that we must have  $\dot{G} = 0$ , implying that departures from equilibrium (dynamic and thermal) only occur if Tolman's conditions (51), are violated. However, since the system was initially at equilibrium, such a violation may only happen at time scales larger than the thermal adjustment time. In other

words, departures from equilibrium, keeping the spherical symmetry, take place at time scales larger than the corresponding to the, general, nonspherical case. Observe that in the purely spherically symmetric case the assumptions  $P = P_{(eq)}$ ,  $\Pi_{(1)(1)} = \Pi_{(1)(1)(eq)}$ ,  $\Pi_{(2)(2)} = \Pi_{(2)(2)(eq)}$  do not hold (since we have to enlarge the time scale in order to observe the first signs of evolution), and of course the onset of evolution is not controlled by the function  $f$  as defined by (61).

We would like to emphasize the appearance of the thermal effect leading to a decreasing of the effective inertial mass density. In this respect, it is worth stressing that the first term on the left, and the  $Ta_\nu$  term on the right, of (44), are directly responsible for the decreasing in the effective inertial mass density. The former should be present in any causal theory of dissipation, whereas the latter is just an expression of the "inertia" of heat already pointed out by Tolman [34].

Therefore any hyperbolic, relativistic dissipative theory yielding a Cattaneo-type equation in the nonrelativistic limit, is expected to give a result similar to the one obtained here. The possible consequences of this effect on the outcome of gravitational collapse have been discussed in some detail in [40,41]. It is also worth noticing that such an effect appears already at the earliest stages of the non-equilibrium (though only along the  $V_{(3)(3)}$  direction).

Finally we would like to conclude with the following remark: In the stationary case one may have a steady rotation around the symmetry axis, leading to nonvanishing (time independent) vorticity  $\Omega_{\mu\nu} \neq 0$ , which of course may be compatible with thermal equilibrium. In this case the spacetime outside the source is described by a metric of the Lewis-Papapetrou family (e.g., Kerr) which as we know admits vorticity in the congruence of the world line of observers (the line element is nondiagonal). The vorticity of the source produces the vorticity in the exterior spacetime. However, in the static situation (the one considered here) you have no vorticity at the outside, which is described by a metric of the Weyl family (e.g., Curzon, Erez-Rosen, etc). In this latter case (nonstationary) we must have  $\Omega_{\mu\nu} = G = 0$  since the metric is diagonal. Since you have no vorticity outside (no frame dragging), you should not expect to have vorticity in the source (see [1] for a discussion on this case).

This last result may be obtained in a more rigorous way, by evaluating (A5) and (A6) in the static case and thermal equilibrium (assuming  $\Omega \neq 0$ ). Then after some lengthy but simple calculations, and using the regularity condition (25), one obtains  $\Omega = 0$ . Thus there is no vorticity associated to the static case. This brings out the difference between the steady vorticity of the stationary case and the vorticity considered here.

Also, the result above, shows that vorticity and heat flux are inherently coupled. This fact was already emphasized in [2].

## ACKNOWLEDGMENTS

L. H. thanks Departament de Física at the Universitat de les Illes Balears, for financial support and hospitality. A. D. P. acknowledges hospitality of the Departament de Física at the Universitat de les Illes Balears. L. H and J. O. acknowledge financial support from the Spanish Ministry of Science and Innovation (Grant No. FIS2009-07238) and Fondo Europeo de Desarrollo Regional (FEDER) (Grant No. FIS2015-65140-P) [Ministerio de Economía

y Competitividad/Fondo Europeo de Desarrollo Regional (MINECO/FEDER)].

## APPENDIX: SOME BASIC EQUATIONS

In what follows we shall deploy only those equations of the formalism which are required for our discussion. The whole set of the equations can be found in [2].

The conservation law  $T_{\beta;\alpha}^\alpha = 0$  leads to the following equations (Eqs. (A6, A7) in [2]):

$$\mu_{;\alpha} V^\alpha + (\mu + P)\Theta + (2\sigma_{(1)(1)} + \sigma_{(2)(2)})\Pi_{(1)(1)} + (2\sigma_{(2)(2)} + \sigma_{(1)(1)})\Pi_{(2)(2)} + q_{;\alpha}^\alpha + q^\alpha a_\alpha = 0, \quad (\text{A1})$$

$$(\mu + P)a_\alpha + h_\alpha^\beta (P_{;\beta} + \Pi_{\beta;\mu}^\mu + q_{\beta;\mu} V^\mu) + \left(\frac{4}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \Omega_{\alpha\beta}\right) q^\beta = 0. \quad (\text{A2})$$

The first of these equations is the ‘‘continuity’’ equation, whereas the second one is the ‘‘generalized Euler’’ equation. This last equation has two components, which, within the time scale under consideration may be written as:

$$(\mu + P + \Pi_{(1)(1)})\dot{V} = -(\mu + P + \Pi_{(1)(1)})\left[4\pi A(P - 2\Pi_{(1)(1)}) - \frac{AB_\theta a_{(2)}}{B^2 r}\right] + \text{‘‘force terms’’}, \quad (\text{A3})$$

and

$$(\mu + P + \Pi_{(2)(2)})\dot{V}_{(3)(3)} = -(\mu + P + \Pi_{(2)(2)})\left[4\pi A(P + 2\Pi_{(1)(1)} + 2\Pi_{(2)(2)}) - \frac{AC' a_{(1)}}{BC}\right] - \frac{AC_\theta}{BCr}\left[\frac{\dot{q}_{(2)}}{A}\right] + \text{‘‘force terms’’}, \quad (\text{A4})$$

where by ‘‘force terms’’ we denote different terms containing pressure gradients and anisotropic stresses.

Next, from the Ricci identities we have (Eqs. (B6, B7) in [2])

$$\begin{aligned} & \frac{2}{3B}\Theta_{,r} - \Omega_{;\mu} e_{(2)}^\mu + \Omega(e_{\beta;\mu}^{(2)} e_{(1)}^\mu e_{(1)}^\beta - e_{(2);;\mu}^\mu) + \sigma_{(1)(1)} a_{(1)} - \Omega a_{(2)} - \sigma_{(1)(1);;\mu} e_{(1)}^\mu \\ & - (2\sigma_{(1)(1)} + \sigma_{(2)(2)})\left(e_{(1);;\mu}^\mu - \frac{a_{(1)}}{3}\right) - (2\sigma_{(2)(2)} + \sigma_{(1)(1)})\left(e_{\beta;\mu}^{(2)} e_{(1)}^\mu e_{(1)}^\beta - \frac{a_{(1)}}{3}\right) = 8\pi q_{(1)}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} & \frac{1}{3\sqrt{A^2 B^2 r^2 + G^2}}\left(\frac{2G}{A}\Theta_{,t} + 2A\Theta_{,\theta}\right) + a_{(2)}\sigma_{(2)(2)} + \Omega_{;\mu} e_{(1)}^\mu + \Omega(e_{(1);;\mu}^\mu + e_{(2)}^\mu e_{(1)}^\beta e_{\beta;\mu}^{(2)}) + \Omega a_{(1)} - \sigma_{(2)(2);;\mu} e_{(2)}^\mu \\ & + (2\sigma_{(1)(1)} + \sigma_{(2)(2)})\left(e_{\beta;\mu}^{(2)} e_{(1)}^\mu e_{(1)}^\beta + \frac{a_{(2)}}{3}\right) - (2\sigma_{(2)(2)} + \sigma_{(1)(1)})\left(e_{(2);;\mu}^\mu - \frac{a_{(2)}}{3}\right) = 8\pi q_{(2)}. \end{aligned} \quad (\text{A6})$$

Finally, from the Bianchi identities, the following two equations describing the evolution of the magnetic part of the Weyl tensor, are obtained (Eqs. (B17, B18) in [2]).

$$\begin{aligned} & -2a_{(2)}\mathcal{E}_{(1)(1)} + 2a_{(1)}\mathcal{E}_{(2)(1)} - E_{2;\delta}^\delta e_{(2)}^2 - \frac{AY_{I,\theta}}{3\sqrt{A^2 B^2 r^2 + G^2}} + \frac{Y_{III,r}}{B} \\ & - \left[\frac{1}{3}(2Y_I + Y_{II})e_{\beta;\delta}^{(1)} + \frac{1}{3}(2Y_{II} + Y_I)e_{(1)}^\nu e_{\nu;\delta}^{(2)} e_{\beta}^{(2)} + Y_{III}(e_{\nu;\delta}^{(2)} e_{(1)}^\nu e_{\beta}^{(1)} + e_{\beta;\delta}^{(2)})\right] e^{\gamma\delta\beta} e_\gamma^{(3)} \\ & + H_{(3)(1),\delta} V^\delta + H_{(3)(1)}(\Theta + \sigma_{(2)(2)} - \sigma_{(1)(1)}) + \Omega H_{(2)(3)} = -\frac{4\pi}{3}\mu_{,\theta} e_{(2)}^2 + 12\pi\Omega q_{(1)} + \frac{4\pi q_{(2)}}{3}(3\sigma_{(1)(1)} + \Theta), \end{aligned} \quad (\text{A7})$$

$$\begin{aligned}
& 2a_{(1)}\mathcal{E}_{(2)(2)} - 2a_{(2)}\mathcal{E}_{(2)(1)} + E_{\beta;\delta}^\delta e_{(1)}^\beta + \frac{Y_{II,r}}{3B} - \frac{AY_{III,\theta}}{\sqrt{A^2 B^2 r^2 + G^2}} \\
& - \left[ -\frac{1}{3}(2Y_I + Y_{II})e_{\nu;\delta}^{(2)} e_{(1)}^\nu e_\beta^{(1)} + \frac{1}{3}(2Y_{II} + Y_I)e_{\beta;\delta}^{(2)} + Y_{III}(e_{\beta;\delta}^{(1)} - e_{(1)}^\nu e_\beta^{(2)} e_{\nu;\delta}^{(2)}) \right] e^{\gamma\delta\beta} e_\gamma^{(3)} \\
& + H_{(2)(3),\delta} V^\delta + H_{(2)(3)}(\Theta + \sigma_{(1)(1)} - \sigma_{(2)(2)}) - \Omega H_{(1)(3)} = \frac{4\pi}{3}\mu_{,\beta} e_{(1)}^\beta - \frac{4\pi q_{(1)}}{3}(3\sigma_{(2)(2)} + \Theta) + 12\pi\Omega q_{(2)}. \quad (\text{A8})
\end{aligned}$$

- 
- [1] L. Herrera, A. Di Prisco, J. Ibáñez, and J. Ospino, *Phys. Rev. D* **87**, 024014 (2013).
- [2] L. Herrera, A. Di Prisco, J. Ibáñez, and J. Ospino, *Phys. Rev. D* **89**, 084034 (2014).
- [3] G. F. R. Ellis, in *Proceedings of the International School of Physics “Enrico Fermi”, Course 47: General Relativity and Cosmology*, edited by R. K. Sachs (Academic Press, New York, 1971).
- [4] G. F. R. Ellis and H. van Elst, [arXiv:gr-qc/9812046v4](https://arxiv.org/abs/gr-qc/9812046v4).
- [5] G. F. R. Ellis *Gen. Relativ. Gravit.* **41**, 581 (2009).
- [6] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, England, 2012).
- [7] R. Kippenhahn and A. Weigert, *Stellar Structure and Evolution*, (Springer Verlag, Berlin, 1990).
- [8] C. Hansen and S. Kawaler, *Stellar Interiors: Physical Principles, Structure and Evolution*, (Springer Verlag, Berlin, 1994).
- [9] L. Herrera, A. Di Prisco, J. Ospino, and J. Carot, *Int. J. Mod. Phys. D* **25**, 1650036 (2016).
- [10] A. Burrows and J. M. Lattimer, *Astrophys. J.* **307**, 178 (1986).
- [11] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Proc. R. Soc. A* **269**, 21 (1962).
- [12] H. Stephani, D. Kramer, M. MacCallum, C. Honselaers, and E. Herlt, *Exact Solutions to Einsteins Field Equations*, 2nd ed. (Cambridge University Press, Cambridge, England, 2003).
- [13] J. Carot, *Classical Quantum Gravity* **17**, 2675 (2000).
- [14] G. T. Carlson, Jr. and J. L. Safko, *Ann. Phys. (N.Y.)* **128**, 131 (1980).
- [15] L. Herrera, J. Ospino, A. Di Prisco, E. Fuenmayor, and O. Troconis, *Phys. Rev. D* **79**, 064025 (2009).
- [16] L. Bel, *C. R. Hebd. Seances Acad. Sci.* **247**, 1094 (1958).
- [17] L. Bel, *Cahiers de Physique* **16**, 59 (1962); *Gen. Relativ. Gravit.* **32**, 2047 (2000).
- [18] A. García-Parrado Gómez-Lobo, *Classical Quantum Gravity* **25**, 015006 (2008).
- [19] L. Herrera, W. Barreto, J. Carot, and A. Di Prisco, *Classical Quantum Gravity* **24**, 2645 (2007).
- [20] R. Sachs, *Proc. R. Soc. A* **270**, 103 (1962).
- [21] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [22] L. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1959).
- [23] D. Joseph and L. Preziosi, *Rev. Mod. Phys.* **61**, 41 (1989).
- [24] R. Maartens, [arXiv:astro-ph/9609119](https://arxiv.org/abs/astro-ph/9609119).
- [25] L. Herrera and D. Pavón, *Physica A (Amsterdam)*, **307**, 121 (2002).
- [26] W. Hiscock and L. Lindblom, *Ann. Phys. (N.Y.)* **151**, 466 (1983).
- [27] I. Müller, *Z. Phys.* **198**, 329 (1967).
- [28] W. Israel, *Ann. Phys. (N.Y.)* **100**, 310 (1976).
- [29] W. Israel and J. Stewart, *Phys. Lett. A* **58**, 213 (1976).
- [30] W. Israel and J. Stewart, *Ann. Phys. (N.Y.)* **118**, 341 (1979).
- [31] D. Pavón, D. Jou, and J. Casas-Vázquez, *Ann. Inst. Henri Poincaré, Sect. A* **36**, 79 (1982).
- [32] B. Carter, in *Journées Relativistes*, edited by M. Cahen, R. Debever, and J. Geheniau (Université Libre de Bruxelles, Bruxelles, 1976).
- [33] C. Cattaneo, *Atti Semin. Mat. Fis. Univ. Modena* **3**, 3 (1948).
- [34] R. Tolman, *Phys. Rev.* **35**, 904 (1930).
- [35] L. Herrera, A. Di Prisco, J. Hernández-Pastora, J. Martín, and J. Martínez, *Classical Quantum Gravity* **14**, 2239 (1997).
- [36] L. Herrera, A. Di Prisco, and J. Martínez, *Phys. Lett. A* **243**, 7 (1998).
- [37] L. Herrera and J. Martínez, *J. Math. Phys. (N.Y.)*, **39**, 3260 (1998).
- [38] L. Herrera, *Phys. Lett. A* **300**, 157 (2002).
- [39] L. Herrera and N. O. Santos, *Phys. Rev. D* **70**, 084004 (2004).
- [40] L. Herrera, A. Di Prisco, and W. Barreto, *Phys. Rev. D* **73**, 024008 (2006).
- [41] L. Herrera, *Int. J. Mod. Phys. D* **15**, 2197 (2006).
- [42] A. Di Prisco, L. Herrera, G. Le Denmat, M. A. H. MacCallum, and N. O. Santos, *Phys. Rev. D* **76**, 064017 (2007).
- [43] L. Herrera, A. Di Prisco, E. Fuenmayor, and O. Troconis, *Int. J. Mod. Phys. D* **18**, 129 (2009).
- [44] M. Sharif and Z. Rehmat, *Gen. Relativ. Gravit.* **42**, 1795 (2010).
- [45] M. Sharif and A. Siddiq, *Gen. Relativ. Gravit.* **43**, 73 (2011).
- [46] M. Sharif and H. Tahir, *Eur. Phys. J. Plus*, **128**, 146 (2013).
- [47] M. Sharif and I. Nawazish, *Astrophys. Space Sci.* **351**, 313 (2014).
- [48] M. Sharif and M. Z. Ul Haq Bhatti, *Mod. Phys. Lett. A* **29**, 1450165 (2014).
- [49] L. Herrera, A. Di Prisco, J. Ospino, and J. Carot, *Phys. Rev. D* **91**, 024010 (2015).
- [50] L. Herrera, *Gen. Relativ. Gravit.* **46**, 1654 (2014).