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Density matrix of radiation of a black hole with a fluctuating horizon

Mikhail Z. Iofa*

Skobeltsyn Institute of Nuclear Physics Moscow State University, Moscow 119991, Russia

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The density matrix of Hawking radiation is calculated in the model of a black hole with a fluctuating horizon. Quantum fluctuations smear the classical horizon of a black hole and modify the density matrix of radiation producing the off-diagonal elements. The off-diagonal elements may store information on correlations between the radiation and the black hole. The smeared density matrix was constructed by convolution of the density matrix calculated with the instantaneous horizon with the Gaussian distribution over the instantaneous horizons. The distribution has the extremum at the classical radius of the black hole and the width of order of the Planck length. Calculations were performed in the model of a black hole formed by the thin collapsing shell which follows a trajectory that is a solution of the matching equations connecting the interior and exterior geometries.

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I. INTRODUCTION

From the time of Hawking's discovery that black holes radiate with the blackbody radiation, the problem of information stored in a black hole [1] has attracted much attention. Different ideas have been discussed, in particular, those of remnants [2–4], "fuzziness" of the black hole [5,6] and references therein, quantum hair [7–9] and Refs. therein, and smearing of the horizon by quantum fluctuations [10–13]. The underlying idea of the last approach is that small fluctuations of the background geometry lead to corrections to the form of the density matrix of the radiation. These corrections are supposed to account for correlations between the black hole and the radiation and contain the imprint of information thrown into the black hole with the collapsing matter.

The idea that the horizon of the black hole is not located at the rigid position naturally follows from the observation that a black hole as a quantum object is described by the wave functional over geometries [14–16]. In particular, the sum over the horizon areas yields the black hole entropy.

In papers [12,13] the density matrix of black hole radiation was calculated in a model with a fluctuating horizon. Horizon fluctuations modify the Hawking density matrix producing off-diagonal elements. Horizon fluctuations were taken into account by convolution the density matrix calculated with the instantaneous horizon radius R with the black hole wave function which was taken in the Gaussian form $\psi(R) = N^{-1/2} e^{-(R-2MG)^2/2\sigma^2}$. Effectively the wave function introduces the smearing of the classical horizon radius $\bar{R} = 2MG$. The width of the distribution, σ, was taken of order the Plank lengths l_p [10,12,13]. In Ref. [10], it was stated that the "horizon fluctuations do not invalidate the semiclassical derivation of the Hawking effect until the black hole mass approaches the Planck mass."

In this note, we reconsider calculation the density matrix of radiation emitted from the black hole formed by the collapsing shell. The shell is supposed to follow the infalling trajectory which is the exact solution to the matching equations connecting the interior (Minkowski) and exterior (Schwarzschild) geometries of the spacetime [17,18]. In this setting one can trace propagation of a ray (we consider only s-modes) through the shell from the past to the future infinity. For the rays propagating in the vicinity of the horizon, we obtain an exact formula connecting $v_{\rm in}$ at the past infinity and $u_{\rm out}$ at the future infinity.

We obtain the expression for the "smeared" density matrix of Hawking radiation of the black hole with the horizon smeared by fluctuations. In the limit $\sigma/MG \to 0$, the smeared density matrix turns to the Hawking density matrix. The smeared density matrix is not diagonal and can be expressed as a sum of the "classical part" and off-diagonal correction which is roughly of order $O(\sigma/MG)$ of the classical part. As a function of frequencies $\omega_{1,2}$ of emitted quanta, the distribution is concentrated around $\omega_1/\omega_2 = 1$ with the width of order $(\sigma/MG)\ln^{1/2}(MG/\sigma)$.

The paper is organized as follows. In Sec. II we review the geometry of the thin collapsing shell which follows a trajectory consisting of two phases. The trajectory is a solution of the matching equations connecting the internal and external geometries of the shell. We trace propagation of a light ray from the past to future infinity. In Sec. III we introduce the wave function of the shell which saturates the uncertainty relations. In Sec. IV, we calculate the density matrix of black hole radiation smeared by horizon fluctuations. Following the approach of paper [19] calculation is performed by two methods: by the "$i\varepsilon$" prescription and by using the normal-ordered two-point function. In Sec. V, using the exact expressions for the smeared radiation

*iofa@theory.sinp. msu.ru

density matrix, we study the diagonal "classical" part of the density matrix and the off-diagonal elements.

II. GEOMETRY OF THE THIN COLLAPSING SHELL

In this section, we introduce notations and review the geometry of space with collapsing thin spherical shell [17,18]. Outside of the shell the exterior geometry is Schwarzschild spacetime, the interior geometry is Minkowsky spacetime. In the Eddington-Finkelstein coordinates, the metric of the exterior spacetime is

$$ds^2_{(\mathrm{ext})} = -(1 - R/r)dv^2 + 2dvdr + r^2 d\Omega^2, \quad r > R, \tag{1}$$

where

$$v = t + x(r),$$
$$u = t - x(r),$$
$$x(r) = r + R\ln(r/R - 1),$$

and

$$v - u = 2x(r).$$

The metric of the interior spacetime is

$$ds^2_{(\mathrm{int})} = -dV^2 + 2dVdr + r^2 d\Omega^2, \tag{2}$$

where

$$V = T + r, \qquad U = T - r.$$

The light rays propagate along the cones $v, u = \mathrm{const}$ in the exterior and along $V, U = \mathrm{const}$ in the interior regions.

The trajectory of the shell is $r = R_{\mathrm{shell}}(\tau)$, where τ is proper time on the shell. The matching conditions of geometries on the shell, at $r = R_{\mathrm{shell}}$, are

$$dV - dU = 2dR_{\mathrm{shell}}, \quad dv - du = \frac{2dR_{\mathrm{shell}}}{1 - R/R_{\mathrm{shell}}},$$
$$dUdV = (1 - R/R_{\mathrm{shell}})dudv, \tag{3}$$

where the differentials are taken along the trajectory. From the matching conditions, follow the equations

$$2R'_s(1 - U') = U'^2 - (1 - R/R_{\mathrm{shell}}), \tag{4}$$

$$2\dot{R}_s(1 - \dot{V}) = -\dot{V}^2 + (1 - R/R_{\mathrm{shell}}). \tag{5}$$

Here the prime and dot denote derivatives over u and v along the trajectory. The trajectory of the shell consists of two phases [18]:

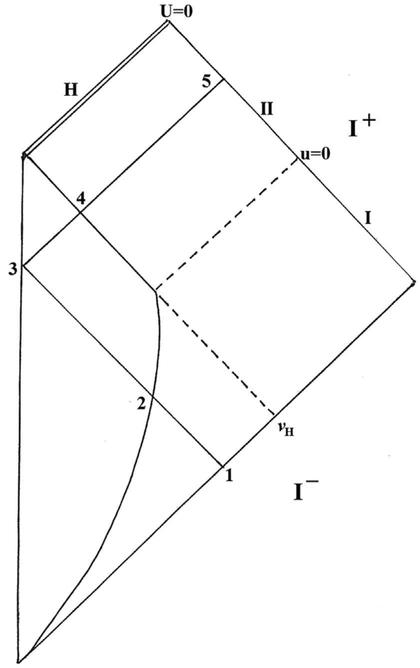

FIG. 1. Penrose diagram for the collapsing shell. For $u < 0$, the shell is in phase I, for $u > 0$, it is in phase II. v_H is the point of horizon formation.

$$I. \quad u < 0: \ R_{\mathrm{shell}}(u) = R_0 = \mathrm{const}$$
$$II. \quad u > 0: \ v = \mathrm{const}, \qquad V = \mathrm{const}.$$

From Eqs. (4), (5), we obtain the following expressions for the trajectory:

In phase I,

$$U(u) = L_0 u - 2R_0 + 2R,$$
$$V(v) = L_0(v - 2x(R_0)) + 2R, \tag{6}$$

where $L_0 = (1 - R/R_0)^{1/2}$.

In phase II,

$$V = 2R,$$
$$U = 2R - 2R_{\mathrm{shell}},$$
$$v = 2x(R_0)$$
$$u = 2x(R_0) - 2x(R_{\mathrm{shell}}). \tag{7}$$

The horizon is formed at $U_H = 0$, $u \to \infty$ and $V_H = 2R$, $v_H = 2x(R_0)$.

We consider the modes propagating backwards in time (see Fig. 1). At I^-, the ray is in phase I, and after crossing the shell, it reaches I^+ in phase II. We let the in-falling ray be at I^- at $v_1 < v_H$. Between points 1 and 2, the ray propagates outside the shell in phase I with $v = v_1$. At point 2, the ray crosses the shell, and we have $v_2 = v_1$ and $V_2(v_1) = L_0(v_1 - 2x(R_0)) + 2R$. The ray propagates in the interior of the shell, and at point 3, $V_3 = V_2$. The reflection condition at $r = 0$, at point 3, is $V_3 = U_3 + 2R$. At the crossing point with shell 4, we have $U_4 = U_3$, where

$$U_4 = -2R_{\rm shell}(4) + 2R,$$
$$u_4 = -2x(R_{\rm shell}(4)) + 2x(R_0).$$

Here $R_{\rm shell}(4)$ is the radial position of the shell trajectory at point 4. The equation for u_4 can be written as

$$\frac{u_4}{2R} = \frac{(R_0 - R) - (R_{\rm shell}(4) - R)}{R} + \ln\frac{R_0 - R}{R_{\rm shell}(4) - R}.$$

In the region $R_{\rm shell}(4) \sim R$, and $R_{\rm shell}(4) \ll R_0$, where $U_4 \ll R$, neglecting in the first term $R_{\rm shell} - R$ as compared with $R_0 - R$, we obtain the approximate equation for u_4:

$$\frac{u_4}{2R} = \frac{R_0 - R}{R} - \ln\frac{-U_4/2}{R_0 - R}. \tag{8}$$

Thus, we have

$$v_1 - v_H = L_0^{-1}(V_2 - 2R) = L_0^{-1}(V_3 - 2R) = L_0^{-1}U_4(u_4)$$
$$= -L_0^{-1}2(R_0 - R)e^{-(u_4 - 2R_0)/2R - 1}. \tag{9}$$

Removing the indices, we obtain our final result as

$$v = v_H - 2(eL_0)^{-1}(R_0 - R)e^{-(u-2R_0)/2R}. \tag{10}$$

The above formulas are purely classical; modifications due to backreaction of Hawking radiation are neglected.

III. QUANTUM BLACK HOLE

The quantum nature of horizons of the black holes was discussed in the work of Carlip and Teitelboim [14], where it was shown that the area of horizon A and the opening angle, Θ, or equivalently the deficit angle $2\pi - \Theta$, form the canonical pair. In Ref. [13], it was shown that the canonical pair is formed by the opening angle and the Wald entropy S_W [20]:

$$\left\{\Theta, \frac{S_W}{2\pi}\right\} = 1. \tag{11}$$

When the black hole is quantized, the Poisson bracket is promoted to the commutation relation

$$[\hat\Theta, \hat S_W] = i\hbar. \tag{12}$$

The wave function of the black hole satisfies the relation

$$-i\frac{\partial\Psi}{\partial S_W} = 2\pi\hbar\Theta\Psi. \tag{13}$$

The minimal uncertainty $\Delta S_W \Delta\Theta = \hbar/2$ wave function is

$$\Psi(\Theta) \sim e^{C(\Theta - 2\pi)^2} e^{\frac{i}{\hbar}\langle S_W\rangle\Theta}, \tag{14}$$

where $C \sim \langle S_W\rangle$. For the spherically symmetric configurations which we consider, the wave function written through the instantaneous horizon radius R is

$$|\Psi(R)| = N^{-1} e^{-\frac{(R-\bar R)^2}{4\sigma^2}}. \tag{15}$$

The scale of the horizon fluctuations is $\sigma \sim l_p$ [10], where $l_p^2 = \hbar G$ is the Planck length and $\bar R = 2MG$ is the classical horizon radius of the black hole of the mass M. The normalization factor N is

$$N^{-2} = \int_0^\infty 4\pi dR R^2 e^{\frac{(R-\bar R)^2}{2\sigma^2}} \simeq \sigma \bar R^2. \tag{16}$$

IV. HAWKING RADIATION FROM THE BLACK HOLE FORMED BY THE SHELL

Let us turn to the calculation of Hawking radiation of the massless real scalar field in the background of the black hole formed by the shell. To perform quantization of the field, we restrict ourselves to the s-wave modes. Expanding the scalar field in the orthonormal set of solutions u_i^- of the Klein-Gordon equation, which at the past null infinity I^- has only positive frequency modes, we have

$$\varphi = \sum_i (a_i u_i^{(-)} + a_i^+ u_i^{(-)*}). \tag{17}$$

The scalar product of the fields is

$$(\varphi_1, \varphi_2) = i\int_\Sigma d\Sigma^\mu \varphi_2^* \overleftrightarrow{\partial}_\mu \varphi_1. \tag{18}$$

Alternatively, the field φ can be expanded at the hypersurface $\Sigma^+ = I^+ \oplus H^+$, where I^+ is the future null infinity and H^+ is the event horizon:

$$\varphi = \sum_i (b_i u_i^{(+)} + b_i^+ u_i^{(+)*} + c_i q_i + c_i^+ q_i^*). \tag{19}$$

Here $\{u_i^{(+)}\}$ is the orthonormal set of modes which contain at the I^+ only positive frequencies, and $\{q_i\}$ is the orthonormal set of solutions of the wave equation which contains no outgoing components [1]. The operators a_i, a_i^+ and b_i, b_i^+ are quantized with respect to the vacua $|{\rm in}\rangle$ and $|{\rm out}\rangle$, respectively.

The modes $u_i^{(+)}$ can be expanded in terms of the modes $u_i^{(-)}$,

$$u_i^{(+)} = \sum_j (\alpha_{ij} u_j^{(-)} + \beta_{ij} u_j^{(-)*}), \tag{20}$$

where α_{ij} and β_{ij} are given by the scalar products

$$\alpha_{ij} = (u_i^{(+)}, u_j^{(-)}), \qquad \beta_{ij} = -(u_i^{(+)}, u_j^{(-)*}).$$

For the spherically symmetric collapse, the basis for the in- and outgoing modes is

$$u_{\omega lm}^{(-)}|_{I^-} \sim \frac{1}{\sqrt{4\pi\omega}} \frac{e^{-i\omega v}}{r} Y_{lm}(\theta, \varphi),$$
$$u_{\omega lm}^{(-)}|_{I^+} \sim \frac{1}{\sqrt{4\pi\omega}} \frac{e^{-i\omega u}}{r} Y_{lm}(\theta, \varphi).$$

Omitting the angular parts, the modes $u_\omega^{(-)}$ and $u_\omega^{(+)}$ are

$$u_\omega^{(-)}(v)|_{I^-} \sim \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v},$$
$$u_\omega^{(+)}(u)|_{I^+} \sim \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u}. \tag{21}$$

From (10), we find

$$u(v) = 2R_0 + 2R\left[-\ln(eL_0) + \ln\frac{R_0 - R}{R} - \ln\frac{v_H - v}{2R}\right]$$
$$= F(R) - 2R\ln\frac{v_H - v}{2R},$$

where $F(R) = 2R_0 + 2R[C + \ln((R_0 - R)/R)]$, $C = -\ln(eL_0)$. To simplify formulas, we consider the case $R_0 \gg R$, so $\ln((R_0 - R)/R) \simeq \ln R_0/R$, and

$$F(R) \simeq 2R_0 + 2R[C + \ln(R_0/R)].$$

Note that both v_H and $u(v)$ have explicit dependence on R.

The Bogolubov coefficient,[1]

$$\beta_{\omega_1\omega_2} = i\int_{-\infty}^{v_H} dv u_{\omega_1}^{(+)}(v) \overleftrightarrow{\partial}_v u_{\omega_2}^{(-)}(v)$$
$$\sim i\int \frac{dv}{\sqrt{\omega_1\omega_2}} e^{-i\omega_1 u(v)} \overleftrightarrow{\partial}_v e^{-i\omega_2 v},$$

smeared by the horizon fluctuations is obtained by convoluting it with the function $|\Psi^2|$:

[1]Hereafter, we abandon the numerical and greybody factors.

$$\bar{\beta}_{\omega_1\omega_2} = \int_0^\infty dRR^2 e^{-(R-\bar{R})^2/2\sigma^2} N^2 \beta_{\omega_1\omega_2}$$
$$\sim (\omega_2/\omega_1)^{1/2} \int_{-\infty}^{v_H} dv e^{-i\omega_1(F(R) - 2R\ln((v_H - v)/2R) - i\omega_2 v}$$
$$\times dRN^2R^2 e^{-(R-\bar{R})^2/2\sigma^2}. \tag{22}$$

Direct evaluation of the smeared Bogolubov coefficient (22) yields (cf. [19])

$$\bar{\beta}_{\omega_1\omega_2} \sim \int dRR^2N^2 e^{-(R-\bar{R})^2/2\sigma^2} R$$
$$\times \sqrt{\frac{\omega_1}{\omega_2}} \frac{e^{-i\omega_1 F(R) + i\omega_2 v_H} \Gamma(2Ri\omega_1)}{(-2Ri\omega_2 + \varepsilon)^{2Ri\omega_1}}. \tag{23}$$

A. Method 1

Following Ref. [19], we consider two ways of calculating the density matrix:

$$\rho_{\omega_1\omega_2} = i\int_{I^-} dv u_{\omega_1}^{(+)}(v) \overleftrightarrow{\partial}_v u_{\omega_2}^{(+)*}(v)$$
$$= i\int_{I^-} dv \int d\omega' d\omega'' \beta_{\omega_1\omega'} \beta^*_{\omega_2\omega''} u_{\omega'}^{(-)}(v) \overleftrightarrow{\partial}_v u_{\omega''}^{(-)*}(v)$$
$$= \int_0^\infty d\omega' \beta_{\omega_1\omega'} \beta^*_{\omega_2\omega'}, \tag{24}$$

where in the last equality we used that the modes $\{u_\omega^{(-)}\}$ form the orthonormal set of functions on I^-. The density matrix smeared by horizon fluctuations is

$$\bar{\rho}_{\omega_1\omega_2} = \int_0^\infty d\omega' \beta_{\omega_1\omega'} \beta^*_{\omega_2\omega'} \int_0^\infty dR_1 R_1^2 e^{-(R_1-\bar{R})^2/2\sigma^2} N^2$$
$$\times \int_0^\infty dR_2 R_2^2 e^{-(R_2-\bar{R})^2/2\sigma^2} N^2. \tag{25}$$

Substituting (23), we have

$$\bar{\rho}_{\omega_1\omega_2} \sim \int_0^\infty d\omega' \frac{\sqrt{\omega_1\omega_2}}{\omega'} R_1 R_2 (-2iR_1\omega' + \varepsilon)^{-2iR_1\omega_1}$$
$$\times (2iR_2\omega' + \varepsilon)^{2iR_2\omega_2} \Gamma(-2iR_1\omega_1)\Gamma(2iR_2\omega_2)$$
$$\times e^{-i\omega_1 F(R_1) + i\omega_2 F(R_2)} dR_1 dR_2 R_1^2 R_2^2 N^4$$
$$\times e^{-(R_1-\bar{R})^2/2\sigma^2} e^{-(R_2-\bar{R})^2/2\sigma^2}. \tag{26}$$

The terms with v_0 have canceled. Integrating over ω', we obtain

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2} &\sim \sqrt{\omega_1\omega_2}\int dR_1 dR_2 R_1^2 R_2^2 \delta(R_1\omega_1 - R_2\omega_2)\\ &\quad\times\left(-\frac{R_1}{R_2}\right)^{-2iR_1\omega_1}\frac{1}{R_1\omega_1\sinh(2\pi R_1\omega_1)}\\ &\quad\times e^{-i\omega_1 F(R_1)+i\omega_2 F(R_2)}N^4 e^{-(R_1-\bar{R})^2/2\sigma^2}e^{-(R_2-\bar{R})^2/2\sigma^2}\\ &= \sqrt{\omega_1\omega_2}\int dR_1 dR_2 R_1^3 R_2^3 \frac{1}{R_1\omega_1\omega_2}\delta\left(R_2 - R_1\frac{\omega_1}{\omega_2}\right)\\ &\quad\times (e^{4\pi R_1\omega_1}-1)^{-1}e^{-2iR_0(\omega_1-\omega_2)}N^4 e^{-(R_1-\bar{R})^2/2\sigma^2}\\ &\quad\times e^{-(R_2-\bar{R})^2/2\sigma^2}, \qquad (27)\end{aligned}$$

where, taking into account the δ function, we substituted

$$\begin{aligned}(-2iR_1)^{-2iR_1\omega_1}(2iR_2)^{2iR_2\omega_2} &\Rightarrow (-R_1/R_2)^{-2iR_1\omega_1}\\ &= e^{2\pi R_1\omega_1}(R_1/R_2)^{-2iR_1\omega_1}\end{aligned}$$

$$\Gamma(-2iR_1\omega_1)\Gamma(2iR_2\omega_2) \Rightarrow \frac{1}{2R_1\omega_1\sinh(2\pi R_1\omega_1)}$$

and

$$\begin{aligned}e^{-i\omega_1 F(R_1)+i\omega_2 F(R_2)} &\Rightarrow e^{2iR_1\omega_1\ln R_1 - 2iR_2\omega_2\ln R_2}\\ &= (R_1/R_2)^{2iR_1\omega_1}.\end{aligned}$$

Integration over R_2 yields

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2} &\sim \frac{1}{\sqrt{\omega_1\omega_2}}\int dR_1 R_1^5\left(\frac{\omega_1}{\omega_2}\right)^3 (e^{4\pi R_1\omega_1}-1)^{-1}e^{-2iR_0(\omega_1-\omega_2)}\\ &\quad\times\frac{1}{\bar{R}^4\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left(R_1\sqrt{1+\omega_1^2/\omega_2^2} - \bar{R}\frac{1+\omega_1/\omega_2}{\sqrt{1+\omega_1^2/\omega_2^2}}\right)^2\right\}\exp\left\{-\frac{\bar{R}^2(\omega_1-\omega_2)^2}{2\sigma^2(\omega_1^2+\omega_2^2)}\right\}. \qquad (28)\end{aligned}$$

Because $\bar{R}/\sigma \gg 1$, both exponents have sharp extrema. Integrating over R_1, we arrive at the density matrix of the form

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2} &\sim \frac{1}{\sqrt{\omega_1\omega_2}}\left(\frac{\omega_1}{\omega_2}\right)^3\left(\frac{1+\omega_1/\omega_2}{1+\omega_1^2/\omega_2^2}\right)^5\\ &\quad\times\frac{\bar{R}}{\sigma(1+\omega_1^2/\omega_2^2)^{1/2}}(e^{4\pi\bar{R}(\omega_1+\omega_2)\omega_1\omega_2/(\omega_1^2+\omega_2^2)}-1)^{-1}\\ &\quad\times e^{-2iR_0(\omega_1-\omega_2)}\exp\left\{-\frac{\bar{R}^2(\omega_1-\omega_2)^2}{2\sigma^2(\omega_1^2+\omega_2^2)}\right\}. \qquad (29)\end{aligned}$$

B. Method 2

Alternatively, the density matrix can be presented in the following form,

$$\begin{aligned}\bar{\rho}_{\omega\omega'} &= \int d\omega_1\int_\Sigma d\Sigma_1^\mu u^{(+)*}_{\omega lm}(x_1)\overleftrightarrow{\partial}_\mu u^{(-)}_{\omega_1 l_1 m_1}(x_1)\\ &\quad\times\int_\Sigma d\Sigma_2^\nu u^{(+)}_{\omega' l' m'}(x_2)\overleftrightarrow{\partial}_\nu u^{(-)*}_{\omega_1 l_1 m_1}(x_2), \qquad (30)\end{aligned}$$

where for the initial value the hypersurface can be taken as either I^- or I^+. Expanding φ in the basis $\{u^{(-)}\}$,

$$\langle \mathrm{in}|\varphi(x_1)\varphi(x_2)|\mathrm{in}\rangle = \int d\omega_1 u^{(-)}_{\omega_1}(x_1)u^{(-)*}_{\omega_1}(x_2),$$

where $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ are vacuum states at I^- and I^+, and using the relation

$$\begin{aligned}\langle \mathrm{in}|{:}\varphi(x_1)\varphi(x_2){:}|\mathrm{in}\rangle &= \langle \mathrm{in}|\varphi(x_1)\varphi(x_2)|\mathrm{in}\rangle\\ &\quad - \langle \mathrm{out}|\varphi(x_1)\varphi(x_2)|\mathrm{out}\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}\rho_{\omega_1\omega_2} &= \int_\Sigma d\Sigma_1^\mu\int_\Sigma d\Sigma_2^\nu [u^{(+)*}_{\omega_1}(x_1)\overleftrightarrow{\partial}_\mu][u^{(+)}_{\omega_2}(x_2)\overleftrightarrow{\partial}_\nu]\\ &\quad\times\langle \mathrm{in}|{:}\varphi(x_1)\varphi(x_2){:}|\mathrm{in}\rangle. \qquad (31)\end{aligned}$$

To perform the calculation of (31), one can use the expansion of the two-point function $\langle \mathrm{in}|{:}\varphi(x_1)\varphi(x_2){:}|\mathrm{in}\rangle$ on I^+ to obtain [19]

$$\begin{aligned}\rho_{\omega_1\omega_2} &\sim (\omega_1\omega_2)^{-1/2}\int_{I^+} du_1 du_2 e^{-iu_1\omega_1+iu_2\omega_2}\\ &\quad\times\left(\frac{(dv/du)(u_1)(dv/du)(u_2)}{(v(u_1)-v(u_2)-i\varepsilon)^2} - \frac{1}{(u_1-u_2-i\varepsilon)^2}\right), \qquad (32)\end{aligned}$$

where, for $v(u)$, we take the function (10). For the density matrix modified by horizon fluctuations, we obtain

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2} &\sim (\omega_1\omega_2)^{-1/2}\int_{I^+} du_1\int_{I^+} du_2\frac{e^{-iu_1\omega_1+iu_2\omega_2}}{R_1R_2}\\ &\quad\times\frac{e^{-(u_1-2R_0)/2R_1}e^{-(u_2-2R_0)/2R_2}}{(e^{-(u_1-2R_0)/2R_1}-e^{-(u_2-2R_0)/2R_2}-i\varepsilon)^2}\\ &\quad\times e^{-(R_1-\bar{R})^2/2\sigma^2-(R_2-\bar{R})^2/2\sigma^2}N^4 R_1^2R_2^2 dR_1 dR_2. \qquad (33)\end{aligned}$$

Extracting in the denominator the factor $(e^{-(u_1-2R_0)/4R_1-(u_2-2R_0)/4R_2})^2$, shifting $u_i-2R_0\Rightarrow u_i$ and changing variables $u_i/4R_i\Rightarrow u_i$, we obtain

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2}\sim(\omega_1\omega_2)^{-1/2}\int_{-\infty}^{\infty}du_1\\ \times\int_{-\infty}^{\infty}du_2 e^{-4i\omega_1u_1R_1+4i\omega_2u_2R_2-2iR_0(\omega_1-\omega_2)}\\ \times\sinh^{-2}(u_1-u_2-i\varepsilon)\\ \times e^{-(R_1-\bar{R})^2/2\sigma^2-(R_2-\bar{R})^2/2\sigma^2}N^4R_1^2R_2^2dR_1dR_2.\end{aligned}\qquad(34)$$

Performing the contour integration over u_1 around the pole in the upper half plane using the formula

$$\int_{-\infty}^{\infty}dy\frac{e^{-i\omega y}}{\sinh^2(y-z-i\varepsilon)}=2\pi\frac{e^{-i\omega z}}{e^{\pi\omega}-1},$$

we have

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2}\sim(\omega_1\omega_2)^{-1/2}\int dR_1dR_2R_1^2R_2^2N^4\omega_1R_1\frac{1}{e^{4\pi\omega_1R_1}-1}\\ \times\int du_2e^{-4i\omega_1R_1u_2+4i\omega_2R_2u_2}\\ \times e^{-2iR_0(\omega_1-\omega_2)}e^{-(R_1-\bar{R})^2/2\sigma^2-(R_2-\bar{R})^2/2\sigma^2}.\end{aligned}\qquad(35)$$

Integration over u_2 yields

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2}\sim(\omega_1\omega_2)^{-1/2}\int dR_1dR_2R_1^2R_2^2N^4\omega_1R_1\frac{1}{e^{4\pi\omega_1R_1}-1}\delta(\omega_1R_1-\omega_2R_2)\\ \times e^{-2iR_0(\omega_1-\omega_2)}e^{-(R_1-\bar{R})^2/2\sigma^2-(R_2-\bar{R})^2)/2\sigma^2}.\end{aligned}\qquad(36)$$

Integrating over R_2 and removing the δ function, we obtain

$$\begin{aligned}\bar{\rho}_{\omega_1\omega_2}\sim\frac{1}{\sqrt{\omega_1\omega_2}}\int dR_1R_1^5\left(\frac{\omega_1}{\omega_2}\right)^3(e^{4\pi R_1\omega_1}-1)^{-1}e^{-2iR_0(\omega_1-\omega_2)}\\ \times\frac{1}{\bar{R}^2\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left(R_1\sqrt{1+\omega_1^2/\omega_2^2}-\bar{R}\frac{1+\omega_1/\omega_2}{\sqrt{1+\omega_1^2/\omega_2^2}}\right)^2\right\}\exp\left\{-\frac{\bar{R}^2(\omega_1-\omega_2)^2}{2\sigma^2(\omega_1^2+\omega_2^2)}\right\}.\end{aligned}\qquad(37)$$

The expression (37) is identical to that obtained by method 1 in (28).

V. DIAGONAL AND OFF-DIAGONAL PARTS OF THE DENSITY MATRIX

In the limit $\bar{R}/\sigma\to\infty$, the expression

$$\frac{\bar{R}}{\sigma\sqrt{1+\omega_1^2/\omega_2^2}}\exp\left\{-\frac{\bar{R}^2}{2\sigma^2}\frac{(1-\omega_1/\omega_2)^2}{1+\omega_1^2/\omega_2^2}\right\}\qquad(38)$$

becomes the delta function $\delta(1-\omega_1/\omega_2)$. The density matrix $\bar{\rho}_{\omega_1\omega_2}$ Eq. (37) turns into the formula for the Hawking spectrum:

$$\rho_{\omega_1\omega_2}\sim\sqrt{\frac{\omega_1}{\omega_2}}\delta(\omega_1-\omega_2)(e^{2\pi\bar{R}(\omega_1+\omega_2)}-1)^{-1}.\qquad(39)$$

The smeared density matrix contains the off-diagonal elements. Because the density matrix has the sharp maximum at $\omega_1/\omega_2=1$, it is natural to divide it into the "classical" contribution,

$$\begin{aligned}\rho^{cl}_{\omega_1\omega_2}\sim\Theta\left(2\frac{\sigma}{\bar{R}}\ln^{1/2}\frac{\bar{R}}{\sigma}-\left|1-\frac{\omega_1}{\omega_2}\right|\right)\frac{\bar{R}}{\sigma}\\ \times\exp\left\{-\frac{\bar{R}^2(1-\omega_1/\omega_2)^2}{4\sigma^2}\right\}\sqrt{\frac{\omega_1}{\omega_2}}(e^{2\pi\bar{R}(\omega_1+\omega_2)}-1)^{-1},\end{aligned}\qquad(40)$$

and the off-diagonal correction. As mentioned above, in the classical contribution the factor multiplying $(e^{2\pi\bar{R}(\omega_1+\omega_2)}-1)^{-1}$ in the limit $\bar{R}/\sigma\to\infty$ turns into the δ function.

At $\omega_1/\omega_2=1$, the expression (38) equals $\bar{R}/(\sigma\sqrt{2})$. At $\omega_1/\omega_2=1\pm2(\sigma/\bar{R})\ln^{1/2}(\bar{R}/\sigma)$, (38) is of order unity. Stated differently, at the distance $2(\sigma/\bar{R})\ln^{1/2}(\bar{R}/\sigma)$ from the extremum, the off-diagonal part is of order $O(\sigma/\bar{R})$ of the classical expression at the point of extremum. To make this difference explicit, we extract the factor $\sigma/\bar{R}$,

$$\rho_{\omega_1\omega_2}=\rho^{cl}_{\omega_1\omega_2}+\frac{\sigma}{\bar{R}}\Delta\rho_{\omega_1\omega_2},\qquad(41)$$

where

$$\frac{\sigma}{\overline{R}}\Delta\rho_{\omega_1\omega_2} \sim \rho_{\omega_1\omega_2}\Theta\left(\left|1-\frac{\omega_1}{\omega_2}\right| - 2\frac{\sigma}{\overline{R}}\ln^{1/2}\frac{\overline{R}}{\sigma}\right). \quad (42)$$

It is of interest to evaluate the contribution of small distances to the smeared density matrix (cf. [19]). It is convenient to use method 2. Starting from (33) and making the change of variables $u_i \to u_i R_i/\overline{R}$, we have

$$\begin{aligned}\overline{\rho}_{\omega_1\omega_2} \sim &(\omega_1\omega_2)^{-1/2}\int du_1 \\ &\times \int du_2 e^{-4i\omega_1 u_1 R_1/\overline{R}+4i\omega_2 u_2 R_2/\overline{R}}\sinh^{-2}\left(\frac{(u_1-u_2)}{\overline{R}}-i\varepsilon\right) \\ &\times N^4 e^{-(R_1-\overline{R})^2/2\sigma^2-(R_2-\overline{R})^2/2\sigma^2}R_1^2R_2^2 dR_1 dR_2, \quad (43)\end{aligned}$$

where we omitted the irrelevant for the estimate terms.

Introducing $z=(u_1-u_2)/2$, $y=(u_1+u_2)/2$, we integrate over y in the interval $(-\infty,\infty)$ and over z in the interval $(-\alpha,\alpha)$:

$$\begin{aligned}\overline{\rho}_{\omega_1\omega_2} \sim &(\omega_1\omega_2)^{-1/2}\int_{-\alpha}^{\alpha} dz\frac{R_1R_2}{\overline{R}^2}e^{-iz(R_1\omega_1+R_2\omega_2)/\overline{R}} \\ &\times \overline{R}\delta(R_1\omega_1-R_2\omega_2)\frac{1}{\sinh^2(z/R-i\varepsilon)} \\ &\times N^4 e^{-(R_1-\overline{R})^2/2\sigma^2-(R_2-\overline{R})^2/2\sigma^2}R_1^2R_2^2 dR_1 dR_2. \quad (44)\end{aligned}$$

Integrating over R_2, we obtain the expression structurally similar to (28) and (37). Because this expression has a sharp extremum at $R_1=\overline{R}$ and $\omega_1/\omega_2=1$, for our estimates we can set in the integrand R_1 and ω_1/ω_2 equal to the extremal values.

The resulting density matrix is

$$\begin{aligned}I \sim &(\omega_1\omega_2)^{-1/2}\int_{-\alpha}^{\alpha} dz e^{-i\omega z}\frac{\overline{R}^2}{\sinh^2(2\overline{R}z)} \\ &\times \Theta\left(\frac{\sigma}{\overline{R}}\ln\frac{\overline{R}}{\sigma}-\left|1-\frac{\omega_1}{\omega_2}\right|\right). \quad (45)\end{aligned}$$

The integral in (45) was estimated in [19] for $\omega\overline{R}<1$, and it was shown that the ratio of (45) to the Hawking spectrum is

$$\frac{I(\omega\overline{R},\alpha/\overline{R})}{(e^{4\pi\omega\overline{R}}-1)^{-1}} \sim \alpha/\overline{R}. \quad (46)$$

Taking $\alpha \sim \sigma\ln\overline{R}/\sigma$ and assuming for an estimate that the mass of the black hole is of order of several solar masses, we obtain that $\alpha/\overline{R} \sim (\sigma/M)\ln(M/\sigma) \ll 1$.

VI. DISCUSSION

In this paper, we discussed modifications of the density matrix of the radiation of the black hole formed by the collapsing shell resulting from the horizon fluctuations of the black hole. Horizon fluctuations are inherent to the black hole considered as a quantum object. Distinct from the original Hawking calculation based on the rigid horizon, horizon fluctuations provide the off-diagonal matrix elements of the density matrix. Qualitatively, the off-diagonal matrix elements account for correlations between the particles in the radiation and the information stored in these correlations.

The construction of the density matrix discussed in the present note is parallel to that of Refs. [12,13], where the density matrix with the off-diagonal corrections was obtained in the form $\rho_H(\omega,\tilde{\omega})+C_{\rm BH}^{1/2}\Delta\rho(\omega,\tilde{\omega})$, where ρ_H is the original Hawking matrix and the off-diagonal correction is of order $C^{1/2}$, where $C_{\rm BH}=l_p^2/4\pi M^2G^2$, where M is the mass of the shell. The fact that the expansion parameter in both approaches is the same is rather obvious because σ/M is the only dimensionless parameter connecting the horizon radius and the scale of fluctuations.

In [12], it was supposed that the shell is assigned a wave function $\Psi_{\rm shell}(R_{\rm shell})$, and the already formed black hole is described by the wave function of the Gaussian form $\Psi_{\rm BH} \sim \exp\{-(R-R_S)^2/2\sigma^2\}$, where $R_S=2M$ is the Schwarzschild radius. Next, it was supposed that, as the shell approaches the Schwarzschild radius, the shell wave function approaches the black hole wave function

$$\Psi_{\rm shell}(R_{\rm shell})|_{R_{\rm shell}\to R_S} \to \Psi_{\rm BH}(R_{\rm shell}).$$

Using the approximate relation $R_{\rm shell}-R_S \simeq v_H - v_{\rm shell}$, the expressions for the expectation values of operators $O(R_{\rm shell})$ smeared with the shell wave function are written in the form of integrals over $v_{\rm shell}$.

In the present paper it was assumed that the fluctuations of the horizon radius R are distributed as the Gaussian function around the Schwarzschild radius. The Bogolubov coefficients smeared by fluctuations of the horizon radius were obtained in a convoluted form of the standard Bogolubov coefficients calculated with radius R with the distribution of horizon radius. The density matrix was defined by the standard expression, but with the Bogolubov coefficients smeared by fluctuations of horizon radius.

Technically, the details of calculations and the actual form of the off-diagonal terms in the density matrix obtained in the present paper and in the paper [12] are different.

Because the structural form (but not the explicit form) of the smeared density matrix obtained in the present paper is similar to that in papers [12,13], we arrive at the same qualitative conclusions concerning the information problem as in these papers. It is possible to construct the N-particle density matrix $\rho^{(N)}$ having dimensionality $N\times N$ and to calculate the entropy of radiation $S/N=-{\rm Tr}(\rho^{(N)}\ln\rho^{(N)}$. Calculating the information contained in the radiation, which is defined as the difference between the thermal Bekenstein-Hawking entropy S_H of radiation, $I = S_H - S$, one obtains the qualitatively correct Page purification curve [21–23].

In Ref. [24], it was shown by another method that off-diagonal terms in the density matrix which are responsible for subtle correlations in the radiation can convert a maximally mixed state into a pure state. An important difference between Ref. [24] and the approaches discussed above is that distinct from the "usual" approaches in which the density matrix is calculated for particles propagating to the infinity I^+, in this paper no degrees of freedom (infalling modes) are traced out.

However, the above results pose a question. In [4,5], it was shown that the Schwarzschild metric admits construction of "nice slices." The nice slices are at $r \simeq \text{const}$ inside the horizon, and one can take $r \sim M/2$. For $M \gg l_p$, the horizon fluctuations which are on the scale l_p are insignificant for particle production on the nice slices. If, however, the horizon fluctuations are somehow connected with hair (in the spirit of [8,9] and Refs. therein), then the niceness is broken and the horizon fluctuations can be connected with the release of information from the black hole.

The expressions for the density matrix discussed in the present paper refer to eternal black holes. Because of the outgoing flux of particles, the mass of the collapsing shell is not constant, but decreases with time

$$\frac{\partial M(u)}{\partial t} = -\langle T_{uu}\rangle \equiv -L_H.$$

Here T_{uu} is the uu component of the radiation stress tensor. In Refs. [17,18] it was found that for the mass of the black hole $M(u) \gg m_p$, where m_p is the Planck mass, the backreaction of black hole radiation does not prevent formation of the event horizon. When the outgoing flux is small and slowly varying, the calculation is self-consistent. The metric of the exterior geometry of the shell at large distances r becomes

$$ds^2 \simeq -\left(1 - \frac{2M(u)}{r}\right)dv^2 + 2dudr + r^2 d\Omega^2,$$

where $dM(u)/du = -L_H$, and $L_H \sim 1/M^2(u)$.

For the case considered in Sec. II at leading order,

$$L_H = \frac{1}{48\pi}\frac{M}{R_{\text{shell}}^3}\left(2 - \frac{3M}{R_{\text{shell}}}\right),$$

where M is the mass of the shell. Substituting $R_{\text{shell}}(U) = -(U - 4M)/2$, we have

$$L_H = \frac{M}{3\pi}\frac{U - M}{(U - 4M)^4}.$$

In the near-horizon region $U \to 0$, and we obtain $L_H \simeq 1/(768\pi M^2)$. The smallness of L_H shows that our semi-classical treatment is valid.

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