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Gravitational wave memory in de Sitter spacetime

Lydia Bieri[*]
Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1120, USA

David Garfinkle[†]
Department of Physics, Oakland University, Rochester, Michigan 48309, USA and Michigan Center for Theoretical Physics, Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan 48109-1120, USA

Shing-Tung Yau[‡]
Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA
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We examine gravitational wave memory in the case where sources and detector are in an expanding cosmology. For simplicity, we treat the case where the cosmology is de Sitter spacetime, and discuss the possibility of generalizing our results to the case of a more realistic cosmology. We find results very similar to those of gravitational wave memory in an asymptotically flat spacetime, but with the magnitude of the effect multiplied by a redshift factor.

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I. INTRODUCTION

Gravitational wave memory, a permanent displacement of the gravitational wave detector after the wave has passed, has been known since the work of Zel'dovich and Polnarev [1], extended to the full nonlinear theory of general relativity by Christodoulou [2], and treated by several authors [3–15]. As is usual in the treatment of isolated systems, all these works considered asymptotically flat spacetimes. However, we do not live in an asymptotically flat spacetime, but rather in an expanding universe. For sources of gravitational waves whose distance from the detector is small compared to the Hubble radius, modeling the system as an asymptotically flat spacetime should be sufficient. However, some of the most powerful sources of gravitational waves (e.g. the collision of two supermassive black holes following the merger of their two host galaxies) are at cosmological distances where the asymptotically flat treatment is not sufficient.

In this paper we will treat gravitational wave memory in an expanding universe. To avoid the complications of the full nonlinear Einstein equations, our treatment will use perturbation theory. There is a well-developed theory of cosmological perturbations (see e.g. the textbook treatment in Ref. [16]). However, this standard cosmological perturbation theory uses metric perturbations, and we have found [11] that the properties of gravitational memory are made more clear when using a manifestly gauge-invariant perturbation theory based on the Weyl tensor. Cosmological perturbation theory using the Weyl tensor was developed by Hawking [17]. We will use a treatment similar to that of Ref. [17], but also, using the conformal flatness of Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes, a treatment that draws heavily on the techniques used in Ref. [11].

Cosmological perturbations depend on the equation of state of the matter. The universe, both at the current time and at any previous times from which a realistic source of gravitational wave memory could come, is dominated by dust and a cosmological constant. For simplicity, in this treatment we will treat only the case of a cosmological constant, leaving the more general case of dust and a cosmological constant for subsequent work. Thus this work treats gravitational waves in an expanding de Sitter spacetime. The perturbation equations are developed in Sec. II, the cosmological memory effect is calculated in Sec. III, and the implications of the results are discussed in Sec. IV.

II. EQUATIONS OF MOTION

From the Bianchi identity $\nabla_{[\epsilon}R_{\alpha\beta]\gamma\delta} = 0$ we have

$$g^{\epsilon\alpha}\nabla_\epsilon C_{\alpha\beta\gamma\delta} = \nabla_{[\gamma}S_{\delta]\beta}, \tag{1}$$

where $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{6}Rg_{\alpha\beta}$ and $R = g^{\alpha\beta}R_{\alpha\beta}$. Using the Einstein field equation with a cosmological constant

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2}$$

we find that Eq. (1) becomes

$$g^{\epsilon\alpha}\nabla_\epsilon C_{\alpha\beta\gamma\delta} = 8\pi\nabla_{[\gamma}X_{\delta]\beta}. \tag{3}$$

Here $X_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{3}Tg_{\alpha\beta}$ and $T = g^{\alpha\beta}T_{\alpha\beta}$.

[*]lbieri@umich.edu
[†]garfinkl@oakland.edu
[‡]yau@math.harvard.edu

Both the Weyl tensor, and $T_{\alpha\beta}$ vanish in de Sitter spacetime. It then follows that when we perturb Eq. (3) from a de Sitter background, the perturbed equation takes the same form with the Weyl tensor and stress-energy replaced by their (gauge-invariant) perturbations and the metric and derivative operator replaced with their background values. We will rewrite this perturbed equation in a convenient form making use of the conformal flatness of de Sitter spacetime. Recall that the line element in a spatially flat FLRW spacetime takes the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{4}$$

Then introducing the usual conformal time η by $\eta \equiv \int dt/a$, we find that the line element takes the form

$$ds^2 = a^2[-d\eta^2 + dx^2 + dy^2 + dz^2]. \tag{5}$$

That is, the de Sitter metric takes the form $g_{\alpha\beta} = a^2\eta_{\alpha\beta}$ where $\eta_{\alpha\beta}$ is the Minkowski metric with Cartesian coordinates (η, x, y, z). It then follows that the perturbed Eq. (3) takes the form

$$\partial^\alpha(a^{-1}C_{\alpha\beta\gamma\delta}) = 8\pi[a\partial_{[\gamma}X_{\delta]\beta} + X_{\beta[\gamma}\partial_{\delta]}a + \eta_{\beta[\gamma}X_{\delta]\lambda}\partial^\lambda a]. \tag{6}$$

Here ∂_α is the coordinate derivative operator with respect to the Cartesian coordinates (η, x, y, z). Also here and in what follows we use the convention that indices are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$.

Following the method of Ref. [11] we now decompose all quantities in terms of spatial tensors as follows, using latin letters for spatial indices:

$$E_{ab} \equiv a^{-1}C_{a\eta b\eta}, \tag{7}$$

$$B_{ab} \equiv (a^{-1})\frac{1}{2}\epsilon^{ef}{}_a C_{efb\eta}, \tag{8}$$

$$\mu = T_{\eta\eta}, \tag{9}$$

$$q_a = T_{\eta a}, \tag{10}$$

$$U_{ab} = T_{ab}. \tag{11}$$

Here $\epsilon_{abc} = \epsilon_{\eta abc}$ where $\epsilon_{\alpha\beta\gamma\delta}$ is the Minkowski spacetime volume element. Then Eq. (6) yields two constraint equations

$$\partial^b E_{ab} = 4\pi a\left(\frac{1}{3}\partial_a(2\mu + U^c{}_c) - \partial_\eta q_a\right), \tag{12}$$

$$\partial^b B_{ab} = 4\pi a\epsilon^{ef}{}_a\partial_e q_f, \tag{13}$$

and two equations of motion

$$\begin{aligned}\partial_\eta E_{ab} &- \frac{1}{2}\epsilon_a{}^{cd}\partial_c B_{db} - \frac{1}{2}\epsilon_b{}^{cd}\partial_c B_{da}\\ &= 4\pi a\left[\partial_{(a}q_{b)} - \frac{1}{3}\delta_{ab}\partial_c q^c - \partial_\eta\left(U_{ab} - \frac{1}{3}\delta_{ab}U^c{}_c\right)\right]\\ &\quad + 4\pi a'\left(U_{ab} - \frac{1}{3}\delta_{ab}U^c{}_c\right),\end{aligned} \tag{14}$$

$$\begin{aligned}\partial_\eta B_{ab} &+ \frac{1}{2}\epsilon_a{}^{cd}\partial_c E_{db} + \frac{1}{2}\epsilon_b{}^{cd}\partial_c E_{da}\\ &= 2\pi a(\epsilon_a{}^{cd}\partial_c U_{db} + \epsilon_b{}^{cd}\partial_c U_{da}).\end{aligned} \tag{15}$$

Here $a' = da/d\eta$ and δ_{ab}, the Kronecker delta, is the spatial metric of Minkowski spacetime.

We now want to decompose the spatial tensors into tensors on the two-sphere. We introduce the usual spherical polar coordinates (r, θ, ϕ) with the usual relation to the Cartesian coordinates (x, y, z). We use capital latin letters to denote two-sphere components. From the electric part of the Weyl tensor E_{ab} we obtain a scalar E_{rr} as well as a vector and a symmetric, trace-free tensor given by

$$X_A = E_{Ar}, \tag{16}$$

$$\tilde{E}_{AB} = E_{AB} - \frac{1}{2}H_{AB}E_C{}^C. \tag{17}$$

Here H_{AB} is the metric on the unit two-sphere, and all two-sphere indices are raised and lowered with this metric. Similarly, the decomposition of the magnetic part of the Weyl tensor yields B_{rr} and

$$Y_A = B_{Ar}, \tag{18}$$

$$\tilde{B}_{AB} = B_{AB} - \frac{1}{2}H_{AB}B_C{}^C. \tag{19}$$

The decomposition of the spatial vector q_a yields a two-sphere scalar q_r and vector q_A, while the decomposition of the spatial tensor U_{ab} yields two-sphere scalars U_{rr} and $N \equiv U^c{}_c$, vector $V_A \equiv U_{Ar}$ and a symmetric trace-free tensor

$$W_{AB} = U_{AB} - \frac{1}{2}H_{AB}U_C{}^C. \tag{20}$$

Then the constraint equations [Eqs. (12) and (13)] become

$$\begin{aligned}\partial_r E_{rr} &+ 3r^{-1}E_{rr} + r^{-2}D^A X_A\\ &= 4\pi a\left(\frac{1}{3}\partial_r(2\mu + N) - \partial_\eta q_r\right),\end{aligned} \tag{21}$$

$$\partial_r B_{rr} + 3r^{-1}B_{rr} + r^{-2}D^A Y_A = 4\pi a r^{-2}\epsilon^{AB}D_A q_B, \tag{22}$$

$$\begin{aligned}\partial_r X_A &+ 2r^{-1}X_A - \frac{1}{2}D_A E_{rr} + r^{-2}D^B\tilde{E}_{AB}\\ &= 4\pi a\left(\frac{1}{3}D_A(2\mu + N) - \partial_\eta q_A\right),\end{aligned} \tag{23}$$

$$\partial_r Y_A + 2r^{-1}Y_A - \frac{1}{2}D_A B_{rr} + r^{-2}D^B \tilde{B}_{AB} = 4\pi a \epsilon_A{}^B (D_B q_r - \partial_r q_B). \tag{24}$$

Here D_A is the derivative operator and ϵ_{AB} is the volume element of the unit two-sphere. The evolution equations [Eqs. (14) and (15)] become

$$\partial_\eta B_{rr} + r^{-2}\epsilon^{AB} D_A X_B = 4\pi a r^{-2}\epsilon^{AB} D_A V_B, \tag{25}$$

$$\partial_\eta E_{rr} - r^{-2}\epsilon^{AB} D_A Y_B = 4\pi a \left(\partial_r q_r - \partial_\eta U_{rr} + \frac{1}{3}\partial_\eta (N-\mu)\right) + 4\pi a' \left(U_{rr} - \frac{1}{3}(2N+\mu)\right), \tag{26}$$

$$\partial_\eta Y_A + \frac{1}{2}r^{-2}\epsilon^{CD} D_C \tilde{E}_{DA} + \frac{1}{4}\epsilon_A{}^C (3D_C E_{rr} - 2\partial_r X_C) = 2\pi a \left(\epsilon_A{}^C \left(\frac{1}{2}D_C(3U_{rr} - N) - \partial_r V_C\right) + r^{-2}\epsilon^{BC} D_B W_{CA}\right), \tag{27}$$

$$\partial_\eta X_A - \frac{1}{2}r^{-2}\epsilon^{CD} D_C \tilde{B}_{DA} - \frac{1}{4}\epsilon_A{}^C (3D_C B_{rr} - 2\partial_r Y_C) = 2\pi a (D_A q_r + \partial_r q_A) - 4\pi a (r^{-1}q_A + \partial_\eta V_A) + 4\pi a' V_A, \tag{28}$$

$$\begin{aligned}\partial_\eta \tilde{B}_{AB} &+ \frac{1}{2}\epsilon_A{}^C (D_C X_B + r^{-1}\tilde{E}_{CB} - \partial_r \tilde{E}_{CB}) + \frac{1}{2}\epsilon_B{}^C (D_C X_A + r^{-1}\tilde{E}_{CA} - \partial_r \tilde{E}_{CA}) + \frac{1}{2}H_{AB}\epsilon^{CD} D_C X_D \\ &= 2\pi a \epsilon_A{}^C (D_C V_B + r^{-1}W_{CB} - \partial_r W_{CB}) + 2\pi a \epsilon_B{}^C (D_C V_A + r^{-1}W_{CA} - \partial_r W_{CA}) + 2\pi a H_{AB}\epsilon^{CD} D_C V_D,\end{aligned} \tag{29}$$

$$\begin{aligned}\partial_\eta \tilde{E}_{AB} &- \frac{1}{2}\epsilon_A{}^C (D_C Y_B - \partial_r \tilde{B}_{CB} + r^{-1}\tilde{B}_{CB}) - \frac{1}{2}\epsilon_B{}^C (D_C Y_A - \partial_r \tilde{B}_{CA} + r^{-1}\tilde{B}_{CA}) - \frac{1}{2}H_{AB}\epsilon^{CD} D_C Y_D \\ &= 4\pi a \left(D_{(A}q_{B)} - \frac{1}{2}H_{AB} D_C q^C\right) - \partial_\eta W_{AB}) + 4\pi a' W_{AB}.\end{aligned} \tag{30}$$

III. CALCULATION OF MEMORY

We now consider the behavior of the fields at large distances from the source. Unlike the asymptotically flat case, we cannot make use of the formal definition of null infinity: de Sitter conformal infinity is spacelike, and all gravitational radiation is negligible there. Instead we define the optical scalar $u = \eta - r$ and consider the case of large r and moderate values of u. Note that in this case "large r" means large compared to the wavelength of the gravitational waves emitted by the source, but *not* large compared to the Hubble length. That is, we treat the case where ra' is of order 1. In the case of Minkowski spacetime, it was shown in Ref. [11] that stress-energy gets to large r and moderate u by traveling in null directions: that is, the dominant component of the stress-energy takes the form

$$T_{\alpha\beta} = A\partial_\alpha u \partial_\beta u. \tag{31}$$

In the Appendix we will show that in our case Eq. (31) continues to hold, but that now A takes the form $A = La^{-2}r^{-2}$ where L is a function of u and the two-sphere coordinates. In physical terms, the quantity L is the power radiated per unit solid angle. That is we have

$$\mu = N = U_{rr} = -q_r = La^{-2}r^{-2} + \ldots \tag{32}$$

with all other components of the stress-energy falling off more rapidly. Here … means "terms higher order in r^{-1}." It was shown in Ref. [11] that in the asymptotically flat case, the electric and magnetic parts of the Weyl tensor behave as follows:

$$\tilde{E}_{AB} = e_{AB}r + \ldots, \tag{33}$$

$$\tilde{B}_{AB} = b_{AB}r + \ldots, \tag{34}$$

$$X_A = x_A r^{-1} + \ldots, \tag{35}$$

$$Y_A = y_A r^{-1} + \ldots, \tag{36}$$

$$E_{rr} = Pr^{-3} + \ldots, \tag{37}$$

$$B_{rr} = Qr^{-3} + \ldots \tag{38}$$

where the coefficient tensor fields are functions of u and the two-sphere coordinates; and that in the limit as $|u| \to \infty$ the only one of these coefficient tensor fields that does not vanish is P. In the Appendix we show that these relations continue to hold. Note that because of the relation between Cartesian and spherical coordinates $\tilde{E}_{AB}$ behaving like r corresponds to Cartesian components of the electric part of the Weyl tensor behaving like r^{-1}. The conditions of Eqs. (33)–(38) are reminiscent of the usual peeling theorem for asymptotically flat spacetimes; however the context is somewhat different. The peeling theorem of Ref. [18] is a consequence of conformal compactification at null infinity. In contrast, de Sitter spacetime has a spacelike conformal completion, and the corresponding peeling theorem [19] essentially says that at sufficiently late times all physical curvatures decay exponentially with time. In contrast, we want to look at the behavior of fields in null directions in de Sitter spacetime at a time that is not large compared to the Hubble time. In-depth treatments of this issue are contained in recent works of Ashtekar *et al.* [20–23].

Now keeping only the dominant terms in Eqs. (21)–(24) and using the fact that ra' is of order unity, we obtain

$$-\dot{P} + D^A x_A = -8\pi L a^{-2}(a + ra'), \tag{39}$$

$$-\dot{Q} + D^A y_A = 0, \tag{40}$$

$$-\dot{x}_A + D^B e_{AB} = 0, \tag{41}$$

$$-\dot{y}_A + D^B b_{AB} = 0. \tag{42}$$

Here an overdot means a derivative with respect to u. Similarly, keeping only the dominant terms in Eqs. (25)–(30) yields

$$\dot{Q} + \epsilon^{AB} D_A x_B = 0, \tag{43}$$

$$\dot{P} - \epsilon^{AB} D_A y_B = 8\pi L a^{-2}(a + ra'), \tag{44}$$

$$\dot{y}_A + \frac{1}{2}\epsilon^{CD} D_C e_{DA} + \frac{1}{2}\epsilon_A{}^C \dot{x}_C = 0, \tag{45}$$

$$\dot{x}_A - \frac{1}{2}\epsilon^{CD} D_C b_{DA} - \frac{1}{2}\epsilon_A{}^C \dot{y}_C = 0, \tag{46}$$

$$\dot{b}_{AB} + \epsilon_A{}^C \dot{e}_{CB} = 0, \tag{47}$$

$$\dot{e}_{AB} - \epsilon_A{}^C \dot{b}_{CB} = 0. \tag{48}$$

From here, the analysis proceeds essentially as in Ref. [11]. By convention, the scale factor a is unity at the present time. Therefore at the position of the detector, η is the same as the usual time, E_{ab} [despite the factor of a^{-1} in Eq. (7)] is equal to the physical electric part of the Weyl tensor and thus is directly related to the tidal force, and a' is equal to H_0, the Hubble constant. Note that Eq. (48) is redundant, since it is equivalent to Eq. (47). Since e_{AB} and b_{AB} vanish as $u \to -\infty$, it follows from Eq. (47) that $b_{AB} = -\epsilon_A{}^C e_{CB}$. This can be used to eliminate b_{AB} from Eqs. (42) and (46) which then become

$$\dot{y}_A + \epsilon^{CD} D_C e_{DA} = 0, \tag{49}$$

$$\dot{x}_A - \frac{1}{2} D^C e_{CA} - \frac{1}{2}\epsilon_A{}^C \dot{y}_C = 0. \tag{50}$$

Combining Eq. (49) with Eq. (45) then yields

$$\dot{y}_A + \epsilon_A{}^B \dot{x}_B = 0. \tag{51}$$

However, since x_A and y_A vanish as $u \to -\infty$, it then follows from Eq. (51) that

$$y_A = -\epsilon_A{}^B x_B. \tag{52}$$

Thus, we can eliminate y_A from Eqs. (40) and (44) which then become

$$\dot{Q} + \epsilon^{AB} D_A x_B = 0, \tag{53}$$

$$\dot{P} - D^A x_A = 8\pi L(1 + rH_0). \tag{54}$$

But these equations are then redundant, since they are equivalent to Eqs. (43) and (39) respectively. Thus the only independent quantities are e_{AB}, x_A, P, Q and L. These quantities satisfy the following equations:

$$D^B e_{AB} = \dot{x}_A, \tag{55}$$

$$\epsilon^{BC} D_B e_{CA} = \epsilon_A{}^C \dot{x}_C, \tag{56}$$

$$D_A x^A = \dot{P} - 8\pi L(1 + rH_0), \tag{57}$$

$$\epsilon^{AB} D_A x_B = -\dot{Q}. \tag{58}$$

Now let us consider how to use Eqs. (55)–(58) to find the memory. Recall that e_{AB} is (up to a factor involving the distance and the initial separation) the second time derivative of the separation of the masses. Thus we want to integrate e_{AB} twice with respect to u. Define the velocity tensor v_{AB}, memory tensor m_{AB} and a tensor z_A by

$$v_{AB} \equiv \int_{-\infty}^{u} e_{AB} du, \tag{59}$$

$$m_{AB} \equiv \int_{-\infty}^{\infty} v_{AB} du, \tag{60}$$

$$z_A \equiv \int_{-\infty}^{\infty} x_A du. \tag{61}$$

Now consider two masses in free fall whose initial separation is d in the B direction. Then after the wave has passed they will have an additional separation. Call the component of that additional separation in the A direction Δd. Then it follows from the geodesic deviation equation that

$$\Delta d = -\frac{d}{r} m^A{}_B. \tag{62}$$

To find m_{AB} we first integrate Eqs. (55) and (56) to obtain

$$D^B v_{AB} = x_A, \tag{63}$$

$$\epsilon^{BC} D_B v_{CA} = \epsilon_A{}^C x_C. \tag{64}$$

Then by integrating again from $-\infty$ to ∞ we obtain

$$D^B m_{AB} = z_A, \tag{65}$$

$$\epsilon^{BC} D_B m_{CA} = \epsilon_A{}^C z_C. \tag{66}$$

Now integrating Eqs. (57) and (58) from $-\infty$ to ∞ yields

$$D_A z^A = \Delta P - 8\pi F(1 + rH_0), \tag{67}$$

$$\epsilon^{AB} D_A z_B = 0, \tag{68}$$

where the quantities ΔP and F are defined by $\Delta P = P(\infty) - P(-\infty)$ and $F = \int_{-\infty}^{\infty} L du$. In physical terms, F is the amount of energy radiated per unit solid angle. In deriving Eq. (68) we have used the fact that Q vanishes in the limit as $|u| \to \infty$. Since z_A is curl-free, there must be a scalar Φ such that $z_A = D_A \Phi$. Then by using Eqs. (67) and (65) we find

$$D_A D^A \Phi = \Delta P - 8\pi F(1 + rH_0), \tag{69}$$

$$D^B m_{AB} = D_A \Phi. \tag{70}$$

IV. DISCUSSION

We now consider the physical implications of these results, and in particular of Eqs. (69)–(70). As in the asymptotically flat case, there are two kinds of gravitational wave memory: an ordinary memory due to sources that do not get out to infinity and a null memory due to sources that do get out to infinity. The ordinary memory is sourced by ΔP, that is the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by F, the energy per unit solid angle radiated to infinity. However, in contrast to the asymptotically flat case, there is a factor of $1 + rH_0$ multiplying F. Note that in cosmology, the wavelength of light from distant sources is redshifted by a factor of $1 + z$ and that to first order in z we have $1 + z = 1 + rH_0$. Thus, expressed in terms of F and r it seems that the null memory is enhanced by a factor of $1 + z$. However, r is not a directly observed property of a distant object: instead we observe the luminosity of the object and infer a luminosity distance d_L related to r by $d_L = r(1 + z)r$. Since the observed memory is given by Eq. (62) which has a factor of r^{-1}, it follows that when expressed in terms of F and d_L, the null memory is enhanced by a factor of $(1 + z)^2$. However, F is the energy radiated per unit solid angle as measured by the observer, who is at cosmological distance from the source. Instead, one might want to calculate the local F (which we will call F_{loc}) as measured by an observer who is sufficiently far from the source to be in its wave zone but still at a distance small compared to the Hubble radius. That is, F_{loc} is the F that source would have in Minkowski spacetime. Because the energy of radiation is diminished by a factor of $1/(1 + z)$, it follows that $F = F_{\mathrm{loc}}/(1 + z)$. Therefore, when expressed in terms of F_{loc} and the luminosity distance, the memory is enhanced by a factor of $1 + z$.

This behavior of memory in the cosmological setting is what would be expected from the properties of gravitational waves treated in the geometric optics (i.e. short-wavelength) approximation, as done by Thorne [24]. This issue will be covered in depth elsewhere [25]. In particular, in Ref. [25] we use the geometric optics approximation to extend our treatment of cosmological memory to the case with both dust and a cosmological constant. As shown by Hawking [17] the inclusion of dust leads to a set of coupled equations involving the Weyl tensor and the shear of the fluid. However, we show that in the geometric optics approximation these equations decouple leading to a simple result for the gravitational wave memory.

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APPENDIX: STRESS-ENERGY AND WEYL TENSORS AT LARGE r

We now consider the behavior of the stress-energy tensor and the Weyl tensor at large r. Since the metric can be written as $g_{\alpha\beta} = a^2 \eta_{\alpha\beta}$ it follows that for any vector ω_α we have

$$\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\gamma_{\alpha\beta} \omega_\gamma, \tag{A1}$$

where the Christoffel symbols are given by

$$\Gamma^\gamma_{\alpha\beta} = \frac{a'}{a} (\delta^\gamma{}_\beta \partial_\alpha \eta + \delta^\gamma{}_\alpha \partial_\beta \eta - \eta_{\alpha\beta} \partial^\gamma \eta). \tag{A2}$$

It then follows that the conservation of stress-energy $g^{\alpha\gamma} \nabla_\gamma T_{\alpha\beta} = 0$ becomes

$$\partial^\alpha (a^2 T_{\alpha\beta}) - a a' T \partial_\beta \eta = 0. \tag{A3}$$

where $T = \eta^{\alpha\beta} T_{\alpha\beta}$. Define $u \equiv \eta - r$ and $\ell_\alpha \equiv -\nabla_\alpha u$. In Ref. [11] it was shown that in the asymptotically flat case, the stress-energy tensor at large distances takes the form $T_{\alpha\beta} = A \ell_\alpha \ell_\beta + \ldots$ where $+\ldots$ means plus terms that are higher order in powers of r^{-1}. Flat space conservation of energy (i.e. $\partial^\alpha T_{\alpha\beta} = 0$) implies that A takes the form $r^{-2} L$ where L is a function of u and the angular coordinates.

However, in de Sitter spacetime conservation of stress-energy takes the form given in Eq. (A3). This is still compatible with $T_{\alpha\beta} = A\ell_\alpha\ell_\beta + \ldots$ but now implies that A takes the form $A = a^{-2}r^{-2}L$ where L is a function of u and the angular coordinates. That is, the stress-energy has the form

$$T_{\alpha\beta} = a^{-2}r^{-2}L\ell_\alpha\ell_\beta + \ldots \tag{A4}$$

and therefore its components have the properties given in Eq. (32).

We now turn to the properties of the Weyl tensor. Denote the right-hand sides of Eqs. (12)–(15) as respectively α_a, β_a, γ_{ab} and λ_{ab}. Then by applying ∂_η to Eq. (14) and using the other three equations we obtain

$$\partial^\mu\partial_\mu E_{ab} = \frac{3}{2}\left(\partial_{(a}\alpha_{b)} - \frac{1}{3}\delta_{ab}\partial^c\alpha_c\right) - \frac{1}{2}\epsilon_a{}^{cd}\partial_c\lambda_{db} - \frac{1}{2}\epsilon_b{}^{cd}\partial_c\lambda_{da} - \partial_\eta\gamma_{ab}. \tag{A5}$$

Similarly, by applying ∂_η to Eq. (15) and using the other three equations we obtain

$$\partial^\mu\partial_\mu B_{ab} = \frac{3}{2}\left(\partial_{(a}\beta_{b)} - \frac{1}{3}\delta_{ab}\partial^c\beta_c\right) + \frac{1}{2}\epsilon_a{}^{cd}\partial_c\gamma_{db} + \frac{1}{2}\epsilon_b{}^{cd}\partial_c\gamma_{da} - \partial_\eta\lambda_{ab}. \tag{A6}$$

Thus we have that the electric and magnetic parts of the Weyl tensor satisfy the flat spacetime wave equation with a source that consists of derivatives of components of the stress-energy tensor. It then follows that the Cartesian components of the electric and magnetic parts of the Weyl tensor go like r^{-1} at large r and moderate u. Note that the relation between Cartesian coordinates and angular coordinates then implies that the angular components E_{AB} and B_{AB} go like r. We now demonstrate that radial components of the electric and magnetic parts of the Weyl tensor must fall off faster than Cartesian components. First note that $\partial_a u = -r_a$ where r_a is a unit vector in the radial direction. Now consider E_{ab} as a power series in r^{-1} with coefficients that depend on u and the angular coordinates. Then it follows that

$$\partial_c E_{ab} = -r_c \frac{\partial}{\partial u} E_{ab} + O(r^{-2}). \tag{A7}$$

But the electric part of the Weyl tensor satisfies $\partial^a E_{ab} = \alpha_b$ from which [using Eq. (A4) to establish the appropriate falloff of α_b] it follows that E_{ra} goes like r^{-2} and therefore that E_{rA} goes like r^{-1}. Now defining $v_a = E_{ra}$ the same line of reasoning shows that $r^a v_a$ must fall off one power of r faster than v_a and therefore that E_{rr} goes like r^{-3}. Finally, the same argument that we have used for the electric part of the Weyl tensor also applies to the magnetic part. Thus, we have established the falloff rates given in Eqs. (33)–(38).

We now consider the behavior of the asymptotic Weyl tensor at large $|u|$. We will assume that at both early and late times the matter consists of widely separated objects moving at constant velocity. Therefore the Weyl tensor is a linear combination of translated and boosted Schwarzschild perturbations of de Sitter spacetime (essentially Schwarzschild–de Sitter spacetime with small mass). But in our coordinates, the Weyl tensor of Schwarzschild–de Sitter falls off like r^{-3}. This property also holds under translations and boosts. It then follows that e_{AB}, b_{AB}, x_A and y_A vanish as $|u| \to \infty$. In the rest frame, Schwarzschild–de Sitter has a purely electric Weyl tensor, and while this changes under boosts, the B_{rr} component remains zero. Therefore as $|u| \to \infty$, we have that Q vanishes but P does not. In principle, our widely separated objects should be treated as boosted Kerr perturbations of de Sitter spacetime if they are black holes, or as boosted perturbations with some other gravitational multipole structure if they are not black holes. However, in general higher multipoles (whether of Kerr black holes or of nonspherical objects) give effects that are higher order in powers of r^{-1}. Therefore we expect that our treatment using boosted Schwarzschild perturbations of de Sitter is sufficient for our purposes.

[1] Y. B. Zel'dovich and A. G. Polnarev, Sov. Astron. **18**, 17 (1974).
[2] D. Christodoulou, Phys. Rev. Lett. **67**, 1486 (1991).
[3] V. B. Braginsky and K. S. Thorne, Nature (London) **327**, 123. (1987).
[4] R. Epstein, Astrophys. J. **223**, 1037 (1978).
[5] M. Turner, Astrophys. J. **216**, 610 (1977).
[6] A. G. Wiseman and C. M. Will., Phys. Rev. D **44**, R2945 (1991).
[7] M. Favata, Classical Quantum Gravity **27**, 084036 (2010).
[8] L. Bieri, P. Chen, and S.-T. Yau, Adv. Theor. Math. Phys. **15**, 1085 (2011).
[9] L. Bieri, P. Chen, and S.-T. Yau., Classical Quantum Gravity **29**, 215003 (2012).
[10] L. Bieri and D. Garfinkle, Annales Henri Poincaré **16**, 801 (2015).
[11] L. Bieri and D. Garfinkle, Phys. Rev. D **89**, 084039 (2014).

[12] A. Tolish and R. M. Wald., Phys. Rev. D **89**, 064008 (2014).
[13] J. Winicour, Classical Quantum Gravity **31**, 205003 (2014).
[14] A. Strominger and A. Zhiboedov, J. High Energy Phys. 01 (2016) 086.
[15] E. Flanagan and D. Nichols, Phys. Rev. D **92**, 084057 (2015).
[16] S. Weinberg, *Cosmology* (Oxford University, New York, 2008).
[17] S. W. Hawking, Astrophys. J. **145**, 544 (1966).
[18] R. Penrose, Proc. R. Soc. A **284**, 159 (1965).
[19] H. Friedrich, J. Geom. Phys. **3**, 101 (1986).
[20] A. Ashtekar, B. Bonga, and A. Kesavan, Classical Quantum Gravity **32**, 025004 (2015).
[21] A. Ashtekar, B. Bonga, and A. Kesavan, Phys. Rev. D **92**, 044011 (2015).
[22] A. Ashtekar, B. Bonga, and A. Kesavan, Phys. Rev. D **92**, 104032 (2015).
[23] A. Ashtekar, B. Bonga, and A. Kesavan, Phys. Rev. Lett. **116**, 051101 (2016).
[24] K. Thorne, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North Holland, Amsterdam, 1983).
[25] L. Bieri, D. Garfinkle, and N. Yunes (to be published).