

**Self-similar motion of a Nambu-Goto string**Takahisa Igata,<sup>1,\*</sup> Tsuyoshi Houri,<sup>2,†</sup> and Tomohiro Harada<sup>1,‡</sup><sup>1</sup>*Department of Physics, Rikkyo University, Toshima, Tokyo 175-8501, Japan*<sup>2</sup>*Department of Physics, Kobe University, Kobe, Hyogo 657-8501, Japan*

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We study the self-similar motion of a string in a self-similar spacetime by introducing the concept of a self-similar string, which is defined as the world sheet to which a homothetic vector field is tangent. It is shown that in Nambu-Goto theory, the equations of motion for a self-similar string reduce to those for a particle. Moreover, under certain conditions such as the hypersurface orthogonality of the homothetic vector field, the equations of motion for a self-similar string simplify to the geodesic equations on a (pseudo)Riemannian space. As a concrete example, we investigate a self-similar Nambu-Goto string in a spatially flat Friedmann-Lemaître-Robertson-Walker expanding universe with self-similarity and obtain solutions of open and closed strings, which have various nontrivial configurations depending on the rate of the cosmic expansion. For instance, we obtain a circular solution that evolves linearly in the cosmic time while keeping its configuration by the balance between the effects of the cosmic expansion and string tension. We also show the instability for linear radial perturbation of the circular solutions.

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**I. INTRODUCTION**

In the context of modern physics, a string has been of strong interest for a long time. At large scales, one-dimensional topological defects formed in the early Universe are referred to as cosmic strings, which have sizes as large as each cosmological horizon [1]. At small scales, strings are thought of as elementary components in string theories [2]. Under these circumstances, it is important to develop our understanding of classical string dynamics in curved spacetimes.

In Nambu-Goto string theory, string dynamics is governed by second-order nonlinear partial differential equations in two variables. Although some solutions are not always analytically tractable, some string solutions have been constructed with the aid of symmetry in the following target spacetimes: Minkowski [3–8], black hole [9–13], and cosmological spacetimes [14–18]. For example, stationarity of a string is defined by a timelike Killing vector field in a target spacetime, which is tangent to the world sheet. Since the Killing vector field generates the time evolution of the string, the equation of motion only determines its configuration on a time slice. This idea was formulated as the stationary strings [19], for which the Nambu-Goto equation reduces to a geodesic equation, which is ordinary differential equations in a single variable. It was generalized to the cohomogeneity-one strings [20], which are defined as the world sheet to which any Killing vector field (not necessarily stationary) is tangent. In physically

realistic systems, these are candidates for final states after radiating gravitational waves [21].

The Killing vector field can be generalized to the homothetic vector field, which is associated with self-similarity of a spacetime. A spacetime that admits a homothetic vector field is called a self-similar spacetime. This has been widely studied [22–34] and was highlighted as the self-similar hypothesis [35], which states that solutions in general relativity naturally develop toward a self-similar form asymptotically under some physical circumstances. It, therefore, may be reasonable to consider self-similar spacetimes to be physically realistic in cosmology and astrophysics.

We are now in the position of having an interest in the dynamics of a string with self-similarity on a self-similar spacetime, that is, a self-similar string. Since this is a generalization of a stationary string on a stationary spacetime, we can expect a self-similar string to be a candidate for a final state in a self-similar spacetime. These solutions will model cosmic strings in the late time of an expanding universe. The purpose of this paper therefore is to formulate a self-similar string and to demonstrate its qualitative behavior.

This paper is organized as follows. In the following section, we propose a definition of a self-similar string in terms of self-similarity on a self-similar target spacetime. On the basis of Nambu-Goto string theory, we obtain the equation of motion for a self-similar string. In Sec. III, we apply our formalism to a self-similar Nambu-Goto string in a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) expanding universe with self-similarity, which includes the Minkowski spacetime as a special case. Solving it in some integrable cases, we consider its dynamics through analytical models. Section IV is devoted

\*igata@rikkyo.ac.jp

†hour@phys.sci.kobe-u.ac.jp

‡harada@rikkyo.ac.jp

to a summary. Throughout this paper, we use geometrized units, in which  $G = 1$  and  $c = 1$ .

## II. FORMULATION OF A SELF-SIMILAR STRING

Let  $(M, g_{\mu\nu})$  be a  $D$ -dimensional self-similar spacetime, i.e., a spacetime that admits a homothetic vector field  $\xi^\mu$ , which is defined by

$$\mathfrak{L}_\xi g_{\mu\nu} = 2Cg_{\mu\nu}, \quad (1)$$

where  $\mu$  and  $\nu$  have the range  $0, 1, \dots, D-1$ , the left-hand side denotes the Lie derivative of  $g_{\mu\nu}$  with respect to  $\xi^\mu$ , and  $C$  is a constant. A homothetic vector field is said to be proper if  $C \neq 0$ , which is unique up to a Killing vector field. In the case where  $C = 0$ , it is nothing but a Killing vector field. Given a homothetic vector field  $\xi^\mu$ , we are able to introduce a local coordinate system  $x^\mu = (\eta, \mathbf{x})$  on  $M$  such that

$$\xi^\mu \partial_\mu = \partial_\eta, \quad (2)$$

where  $\eta$  is called a homothetic coordinate and  $\mathbf{x} = (x^1, x^2, \dots, x^{D-1})$ . In this coordinate system, if  $\xi^\mu$  is timelike or spacelike,  $g_{\mu\nu}$  is generally written in the  $1 + (D-1)$  form

$$ds^2 = \Omega^2(\eta)[f(\mathbf{x})(d\eta + B_i(\mathbf{x})dx^i)^2 + h_{ij}(\mathbf{x})dx^i dx^j], \quad (3)$$

where the Latin indices  $i, j, \dots$  have the range  $1, 2, \dots, D-1$ . Note that the functions  $f$ ,  $B_i$ , and  $h_{ij}$  do not depend on  $\eta$ , and  $\Omega(\eta) = e^{C\eta}$  by virtue of Eq. (1). For a null  $\xi^\mu$ , the form (3) does not apply.<sup>1</sup>

Now let us define a self-similar string in terms of self-similarity of  $(M, g_{\mu\nu})$  by employing the way that was introduced to define a cohomogeneity-one string [19,20]. Let  $X^\mu(\tau, \sigma)$  be embedding functions of a string, which describes a two-dimensional world sheet. Then, we define a self-similar string as a world sheet  $\Sigma$  to which a homothetic vector field  $\xi^\mu$  is tangent, that is,

$$\xi^{[\mu} \dot{X}^\nu X'^{\lambda]} = 0, \quad (4)$$

where  $\lambda = 0, 1, \dots, D-1$ , the square bracket denotes antisymmetrization, and the dot and prime are derivatives with respect to  $\tau$  and  $\sigma$ , respectively. Since a homothetic vector field  $\xi^\mu$  is tangent to  $\Sigma$ , the homothetic coordinate  $\eta$  associated with  $\xi^\mu$  can be introduced as a coordinate on  $\Sigma$ . Therefore, we may parametrize the embedding functions in the homothetic coordinate system  $(\eta, \mathbf{x})$  as

$$X^\mu(\tau, \sigma) = (\tau, \mathbf{X}(\sigma)), \quad (5)$$

where  $\mathbf{X}(\sigma) = (X^1(\sigma), \dots, X^{D-1}(\sigma))$ . This expression indeed satisfies Eq. (4). Once a theory is specified, the equations of motion for  $\mathbf{X}(\sigma)$  reduce to ordinary differential equations. Thus, the problem of determining the dynamics of a self-similar string reduces to that of ‘‘a particle.’’ However, it is still uncertain whether all the equations are compatible because the equations for  $\mathbf{X}(\sigma)$  can be over-determined. In what follows, we restrict ourselves to consider a self-similar string in Nambu-Goto theory. Then, we see that its equations of motion result in those of a particle motion.

### A. Nambu-Goto equation for a self-similar string

Let us assume that the dynamics of a self-similar string on  $(M, g_{\mu\nu})$  is governed by Nambu-Goto string theory. The Nambu-Goto action is given by

$$S = -\mu \int_\Sigma \sqrt{-\gamma} d\tau d\sigma, \quad (6)$$

where  $\mu$  is the string tension, and  $\gamma$  is defined as the determinant of the induced metric  $\gamma_{ab} = g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu$  on  $\Sigma$ , where  $a, b$  have  $\tau, \sigma$ . When no confusion arises from abbreviation, we omit the arguments  $X^\mu$  from any function in what follows. The variation of Eq. (6) with respect to  $X^\mu$  yields the Nambu-Goto equation

$$\partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu) + \sqrt{-\gamma} \gamma^{ab} \Gamma^\mu_{\nu\lambda} \partial_a X^\nu \partial_b X^\lambda = 0, \quad (7)$$

where  $\gamma^{ab}$  is the inverse metric of  $\gamma_{ab}$ , and  $\Gamma^\mu_{\nu\lambda}$  denotes the Christoffel symbol of  $g_{\mu\nu}$ . Under the settings of Eqs. (3)–(5), the induced metric has the form

$$\gamma_{ab} d\zeta^a d\zeta^b = \Omega^2 [f(d\tau + B d\sigma)^2 + h d\sigma^2], \quad (8)$$

where  $\zeta^a = (\tau, \sigma)$  and

$$B = B_i X'^i, \quad (9)$$

$$h = h_{ij} X'^i X'^j. \quad (10)$$

The determinant of  $\gamma_{ab}$  is given by  $\gamma = \Omega^4 f h$ . Then, the components of  $\sqrt{-\gamma} \gamma^{ab}$  in terms of  $\zeta^a$  are calculated as

$$\sqrt{-\gamma} \gamma^{ab} = -\text{sgn}(f) \sqrt{-f/h} \begin{pmatrix} B^2 + h/f & -B \\ -B & 1 \end{pmatrix}. \quad (11)$$

Substituting Eq. (11) into Eq. (7) and dividing it by  $\sqrt{-\gamma} \gamma^{\sigma\sigma}$ , we obtain  $E^\mu = 0$ , where

<sup>1</sup>We can construct a self-similar null string in the same way as in Ref. [36] because a null  $\xi^\mu$  is tangent to a null geodesic.

$$\begin{aligned}
 E^\mu &= X^\nu \nabla_\nu X^\mu + (\ln \sqrt{-f/h})' X^\mu \\
 &+ (B^2 + h/f) \dot{X}^\nu \nabla_\nu \dot{X}^\mu - \frac{(B\sqrt{-f/h})'}{\sqrt{-f/h}} \dot{X}^\mu \\
 &- 2B\Gamma^\mu{}_{\nu\lambda} \dot{X}^\nu X'^\lambda,
 \end{aligned} \tag{12}$$

and  $\nabla_\mu$  is the Levi-Civita covariant derivative associated with  $g_{\mu\nu}$ . Each component of  $E^\mu = 0$  has the form

$$E^\eta = -B_i E^i = 0, \tag{13}$$

$$E^i = \mathcal{E}^i + 2C(B^i h_{jk} - \delta^i_j B_k) X'^j X'^k = 0, \tag{14}$$

where  $B^i = h^{ij} B_j$ ,  $h^{ij}$  is the inverse of  $h_{ij}$ , and we have defined  $\mathcal{E}^i$  as

$$\mathcal{E}^i = X'^j D_j X'^i - \frac{d \ln \sqrt{-f/h}}{d\sigma} X'^i, \tag{15}$$

where  $D_i$  denotes the Levi-Civita covariant derivative associated with  $|f|h_{ij}$ . It immediately follows that a solution of Eq. (14) always solves Eq. (13). This shows that the assumptions for a self-similar string are compatible with Nambu-Goto theory. Thus, we have obtained the reduced Nambu-Goto equation (14) for a self-similar string in a self-similar spacetime, which is identical to the equations of motion for a particle in  $D - 1$  dimensions, as was expected.

In Refs. [19,20], the equations of motion for a cohomogeneity-one string were obtained by just taking the variation of the reduced action. However, the present analysis does not rely on this method. In fact, the reduced action for a self-similar string, along with the assumptions (2)–(5), is given by

$$S = -\mu \int \Omega^2 \sqrt{-f h} d\tau d\sigma. \tag{16}$$

The variation of this reduced action leads to  $\mathcal{E}^i = 0$ , which is not equivalent to the equations of motion (14) in general. This difference commonly happens when we consider a reduced action, depending on how compatible the assumptions we consider are. Our calculation shows that if the second term in Eq. (14) vanishes the equations obtained by taking the variation of the reduced action (16) coincide with the equations of motion for a self-similar string. In the next section, we investigate in detail the conditions that the second term in Eq. (14) vanishes.

### B. Reduction to a geodesic equation

In particular situations where

$$C(B^i h_{jk} - \delta^i_j B_k) X'^j X'^k = 0, \tag{17}$$

the reduced Nambu-Goto equation (14) further simplifies to  $\mathcal{E}^i = 0$ , i.e.,

$$X'^j D_j X'^i = \frac{d \ln \sqrt{-f/h}}{d\sigma} X'^i. \tag{18}$$

This equation describes a geodesic flow on the orbit space  $O$  of  $\xi^\mu$  with the metric  $|f|h_{ij}$ . Hence, the problem of finding a self-similar Nambu-Goto string on  $(M, g_{\mu\nu})$  is simplified to that of solving the geodesic equation on a (pseudo)Riemannian space  $(O, |f|h_{ij})$ .

This situation can occur if any one of the conditions

- (i)  $C = 0$ ,
- (ii)  $B_i = 0$ ,
- (iii)  $B^i \parallel X'^i$  (19)

is satisfied. Since Condition (i) corresponds to the case of a cohomogeneity-one string, our formulation justifies the reduced action method by use of a Killing vector field developed in Refs. [19,20]. Condition (ii) means that the homothetic vector field is hypersurface orthogonal and certainly occurs as is seen in Sec. III. Condition (iii) can be rewritten as  $\xi_i \parallel h_{ij} X'^j$ .

## III. SELF-SIMILAR NAMBU-GOTO STRING

### A. Formulation in a self-similar expanding universe

In this section, we consider the dynamics of a self-similar Nambu-Goto string in an expanding flat universe described by the FLRW metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{20}$$

where  $(t, x, y, z)$  are the comoving Cartesian coordinates, and  $a(t)$  is the scale factor. For the metric to admit a proper homothetic vector field  $\xi^\mu$ , the scale factor is restricted to have the form [37]

$$a(t) = (t/t_0)^{1-1/C}, \tag{21}$$

where  $C$  and  $t_0$  are nonzero constants, and then  $\xi^\mu$  is given by

$$\xi^\mu \partial_\mu = Ct \partial_t + r \partial_r. \tag{22}$$

The metric coincides with the Minkowski one when  $C = 1$ . For  $C \neq 1$ , we take the sign of  $t_0$  to be  $t/t_0 > 0$ . Furthermore, we assume that a universe is expanding,  $da/dt > 0$ . Thus, under the choice of  $\partial_t$  to be future directed, we take the coordinate range  $0 < t < \infty$  for  $C > 1$  or  $C < 0$  and  $-\infty < t < 0$  for  $0 < C < 1$ . If we consider that a universe is filled with a perfect fluid with the equation of state  $p = w\rho$ , the parameters  $C$  and  $w$  are related to each other as

$$w = \frac{3 - C}{3(C - 1)}, \quad (23)$$

where  $w$  is a constant other than  $-1$  and  $-1/3$ .

Moreover, we introduce a conformal time  $\lambda$  defined by

$$\lambda = \lambda_0(t/t_0)^{1/C}, \quad (24)$$

where  $\lambda_0 = Ct_0$ . The metric then takes the form

$$ds^2 = (\lambda/\lambda_0)^{2(C-1)}(-d\lambda^2 + dr^2 + r^2 d\Omega^2), \quad (25)$$

where we have introduced the spherical-polar coordinates  $(r, \theta, \phi)$  in the spatial part and  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ . The homothetic vector field is given by  $\xi^\mu \partial_\mu = \lambda \partial_\lambda + r \partial_r$ . In the following subsections, we introduce a local homothetic coordinate system in Region I ( $r < |\lambda|$ ) and Region II ( $|\lambda| < r$ ), where the homothetic vector field is timelike and spacelike, respectively, and derive the equations of motion for a self-similar string associated with  $\xi^\mu$ . Hereafter, we adopt the units in which  $|\lambda_0| = 1$ .

### I. Region I ( $r < |\lambda|$ )

Let us make the coordinate transformation in Region I defined by

$$\lambda = \kappa e^\eta \cosh \chi, \quad (26)$$

$$r = e^\eta \sinh \chi, \quad (27)$$

where  $\kappa = \text{sgn}(\lambda_0)$ , and the coordinates  $\eta$  and  $\chi$  have the range  $-\infty < \eta < \infty$  and  $0 < \chi < \infty$ , respectively. The metric is then transformed to

$$ds^2 = e^{2C\eta} (\cosh \chi)^{2(C-1)} (-d\eta^2 + d\chi^2 + \sinh^2 \chi d\Omega^2), \quad (28)$$

which in particular yields the Milne metric in  $C = 1$ .

Since the form (28) fits into Eq. (3), the homothetic vector field (22) has the same form as Eq. (2). We also notice that the metric satisfies Condition (ii) discussed in Sec. II B, so that the dynamics of a self-similar string is determined by the geodesic equation (18). In the present homothetic coordinate system  $(\eta, \chi, \theta, \phi)$ , the embedding functions are written as

$$X^\mu(\tau, \sigma) = (\tau, \mathcal{X}(\sigma), \Theta(\sigma), \Phi(\sigma)), \quad (29)$$

where  $X^i(\sigma)$  are determined by the geodesic equation with respect to the metric

$$|f|h_{ij} dx^i dx^j = (\cosh \chi)^{4(C-1)} (d\chi^2 + \sinh^2 \chi d\Omega^2), \quad (30)$$

which is conformal to the three-dimensional hyperbolic space  $\mathbb{H}^3$ . In particular, a solution  $X^i(\sigma)$  for  $C = 1$  is identified with a geodesic on  $\mathbb{H}^3$ .

Let us solve the geodesic equation by using the Hamiltonian formalism. To rewrite this system in canonical variables, we employ the Polyakov-type action with the Lagrangian in the form

$$L = \frac{1}{2N} |f| h_{ij} \frac{dX^i}{d\sigma} \frac{dX^j}{d\sigma} + \frac{N}{2}, \quad (31)$$

where  $N$  is an auxiliary variable. Defining the canonical momentum  $p_i = N^{-1} |f| h_{ij} X'^j$  conjugate to  $X^i$ , we obtain the Hamiltonian

$$H = \frac{N}{2} \left( \frac{p_\chi^2}{(\cosh \chi)^{4(C-1)}} + \frac{l^2}{(\cosh \chi)^{4(C-1)} \sinh^2 \chi} - 1 \right), \quad (32)$$

where, without loss of generality, we have assumed  $\Theta(\sigma) = \pi/2$ ,  $p_\theta(\sigma) = 0$ , and  $p_\phi = l \geq 0$ , because spherical symmetry is induced on  $(O, |f|h_{ij})$ . Since this Hamiltonian does not depend on  $\phi$ , the quantity  $l$  is a constant of motion and is related to the strength of the conserved angular momentum flux on the world sheet.

The Hamilton equation and the constraint  $H = 0$  yield

$$\mathcal{X}'^2 + V(\mathcal{X}) = 0, \quad (33)$$

$$V = \frac{l^2}{\sinh^2 \chi} - (\cosh \chi)^{4(C-1)}, \quad (34)$$

$$\Phi' = \frac{l}{\sinh^2 \chi}, \quad (35)$$

for which the gauge is fixed by  $N = (\cosh \chi)^{4(C-1)}$ . Equations (33) and (34) give us a one-dimensional problem with the potential  $V$ . The first term of  $V$  is related to the angular momentum flux and makes a potential barrier near  $\chi = 0$ . The second term represents the effect of the cosmic expansion because this includes  $C$  and turns to be flat in the Minkowski background,  $C = 1$ .

Let us classify the behavior of solutions in terms of  $l$  and  $C$ . Since the  $C = 1$  case is analyzed in detail in Sec. III B 1, we examine the case  $C \neq 1$  in what follows.

We consider qualitative properties of a solution with  $l = 0$ . Since  $l = 0$  leads to  $V < 0$  and  $\Phi' = 0$ , we obtain a straight string on each of the constant  $\eta$  slices that passes through  $\chi = 0$  and asymptotically approaches  $\chi = \infty$ . On the other hand, the configuration on each of the constant  $t$  slices is a finite straight segment with the end points moving at the speed of light. We can explicitly see the dependence of string configuration on time slices in Figs. 1(a)–1(c), which show these embeddings into the

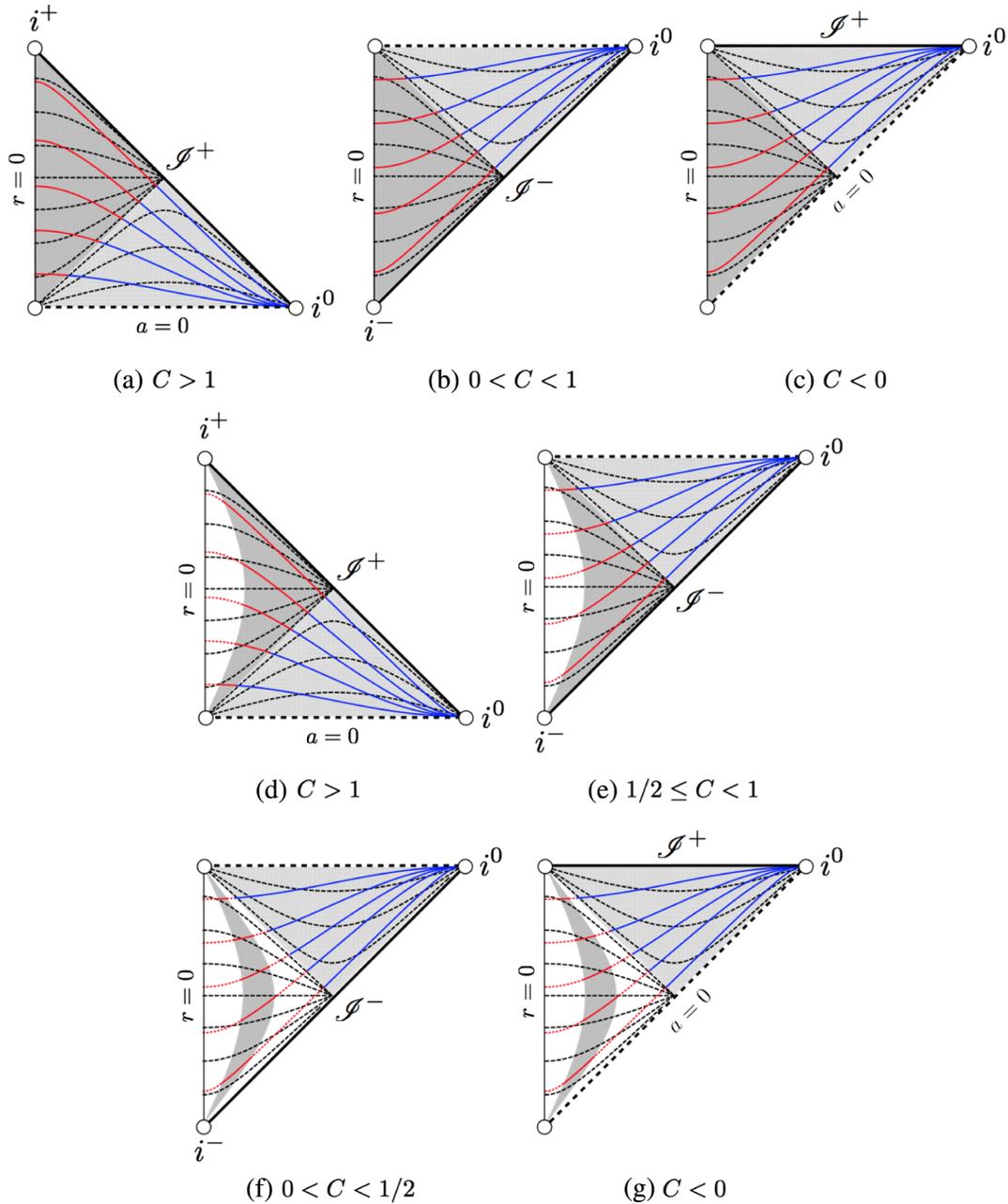


FIG. 1. Embeddings of a self-similar Nambu-Goto string into the conformal diagram of the spatially flat FLRW spacetime with self-similarity. Figures (a)–(c) are the case  $l = 0$  and  $\bar{l} = 0$ , and Figs. (d)–(g) are the case  $l \neq 0$  and  $\bar{l} \neq 0$ , in which  $l$  is restricted to  $0 < l < l_c$  for  $C \leq 1/2$ . The symbols  $i^0$ ,  $i^\pm$ , and  $\mathcal{I}^\pm$  are spatial, timelike, and null infinity, respectively, + and – of which indicate future and past, respectively. The black thick dashed line is a singularity. The dark and light gray regions show a world sheet in Region I and in Region II, respectively. The black thin dashed lines denote the constant  $\eta$  slices in Region I and the constant  $\chi$  slices in Region II. The red and blue solid lines are the constant  $t$  slices.

conformal diagrams of the spatially flat FLRW spacetimes. The dark gray region denotes the world sheet, and black dashed and red solid lines in Region I denote the constant  $\eta$  and constant  $t$  slices, respectively. Note that this solution is eventually identified with a cohomogeneity-one string that possesses spatial homogeneity, and in addition, its analytic continuation to Region II provides a straight line with

infinite length passing through  $r = 0$  on the constant  $t$  slices. As shown in Ref. [14], a straight line solution is stable in linear perturbations. To focus on a self-similar string, we assume  $l > 0$  in what follows.

For  $C > 1/2$  (i.e.,  $w < -5/3$  or  $w > -1/3$ ),  $V$  always has a zero at  $\chi = \chi_*$ , where  $V(\chi_*) = 0$  and is negative for all  $\chi \geq \chi_*$ . It follows that a solution on each of the constant

$\eta$  slices extends from  $\chi = \chi_*$  to  $\chi = \infty$  (null infinity). In Figs. 1(d) and 1(e), the world sheets are depicted with dark gray in  $\chi \geq \chi_*$  of Region I.

For  $C < 1/2$  (i.e.,  $-5/3 < w < -1/3$ ),  $V$  has at most two zeros,  $\chi_{\min}$  and  $\chi_{\max}$ . The necessary and sufficient condition for  $V$  to have zeros is

$$0 < l \leq l_c = \frac{(1-2C)^{(1-2C)/2}}{[2(1-C)]^{1-C}}, \quad (36)$$

where the equality  $l = l_c$  is achieved if  $\chi_{\min} = \chi_{\max}$ . The allowed range of a solution satisfying Eq. (36) is restricted to  $\chi_{\min} \leq \chi \leq \chi_{\max}$ , which implies that a closed string configuration can be included. Figures 1(f) and 1(g) show a string extended over a finite spatial section in Region I. The special case  $\chi_{\min} = \chi_{\max}$  is analyzed in detail in Sec. III B 3.

The remaining is  $C = 1/2$  (i.e.,  $w = -5/3$ ), of which the FLRW universe is filled with phantom energy with special properties (see, for example, [38]). Because  $V$  can be negative for all  $\chi \geq \tanh^{-1} l$ , where  $0 < l < 1$ , a solution on the constant  $\eta$  slices extends from  $\chi = \tanh^{-1} l$  to  $\chi = \infty$  (past null infinity), as can be seen in Fig. 1(e). We are able to obtain an analytical solution in Sec. III B 2.

The end of this section is devoted to reconsidering the dynamical evolution of a self-similar string in Region I in the comoving coordinate system. If we synchronize the string time coordinate to the comoving time  $t$ , then in the comoving spherical-polar coordinate system a radial solution in the proper length is written as  $a(t)r(t, \sigma) = \kappa t \tanh \mathcal{X}(\sigma)$ . Hence, in the comoving Cartesian coordinate system, we have

$$a(t)X^i(t, \sigma) = tQ^i(\sigma), \quad (37)$$

where  $Q^i(\sigma)$  are functions of  $\mathcal{X}(\sigma)$ ,  $\Theta(\sigma)$ , and  $\Phi(\sigma)$ . This means the homothetic scaling of a self-similar string configuration as  $t$  proceeds.

## 2. Region II ( $|\lambda| < r$ )

Let us turn our attention to a self-similar Nambu-Goto string in Region II. We define new coordinates  $\bar{\chi}$  and  $\bar{\eta}$  as

$$\lambda = \kappa e^{\bar{\eta}} \sinh \bar{\chi}, \quad (38)$$

$$r = e^{\bar{\eta}} \cosh \bar{\chi}, \quad (39)$$

where  $0 < \bar{\chi} < \infty$  and  $-\infty < \bar{\eta} < \infty$ , so that Eq. (25) is transformed as

$$ds^2 = e^{2C\bar{\eta}} (\sinh \bar{\chi})^{2(C-1)} (-d\bar{\chi}^2 + d\bar{\eta}^2 + \cosh^2 \bar{\chi} d\bar{\Omega}^2). \quad (40)$$

Since this form of the metric (40) fits into Eq. (3) and also satisfies Condition (ii) in Sec. II B, the dynamics of a

self-similar string associated with the homothetic vector field (22) is determined by the geodesic equation (18). In this coordinate system,  $(\bar{\eta}, \bar{\chi}, \bar{\theta}, \bar{\phi})$ , the embedding functions are written as

$$X^\mu(\tau, \sigma) = (\sigma, \bar{\mathcal{X}}(\tau), \bar{\Theta}(\tau), \bar{\Phi}(\tau)), \quad (41)$$

where we have interchanged  $\tau$  and  $\sigma$  in comparison to Eq. (5) because  $\xi^\mu$  is spacelike in Region II. The embedding functions  $\bar{X}^i(\sigma)$  are determined by the geodesic equation with respect to the metric

$$|f|h_{ij}dx^i dx^j = (\sinh \bar{\chi})^{2(C-1)} (-d\bar{\chi}^2 + \cosh^2 \bar{\chi} d\bar{\Omega}^2). \quad (42)$$

In the same manner as used in Sec. III A 1, we obtain the Hamiltonian for a geodesic in  $(O, |f|h_{ij})$

$$\bar{H} = \frac{\bar{N}}{2} \left( -\frac{p_{\bar{\chi}}^2}{(\sinh \bar{\chi})^{4(C-1)}} + \frac{\bar{l}^2}{(\sinh \bar{\chi})^{4(C-1)} \cosh^2 \bar{\chi}} + 1 \right), \quad (43)$$

where  $\bar{N}$  is a Lagrange multiplier enforcing a constraint, and without loss of generality, we have assumed that  $\bar{\Theta}(\tau) = \pi/2$ ,  $p_{\bar{\theta}}(\sigma) = 0$ , and  $\bar{l} = p_{\bar{\phi}} \geq 0$ . Then the Hamilton equation and the constraint  $\bar{H} = 0$  lead to

$$\dot{\bar{\mathcal{X}}}^2 + \bar{V}(\mathbf{X}) = 0, \quad (44)$$

$$\bar{V} = -\frac{\bar{l}^2}{\cosh^2 \bar{\chi}} - (\sinh \bar{\chi})^{4(C-1)}, \quad (45)$$

$$\dot{\bar{\Phi}} = \frac{\bar{l}}{\cosh^2 \bar{\chi}}, \quad (46)$$

where we have chosen  $\bar{N} = (\sinh \bar{\chi})^{4(C-1)}$  in this gauge. Equations (44) and (45) give us a one-dimensional problem with the potential  $\bar{V}$ . Let us classify the behavior of the solutions in terms of  $\bar{l}$  and  $C$  in what follows.

For  $C > 1$  (i.e.,  $w > -1/3$ ), a self-similar string on each of the constant  $\bar{\chi}$  slices has an end point at  $\bar{\eta} = 0$  on the initial singularity and asymptotically approaches  $\bar{\eta} = \infty$  (spatial infinity). In the limit  $\bar{\chi} \rightarrow 0$ , this string reaches the initial singularity, and in the limit  $\bar{\chi} \rightarrow \infty$ , it approaches the null hypersurface to consist of  $\lambda = r$  and part of future null infinity. In Figs. 1(a) and 1(d), we can see the embedding of such solutions into the conformal diagrams of the spatially flat FLRW spacetimes, where the light gray region denotes the world sheet in Region II. The black thin dashed lines denote the constant  $\bar{\chi}$  slices, and the blue solid lines denote the constant  $t$  slices. These figures explicitly show that a string on the constant  $t$  slices in Region II has an end point at  $r = |\lambda|$ , which moves at the speed of light, and a boundary at spatial infinity.

For  $0 < C < 1$  (i.e.,  $w < -1$ ), a self-similar string on the constant  $\bar{\chi}$  slices has an end point at  $\bar{\eta} = 0$  (big rip singularity), whereas the other asymptotically approaches  $\bar{\eta} = \infty$  (spatial infinity), as seen in Figs. 1(b), 1(e), and 1(f). In the limit  $\bar{\chi} \rightarrow \infty$ , this string approaches the null hypersurface made up of  $\lambda = -r$  and part of past null infinity and encounters the big rip singularity at  $\bar{\chi} = 0$ .

For  $C < 0$  (i.e.,  $-1 < w < -1/3$ ), a solution on the constant  $\bar{\chi}$  slices shows that a string extends to two pieces of spatial infinity, as seen in Figs. 1(c) and 1(g). In the limit  $\bar{\chi} \rightarrow \infty$ , this string reaches the null hypersurface composed of  $\lambda = -r$  and part of the initial null singularity and asymptotically approaches  $\bar{\chi} = 0$  (future null infinity).

For  $\bar{l} = 0$ , a self-similar string results in a cohomogeneity-one string of half line with the end point moving at the speed of light. This analytical continuation to Region I can be an infinite straight line passing through  $r = 0$ . We assume  $\bar{l} > 0$  in what follows.

## B. Analytical solutions

In the following sections, we investigate analytical solutions to describe a self-similar string in an FLRW expanding universe with self-similarity.

### 1. $C=1$ (Minkowski spacetime)

Let us analyze a self-similar Nambu-Goto string associated with  $\xi^\mu$  in  $C = 1$  (i.e., Minkowski spacetime). In Region I, we can obtain a solution from Eqs. (33)–(35) in the form

$$\cosh \mathcal{X} = \sqrt{1 + l^2} \cosh \sigma, \quad (47)$$

$$\cos \Phi = \frac{l}{\sqrt{l^2 + \tanh^2 \sigma}}, \quad (48)$$

where, without loss of generality, we have fixed each integral constant as  $\cosh \mathcal{X}(0) = (1 + l^2)^{1/2}$  and  $\Phi(0) = 0$  and have taken a branch in Eq. (48). In the Cartesian coordinate system, this solution is rewritten in the form  $X^\mu(\tau, \sigma) = (T(\tau, \sigma), X(\tau, \sigma), Y(\tau, \sigma), 0)$ , where

$$T = \kappa e^\tau \sqrt{1 + l^2} \cosh \sigma, \quad (49)$$

$$X = l e^\tau \cosh \sigma, \quad (50)$$

$$Y = e^\tau \sinh \sigma. \quad (51)$$

These functions indeed solve the two-dimensional wave equation  $-\ddot{X}^\mu + X'^{\mu} = 0$ , which is derived from Eq. (7) because the gauge choice of this solution is the conformal gauge,  $\sqrt{-\gamma} \gamma^{ab} = \eta^{ab}$ .

As expected from the fact that Eqs. (47) and (48) are identical to a geodesic equation in  $\mathbb{H}^3$ , the solution

describes a straight line on each of the constant  $t$  slices such that

$$X(t, y) = \kappa \frac{lt}{\sqrt{1 + l^2}}, \quad (52)$$

where we have introduced  $t$  and  $y$  as new string parameters that must satisfy the inequality

$$\left| \frac{y}{t} \right| < \frac{1}{\sqrt{1 + l^2}}. \quad (53)$$

This is required from the fact that  $\xi^\mu$  is timelike in Region I.

Since the string moves in  $x$ -direction uniformly, we can always find the static frame of the string by applying the Lorentz transformation. Hence, without loss of generality, we may choose  $l = 0$ . In the region  $r < -t$ , this solution shows a physical picture as follows: At an initial time, e.g.,  $t = -1$ , the string is placed on the  $y$  axis with the proper length 2, shrinks to the length  $2|t|$  at time  $t$ , which means that the end points approach each other at the speed of light, and finally collapses to a point in the limit  $t \rightarrow 0$ , whereas in the region  $r < t$ , we obtain its time reversal picture.

Let us focus on a self-similar Nambu-Goto string in Region II. We can solve Eqs. (44)–(46) as

$$\sinh \bar{\mathcal{X}} = \sqrt{1 + \bar{l}^2} \sinh \tau, \quad (54)$$

$$\cos \bar{\Phi} = \frac{\bar{l} \tanh \tau}{\sqrt{1 + \bar{l}^2 \tanh^2 \tau}}, \quad (55)$$

where  $0 < \tau < \infty$ , without loss of generality; we have chosen that  $\sinh \bar{\mathcal{X}}(0) = 0$  and  $\cos \bar{\Phi}(0) = 0$  and have taken a branch in Eq. (55). In the Cartesian coordinates, this solution is written as  $X^\mu(\tau, \sigma) = (\bar{T}(\tau, \sigma), \bar{X}(\tau, \sigma), \bar{Y}(\tau, \sigma), 0)$ , where

$$\bar{T} = \kappa e^\sigma \sqrt{1 + \bar{l}^2} \sinh \tau, \quad (56)$$

$$\bar{X} = \bar{l} e^\sigma \sinh \tau, \quad (57)$$

$$\bar{Y} = e^\sigma \cosh \tau. \quad (58)$$

After the reparametrization to  $t$  and  $y$ , the form of this solution coincides with Eq. (52) under identifying  $l$  to  $\bar{l}$ , where these string parameters are restricted by the condition that  $\xi^\mu$  is spacelike,  $|y/t| > 1/(1 + \bar{l}^2)^{1/2}$ . Hence, this solution describes a half line on each constant  $t$  and moves in  $x$ -direction uniformly. The end point moves at the speed of light.

These results explicitly show that the solution obtained in Region I can be analytically continued to Region II through the boundary between these regions. The

maximally extended string shows a straight line, which is cohomogeneity-one, with boundaries at spatial infinity.

## 2. $C = 1/2$

Let us consider a self-similar Nambu-Goto string associated with  $\xi^\mu$  in the expanding universe with  $C = 1/2$ , which is filled with the phantom energy of  $w = -5/3$ . Assuming  $0 < l < 1$ , in which Eq. (33) has a real solution as examined in Sec. III A 1, we obtain a solution in Region I

$$\cosh \mathcal{X} = \sqrt{(1 - l^2)\sigma^2 + \frac{1}{1 - l^2}}, \quad (59)$$

$$\tan \Phi = -\frac{l}{1 - l^2}\sigma^{-1}, \quad (60)$$

where, without loss of generality, we have chosen each integral constant as  $\cosh \mathcal{X}(0) = 1/(1 - l^2)^{1/2}$  and  $\Phi(0) = \pi/2$ . Since  $\mathcal{X} \rightarrow \infty$  as  $\sigma \rightarrow \pm\infty$ , this string on the constant  $\eta$  slices possesses boundaries at  $\chi = \infty$  (past null infinity) and, hence, is an open string with infinite length. In the comoving Cartesian coordinate system, on the other hand, the solution is of the form

$$Y(t, x) = l\sqrt{(t/t_0)^4 - x^2}, \quad (61)$$

where  $t_0 = -2$  in our units, and we have introduced  $t$  and  $x$  as new string parameters. This shows a semiellipse on each constant  $t$  and includes end points at  $(x, y, z) = (\pm(t/t_0)^2, 0, 0)$ , where each segment moves at the speed of light. These two points of view are illustrated in Fig. 1(e), where the dark gray region shows the embedding of this solution, and black dashed and red solid lines denote the constant  $\eta$  and  $t$  slices, respectively.

It is noteworthy that the analytic extension of the solution through  $y = 0$  is an ellipse with the vertices moving at the speed of light. As discussed at the end of Sec. III A 1, the major and minor axes in proper length are proportional to  $t$  and are getting smaller and smaller as  $t$  increases from  $-\infty$  to 0.

Let us turn our attention to a solution in Region II. Integrating Eqs. (44)–(46), we obtain

$$\sinh \bar{\mathcal{X}} = \sqrt{(1 + \bar{l}^2)\tau^2 - \frac{1}{1 + \bar{l}^2}} \quad (62)$$

$$\tan \bar{\Phi} = -\frac{l}{1 + \bar{l}^2}\tau^{-1}, \quad (63)$$

where, without loss of generality,  $\tau_0 < \tau < \infty$ , in which  $\tau_0 = 1/(1 + \bar{l}^2)$ , and we have chosen  $\sinh \bar{\mathcal{X}}(\tau_0) = 0$ ,  $\tan \bar{\Phi}(\tau_0) = -\bar{l}$ , and the branch  $\tan^{-1} \bar{l} < \bar{\Phi} < \pi$ , where  $\pi/2 < \tan^{-1} \bar{l} < \pi$ . Then, the solution describes an open string on the constant  $\bar{\eta}$  slices with boundaries at spatial

infinity. This solution in the comoving Cartesian coordinate system takes the form

$$\bar{\mathcal{X}}(t, y) = -\sqrt{(t/t_0)^4 + y^2/\bar{l}^2}, \quad (64)$$

where  $t$  and  $y$  have been taken to new parameters. Hence, this string on the constant  $t$  slices shows a hyperbola in the second quadrant on  $z = 0$  and includes an end point at  $(x, y, z) = (-(t/t_0)^2, 0, 0)$  moving at the speed of light. In Fig. 1(e), the light gray region shows the world sheet of this solution, on which black dashed and blue solid lines denote the constant  $\chi$  and  $t$  slices, respectively.

The analytic extension of the solution through  $y = 0$  is a half of hyperbola including the vertex with null trajectory. Note that  $\bar{l}$  determines the curvature of the string. For example, the string in the limit  $\bar{l} \rightarrow \infty$  is a straight line, and the one with  $\bar{l} \ll 1$  has high curvature at the vertex.

## 3. $C < 1/2$

Let us investigate the solution that has constant  $\chi$  in the expanding universe with  $C < 1/2$  (i.e.,  $-5/3 < w < -1/3$ ), which is filled with dark or phantom energy. We call this the self-similar circular string in what follows. In the case  $l = l_c$ , where  $l_c$  is defined in Eq. (36), this is realized with the radius  $\chi = \chi_c$ , where

$$\chi_c = \tanh^{-1} \frac{1}{\sqrt{2(1 - C)}}. \quad (65)$$

Substituting  $\chi_c$  and  $l_c$  into Eq. (35), we have the solution  $\Phi(\sigma) = \Phi_c$ , where

$$\Phi_c = \frac{l_c \sigma}{1 - 2C}, \quad (66)$$

where, without loss of generality, we have determined a constant of integration as  $\Phi(0) = 0$ .

The circumferential radius in proper length is

$$L_c = -\frac{Ct}{\sqrt{2(1 - C)}}, \quad (67)$$

where  $Ct < 0$ . This depends linearly on  $t$ , as demonstrated at the last of Sec. III A 1. For  $C < 0$  (i.e.,  $-1 < w < -1/3$ ),  $L_c$  increases as the time  $t$  proceeds because of radially outward initial condition. For  $0 < C < 1/2$  (i.e.,  $-5/3 < w < -1$ ),  $L_c$  decreases as the time  $t$  proceeds because of radially inward initial condition. Furthermore,  $L_c$  satisfies the relation

$$L_c H = \sqrt{\frac{1 - C}{2}}, \quad (68)$$

where  $H = a^{-1} da/dt = (1 - C^{-1})t^{-1}$  is the Hubble parameter. Hence, the size of the self-similar circular string

is at least larger than a half of the Hubble radius, i.e.,  $L_c > H^{-1}/2$ .

Since  $dL_c/dt$  is constant, we can interpret that the self-similar circular string is realized by the balance between the string tension and the effect of the cosmic accelerated expansion. The string tension acts as an attractive force to the self-similar circular string and cancels the effect of the cosmic accelerated expansion acting as a repulsive force to the string. We can find a circular string similar to the self-similar circular string in the de Sitter spacetime (i.e.,  $w = -1$ ), where there exists no proper homothetic vector field. Although the circular string is not a self-similar string, this keeps its proper circumferential radius constant [39], which is realized by the balance between the string tension and the effect of the de Sitter expansion.

In the limit  $C \rightarrow 1/2$ , the world sheet of a self-similar circular string approaches the null surface  $r = -\lambda$ . We could find a self-similar circular null string in the case  $C = 1/2$ , if we generalized our formalism to the case of a null homothetic vector field. For  $C > 1/2$  (i.e.,  $w < -5/3$ ), there is no circular self-similar string as discussed by use of the potential  $V$  in Sec. III A 1. Such a string would be physically forbidden because its world sheet would become spacelike.

To conclude whether this model is physically realistic, we analyze stability of the self-similar circular string under a linear perturbation. Let  $L(t)$  be the proper radius of the circular string that is radially disturbed around  $L_c$  given by Eq. (67) in the form

$$L(t) = L_c(1 + \delta(t)), \quad (69)$$

where  $\delta(t) \ll 1$ . Substitution of Eq. (69) into Eq. (7) yields the linearized equation in  $\delta$

$$\ddot{\delta} + \frac{3\dot{\delta}}{t} - \frac{2(1-2C)}{C^2 t^2} \delta = 0. \quad (70)$$

The perturbation evolves with  $t$  as

$$\delta = \alpha_+(t/t_0)^{-1+\sqrt{1+2(1-2C)/C^2}} + \alpha_-(t/t_0)^{-1-\sqrt{1+2(1-2C)/C^2}}, \quad (71)$$

$$\alpha_{\pm} = \frac{\delta_0}{2} \pm \frac{\delta_0 + \beta_0}{2\sqrt{1+2(1-2C)/C^2}}, \quad (72)$$

where  $\delta_0 = \delta(t_0)$  and  $\beta_0 = t_0\dot{\delta}(t_0)$ . The first term is a growing mode for  $C < 0$  (i.e.,  $-1 < w < 1/3$ ), and the second term is a growing mode for  $0 < C < 1/2$  (i.e.,  $-5/3 < w < -1$ ). Thus, we can conclude that a self-similar circular string is an unstable equilibrium solution.

#### IV. SUMMARY

In this paper, we have proposed a self-similar string in a self-similar spacetime. The self-similar string is defined by

the world sheet to which a homothetic vector field in a self-similar target spacetime is tangent. We have investigated the dynamics on the basis of Nambu-Goto string theory and have demonstrated the equation of motion to be an ordinary differential equation identified with the equation of motion in particle mechanics. The equation further reduces to a geodesic equation in the following cases: (i) The homothetic vector field is a Killing vector field (i.e., a cohomogeneity-one string). (ii) The homothetic vector field is hypersurface orthogonal. (iii) It is the parallel condition [see Eq. (17) for details]. Hence, at least in these cases, we have obtained the formalism for a self-similar string in a similar manner as formulated in the cohomogeneity-one string.

We have applied our formalism to a self-similar string in the Minkowski spacetime or spatially flat FLRW expanding spacetime with self-similarity. In the Minkowski spacetime, a self-similar string becomes a straight segment or line, which is eventually identified with a cohomogeneity-one string. In the expanding spacetime, however, a self-similar string can have nontrivial configuration, which is classified into two types: extended to spacetime boundary and confined in a finite region. The former includes a straight line solution, which also has spatial homogeneity and is linearly stable. The latter includes analytically tractable solutions, so that we have obtained geometrically simple configuration such as an ellipse and a hyperbola in the case where  $C = 1/2$ . In addition, a circular self-similar string for  $C < 1/2$  explicitly provides us instructive pictures. We have found that the solution is realized by the balance between the effect of the cosmic accelerated expansion and the string tension. These kinds of solutions evolve linearly in the cosmic time. Note that, however, a circular self-similar string is an unstable equilibrium solution. The result suggests that all self-similar string confined in a finite region are unstable and cannot be a candidate for a final state.

We are able to investigate a self-similar string in these expanding universes further by using a numerical integration and then obtain many nontrivial configurations. In addition, it will provide a deeper insight to investigate the other choices of a homothetic vector field defining a self-similar string (e.g., with twist) or a target spacetime (e.g., gravitationally collapsing backgrounds). Then, we are able to verify the validity of Condition (iii).

We can generalize this definition by means of the other self-similarity, e.g., kinematic self-similarity. It is interesting for future work to examine self-similar strings in the other string theories or self-similar membranes (which might be a generalization of Ref. [40]).

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