

Classical and quantum cosmology of the little rip abrupt eventImanol Albarran,^{1,2,*} Mariam Bouhmadi-López,^{1,2,3,4,†} Claus Kiefer,^{5,‡} João Marto,^{1,2,§} and Paulo Vargas Moniz^{1,2,||}¹*Departamento de Física, Universidade da Beira Interior, 6200 Covilhã, Portugal*²*Centro de Matemática e Aplicações da Universidade da Beira Interior (CMA-UBI), 6200 Covilhã, Portugal*³*Department of Theoretical Physics University of the Basque Country UPV/EHU, P.O. Box 644, 48080 Bilbao, Spain*⁴*IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain*⁵*Institut für Theoretische Physik, Universität zu Köln, Zùlpicher Strasse 77, 50937 Köln, Germany*

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We analyze from a classical and quantum point of view the behavior of the Universe close to a little rip, which can be interpreted as a big rip sent towards the infinite future. Like a big rip singularity, a little rip implies the destruction of all bounded structures in the Universe and is thus an event where quantum effects could be important. We present here a new phantom scalar field model for the little rip. The quantum analysis is performed in quantum geometrodynamics, with the Wheeler-DeWitt equation as its central equation. We find that the little rip can be avoided in the sense of the DeWitt criterion, that is, by having a vanishing wave function at the place of the little rip. Therefore our analysis completes the answer to the question: can quantum cosmology smoothen or avoid the divergent behavior genuinely caused by phantom matter? We show that this can indeed happen for the little rip, similar to the avoidance of a big rip and a little sibling of the big rip.

DOI: [10.1103/PhysRevD.94.063536](https://doi.org/10.1103/PhysRevD.94.063536)**I. INTRODUCTION**

One of the most challenging problems in theoretical physics is the formulation of a consistent quantum theory of gravity [1,2]. Such a theory is needed not only for conceptual reasons, but also for understanding the origin of the Universe and the structure of black holes. In our paper, we shall deal with quantum cosmology, that is, the application of quantum theory to the Universe as a whole. For this purpose, we shall use the conservative framework called quantum geometrodynamics, with the Wheeler-DeWitt equation as its central equation. This framework is straightforwardly obtained by constructing quantum wave equations from which the Einstein equations can be recovered in the semiclassical (WKB) limit [3].

Besides these fundamental issues, we also encounter the problem of explaining the observed acceleration of the Universe. Phenomenologically, this is done by adding an ingredient called dark energy (DE) [4]. Some of the models describing DE predict the occurrence of singularities beyond big bang (or big crunch) occurring for example in the finite future. Aside from DE singularities, there are also DE abrupt events like the little rip [5–12]. We name them abrupt events rather than singularities because they

occur at an infinite future cosmic time. Some of these models are in accordance with current data [13]. Since the presence of singularities and abrupt events in a theoretical framework is an indication of its breakdown, we expect quantum effects to be important there, too. A central question is then whether those future singularities and abrupt events can be avoided in quantum cosmology or not [14]. This question will also be addressed (and answered) for the models discussed in our paper. Naively, we would expect that at cosmological scales quantum effects are important only in the early Universe, that is, on time scales of the order of the Planck time, t_p , and for distances related to the Planck length l_p . This naive belief is based on the fact that quantum theory is usually important for small systems such as atoms or molecules. Assuming the universality of the superposition principle, quantum effects can occur at any scale, whenever decoherence is negligible. This can happen even for the Universe as a whole, for example, in the case of a classically recollapsing Universe [15], or in cases where singularities or abrupt events are present in the classical theory, as is the case here.

Before proceeding further, we should clarify that among all DE singularities and abrupt events, only three of them are intrinsic to phantom DE, that is, within a relativistic model they happen if and only if suitable phantom matter is present. These are the big rip, the little rip, and the little sibling of the big rip. Consequently, if we want to address the question: *can quantum cosmology smoothen or avoid divergent behaviors caused by phantom matter*, we need to quantize models that induce in the classical picture a big

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rip, a little rip, or a little sibling of the big rip. These questions have been partially addressed in the quantum theory of cosmological models with a big rip [16,17] or a little sibling of the big rip [18]. In this paper, we will complete the answer to these questions by quantizing a classical model for the little rip. In addition and for completeness, we now recall the definition of the big rip, the little rip, and the little sibling of the big rip.

- (i) Big rip singularity: It takes place at a finite cosmic time with an infinite scale factor where the Hubble parameter and its cosmic time derivative diverge [19–26].
- (ii) Little rip: This case corresponds to an abrupt event rather than a future space-time singularity. The radius of the Universe, the Hubble parameter, and its cosmic time derivative all diverge at an infinite cosmic time [5–12]. In addition, all the structures in the Universe would be ripped apart in a finite cosmic time [10]. This kind of behavior was first found in [5] within the context of a four-dimensional modified gravity model and later on in [9] in an induced gravity brane-world model. In [5,9] the little rip was induced by pure geometrical effects. The name little rip was coined in [10], where the abrupt event was induced by matter with the equation of state (2.1). This equation of state was analyzed previously, as far as we know, in [8] (see also [6,7]).
- (iii) Little sibling of the big rip: This case again corresponds to an abrupt event rather than a future space-time singularity. At this event, the Hubble rate and the scale factor blow up but the cosmic derivative of the Hubble rate does not [27]. Therefore, this abrupt event takes place at an infinite cosmic time where the scalar curvature diverges. In addition, even though the event seems to be harmless as it takes place in the infinite future, the bound structures in the Universe would be unavoidably destroyed in a finite cosmic time from now. This was first analyzed in [27] within a classical setup.

Our paper is organized as follows. In Sec. II, we review and summarize some of the known results about the little rip abrupt event from a classical point of view and as induced by a specific equation of state [cf. Eq. (2.1)]. In Sec. III and for later convenience, we introduce as well a scalar field suitable to describe the nowadays late-time acceleration of the Universe and are simultaneously able to induce a little rip asymptotically in the presence and absence of dark matter (DM). In Sec. IV, we present and solve the Wheeler-DeWitt equation for the models given in Secs. II and III. We show in both cases the existence of solutions to the Wheeler-DeWitt equation that avoid the little rip. Finally, in Sec. V we present our conclusions. In addition, we include Appendixes A and B, in which we prove the validity of the approximations used in Sec. IV, and Appendix C, where the Symanzik scaling behavior is

presented as an alternative method to analyze the scalar field eigenstates.

II. A BRIEF REVIEW OF THE LITTLE RIP EVENT

In our paper, we shall employ a Friedmann-Lemaître-Robertson-Walker (FLRW) model with flat spatial sections (that is, choosing $k = 0$). The little rip is obtained in this framework by introducing a perfect fluid with equation of state [8,10]¹

$$p_d = -\rho_d - A\sqrt{\rho_d}, \quad (2.1)$$

where A is a positive constant, and ρ_d and p_d are the energy density and the pressure of this fluid, respectively. The subscript d stands for DE, since the observed acceleration of the Universe could be described by (2.1) [10]. The constant A has the physical dimension of a square root of density and can thus be written as

$$A =: \sqrt{\rho_*}, \quad (2.2)$$

with a characteristic density ρ_* .

Since we are interested in the description of the little rip, and since this event occurs in the infinite future, we are mainly interested in the asymptotic behavior of this model. This can be addressed using standard cosmological equations. We first employ the Friedmann-Lemaître equation

$$H^2 = \frac{8\pi G}{3}\rho, \quad (2.3)$$

where $H \equiv \dot{a}/a$. Here and in the following, we use units with $c = 1$. The energy density can be described as the sum of the contribution of the different matter components in the Universe:

$$\begin{aligned} \rho &= \frac{3H_0^2}{8\pi G} \sum_j \Omega_j(a) \equiv \rho_c \sum_j \Omega_j(a), \\ \Omega_{j0} &= \frac{8\pi G}{3H_0^2} \rho_{j0} \equiv \frac{\rho_{j0}}{\rho_c}, \end{aligned} \quad (2.4)$$

where Ω_j denotes the energy density fraction of each component of the Universe and ρ_c is the critical density; the index 0 means that the corresponding quantity is evaluated at present time. At very late times, we can disregard the contribution of DM to the total energy density; however, it corresponds to a significant part of the present content. Fixing accurate values for the model parameters A and Ω_{d0} requires the imposition of observational constraints. We will assume simply that $0 < \Omega_{d0} < 1$, for the precise value has no effects on our analysis. Within this approximation,

¹For previous work on this kind of abrupt event, see [5–12].

(2.3) contains just a single term corresponding to the energy density of DE.

The second equation that we use is the energy conservation equation, from which one gets [10]

$$\rho_d = \rho_{d0} \left[\frac{3A}{2\sqrt{\rho_{d0}}} \ln\left(\frac{a}{a_0}\right) + 1 \right]^2, \quad (2.5)$$

where a_0 is an integration constant that we set equal to the current size of the Universe. Therefore, after integrating (2.3), the asymptotic behavior of the scale factor with respect to cosmic time reads [10]

$$\frac{a}{a_0} \sim \exp[\beta(e^{at} - 1)], \quad \text{where } \alpha \equiv \sqrt{6\pi G}A, \\ \beta \equiv \sqrt{\frac{\Omega_{d0}}{6\pi G} \frac{H_0}{A}}. \quad (2.6)$$

The little rip event happens for large values of a , where ρ and p blow up, and therefore also H and \dot{H} . Notice that, unlike the big rip, the little rip event is reached in infinite cosmic time.

III. THE LITTLE RIP AS INDUCED BY A SCALAR FIELD

For later convenience, we map the perfect fluid with equation of state (2.1) to a scalar field, ϕ . As the constant A must be positive for (2.1) to induce a little rip, the mapping to a scalar field entails a phantom character for the field. Consequently, we can write the kinetic energy and potential of the scalar field as

$$\dot{\phi}^2 = -(\rho + p), \quad (3.1)$$

$$V = \frac{1}{2}(\rho - p). \quad (3.2)$$

Inserting (2.5) and (2.1) in (3.1), we get

$$\dot{\phi}^2 = A\rho_d^{\frac{1}{2}} = \left| \frac{3A^2}{2} \ln\left(\frac{a}{a_0}\right) + A\sqrt{\rho_{d0}} \right|. \quad (3.3)$$

Introducing the new variable

$$x \equiv \ln\left(\frac{a}{a_0}\right), \quad (3.4)$$

we can express $\dot{\phi}$ as

$$\dot{\phi} = \frac{d\phi}{dx} H. \quad (3.5)$$

We now treat in separate subsections the cases without and with DM.

A. Disregarding dark matter

Using (3.3) and (2.3), we can write

$$d\phi = \frac{\dot{\phi}}{H} dx = \pm \frac{\sqrt{3}}{\kappa} \left(\frac{\Omega_*}{\Omega_{d0}} \right)^{\frac{1}{4}} \frac{dx}{\left| \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right|^{\frac{1}{2}}}, \quad (3.6)$$

where $\kappa^2 \equiv 8\pi G$ and $\Omega_* \equiv (A\kappa/\sqrt{3}H_0)^2 \equiv \rho_*/\rho_c$.² The latter denotes a critical energy density fraction which is related with the model parameter A and quantifies the deviation of a DE model based on (2.1) from the standard Λ CDM model, that is, the smaller Ω_* is, the closer we are to the Λ CDM model. Notice that the expression (3.6) is only valid asymptotically, for we have disregarded the contribution of DM which will redshift quickly in the future and thus become negligible compared to DE. Finally, from integrating (3.6) we find (for $\Omega_* \neq 0$),

$$\phi(x) = \pm \frac{4}{\sqrt{3}\kappa} \left(\frac{\Omega_{d0}}{\Omega_*} \right)^{\frac{1}{4}} \left| \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right|^{\frac{1}{2}} \\ \times \text{sign} \left(\frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right). \quad (3.7)$$

We have chosen the integrations constants, ϕ_* and x_* , such that

$$\phi_* = \pm \frac{4}{\sqrt{3}\kappa} \left(\frac{\Omega_{d0}}{\Omega_*} \right)^{\frac{1}{4}} \left| \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x_* + 1 \right|^{\frac{1}{2}} \\ \times \text{sign} \left(\frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x_* + 1 \right). \quad (3.8)$$

In addition, we have selected x_* to be large enough to ensure the validity of the approximation made in (3.6); that is, we are far enough in the future such that the DM component can be ignored in the Friedmann equation. For practical purpose, we select $x_* = 1.17$, where the matter energy density is 2 orders of magnitude smaller than the DE density. Therefore, x_* is large enough for the Universe to be in an almost total DE domination phase. This numerical value is not crucial for this subsection, but it has to be fixed in the next subsection where numerical calculations are required and therefore a fixed value of x_* is needed. In addition, our results do not change by imposing larger values of x_* . Finally, the function $\text{sign}(x)$ is the sign function, that is

²The Λ CDM model is a homogeneous and isotropic Universe filled with CDM and a cosmological constant which plays the role of DE.

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}. \quad (3.9)$$

As mentioned before, the equation of state shown in Eq. (2.1) describes a deviation from the standard Λ CDM model through the parameter A . Therefore, for a vanishing parameter A , the expected classical trajectory $\phi(x)$ is characterized by a constant, i.e., $d\phi = 0$. This result can be recovered by taking the limit $\Omega_* \rightarrow 0$ in Eq. (3.6); however, after the integration done in Eq. (3.7) the outcome is not well defined for the limit $\Omega_* \rightarrow 0$ (notice that ϕ_* could blow up in this case). To get a suitable expression for small values of Ω_* , we perform a Taylor expansion up to first order of the general integral of Eq. (3.6), which reads

$$\phi(x) - \tilde{\phi}_* \simeq \pm \frac{\sqrt{3}}{\kappa} \left(\frac{\Omega_*}{\Omega_{d0}} \right)^{\frac{1}{4}} (x - \tilde{x}_*), \quad (3.10)$$

where in this case, we have chosen $\tilde{\phi}_*$ and \tilde{x}_* in such way that

$$\tilde{\phi}_* = \pm \frac{\sqrt{3}}{\kappa} \left(\frac{\Omega_*}{\Omega_{d0}} \right)^{\frac{1}{4}} \tilde{x}_*. \quad (3.11)$$

This result will be used later to determine the potential $V(\phi)$.

In the little rip not only does the scale factor get very large, but also the scalar field ϕ ; see Fig. 1. From now on we will focus on this regime.

For the case of $\Omega_* \neq 0$, since the function (3.7) is invertible, we can consequently write $x = x(\phi)$,

$$x = \frac{\kappa^2}{8} \phi^2 - \frac{2}{3} \sqrt{\frac{\Omega_{d0}}{\Omega_*}} \quad \text{for } 0 < \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1, \quad (3.12)$$

$$x = -\frac{\kappa^2}{8} \phi^2 - \frac{2}{3} \sqrt{\frac{\Omega_{d0}}{\Omega_*}} \quad \text{for } \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 < 0. \quad (3.13)$$

Once we have the relation between the potential and the energy density, (3.2), we can write the potential in terms of x , that is

$$V(x) = \rho_{d0} \left[\frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right]^2 + \frac{3H_0^2}{2\kappa^2} \sqrt{\Omega_{d0}\Omega_*} \left| \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right|. \quad (3.14)$$

As can be seen, for a vanishing Ω_* , the potential becomes constant as expected within the Λ CDM paradigm, i.e. $V = \rho_{d0}$. Using Eq. (3.12) in the latter expression, the potential shows a quadratic dependence on the scalar field:

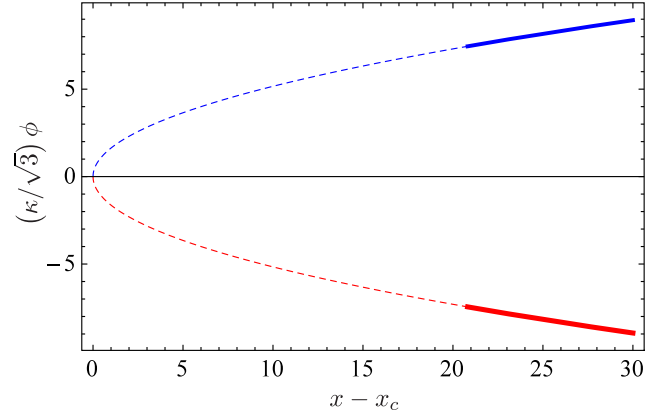


FIG. 1. Plot of the scalar field, ϕ , versus $x \equiv \ln(a/a_0)$ where $x_c = -2\sqrt{\Omega_{d0}}/3\sqrt{\Omega_*}$. This plot is valid for $\Omega_* \neq 0$ since x_c is not well defined for a vanishing Ω_* , i.e. a vanishing A . The solution (3.7) gives two branches, one above $\phi = 0$ (blue) and another below $\phi = 0$ (red). The dashed curve describes a realm where the neglected DM contribution is important, while the solid lines describe a regime where we assume a complete DE domination. We disregard the solutions for $x < x_c$ as our approximation breaks down there. Therefore, only the solid lines are physically relevant for our purpose.

$$V(\phi) = b_1 \phi^4 + b_2 \phi^2, \quad (3.15)$$

where the constants b_1 and b_2 are defined as

$$b_1 \equiv \frac{27}{256} \kappa^2 H_0^2 \Omega_*, \quad b_2 \equiv \frac{9}{32} H_0^2 \Omega_*. \quad (3.16)$$

On the one hand, notice that b_1 has physical dimension of an inverse mass times length (and is thus dimensionless in natural units where $\hbar = 1$ and $c = 1$), while b_2 has the dimension of an inverse length squared (mass squared in natural units). As was mentioned above, (3.7) does not take into account the contribution of DM; therefore, the result shown in Fig. 1 is only valid for very large values of the scale factor.

On the other hand, for a vanishing parameter A the potential given in (3.15) cannot show the expected constant value. This is not surprising as (3.15) was deduced using (3.7) which is not valid for $A = 0$. To recover this solution it is necessary to replace in Eq. (3.14) the expression obtained in Eq. (3.10) for small values of Ω_* . In fact, in that case, we obtain

$$V(\phi) \simeq \rho_{d0} \left[\frac{\sqrt{3}\kappa}{2} \left(\frac{\Omega_*}{\Omega_{d0}} \right)^{\frac{1}{4}} \phi + 1 \right]^2 + \frac{3H_0^2}{2\kappa^2} \sqrt{\Omega_{d0}\Omega_*} \left| \frac{\sqrt{3}\kappa}{2} \left(\frac{\Omega_*}{\Omega_{d0}} \right)^{\frac{1}{4}} \phi + 1 \right|. \quad (3.17)$$

As can be seen from the previous expression when $A \rightarrow 0$, $V(\phi)$ approaches a constant; i.e. the model in this case behaves as Λ CDM.

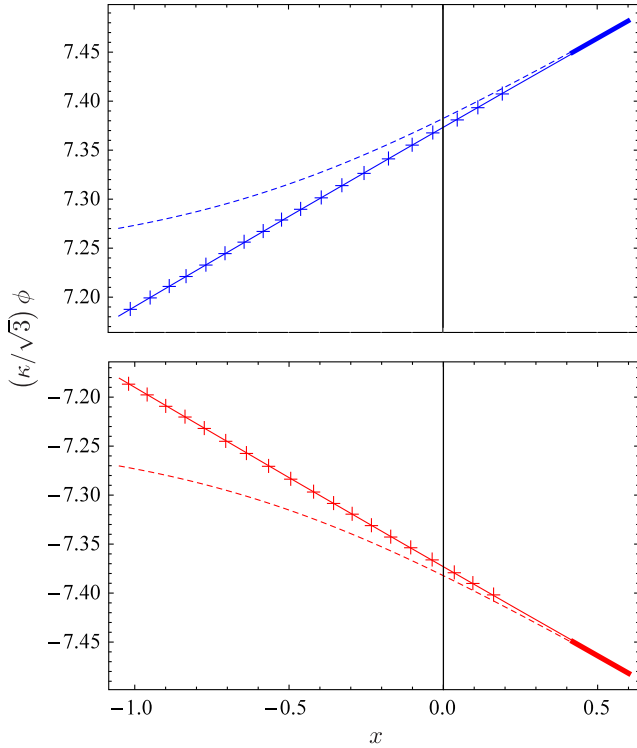


FIG. 2. Plot of the rescaled scalar field, $(\kappa/\sqrt{3})\phi$, versus x , the logarithmic scale factor. The solution (3.19) has two branches which we have drawn as dashed lines in the bottom (red) and upper (blue) panels. These lines take into account DM contribution. The solid blue and red lines correspond to the solution (3.7) where DM is neglected. All the plots have been obtained for $x_* = 1.17$. For practical purpose, we see that for values larger than $x = 0.42$ (i.e. the energy density of DM is 10 times smaller than that of DE), the difference between the two solutions (inclusion of DM and exclusion of DM) is almost negligible. For values of x smaller than $x = 0.42$, the approximated solution starts to show a relevant deviation from the exact solution and we have drawn in this case the approximated solution as a curve with crosses. In addition, we have fixed the other constants as $H_0 = 70.1 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_{m0} = 0.274$, and $A\kappa = 3.46 \times 10^{-3} \text{ Gyr}^{-1}$ according to the best fit obtained in [10].

B. Including dark matter

Just for completeness and to get an accurate solution also for small values of x (but still large enough to be in a matter domination epoch after the radiation dominated epoch), it is necessary to incorporate the DM contribution to the energy density budget of the Universe. Following the same approach we used before, (3.5) can be written as

$$d\phi = \pm \frac{\dot{\phi}}{H} dx = \left\{ \frac{|p_d(x) + \rho_d(x)|}{H^2} \right\}^{\frac{1}{2}} dx. \quad (3.18)$$

The contribution of DM is here included in the Hubble parameter. The equation for $\phi(x)$ is now given by

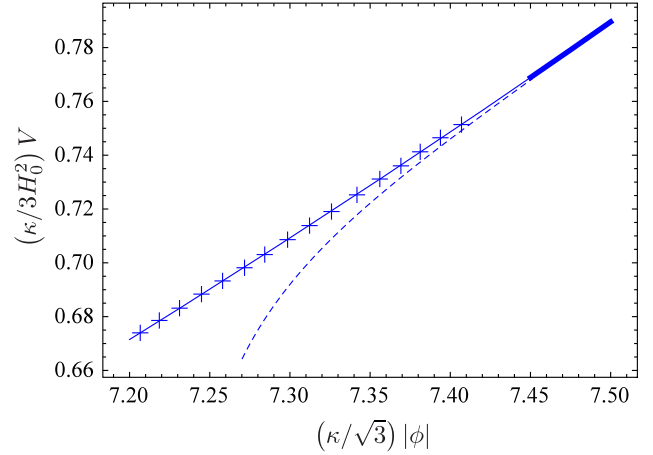


FIG. 3. Plot of the dimensionless potential, $(\kappa/3H_0^2)V$, versus the absolute value of the scaled scalar field, $(\kappa/\sqrt{3})|\phi|$. The dashed curve takes into account the presence of DM, while the solid line neglects it. For consistency with the other plots, we take $x_* = 1.17$. The deviation becomes significant when $(\kappa/\sqrt{3})|\phi| < 7.45$ (drawn as thin curve with crosses), that is, $x < 0.42$, corresponding to the energy density of DM being 10 times smaller than that of DE.

$$\phi(x) = \pm \frac{\sqrt{3}}{\kappa} \int_{x_*}^x \left\{ \frac{\sqrt{\Omega_* \Omega_{d0}} \left| \frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right|}{\Omega_{m0} e^{-3x} + \Omega_{d0} \left(\frac{3}{2} \sqrt{\frac{\Omega_*}{\Omega_{d0}}} x + 1 \right)^2} \right\}^{\frac{1}{2}} dx + \phi_*. \quad (3.19)$$

The integral in (3.19) cannot be solved analytically; therefore, we have performed a numerical integration in which the integration constant ϕ_* was fixed as after (3.7) to the value imposed in Eq. (3.8). In this way, we ensure that the approximated model and the numerical solution are equal at the point x_* as long as x_* is large enough. For practical purpose, we select $x_* = 1.17$, where the matter energy density is 2 orders of magnitude smaller than the DE density. Therefore, x_* is large enough for the Universe to be in an almost total DE domination phase. Figure 2 shows $\phi(x)$.

Once we have obtained the solution for the scalar field, we get the numerical solution for the potential $V(\phi)$, which also takes into account the DM contribution. We compare the obtained potential with the approximated potential (which neglects DM) in Fig. 3.

Because DM is completely negligible at late times, the quantum analysis of the little rip is unaffected by it. We will thus neglect DM from now on.

IV. WHEELER-DEWITT EQUATION

The canonical formulation of general relativity leads to four local constraints. If quantization is performed in the Dirac sense, they turn into the Wheeler-DeWitt (WDW) equation and the quantum diffeomorphism constraints [1]. Making from the outset a FRLW ansatz, as we do here, only

the WDW equation in the form of one partial differential equation remains. To be concrete, the classical metric is (for a flat spatial metric) given by

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) [dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2], \quad (4.1)$$

where N is the lapse function, a is the size of the Universe, and t is the cosmic time. Choosing, in addition, a perfect fluid and a set of minimally coupled scalar fields, the total action consisting of Einstein-Hilbert action and matter action reduces to [1,2,16–18,28]

$$S = \int \mathcal{L} dt, \quad \mathcal{L} = \frac{\mathcal{V}_0}{2\pi} \left[-\frac{3\pi}{4GN} \dot{a}^2 a - 2\pi^2 N a^3 \times \left(\rho - \frac{1}{2N^2} \sum_i l_i \dot{\phi}^2 + V(\phi_1, \dots, \phi_n) \right) \right], \quad (4.2)$$

where $l = 1, (-1)$ for standard (phantom) scalar fields. The constant \mathcal{V}_0 stands for the volume of the three-dimensional spatial sections for $a = 1$. As we are dealing with spatially flat sections, it is implicitly assumed that we either choose a torus compactification with the correct volume or we leave the volume open, that is, choose a reference volume \mathcal{V}_0 . We take the second option. Please note that this leads only to a constant factor multiplying the Lagrangian and, therefore, will not affect the results presented below.

Starting from the Lagrangian (4.2), the conjugate momenta can be calculated and the classical Hamiltonian can be obtained from a Legendre transformation. The classical constraint $\mathcal{H} = 0$ then becomes after quantization [1,2]

$$\hat{\mathcal{H}}\Psi = 0. \quad (4.3)$$

This equation enables an appropriate quantum approach where the wave function of the Universe Ψ depends on the degrees of freedom used to describe the physical system under study, that is, the configuration space. We will address quantization by means of (i) a single degree of freedom corresponding to the scale factor, where the matter content is given by a perfect fluid [with a known equation of state corresponding to (2.1)], and (ii) 2 degrees of freedom corresponding to the scale factor and a scalar field (which portrays the matter content).

In the first approach, the scale factor is the only independent variable. This is certainly a very simple model, but it is interesting enough to study the behavior of the wave function near singularities. In the absence of a full quantum gravity framework it is, of course, an open question what the correct criterion of singularity avoidance is. A useful heuristic criterion is the one introduced by

DeWitt [29] in 1967; it states that the wave function should vanish at the place of the classical singularity. This criterion was successfully applied to a variety of cosmological models; see [1,2,14,17,18,30–32].

In the second approach, an approximation describing the matter content by a scalar field yields a suitable framework with an additional degree of freedom. We move from the classical trajectory, $\phi = \phi(a)$, to the corresponding quantum analog where the wave function is defined over the configuration space (a, ϕ) . In this way, the quantum nature arises and gains significance close to the singularity, once the quantum effects become important. Here, again, the DeWitt criterion is useful as a heuristic device.

A. Wheeler-DeWitt equation with a perfect fluid

In this subsection, we will implement the quantization in the simplest way, which consists in describing the matter content as a perfect fluid with a given equation of state. Therefore, the energy density can be written in terms of the scale factor, which is the single variable within this analysis. The Lagrangian for this model reads [1,2,17,33]

$$\mathcal{L} = -\frac{3\pi}{4GN} \dot{a}^2 a - 2N\pi^2 a^3 \rho(a). \quad (4.4)$$

The conjugate momentum is

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{a}} = -\frac{3\pi}{2G} N a \dot{a}, \quad (4.5)$$

and the Hamiltonian reads

$$\mathcal{H} = -N \frac{G}{3\pi} \frac{\pi_a^2}{a} + 2\pi^2 N a^3 \rho(a). \quad (4.6)$$

For the sake of simplicity, we introduce the new constants

$$\eta \equiv \frac{\pi a_0^3 H_0}{G\hbar}, \quad b \equiv \frac{3}{2} \sqrt{\frac{\Omega_*}{6\Omega_{d0}}}. \quad (4.7)$$

The exact form of the WDW equation depends on the chosen factor ordering. We shall employ two different such orderings in order to study its influence.

1. First quantization procedure: $a\hat{\mathcal{H}}(a, \hat{\pi}_a)\psi(a) = 0$

We choose here [16]

$$\hat{\pi}_a^2 = -\hbar^2 \partial_a^2. \quad (4.8)$$

Employing this in the quantum version of (4.6) and multiplying the result by $[3\pi a/G\hbar^2 N]$, we get

$$\frac{3\pi}{\hbar^2 N} \frac{a}{N} \hat{\mathcal{H}} = \partial_a^2 + \frac{6\pi^3}{\hbar^2 G} a^4 \rho \left(\frac{a}{a_0} \right). \quad (4.9)$$

In order to get a dimensionless WDW equation, we will rescale the scale factor and its partial derivative as

$$u \equiv \frac{a}{a_0}, \quad \partial_u^2 \equiv a_0^2 \partial_a^2. \quad (4.10)$$

After carrying the change of variable introduced above and using (2.5) for the energy density, the WDW equation (4.3) can be written as

$$\left\{ \partial_u^2 + \left(\frac{3}{2} \eta \right)^2 \Omega_{d0} u^4 [1 + \sqrt{6} b \ln(u)]^2 \right\} \Psi_1(u) = 0, \quad (4.11)$$

where we have used the definitions given in (4.7). The approximated WKB solution up to first order reads (see, for example, the method used in [17,18,34] and Appendix A)

$$\begin{aligned} \Psi_1(u) &\approx \Omega_{d0}^{-\frac{1}{4}} \sqrt{\frac{2}{3\eta u}} [1 + \sqrt{6} b \ln(u)]^{-\frac{1}{2}} \\ &\times \{ D_1 e^{i\frac{3\eta}{2} S_0(u)} + D_2 e^{-i\frac{3\eta}{2} S_0(u)} \}, \end{aligned} \quad (4.12)$$

where D_1 and D_2 are constants and

$$\begin{aligned} S_0(u) &= \sqrt{\Omega_{d0}} \int_{u_1}^u y^2 [1 + \sqrt{6} b \ln(y)] dy \\ &= \frac{\sqrt{\Omega_{d0}}}{3} u^3 \left\{ 1 + \sqrt{6} b \left[\ln(u) - \frac{1}{3} \right] \right\} \\ &\quad - \frac{\sqrt{\Omega_{d0}}}{3} u_1^3 \left\{ 1 + \sqrt{6} b \left[\ln(u_1) - \frac{1}{3} \right] \right\}. \end{aligned} \quad (4.13)$$

In addition, u_1 is a large enough constant to ensure not only a positive value of the above integral, but also to guarantee that the system is well inside the DE domination regime, that is, $\ln(u_1) \gg -1/(b\sqrt{6})$. Note that in the quantum treatment, we disregard the contribution of DM by assuming a single component through which the energy density is expressed. This is in full agreement with the fact that by the time the classical abrupt event is approached, DM contribution is negligible; see Sec. III B. From the inspection of (4.12), we see that the wave function vanishes for large values of u . This is exactly the region where in the classical model the little rip takes place. The DeWitt criterion is fulfilled, and the little rip is avoided. It is interesting to note that this criterion is here equivalent to the boundary condition that $\Psi_1 \rightarrow 0$ for $a \rightarrow \infty$ in analogy with the boundary condition usually imposed on the Schrödinger equation for bounded systems.

2. Second quantization procedure (Laplace-Beltrami factor ordering): $\hat{\mathcal{H}}(a, \hat{\pi}_a)\psi(a) = 0$

This quantization procedure is based on the Laplace-Beltrami operator which is the covariant generalization of the Laplacian operator in minisuperspace [1]. The corresponding operator is different depending on the involved degrees of freedom. For the case of a single component described by a perfect fluid, it is written as (cf. for example Ref. [17])

$$\frac{\hat{\pi}_a^2}{a} = -\hbar^2 \left[a^{-\frac{1}{2}} \frac{d}{da} \right] \left[a^{-\frac{1}{2}} \frac{d}{da} \right]. \quad (4.14)$$

To diagonalize the operator, we suggest the following change of variable

$$z \equiv \left(\frac{a}{a_0} \right)^{\frac{3}{2}}, \quad \frac{\hat{\pi}_a^2}{a} = -\frac{9\hbar^2}{4a_0^3} \frac{d^2}{dz^2}. \quad (4.15)$$

Using this operator in the quantum version of (4.6) and multiplying by $[4\pi a_0^3/3G\hbar^2 N]$, we get the following dimensionless expression,

$$\frac{4\pi a_0^3}{3G\hbar^2 N} \hat{\mathcal{H}} = \partial_z^2 + \frac{8\pi^3 a_0^6}{3G\hbar^2} z^2 \rho(z). \quad (4.16)$$

Using (2.5) for the energy density, the fundamental WDW equation given in (4.3) reduces to

$$\left\{ \partial_z^2 + \eta^2 \Omega_{d0} z^2 \left[1 + \sqrt{\frac{8}{3}} b \ln(z) \right]^2 \right\} \Psi_2(z) = 0, \quad (4.17)$$

where the constants η and b are defined in (4.7). The approximated WKB solution up to first order reads (see, for example, [17,34]; for a summary, see also Appendix A)

$$\begin{aligned} \Psi_2(a) &\approx \Omega_{d0}^{-\frac{1}{4}} \sqrt{\frac{1}{\eta}} z^{-\frac{1}{2}} \left[1 + \sqrt{\frac{8}{3}} b \ln(z) \right]^{-\frac{1}{2}} \\ &\times \{ C_1 e^{i\eta Q_0(z)} + C_2 e^{-i\eta Q_0(z)} \}, \end{aligned} \quad (4.18)$$

where C_1 and C_2 are constants and

$$\begin{aligned} Q_0(z) &= \sqrt{\Omega_{d0}} \int_{z_1}^z y \left[1 + \sqrt{\frac{8}{3}} b \ln(y) \right] dy \\ &= \frac{\sqrt{\Omega_{d0}}}{2} z^2 \left\{ 1 + \sqrt{\frac{8}{3}} b \left[\ln(z) - \frac{1}{2} \right] \right\} \\ &\quad - \frac{\sqrt{\Omega_{d0}}}{2} z_1^2 \left\{ 1 + \sqrt{\frac{8}{3}} b \left[\ln(z_1) - \frac{1}{2} \right] \right\}. \end{aligned} \quad (4.19)$$

Like in the first quantization procedure, we assume that z_1 is large enough to ensure a positive value of the above

integral; in fact, it corresponds to the same scale factor u_1 that we used in the previous quantization.

As can be seen, the wave function vanishes for large values of z , where the little rip takes place. Therefore, the DeWitt criterion is again fulfilled; this can be seen as an indication that our results do not depend on the chosen factor ordering.

Before concluding, we would like to highlight that both WKB solutions at first order can be related by

$$\frac{|\Psi_1(u)|^2}{|\Psi_2(z)|^2} = \frac{du}{dz}, \quad \text{where } z = u^{\frac{3}{2}}, \quad (4.20)$$

where the equalities $D_1 = C_1$ and $D_2 = C_2$ have been assumed. At zero order, the two WKB solutions coincide as $(3/2)S_0(u) = Q_0(z)$.

B. Wheeler-DeWitt equation with a phantom scalar field

For a system with a single (phantom) scalar field and a given potential, the quantum Hamiltonian is written as [1,2,16,18,28]

$$\hat{\mathcal{H}} = Na_0^{-3} e^{-3x} \left\{ \frac{\hbar^2}{4\pi^2} \left[\frac{\kappa^2}{6} \partial_x^2 + \partial_\phi^2 \right] + 2\pi^2 a_0^6 e^{6x} V(\phi) \right\}. \quad (4.21)$$

Close to the little rip we can approximate the potential as $V(\phi) \simeq b_1 \phi^4$. Since $\hat{\mathcal{H}}\Psi(x, \phi) = 0$, we have

$$\left\{ \frac{\hbar^2}{4\pi^2} \left[\frac{\kappa^2}{6} \partial_x^2 + \partial_\phi^2 \right] + \sigma e^{6x} \phi^4 \right\} \Psi(x, \phi) = 0, \quad (4.22)$$

where we have gathered all parameters in a single one called σ which reads³

$$\sigma \equiv 2\pi^2 a_0^6 b_1 = \frac{27}{128} \pi^2 a_0^6 \kappa^2 H_0^2 \Omega_*. \quad (4.23)$$

We next apply the following change of variables:

$$\phi = r(z)\varphi, \quad x = z, \quad (4.24)$$

where $r = r(z)$ is a function that only depends on the new variable z . Consequently, we have

³This Wheeler-DeWitt equation can be solved following the method introduced in Refs. [35–37] and, in particular, invoking the Symanzik scaling law. We briefly summarize this method in Appendix C. We thank the referee for pointing out this method to us.

$$\partial_\phi^2 = r^{-2} \partial_\varphi^2,$$

$$\begin{aligned} \partial_x^2 &= \left(\frac{r'}{r} \right)^2 [\varphi^2 \partial_\varphi^2 + \varphi \partial_\varphi] - 2 \frac{r'}{r} \varphi \partial_\varphi \partial_z \\ &+ \left[\left(\frac{r'}{r} \right)^2 - \frac{r''}{r} \right] \varphi \partial_\varphi + \partial_z^2, \end{aligned} \quad (4.25)$$

where the prime stands for derivatives with respect to z . Applying this change of variable and multiplying (4.22) by r^2 , we get

$$\begin{aligned} &\left\{ \frac{\hbar^2 \kappa^2}{24\pi^2} r^2 \left[\left(\frac{r'}{r} \right)^2 [\varphi^2 \partial_\varphi^2 + \varphi \partial_\varphi] - 2 \frac{r'}{r} \varphi \partial_\varphi \partial_z \right. \right. \\ &+ \left. \left. \left[\left(\frac{r'}{r} \right)^2 - \frac{r''}{r} \right] \varphi \partial_\varphi + \partial_z^2 \right] + \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 + \sigma e^{6z} r^6 \varphi^4 \right\} \\ &\times \Psi(z, \varphi) = 0. \end{aligned} \quad (4.26)$$

Now, we choose $r(z) = e^{-z}$ with the aim to leave the potential term with a single dependence on the variable φ ,

$$\begin{aligned} &\left\{ \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} [\varphi^2 \partial_\varphi^2 + \varphi \partial_\varphi + 2\varphi \partial_\varphi \partial_z + \partial_z^2] + \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 + \sigma \varphi^4 \right\} \\ &\times \Psi(z, \varphi) = 0. \end{aligned} \quad (4.27)$$

We next assume that in Eq. (4.27) some terms can be neglected under the presumption

$$\begin{aligned} &\frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} [\varphi^2 \partial_\varphi^2 + \varphi \partial_\varphi + 2\varphi \partial_\varphi \partial_z] \Psi(z, \varphi) \\ &\ll \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} \partial_z^2 \Psi(z, \varphi), \quad \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 \Psi(z, \varphi), \quad \sigma \varphi^4 \Psi(z, \varphi), \end{aligned} \quad (4.28)$$

for large values of z and φ which is the regime where we want to solve the partial differential equation (4.27). This approximation must be justified after obtaining the solutions for $\Psi(z, \varphi)$ (see Appendix B for details). As can be seen, after disregarding these elements in Eq. (4.27) we have two terms whereby each of them depends on a single variable. Therefore, we can employ a separation ansatz, and the wave function can be written as a sum over products of two functions,

$$\Psi(z, \varphi) = \sum_k U_k(\varphi) C_k(z) q_k, \quad (4.29)$$

where q_k denotes the amplitude for each solution and k is a constant related to the “energy” of the system which characterizes the states described through the functions $C_k(z)$ and $U_k(\varphi)$. These functions, in turn, are the solutions of the following differential equations

$$\left\{ \frac{\hbar^2 \kappa^2}{24\pi^2} \partial_z^2 + k e^{2z} \right\} C_k(z) = 0, \quad (4.30)$$

$$\left\{ \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 + \sigma \varphi^4 - k \right\} U_k(\varphi) = 0. \quad (4.31)$$

Equation (4.31) corresponds to the inverted anharmonic oscillator in quantum mechanics; see, for example, [38]. For $C_k(z)$, we get exact solutions corresponding to Bessel functions with vanishing order:

(i) for $k > 0$,

$$C_k(z) = C_{k1} J_0 \left[\frac{2\pi}{\hbar \kappa} \sqrt{6k} e^z \right] + C_{k2} Y_0 \left[\frac{2\pi}{\hbar \kappa} \sqrt{6k} e^z \right], \quad (4.32)$$

(ii) for $k < 0$,

$$C_k(z) = \tilde{C}_{k1} I_0 \left[\frac{2\pi}{\hbar \kappa} \sqrt{6|k|} e^z \right] + \tilde{C}_{k2} K_0 \left[\frac{2\pi}{\hbar \kappa} \sqrt{6|k|} e^z \right], \quad (4.33)$$

where C_{k1} , C_{k2} , \tilde{C}_{k1} , and \tilde{C}_{k2} are constants. Since the functions $I_0(z)$ diverge for $z \rightarrow \infty$ [39], we choose $\tilde{C}_{k1} = 0$ to ensure that the wave function vanishes close to the little rip. For large values of z , we then get

(i) for $k > 0$,

$$C_k(z) \sim \left[\frac{\hbar^2 \kappa^2}{6\pi^4 k} \right]^{\frac{1}{4}} e^{-\frac{z}{2}} \left\{ C_{k1} \cos \left(\frac{2\pi}{\hbar \kappa} \sqrt{6|k|} e^z - \frac{\pi}{4} \right) + C_{k2} \sin \left(\frac{2\pi}{\hbar \kappa} \sqrt{6|k|} e^z - \frac{\pi}{4} \right) \right\}, \quad (4.34)$$

(ii) for $k < 0$,

$$C_k(z) \sim \tilde{C}_{k2} \left[\frac{\hbar^2 \kappa^2}{96k} \right]^{\frac{1}{4}} e^{-\frac{z}{2}}. \quad (4.35)$$

The second order differential equation for $U_k(\varphi)$ is more difficult to solve. Disregarding the constant term k in (4.31), which is equivalent to finding the solution for $k = 0$, it can be written as (see the Appendix A)⁴

$$U(\varphi) = \sqrt{\varphi} \left\{ U_1 J_{\frac{1}{6}} \left[\frac{2\pi\sqrt{\sigma}}{3\hbar} \varphi^3 \right] + U_2 J_{-\frac{1}{6}} \left[\frac{2\pi\sqrt{\sigma}}{3\hbar} \varphi^3 \right] \right\}, \quad (4.36)$$

where U_1 and U_2 are integration constants. For large values of φ , we have

⁴Naively, we expect k to be irrelevant close to the little rip where φ gets very large values; see below for a rigorous justification of this observation.

$$U(\varphi) \sim \sqrt{\frac{6\hbar}{2\pi^2 \sigma^{\frac{1}{2}} \varphi}} \left\{ U_1 \cos \left(\frac{2\pi\sqrt{\sigma}}{3\hbar} \varphi^3 - \frac{\pi}{3} \right) + U_2 \sin \left(\frac{2\pi\sqrt{\sigma}}{3\hbar} \varphi^3 - \frac{\pi}{3} \right) \right\}, \quad (4.37)$$

and the wave function vanishes asymptotically. It is worth noticing that for small values of the argument $(2\pi\sqrt{\sigma}/3\hbar)\varphi^3$ in Eq. (4.37) we have

$$U(\varphi) \sim U_1 \left(\frac{\pi\sqrt{\sigma}}{3\hbar} \right)^{\frac{1}{6}} \frac{\varphi}{\Gamma(\frac{7}{6})} - U_2 \left(\frac{\pi\sqrt{\sigma}}{3\hbar} \right)^{-\frac{1}{6}} \frac{\Gamma(\frac{1}{6})}{\pi}. \quad (4.38)$$

This limit seems to correspond to a regime where σ (which is proportional to the parameter Ω_* , i.e. quadratic in A^2) is small enough to ensure infinitesimal values of the argument in Eq. (4.36) even for large values of φ . It turns out that the term proportional to U_2 in Eq. (4.38) is not well defined when σ or A vanishes. This might indicate that the wave function for the Λ CDM universe is not well defined. However, this is not the case because when σ or A approaches zero $V(\varphi)$ should be the one given in Eq. (3.17) rather than what we used and defined in Eq. (3.15). In addition, the solution (4.38) was obtained after disregarding the term k in Eq. (4.31) which cannot be ignored in the case of small $\sigma\varphi^4$.

After performing the approximation ($k \ll [4\pi^2\sigma/\hbar^2]\varphi^4$) in (4.31), we can find an exact solution, but in return, we lose the information of k in $U_k(\varphi)$. A simple way to obtain an approximated wave function keeping the contribution of k is via the WKB approximation; the expression for the approximated wave function up to first order is given by

$$U(\varphi) \simeq \left[\frac{4\pi^2}{\hbar^2} (\sigma\varphi^4 - k) \right]^{-\frac{1}{4}} \{ U_{k1} e^{iS_0(\varphi)} + U_{k2} e^{-iS_0(\varphi)} \}, \quad (4.39)$$

where U_{k1} and U_{k2} are constants and

$$S_0(\varphi) = \frac{2\pi}{\hbar} \int_{\varphi_1}^{\varphi} \sqrt{\sigma y^4 - k} dy, \quad (4.40)$$

where φ_1 is large enough to ensure a purely real solution even for positive values of k ($0 < \sigma\varphi_1^4 - k$). The latter integral can be expressed as follows (see pp. 128 and 129 of [40]):

(i) for $0 < k$,

$$\int_{\varphi_1}^{\varphi} \sqrt{\sigma y^4 - k} dy = \frac{2\pi}{3\hbar} \left\{ y(\sigma y^4 - k)^{\frac{1}{2}} - \sqrt{2} \frac{k^{\frac{3}{2}}}{\sigma^{\frac{1}{4}}} F \times \left[\arccos \left(\frac{k^{\frac{1}{4}}}{\sigma^{\frac{1}{4}} y} \right), \frac{1}{\sqrt{2}} \right] \right\} \Big|_{\varphi_1}^{\varphi}, \quad (4.41)$$

(ii) for $k < 0$,

$$\begin{aligned} & \int_{\varphi_1}^{\varphi} \sqrt{\sigma y^4 - k} dy \\ &= \frac{2\pi}{3\hbar} \left\{ y(\sigma y^4 + |k|)^{\frac{1}{2}} + \frac{|k|^{\frac{3}{2}}}{\sigma^{\frac{1}{4}}} F \right. \\ & \quad \left. \times \left[\arccos \left(\frac{\sqrt{|k|} - \sqrt{\sigma} y^2}{\sqrt{|k|} + \sqrt{\sigma} y^2} \right), \frac{1}{\sqrt{2}} \right] \right\} \Big|_{\varphi_1}^{\varphi}, \end{aligned} \quad (4.42)$$

where the function $F[h(y), d]$ is an elliptic integral of the first kind with argument $h(y)$ and elliptic modulus d . Note that for $k = 0$ we recover the asymptotic solution given by the Bessel functions (4.36). For large values of φ the performed WKB approximation and the found Bessel functions have the same asymptotic behavior, in this limit, no matter what the value of k is. Therefore, for very large values of φ we can write

$$\Psi(z, \varphi) \simeq U(\varphi) \sum_k C_k(z) q_k. \quad (4.43)$$

In any case, the resulting wave function has two oscillatory terms modulated by a function which goes to zero for large values. Returning to the initial variables for $z \rightarrow \infty$ and $\varphi \rightarrow \infty$ limits the wave function decreases as

$$\Psi(x, \phi) \sim [\phi e^{\frac{3}{2}x}]^{-1}. \quad (4.44)$$

Therefore, the wave function vanishes close to the little rip, fulfilling the DeWitt boundary condition.

V. CONCLUSIONS

A central issue in any theory of quantum gravity is the avoidance of classical singularities. At the present state of the field, this cannot be done in any sense close to the rigor of the classical singularity theorems. The hope is thus to get some insight from suitable models for which concrete results can be obtained. As a heuristic sufficient (though not necessary) criterion of singularity avoidance, one can employ the DeWitt criterion of the vanishing wave function. The applicability of this criterion has already been studied for a wide class of classical singularities. In the present paper, we have completed the discussion by studying the situation of the little rip, which is strictly speaking not a singularity, like it is the case of a big rip, but an abrupt event, though it shares some features with it. We have studied the two situations of a perfect fluid and of a phantom scalar field, the first being a phenomenological, the second a more fundamental dynamical model. A phantom field (field with negative kinetic energy) is needed in order to implement the equation of state leading to a little rip. We have found that the DeWitt criterion can indeed be applied in both

cases and that the little rip can thus be avoided. We should emphasize that models such as these, although looking purely academic at first glance, are supported by data [13]. If indeed true, the future of the Universe would end in a full quantum era (without classical observers), in full analogy to its quantum beginning.

Some words about the applicability of the DeWitt criterion [[29], Eq. (6.31)] are in order. It is based on the heuristic extrapolation of the quantum mechanical probability interpretation (based on the Schrödinger inner product) to quantum cosmology. But since the Wheeler-DeWitt equation is of hyperbolic nature (with and without matter), and thus resembles a Klein-Gordon equation, one might think that a Klein-Gordon inner product would be more appropriate. This is, however, not the case, because it was proven that one cannot separate positive and negative frequencies in the Wheeler-DeWitt equation, and thus one is faced with the problem of negative probabilities; see, for example, [1], Sec. 5.2.2 for a discussion and references. This problem can perhaps be avoided by going to “third quantization,” but this is a framework different from the present one. Our point of view here is that an inner product of the Schrödinger type can be used in quantum cosmology, even if the situation in the full theory is unclear⁵ and even if this poses the danger of not allowing normalizable solutions. At least in the models hitherto considered, this inner product can be implemented and the DeWitt criterion can be applied.

In our paper, we have restricted ourselves to the minisuperspace approximation. The real Universe is, however, not homogeneous, so one possible extension of our work is the inclusion of (scalar and tensor) perturbations and solving the WDW equation near and at the region of the little rip. The full quantum state then describes an entanglement between the minisuperspace part and the perturbations. Tracing out the perturbation part from the full state leads to a density matrix ρ for the minisuperspace part. If the interaction with the perturbations leads to a suppression of the off-diagonal elements in ρ , one can interpret this as an effective quantum-to-classical transition or decoherence for the background. Decoherence in quantum cosmology was discussed in detail for many situations; see, for example, [41] and the references therein. One might expect that close to the little rip region, decoherence stops and quantum interferences become important, enabling the DeWitt criterion to be fulfilled there, as discussed in our paper. Genuine quantum effects have also shown to be important near the turning point of a classically recollapsing universe [15]. We hope to address these and other issues in future publications.

⁵The formalism of full loop quantum gravity, for example, employs a Schrödinger inner product.

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APPENDIX A: WKB APPROXIMATION

We next review briefly the WKB method for second order differential equation

$$\left[\frac{d^2}{dy^2} + V_{\text{eff}}(y) \right] \psi(y) = 0, \quad (\text{A1})$$

where y is defined such that it is a dimensionless degree of freedom and where the effective potential can be written as

$$V_{\text{eff}}(y) = \tilde{\eta}^2 g(y). \quad (\text{A2})$$

Moreover, $\tilde{\eta}$ is a dimensionless parameter related with the constants of the system. The general expression for the WKB approximated solution (up to first order) reads [34]

$$\psi(y) \approx [-\tilde{\eta}^2 g(y)]^{-\frac{1}{4}} [B_1 e^{iS_0(y)} + B_2 e^{-iS_0(y)}], \quad (\text{A3})$$

where B_1 and B_2 are constants and

$$S_0(y) = \tilde{\eta} \int_{y_1}^y \sqrt{g(y)} dy. \quad (\text{A4})$$

The solution (A3) is valid as long as the inequality

$$\frac{1}{\tilde{\eta}^2} \left| \frac{5\dot{g}^2(y) - 4\ddot{g}(y)g(y)}{16g^3(y)} \right| \ll 1 \quad (\text{A5})$$

$$\frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} [\varphi^2 \partial_\varphi^2 + \varphi \partial_\varphi + 2\varphi \partial_\varphi \partial_z] C(z) U(\varphi) \ll \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} \partial_z^2 C(z) U(\varphi), \quad \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 C(z) U(\varphi), \quad \sigma \varphi^4 C(z) U(\varphi). \quad (\text{B1})$$

As a result of the realized approximation, the last two terms on the rhs of the above inequality have the same order of magnitude for large values of z and φ or ϕ and x [see Eq. (B2)]. In fact, the dominant terms that we keep read

$$\begin{aligned} \frac{\hbar^2}{4\pi^2} \partial_\varphi^2 C(z) U(\varphi), \quad \sigma \varphi^4 C(z) U(\varphi) &\sim \frac{\hbar}{2\pi} \left(\frac{\kappa^2}{6k} \right)^{\frac{1}{4}} \sigma^{\frac{3}{4}} \varphi^3 e^{-\frac{1}{2}z} = \frac{\hbar}{2\pi} \left(\frac{\kappa^2}{6k} \right)^{\frac{1}{4}} \sigma^{\frac{3}{4}} \phi^3 e^{\frac{5}{2}x}, \\ \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} \partial_z^2 C(z) U(\varphi) &\sim \frac{\hbar}{2\pi} \left(\frac{\kappa^2 k^3}{6} \right)^{\frac{1}{4}} \sigma^{-\frac{1}{4}} \varphi^{-1} e^{-\frac{1}{2}z} = \frac{\hbar}{2\pi} \left(\frac{\kappa^2 k^3}{6} \right)^{\frac{1}{4}} \sigma^{-\frac{1}{4}} \phi^{-1} e^{-\frac{3}{2}x}, \end{aligned} \quad (\text{B2})$$

while the neglected terms evolve asymptotically as

is fulfilled. When the left-hand side of Eq. (A5) goes to zero we can be sure that the behavior of the exact solution in this regime matches almost perfectly with the WKB approximation. The approximations used in Sec. IV for the first and the second quantization procedure corresponds to an effective potential whose general shape reads

$$V_{\text{eff}}(y) \equiv \tilde{\eta} y^n [1 + \gamma \ln(y)], \quad (\text{A6})$$

where for each quantization procedure we have the

(i) first quantization procedure,

$$\tilde{\eta} = \frac{2}{3} \sqrt{\Omega_{d0} \eta}, \quad n = 2, \quad \gamma = \sqrt{6} b; \quad (\text{A7})$$

(ii) second quantization procedure,

$$\tilde{\eta} = \sqrt{\Omega_{d0} \eta}, \quad n = 1, \quad \gamma = \sqrt{\frac{8}{3}} b. \quad (\text{A8})$$

The necessary condition for the approximation to be valid reads

$$\begin{aligned} \frac{1}{\tilde{\eta}^2} \left| \frac{1}{16y^{n+2} [1 + \gamma \ln(y)]^2} \left\{ n^2(9 - 4n) + \frac{4\gamma n(1 - 6n - 2\gamma)}{[1 + \gamma \ln(y)]} \right. \right. \\ \left. \left. + \frac{\gamma^2(36 - 32n - 16\gamma)}{[1 + \gamma \ln(y)]^2} \right\} \right| \ll 1. \end{aligned} \quad (\text{A9})$$

For both of the cases mentioned above, the little rip occurs for large values of the variable y , where the latter expression goes to zero when $y \rightarrow \infty$; i.e. the WKB approximated solution is valid.

APPENDIX B: JUSTIFICATION FOR THE APPROXIMATION DONE IN EQ. (4.27)

The approximation done for the differential equation (4.27) consists of disregarding the first three terms after the change of variable realized in (4.24). Once these terms are neglected, the resulting differential equation is separable and the approximation is valid if

$$\begin{aligned}
 \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} \varphi^2 \partial_\varphi^2 C(z) U(\varphi) &\sim \frac{\hbar}{12\pi} \left(\frac{\kappa^{10}}{6k}\right)^{\frac{1}{4}} \sigma^{\frac{3}{4}} \varphi^5 e^{-\frac{5}{2}z} = \frac{\hbar}{12\pi} \left(\frac{\kappa^{10}}{6k}\right)^{\frac{1}{4}} \sigma^{\frac{3}{4}} \phi^5 e^{\frac{5}{2}x}, \\
 \frac{\hbar^2 \kappa^2}{24\pi^2} e^{-2z} \varphi \partial_\varphi C(z) U(\varphi) &\sim \frac{\hbar^2}{24\pi^2} \left(\frac{\kappa^{10}}{6k}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \varphi^2 e^{-\frac{3}{2}z} = \frac{\hbar^2}{24\pi^2} \left(\frac{\kappa^{10}}{6k}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \phi^2 e^{-\frac{1}{2}x}, \\
 \frac{\hbar^2 \kappa^2}{12\pi^2} e^{-2z} \varphi \partial_\varphi \partial_z C(z) U(\varphi) &\sim \frac{\hbar}{6\pi} \left(\frac{3\kappa^6 k}{8}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \varphi^2 e^{-\frac{3}{2}z} = \frac{\hbar}{6\pi} \left(\frac{3\kappa^6 k}{8}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \phi^2 e^{\frac{1}{2}x}.
 \end{aligned} \tag{B3}$$

Therefore, in order to obtain the compliance region of the realized approximation we compare the smallest of the saved terms with the largest between the neglected ones, that is

$$\begin{aligned}
 \frac{\hbar}{12\pi} \left(\frac{\kappa^{10}}{6k}\right)^{\frac{1}{4}} \sigma^{\frac{3}{4}} \phi^5 e^{\frac{5}{2}x} &\ll \frac{\hbar}{2\pi} \left(\frac{\kappa^2 k^3}{6}\right)^{\frac{1}{4}} \sigma^{-\frac{1}{4}} \phi^{-1} e^{-\frac{3}{2}x}, \\
 \Rightarrow \frac{\kappa^2 \sigma}{6k} \phi^6 e^{4x} &\ll 1.
 \end{aligned} \tag{B4}$$

Finally, the realized approximation is valid as long as $(\kappa^2 \sigma / 6k) \phi^6 e^{4x} \ll 1$. This means that for sufficiently small values of σ , i.e. for small values of A , which is indeed the observationally preferred situation, and large values of x , i.e. $x \gg 1$ (but finite), the approximation we have used is valid.

APPENDIX C: SCALAR FIELD EIGENSTATES AND SYMANZIK SCALING BEHAVIOR

In this appendix we analyze Eq. (4.22) in the context of the Symanzik scaling law, following the results found in [35–37]. We start by performing, in the aforementioned equation, the following change of variables,

$$x = c_1 \bar{x}, \quad \phi = c_2 \bar{\phi}, \tag{C1}$$

where c_1 and c_2 are constants. We obtain

$$\left\{ \frac{\hbar^2}{4\pi^2} \left[\frac{\kappa^2}{6c_1^2} \partial_{\bar{x}}^2 + \frac{1}{c_2^2} \partial_{\bar{\phi}}^2 \right] + \sigma e^{6c_1 \bar{x}} c_2^4 \bar{\phi}^4 \right\} \Psi(\bar{x}, \bar{\phi}) = 0, \tag{C2}$$

where by imposing

$$c_1 = \frac{\hbar}{2\sqrt{6\pi}} \kappa, \quad c_2 = i \frac{\hbar}{2\pi}, \tag{C3}$$

we get

$$\left[\partial_{\bar{x}}^2 - \partial_{\bar{\phi}}^2 + \frac{\hbar^4}{16\pi^4} \sigma e^{\frac{\sqrt{6}\hbar\kappa\bar{x}}{2\pi}} \bar{\phi}^4 \right] \Psi(\bar{x}, \bar{\phi}) = 0, \tag{C4}$$

⁶Small values of A imply small deviations of our model from the Λ CDM scenario.

which is precisely given by Eq. (1) of Ref. [35]. As done in [35], we conclude that the general solutions of (C4) can be expressed as

$$\Psi(\bar{x}, \bar{\phi}) = \sum_{n=0}^{+\infty} A_n(\bar{x}) \Phi_n(\bar{x}, \bar{\phi}), \tag{C5}$$

which can be rewritten in a vectorial notation as

$$\Psi(\bar{x}, \bar{\phi}) = \Phi^T(\bar{x}, \bar{\phi}) \cdot \mathbf{A}(\bar{x}), \tag{C6}$$

where the scalar part wave functions satisfy

$$\left[\partial_{\bar{\phi}}^2 + E_n(\bar{x}) - \frac{\hbar^4}{16\pi^4} \sigma e^{\frac{\sqrt{6}\hbar\kappa\bar{x}}{2\pi}} \bar{\phi}^4 \right] \Phi_n(\bar{x}, \bar{\phi}) = 0. \tag{C7}$$

Notice that the scalar part solutions depend on the scale factor. Using the Symanzik scaling law [35–37], we have that

$$\begin{aligned}
 \Phi_n(\bar{x}, \bar{\phi}) &= \left[\frac{\hbar^4}{16\pi^4} \sigma e^{\frac{\sqrt{6}\hbar\kappa\bar{x}}{2\pi}} \right]^{\frac{1}{12}} f_n(\bar{\chi}), \\
 E_n(\bar{x}) &= \left[\frac{\hbar^4}{16\pi^4} \sigma e^{\frac{\sqrt{6}\hbar\kappa\bar{x}}{2\pi}} \right]^{\frac{1}{3}} \varepsilon_n,
 \end{aligned} \tag{C8}$$

where

$$\bar{\chi} = \left[\frac{\hbar^4}{16\pi^4} \sigma e^{\frac{\sqrt{6}\hbar\kappa\bar{x}}{2\pi}} \right]^{\frac{1}{12}} \bar{\phi}. \tag{C9}$$

Furthermore, the vectorial scalar field wave equation can be used to define a coupling matrix Ω as [35–37]

$$\frac{\partial \Phi}{\partial \bar{x}} = \Omega \Phi(\bar{x}, \bar{\phi}). \tag{C10}$$

Given that $\{\Phi_n\}$ are an orthonormal basis, we can conclude that

$$\Omega_{mn} = \frac{\varepsilon_m - \varepsilon_n}{4} \int d\bar{\chi} \bar{\chi}^2 f_m(\bar{\chi}) f_n(\bar{\chi}). \tag{C11}$$

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