PHYSICAL REVIEW D **94,** 063531 (2016)

Cosmological power spectrum in a noncommutative spacetime

Rahul Kothari,[1,*] Pranati K. Rath,[2,†] and Pankaj Jain[1,‡]

[1]*Department of Physics, Indian Institute of Technology, Kanpur 208016, India*

[2]*Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China*

(Received 16 February 2016; published 29 September 2016)

We propose a generalized star product that deviates from the standard one when the fields are considered at different spacetime points by introducing a form factor in the standard star product. We also introduce a recursive definition by which we calculate the explicit form of the generalized star product at any number of spacetime points. We show that our generalized star product is associative and cyclic at linear order. As a special case, we demonstrate that our recursive approach can be used to prove the associativity of standard star products for same or different spacetime points. The introduction of a form factor has no effect on the standard Lagrangian density in a noncommutative spacetime because it reduces to the standard star product when spacetime points become the same. We show that the generalized star product leads to physically consistent results and can fit the observed data on hemispherical anisotropy in the cosmic microwave background radiation.

DOI: 10.1103/PhysRevD.94.063531

I. INTRODUCTION

A remarkable prediction of quantum gravity is that spacetime might be noncommutative at the Planck scale. The basic idea is that in order to probe short distances, we require higher energies. However, at sufficiently high energy, we shall necessarily form black holes and, hence, lose precision about spacetime coordinates. This idea imposes some uncertainty relationships among different coordinates which can be implemented by proposing that these coordinates do not commute [1–6]. It has been argued that this noncommutativity of coordinates might have interesting implications for cosmology [7–18]. In particular, the power spectrum generated during inflation could be modified and may lead to signatures of non-Gaussianity [17–19].

The noncommutative model is rather interesting since it has the potential [19–21] to explain the observed hemispherical anisotropy in cosmic microwave background radiation (CMBR) [22–30]. This is because it produces a dipolar term in the primordial power spectrum. Such an anisotropic model cannot arise in the standard commutative spacetime. Assuming that it is possible to generate the right form of the dipolar power spectrum starting from a noncommutative model, which leads to physically acceptable results, the consequences are striking. It implies that the shortest-distance, perhaps Planck-scale, physics associated with the noncommutativity of spacetime may currently be probed at the largest-distance scales in the Universe. Furthermore, anisotropies (or inhomogeneities) at very early times may be observable today as anisotropies on the largest-distance scales [31,32] and might be responsible for some of the observed violations of the cosmological principle [22–30,33–39].

The hemispherical anisotropy is parametrized in terms of the phenomenological dipole modulation model [40–44]. It has been argued [19–21] that the power spectrum obtained in [18] is not acceptable because it produces imaginary correlations among temperature spherical harmonic coefficients, a_{lm}'s, while they should be real. Clearly there is something wrong with the power spectrum obtained in [18]. Some solutions to this problem have already been proposed in Refs. [19,21]. However these do not really solve the problem. In particular the prescription given in [19] requires us to define the expectation value of different parts of an operator differently. It is not clear how such a prescription might emerge from a fundamental framework. Reference [21] instead suggests that we should take a different product while computing the power spectrum. While this is permissible, it is *ad hoc*. It provides no theoretical justification for why a different product is used in the calculation of the power spectrum. In the present paper we examine some of the assumptions that go into the calculation of the power spectrum and, subsequently, the temperature correlations. This might suggest a generalization of the standard star product that may lead to an acceptable cosmological power spectrum.

In their calculation of temperature correlations, the authors of [17,18] assume that the transfer function which relates the power spectrum in the early Universe is approximately the same as that assumed in commutative spacetimes. This is reasonable because by the end of inflation all effects of noncommutativity are expected to be negligible. Hence, the evolution can be well approximated by neglecting the effects of noncommutativity.

The power spectrum in [17,18] is obtained by assuming that all products in a noncommutative spacetime must be taken as star products. This is also a reasonable assumption since a star product implements the basic commutation relation among different coordinates, given by [1,2,4–6],

*rahulko@iitk.ac.in
†pranati@iopb.res.in
‡pkjain@iitk.ac.in

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}. \quad (1)$$

Here the parameter $\Theta^{\mu\nu}$ is antisymmetric and the coordinate functions, $\hat{x}^\mu(x)$, depend on the choice of coordinate system. Different choices will lead to different models of noncommutative spacetime. The authors of [17,18] consider a scalar field theory in a background expanding Universe. The coordinates $\hat{x}^\mu$ are taken to be the comoving coordinates. They compute the two-point correlations of the scalar field, ϕ, by assuming that their product can be taken as a star product.

We next point out that imposing the commutation relations on comoving coordinates is simply a model. One can consider generalizations of it. Later in this section, we explicitly discuss the deviation in this product if we choose to impose the basic commutator, Eq. (1), on coordinates different than comoving coordinates. Furthermore, it is also possible to have different product rules, such as the Wick-Voros product rule, which lead to different field theories [45]. Finally, in our analysis we require the correlation function,

$$\Delta(\vec{x}, \vec{x}') = \langle 0|\phi(\vec{x}, t) \star \phi(\vec{x}', t)|0\rangle, \quad (2)$$

i.e., the product of fields at two different spatial positions. It is precisely this correlation that leads to the problem with the cosmological power spectrum mentioned above. If we treat coordinates x^μ as operators, then different spacetime positions is not a well-defined concept. However, in the star product representation, this is well defined and we need to specify what is meant by the product of fields given in Eq. (2). In principle, we can consider a generalization of the star product when the fields are evaluated at different positions. This will also lead to an acceptable field theory as long as the action is well defined and does not violate any basic principles.

Because of the ambiguities mentioned above, it is clear that we do not have a unique definition of how to compute the product defined in Eq. (2). Here we treat the resulting theory as an effective field theory that might capture some basic aspects of the underlying noncommutative framework. In our analysis, we shall be interested only in the leading-order term in $\Theta^{\mu\nu}$. Our main interest is to obtain a cosmological power spectrum which can lead to physically acceptable results. Observationally, this power spectrum can be probed to at most first order in the noncommutative parameter $\Theta^{\mu\nu}$. Hence, in our phenomenological approach, we restrict ourselves to this order. For products of field at different spacetime positions, we examine a generalized star product involving a form factor $F(x, x')$ and defined as

$$\phi(\vec{x}, t) \star \phi(\vec{x}', t) = \left(1 + \frac{i}{2} F(\vec{x}, \vec{x}')\Theta^{\mu\nu} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} + \cdots\right)\phi(x)\phi(x')\Big|_{t'\to t}. \quad (3)$$

Later, in Eq. (6), we generalize this for an arbitrary number of spacetime points. We also impose the constraint that in the limit $x' \to x$, $F(\vec{x}, \vec{x}') \to 1$, such that the generalized star product reduces to the standard star product. Hence, the action is not affected due to this generalization.

We next show that a form factor naturally arises if, for example, we choose coordinates different from the comoving coordinates in order to impose the basic commutation rule, Eq. (1). Let these coordinates be denoted by $X^\mu(x)$. We now impose the standard star product with derivatives taken with respect to the coordinates X^μ and X'^μ. In this case the standard star product, to leading order, becomes

$$\phi(\vec{x}, t) \star \phi(\vec{x}', t) = \left(1 + \frac{i}{2}\Theta^{\mu\nu}\frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^\beta}{\partial X'^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta}\right)\phi(x)\phi(x')\Big|_{t'\to t}. \quad (4)$$

Hence, we find that a form factor has effectively appeared in this equation. However, it is not of the kind we require for our purpose. As we shall see later, we also require an imaginary part in the form factor, whereas the one generated by this procedure is real. Furthermore, additional terms generated by this procedure do not vanish even in the limit $x' \to x$, in contrast to the form factor $F(\vec{x}, \vec{x}')$ that reduces to 1 in this limit.

In the next section, we show that the generalized star product [i.e., Eq. (3)] is associative as well as cyclic. In Sec. III we compute the power spectrum of the scalar field after making a suitable choice of the form factor $F(\vec{x}, \vec{x}')$.

II. ASSOCIATIVITY AND CYCLICITY OF THE GENERALIZED STAR PRODUCT

The standard star product satisfies some basic rules, such as associativity and cyclicity. Associativity is a useful mathematical requirement, for if the associativity of an algebra G (with a given product law, say "$\vee$") is established then the product of n elements $\bigvee_{i=1}^n x_i$ where $x_i \in G$, $1 \le i \le n$ is unique (see, for example, Proposition 2.1.4 of [46]). In other words, associativity is a sufficient condition to ensure unique product of n elements of G. In literature, however, authors have also used nonassociative star products (see Ref. [47]). The presence of associativity in the context of the star product implies a unique multiplication of n functions, without which the cyclicity condition [discussed below in Eq. (14)] will not make any sense.

In this section, we show that the generalized star product is associative and cyclic to the order of our calculation. We first perform a more general analysis and then specialize it to our particular case [see Eq. (3)] by choosing a specific form factor [in Eq. (20)]. Let $f_i(x_j)$ represent an indexed function f_i depending upon the indexed spacetime point x_j. Let us consider the generalized m-point star function

$$\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} = f_{i_1}(x_{j_1}) \star (f_{i_2}(x_{j_2}) \star \ldots \star (f_{i_{m-1}}(x_{j_{m-1}}) \star f_{i_m}(x_{j_m}))), \\ = f_{i_1}(x_{j_1}) \star \Omega^{j_2,j_3,\ldots,j_m}_{i_2,i_3,\ldots,i_m}. \quad (5)$$

Here we have placed brackets clubbing a pair of functions. These brackets are important because associativity of the generalized star product has not been established yet. The generalized star product between an m- and n-point star function is defined to be

$$\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} \star \Omega^{l_1,l_2,\ldots,l_n}_{k_1,k_2,\ldots,k_n} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{m}\sum_{q=1}^{n} F_{j_p l_q}\frac{\partial}{\partial x^\mu_{j_p}}\frac{\partial}{\partial x^\nu_{l_q}} + \cdots\right] \times \Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\Omega^{l_1,l_2,\ldots,l_n}_{k_1,k_2,\ldots,k_n}. \quad (6)$$

Here $F_{j_p j_q}$ is an abbreviation for $F(x_{j_p}, x_{j_q})$, i.e., the form factor as a function of spacetime points x_{j_p} and x_{j_q}. In case we desire to make two spacetime points the same, as in the case of two-point star function $\Omega^{1,1}_{i_1 i_2}$, we simply take the limit $x_2 \to x_1$, as shown in the following:

$$\Omega^{1,1}_{i_1 i_2} = \lim_{x_2\to x_1}\Omega^{1,2}_{i_1 i_2} = f_{i_1}(x_1) \star f_{i_2}(x_2)\Big|_{x_2\to x_1}.$$

We emphasize that Eq. (6) is a recursive definition which gives an m-point star function from an $(m-1)$-point equation only says that first we need to calculate m- and n-point star products (recursively) star function, and that nowhere in this equation has the associativity *a priori* been assumed. The and then take the star product between them.

We may compare our generalized star product, i.e., Eq. (6), with the standard definition of the star product, given in [48],

$$f_1(x_1) \star f_2(x_2) \star \ldots \star f_n(x_n) = \prod_{\substack{a,b=1\\ a<b}}^{n} \exp\left[i\frac{\Theta^{ij}}{2}\frac{\partial}{\partial x^i_a}\frac{\partial}{\partial x^j_b}\right] f_1(x_1)f_2(x_2)\ldots f_n(x_n). \quad (7)$$

When $F_{j_p j_q} = 1$ for all spacetime coordinate pairs, we may extend our recursive definition, Eq. (6), to all orders. Assuming an exponential form, we can derive Eq. (7) as shown in the Appendix. We also use this to establish the associativity of the standard star product as a special case. We point out that the proof of associativity of the standard star product in the most general case, which includes the product of fields at different spacetime points, is so far lacking in the literature.

A. The definition's *modus operandi*

We next illustrate our recursive definition's mode of operation using some simple examples. Let us first apply this definition for two- and three-point star functions. From Eq. (5),

$$\Omega^{j_1}_{i_1} = f_{i_1}(x_{j_1}), \quad (8)$$

$$\Omega^{j_1,j_2}_{i_1,i_2} = f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}), \\ = \Omega^{j_1}_{i_1} \star \Omega^{j_2}_{i_2}. \quad (9)$$

Here Eq. (9) follows from Eq. (8). Using Eqs. (6) and (9), $\Omega^{j_1,j_2}_{i_1,i_2}$ is equal to

$$\Omega^{j_1}_{i_1} \star \Omega^{j_2}_{i_2} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}F_{j_1 j_2}\frac{\partial}{\partial x^\mu_{j_1}}\frac{\partial}{\partial x^\nu_{j_2}}\right]\Omega^{j_1}_{i_1}\Omega^{j_2}_{i_2}, \\ = \left[1 + \frac{i}{2}\Theta^{\mu\nu}F_{j_1 j_2}\frac{\partial}{\partial x^\mu_{j_1}}\frac{\partial}{\partial x^\nu_{j_2}}\right] f_{i_1}(x_{j_1})f_{i_2}(x_{j_2}). \quad (10)$$

We emphasize that we are restricting ourselves to the leading-order contributions in $\Theta^{\mu\nu}$. Similarly, we can calculate the quantity $\Omega^{j_1,j_2,j_3}_{i_1,i_2,i_3}$ which according to Eq. (5) is $\Omega^{j_1}_{i_1} \star \Omega^{j_2,j_3}_{i_2,i_3}$. Again using Eq. (6) and neglecting any second-order contribution coming from $\Omega^{j_1,j_3}_{i_2,i_3}$, we obtain

$$\Omega^{j_1,j_2,j_3}_{i_1,i_2,i_3} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\left(F_{j_1 j_2}\frac{\partial}{\partial x^\mu_{j_1}}\frac{\partial}{\partial x^\nu_{j_2}} + F_{j_1 j_3}\frac{\partial}{\partial x^\mu_{j_1}}\frac{\partial}{\partial x^\nu_{j_3}} + F_{j_2 j_3}\frac{\partial}{\partial x^\mu_{j_2}}\frac{\partial}{\partial x^\nu_{j_3}}\right)\right] f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})f_{i_3}(x_{j_3}). \quad (11)$$

As per Eq. (5), $\Omega^{j_1,j_2,j_3}_{i_1,i_2,i_3}$ is equal to $f_{i_1}(x_{j_1}) \star (f_{i_2}(x_{j_2}) \star f_{i_3}(x_{j_3}))$. The same method can also be used to calculate generalized an m-point star function for any number of points; the result is found to be (see Appendix)

$$\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{m} F_{j_p j_q}\frac{\partial}{\partial x^\mu_{j_p}}\frac{\partial}{\partial x^\nu_{j_q}}\right] f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_m}(x_{j_m}). \quad (12)$$

Next we establish the associativity of generalized star product given in Eq. (6), for which we first need to calculate the quantity $\Omega^{j_1,j_2}_{i_1,i_2} \star \Omega^{j_3}_{i_3}$. This quantity can be evaluated using Eq. (6) and can be shown to be the same as in Eq. (11). Thus $\Omega^{j_1,j_2}_{i_1,i_2} \star \Omega^{j_3}_{i_3} = \Omega^{j_1}_{i_1} \star \Omega^{j_2,j_3}_{i_2,i_3}$, or in other words,

$$(f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2})) \star f_{i_3}(x_{j_3}) = f_{i_1}(x_{j_1}) \star (f_{i_2}(x_{j_2}) \star f_{i_3}(x_{j_3})). \tag{13}$$

Hence, the generalized star product is associative at the leading order; therefore, at this order $\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} = f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}) \star \ldots \star f_{i_m}(x_{j_m})$. Now we argue that generalized star product satisfies cyclicity as well. The cyclicity property is (see Refs. [48,49])

$$\int d^4x f_1(x) \star f_2(x) \star \ldots \star f_n(x) = \int d^4x f_n(x) \star f_1(x) \star \ldots \star f_{n-1}(x). \tag{14}$$

Because the result (12) is true for any indexed set i and j, let us take $i_1 = j_1 = 1$, $i_2 = j_2 = 2$, and so on. In this case, the m-point generalized star function $\Omega^{1,2,\ldots,m}_{1,2,\ldots,m}$ in Eq. (12) becomes

$$\Omega^{1,2,\ldots,m}_{1,2,\ldots,m} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{m} F_{pq}\frac{\partial}{\partial x^\mu_p}\frac{\partial}{\partial x^\nu_q}\right] \times f_1(x_1)f_2(x_2)\ldots f_m(x_m).$$

When all spacetime coordinates become the same, the form factor F_{pq} is equal to 1, in which case this equation reduces to a standard star product at linear order. Thus, the generalized star product is cyclic.

As stated before, the form factor introduced will be chosen in order to fit the cosmological data; hence, in this sense our proposed star product in Eq. (6) is purely phenomenological. However, it is mathematically well defined in the sense that it requires only a single product rule [i.e., Eq. (6)] applicable in all cases. As stated above, we treat this as an effective framework, which may in future provide some guidance in constructing a consistent fundamental framework. In the next section we compute the power spectrum of the scalar field, making a suitable choice of the form factor $F(\vec{x}, \vec{x}')$ using expression (3) in Eq. (2).

III. POWER SPECTRUM IN FRIEDMANN-ROBERTSON-WALKER BACKGROUND

In this section we compute the correlation function $\Delta(\vec{x}, \vec{x}')$ defined in Eq. (2) for the case of an expanding de Sitter universe at leading order in $\Theta^{\mu\nu}$. The scalar field may be expressed as

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3}(a_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}\zeta_{\vec{k}}(t) + a^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\zeta^\star_{\vec{k}}(t)), \tag{15}$$

where $\zeta_{\vec{k}} = u_{\vec{k}}/a$ and the mode function $u_{\vec{k}} = \frac{e^{-ik\eta}}{\sqrt{2k}}(1 - \frac{i}{k\eta})$. A direct calculation yields

$$\Delta(\vec{x} - \vec{x}') = \langle 0|\phi(\vec{x}, t)\phi(\vec{x}', t)|0\rangle + \Delta_1(\vec{x} - \vec{x}'), \tag{16}$$

where we have used translational invariance and set $\Delta(\vec{x}, \vec{x}') = \Delta(\vec{x} - \vec{x}')$. Here the first term on the right-hand side is the standard contribution in commutative spacetime and the second term is the leading-order correction. We are interested in its Fourier transform,

$$\delta P(\vec{k}) = \int d^3\vec{R}e^{-i\vec{k}\cdot\vec{R}}\Delta_1(\vec{x} - \vec{x}'), \tag{17}$$

where $\vec{R} = \vec{x} - \vec{x}'$. We obtain

$$\delta P(\vec{k}) = \frac{1}{2}\Theta^{0i}\int \frac{d^3\vec{R}d^3\vec{q}}{(2\pi)^3}e^{i(\vec{q}-\vec{k})\cdot\vec{R}}F(\vec{R})q_i f(q), \tag{18}$$

where

$$f(q) = \dot{\zeta}_{\vec{q}}\zeta^\star_{\vec{q}} + \dot{\zeta}^\star_{\vec{q}}\zeta_{\vec{q}} = -\frac{\eta^2H^3}{q} \tag{19}$$

and H is the Hubble's constant. We next make the following choice for the form factor:

$$F(\vec{R}) = \cos(\vec{\lambda}\cdot\vec{R}/\eta) + iB\vec{R}\cdot\vec{R}\sin^2(\vec{\alpha}\cdot\vec{R}), \tag{20}$$

where B and λ are parameters. The form factor approaches 1 in the limit $\vec{R} \to 0$. Hence, $\delta P(\vec{k})$, upon performing the binomial expansion assuming $|\frac{\vec{\alpha}}{k}| \ll 1$ and keeping the first nonzero term (quadratic terms in α) will be

$$-\frac{1}{2}\Theta^{0i}\eta^2H^3\left[\frac{1}{2}\left(\frac{\eta k_i - \lambda_i}{|\eta\vec{k} - \vec{\lambda}|} + \frac{\eta k_i + \lambda_i}{|\eta\vec{k} + \vec{\lambda}|}\right) + \frac{2iB}{k^5}(3k_i|\vec{\alpha}|^2 - 15k_i(\vec{\alpha}\cdot\hat{k})^2 + 6k\alpha_i\vec{\alpha}\cdot\hat{k})\right].$$

The form factor has to be dimensionless, therefore, $B \sim [L]^{-2}$; we choose it to be $B = \frac{b}{\eta^2}$, $b \in \mathbb{R}$. In the limit $\eta \to 0$ we obtain

$$\delta P(\vec{k}) = i\frac{b\Theta^{0i}H^3}{k^5}(15k_i(\vec{\alpha}\cdot\hat{k})^2 - 3k_i|\vec{\alpha}|^2 - 6k\alpha_i\vec{\alpha}\cdot\hat{k}). \tag{21}$$

Hence, we obtain a power spectrum of the form anticipated in [19,21]; i.e., it will lead to physically acceptable temperature correlations. The imaginary part of the form factor has been chosen so that we obtain the power required to fit the data [44,50]. We point out that the dipolar power spectrum was found to decay by approximately one power of k higher than the standard scale invariant power spectrum [50]. This is exactly what is found in our analysis. However, Eq. (21) contains additional structure that is not present in the power spectrum assumed in [50]. This corresponds to the first and third terms inside the bracket in Eq. (21). These will lead to additional correlations whose structure depends on the choice of the vector $\vec{\alpha}$. Whether these correlations are present in data is so far not known. In any case, these depend on the precise choice of the form factor in Eq. (20). We have considerable freedom in choosing this function; our main motivation in this paper is to show that a form exists that leads to physically sensible results, rather than a detailed fit to data. Furthermore, our model is phenomenological and it remains to be determined whether such a model can arise from a fundamental framework. More detailed analyses and applications to CMBR data are left to future research.

IV. CONCLUSION

We have proposed a generalized star product Eq. (3) that is applicable when the fields are evaluated at two different spacetime positions, x_1 and x_2, and which is derived with the help of a recursive definition, i.e., Eq. (6). The product involves an effective form factor that becomes unity when $x_1 = x_2$. Using a model for this form factor we compute the cosmological primordial power spectrum produced during inflation. The model may be regarded as an effective field theory within the framework of noncommutative coordinates. The formalism is mathematically well defined but introduces an unknown form factor. We are forced to introduce it in order to avoid the physically inconsistent results predicted by the standard star product. We hope our model might provide some guidance in constructing a fundamental theory that may lead to physically sensible results. We find that in Fourier space the power spectrum acquires a dipolar imaginary structure exactly as anticipated in [19,21]. Such a structure is required in order for it to yield an acceptable CMB temperature anisotropy pattern. It has been argued that this might provide an explanation of the observed hemispherical anisotropy [22–30] or, equivalently, the dipole modulation [40–43,51,52] of the CMB temperature field. Thus, our results show that it is possible to explain the hemispherical anisotropy using spacetime noncommutativity.

ACKNOWLEDGMENTS

R. K. sincerely acknowledges the support of a fellowship from CSIR, New Delhi. He is also thankful to Mr. Ambuj Pandey and Professor S. K. Pattanayak for mathematical discussions.

APPENDIX: SOME MATHEMATICAL PROPERTIES OF STANDARD AND GENERALIZED STAR PRODUCTS

In this appendix we deduce the results referred to in the preceding sections. As mentioned before, we now first extend our recursive definition [Eq. (6)] to all orders when $F_{pq} = 1, \forall p$ and q, in which case, assuming an exponential form, we obtain

$$\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} \star \Omega^{l_1,l_2,\ldots,l_n}_{k_1,k_2,\ldots,k_n} = \exp\left(\frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{l_q}}\right)\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\Omega^{l_1,l_2,\ldots,l_n}_{k_1,k_2,\ldots,k_n}. \tag{A1}$$

Next we establish associativity of this product rule. We must point out that our analysis in this appendix also serves to establish the associativity of the standard star product for the most general case, when we consider products of fields which depend on different spacetime positions. It must be noted that even for this case, Eq. (A1) does not have the same form as the standard star product, defined in Eq. (7). However the two are found to be equal, as shown later in this appendix. Finally, we prove Eq. (12).

Theorem 1: The recursive definition given in Eq. (A1) at all orders in the noncommutative parameter $\Theta^{\mu\nu}$ is associative. Thus, in this case the n-point star function can be uniquely written as a product of one-point functions $\Omega^{j_1}_{i_1} \equiv f_{i_1}(x_{j_1})$ in the following way:

$$\Omega^{j_1,j_2,\ldots,j_n}_{i_1,i_2,\ldots,i_n} = f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}) \star \ldots \star f_{i_n}(x_{j_n}).$$

Furthermore, in the same limit it has the following explicit form:

$$f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}) \star \ldots \star f_{i_n}(x_{j_n}) = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{n}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right] f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_n}(x_{j_n}). \tag{A2}$$

Proof.—First, we prove associativity. Let us calculate $\Omega^{j_1}_{i_1} \star \Omega^{j_2,j_3}_{i_2,i_3}$. For this we first need $\Omega^{j_2,j_3}_{i_2,i_3}$, which from Eq. (A1) equals

$$\Omega^{j_2,j_3}_{i_2,i_3} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}_{j_2}}\frac{\partial}{\partial x^{\nu}_{j_3}}\right] f_{i_2}(x_{j_2})f_{i_3}(x_{j_3}). \tag{A3}$$

Again using Eqs. (A1) and (A3), $\Omega_{i_1}^{j_1} \star \Omega_{i_2,i_3}^{j_2,j_3}$ becomes

$$\Omega_{i_1}^{j_1} \star \Omega_{i_2,i_3}^{j_2,j_3} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\left(\frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_2}^{\nu}} + \frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right)\right] f_{i_1}(x_{j_1}) \exp\left[\frac{i}{2}\Theta^{\rho\sigma}\frac{\partial}{\partial x_{j_2}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right] f_{i_2}(x_{j_2}) f_{i_3}(x_{j_3}),$$

but the second exponential does not have any x_{j_1} dependence; therefore, this term can also be written as

$$\Omega_{i_1}^{j_1} \star \Omega_{i_2,i_3}^{j_2,j_3} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\left(\frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_2}^{\nu}} + \frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right)\right] \exp\left[\frac{i}{2}\Theta^{\rho\sigma}\frac{\partial}{\partial x_{j_2}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2}) f_{i_3}(x_{j_3}). \tag{A4}$$

Now, notice that if

$$X_{(p,q)} = \frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x_p^{\mu}}\frac{\partial}{\partial x_q^{\nu}}, \tag{A5}$$

then $[X_{(p,q)}, X_{(r,s)}] = 0$ for all p, q, r, and s, and also that

$$\frac{i}{2}\Theta^{\mu\nu}\left(\frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_2}^{\nu}} + \frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right) = X_{(j_1,j_2)} + X_{(j_1,j_3)}, \frac{i}{2}\Theta^{\rho\sigma}\frac{\partial}{\partial x_{j_2}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}} = X_{(j_2,j_3)},$$

It is concluded that $[X_{(j_1,j_2)} + X_{(j_1,j_3)}, X_{(j_2,j_3)}] = [X_{(j_1,j_2)}, X_{(j_2,j_3)}] + [X_{(j_1,j_3)}, X_{(j_2,j_3)}] = 0$. Therefore, by the Zassenhaus formula (see, for example, [53] according to which if $[X,Y] = 0$ then $e^{X+Y} = e^X e^Y$) Eq. (A4) can be written as

$$\Omega_{i_1}^{j_1} \star \Omega_{i_2,i_3}^{j_2,j_3} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\left(\frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_2}^{\nu}} + \frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}} + \frac{\partial}{\partial x_{j_2}^{\mu}}\frac{\partial}{\partial x_{j_3}^{\nu}}\right)\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2}) f_{i_3}(x_{j_3}). \tag{A6}$$

In a similar manner, using the same kind of reasoning, $\Omega_{i_1,i_2}^{j_1,j_2} \star \Omega_{i_3}^{j_3}$ can be shown to be equal to Eq. (A6). Thus, it is concluded that the generalized star product when $F_{pq} = 1$ for all p, q and at all orders in the noncommutative parameters $\Theta^{\mu\nu}$ is associative. Because of the associativity, Eq. (5) in this case becomes

$$\Omega_{i_1,i_2,\ldots,i_n}^{j_1,j_2,\ldots,j_n} = f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}) \star \ldots \star f_{i_n}(x_{j_n}) = \Omega_{i_1}^{j_1} \star \Omega_{i_2}^{j_2} \star \ldots \Omega_{i_n}^{j_n}.$$

The next part is proved by induction. Let $P(n)$ be the statement that Eq. (A2) is true. Clearly, from Eq. (A1) with $l_1 = j_2$ and $k_1 = i_2$, we have

$$f_{i_1}(x_{j_1}) \star f_{i_2}(x_{j_2}) = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x_{j_1}^{\mu}}\frac{\partial}{\partial x_{j_2}^{\nu}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2}),$$

which is exactly what we get from Eq. (A2) when $n = 2$; hence, $P(2)$ is true. After this we assume the induction hypothesis that $P(m)$ is true, that is, the following equation holds:

$$\Omega_{i_1,i_2,\ldots,i_m}^{j_1,j_2,\ldots,j_m} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\p<q}}^{m}\frac{\partial}{\partial x_{j_p}^{\mu}}\frac{\partial}{\partial x_{j_q}^{\nu}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2}) \ldots f_{i_m}(x_{j_m}),$$

and we show $P(m+1)$ to be true too. Again using Eq. (A1), we get

$$\Omega_{i_1,i_2,\ldots,i_m}^{j_1,j_2,\ldots,j_m} \star \Omega_{i_{m+1}}^{j_{m+1}} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{m}\frac{\partial}{\partial x_{j_p}^{\mu}}\frac{\partial}{\partial x_{j_{m+1}}^{\nu}}\right] \Omega_{i_1,i_2,\ldots,i_m}^{j_1,j_2,\ldots,j_m} \Omega_{i_{m+1}}^{j_{m+1}},$$

the rhs of which is equal to

$$\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{m}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_{m+1}}}\right]\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{m}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_m}(x_{j_m})f_{i_{m+1}}(x_{j_{m+1}}).$$

Again observing that [see Eq. (A5)]

$$\left[\sum_{p=1}^{m}X_{(j_p,j_{m+1})},\sum_{\substack{r,q=1\\ r<q}}^{m}X_{(j_r,j_q)}\right]=\sum_{\substack{r,q=1\\ r<q}}^{m}\sum_{p=1}^{m}[X_{(j_p,j_{m+1})},X_{(j_r,j_q)}]=0,$$

and invoking the Zassenhaus formula, it is concluded that

$$f_{i_1}(x_{j_1})\star f_{i_2}(x_{j_2})\ldots\star f_{i_{m+1}}(x_{j_{m+1}})=\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{m+1}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_m}(x_{j_m})f_{i_{m+1}}(x_{j_{m+1}}).$$

Thus $P(m+1)$ is also true; hence, by the principle of mathematical induction (see for example [54]), $P(n)$ is true for all $n\geq 2$. ∎

We next show the equivalence between the generalized star product for the case $F_{pq}=1$ and the standard star product.

Proposition 2: The standard n-point star product $f_{i_1}(x_{j_1})\star f_{i_2}(x_{j_2})\star\ldots\star f_{i_n}(x_{j_n})$ given in Eq. (7) is the same as

$$\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{n}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_n}(x_{j_n}). \quad (A7)$$

Proof.—The proof is really a corollary to the result that if $[X_i,X_j]=0\,\forall i$ and j, then $\exp(\sum X_i)=\prod\exp(X_i)$. We now prove this result. First, notice that the commutator $[X_1,\sum_{i=2}^{n}X_i]$ is the same as $\sum_{i=2}^{n}[X_1,X_i]$, but this is zero by assumption because $[X_i,X_j]=0$. Therefore by the Zassenhaus formula

$$\exp\left(X_1+\sum_{i=2}^{n}X_i\right)=\exp(X_1)\exp\left(\sum_{i=2}^{n}X_i\right).$$

The same argument can also be used for the second term, and can be continued further till one ends up with only one term in the summation. Now, because in our case $[X_{(j_p,j_q)},X_{(j_r,j_s)}]=0$ [see Eq. (A5) for definition] for all j_p, j_q, j_r, and j_s, and it can be shown that for all p and q, $X_{(j_p,j_q)}$ will form an indexed sequence of operators, hence

$$\prod_{\substack{p,q=1\\ p<q}}^{n}\exp\left[i\frac{\Theta^{\mu\nu}}{2}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_n}(x_{j_n}),$$

is equal to

$$\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{n}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_n}(x_{j_n}).$$

The derivation done until now is valid for any arbitrary index. As a final step, let us take $j_p=p$ and $i_p=p$ to get

$$\prod_{\substack{p,q=1\\ p<q}}^{n}\exp\left[i\frac{\Theta^{\mu\nu}}{2}\frac{\partial}{\partial x^{\mu}_{p}}\frac{\partial}{\partial x^{\nu}_{q}}\right]f_1(x_1)f_2(x_2)\ldots f_n(x_n)=\exp\left[\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{n}\frac{\partial}{\partial x^{\mu}_{p}}\frac{\partial}{\partial x^{\nu}_{q}}\right]f_1(x_1)f_2(x_2)\ldots f_n(x_n).$$

Therefore, it is concluded that the generalized star product in expression (6) reduces to Eq. (7) under appropriate conditions. ∎

Now we prove Eq. (12).

Proposition 3: The explicit form of an m-point generalized star function $\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}$ defined through Eq. (6) is

$$\Omega^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}=\left[1+\frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\ p<q}}^{m}F_{j_pj_q}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right]f_{i_1}(x_{j_1})f_{i_2}(x_{j_2})\ldots f_{i_m}(x_{j_m}). \quad (A8)$$

Proof.—We use induction in this case. Let $P(m)$ be the statement that Eq. (A8) is true. Using Eq. (6) $\Omega^{j_1 j_2}_{i_1 i_2}$ we obtain the same form using Eq. (A8) for $m = 2$; thus, $P(2)$ is true. Let us next assume $P(k)$ to be true where $P(k)$ is the following induction hypothesis:

$$\Omega^{j_1,j_2,\ldots,j_k}_{i_1,i_2,\ldots,i_k} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\p<q}}^{k} F_{j_p j_q}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2})\ldots f_{i_k}(x_{j_k}).$$

We prove that $P(k+1)$ is also true. From Eqs. (5) and (6) we have

$$\Omega^{j_1,j_2,\ldots,j_{k+1}}_{i_1,i_2,\ldots,i_{k+1}} = \Omega^{j_1,j_2,\ldots,j_k}_{i_1,i_2,\ldots,i_k} \star \Omega^{j_{k+1}}_{i_{k+1}},$$

and thus

$$\Omega^{j_1,j_2,\ldots,j_{k+1}}_{i_1,i_2,\ldots,i_{k+1}} = \left(1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{k} F_{j_p j_{k+1}}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_{k+1}}}\right)\Omega^{j_1,j_2,\ldots,j_k}_{i_1,i_2,\ldots,i_k}\Omega^{j_{k+1}}_{i_{k+1}},$$

which, after using the induction hypothesis, becomes

$$\Omega^{j_1,j_2,\ldots,j_{k+1}}_{i_1,i_2,\ldots,i_{k+1}} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{p=1}^{k} F_{j_p j_{k+1}}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_{k+1}}}\right]\left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\p<q}}^{k} F_{j_p j_q}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2})\ldots f_{i_{k+1}}(x_{j_{k+1}}).$$

Keeping only linear order in the noncommutative parameter $\Theta^{\mu\nu}$, we get

$$\Omega^{j_1,j_2,\ldots,j_{k+1}}_{i_1,i_2,\ldots,i_{k+1}} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\left(\sum_{p=1}^{k} F_{j_p j_{k+1}}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_{k+1}}} + \sum_{\substack{p,q=1\\p<q}}^{k} F_{j_p j_q}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right)\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2})\ldots f_{i_{k+1}}(x_{j_{k+1}}),$$

which is equal to

$$\Omega^{j_1,j_2,\ldots,j_{k+1}}_{i_1,i_2,\ldots,i_{k+1}} = \left[1 + \frac{i}{2}\Theta^{\mu\nu}\sum_{\substack{p,q=1\\p<q}}^{k+1} F_{j_p j_q}\frac{\partial}{\partial x^{\mu}_{j_p}}\frac{\partial}{\partial x^{\nu}_{j_q}}\right] f_{i_1}(x_{j_1}) f_{i_2}(x_{j_2})\ldots f_{i_{k+1}}(x_{j_{k+1}}).$$

Thus, $P(k+1)$ is true; hence, by principle of mathematical induction $P(n)$ is true for all $n \geq 2$. ■

---

[1] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Spacetime quantization induced by classical gravity, Phys. Lett. B **331**, 39 (1994).

[2] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, England, 1999).

[3] D. V. Ahluwalia, Quantum measurement, gravitation, and locality, Phys. Lett. B **339**, 301 (1994).

[4] A. Connes, *Noncommutative Geometry* (Elsevier Science, New York, 1995).

[5] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometries*, Lecture Notes in Physics Monographs (Springer-Verlag, Berlin, 2003).

[6] J. M. Gracia-Bondía, J. C. Varilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts Basler Lehrbücher (Birkhäuser, Boston, 2013).

[7] C.-S. Chu, B. R. Greene, and G. Shiu, Remarks on inflation and noncommutative geometry, Mod. Phys. Lett. A **16**, 2231 (2001).

[8] F. Lizzi, G. Mangano, G. Miele, and M. Peloso, Cosmological perturbations and short distance physics from noncommutative geometry, J. High Energy Phys. 06 (2002) 049.

[9] R. Brandenberger and P.-M. Ho, Noncommutative spacetime, stringy spacetime uncertainty principle, and density fluctuations, Phys. Rev. D **66**, 023517 (2002).

[10] Q.-G. Huang and M. Li, CMB power spectrum from noncommutative spacetime, J. High Energy Phys. 06 (2003) 014.

[11] Q. G. Huang and M. Li, Noncommutative inflation and the CMB multipoles, J. Cosmol. Astropart. Phys. 11 (2003) 001.

[12] S. Tsujikawa, R. Maartens, and R. Brandenberger, Noncommutative inflation and the CMB, Phys. Lett. B **574**, 141 (2003).

[13] A. P. Balachandran, A. R. Queiroz, A. M. Marques, and P. Teotonio-Sobrinho, Quantum fields with noncommutative target spaces, Phys. Rev. D **77**, 105032 (2008).

[14] L. Barosi, F. A. Brito, and A. R. Queiroz, Noncommutative field gas driven inflation. J. Cosmol. Astropart. Phys. 04 (2008) 005.

[15] A. H. Fatollahi and M. Hajirahimi, Noncommutative black-body radiation: Implications on cosmic microwave background, Europhys. Lett. **75**, 542 (2006).

[16] A. H. Fatollahi and M. Hajirahimi, Black-body radiation of noncommutative gauge fields, Phys. Lett. B **641**, 381 (2006).

[17] E. Akofor, A. P. Balachandran, S. G. Jo, A. Joseph, and B. A. Qureshi, Direction-dependent CMB power spectrum and statistical anisotropy from noncommutative geometry, J. High Energy Phys. 05 (2008) 092.

[18] E. Akofor, A. P. Balachandran, A. Joseph, L. Pekowsky, and B. A. Qureshi, Constraints from the cosmic microwave background on spacetime noncommutativity and causality violation, Phys. Rev. D **79**, 063004 (2009).

[19] T. S. Koivisto and D. F. Mota, CMB statistics in noncommutative inflation, J. High Energy Phys. 02 (2011) 061.

[20] N. E. Groeneboom, M. Axelsson, D. F. Mota, and T. Koivisto, Imprints of a hemispherical power asymmetry in the seven-year WMAP data due to non-commutativity of space-time, arXiv:1011.5353.

[21] P. Jain and P. K. Rath, Noncommutative geometry and the primordial dipolar imaginary power spectrum, Eur. Phys. J. C **75**, 113 (2015).

[22] H. K. Eriksen, F. K. Hansen, A. J. Banday, K. M. Górski, and P. B. Lilje, Asymmetries in the cosmic microwave background anisotropy field, Astrophys. J. **605**, 14 (2004).

[23] H. K. Eriksen, A. J. Banday, K. M. Górski, F. K. Hansen, and P. B. Lilje, Hemispherical power asymmetry in the third-year Wilkinson Microwave Anisotropy Probe sky maps, Astrophys. J. Lett. **660**, L81 (2007).

[24] F. K. Hansen, A. J. Banday, K. M. Górski, H. K. Eriksen, and P. B. Lilje, Power asymmetry in cosmic microwave background fluctuations from full sky to sub-degree scales: Is the universe isotropic?, Astrophys. J. **704**, 1448 (2009).

[25] J. Hoftuft, H. K. Eriksen, A. J. Banday, K. M. Górski, F. K. Hansen, and P. B. Lilje, Increasing evidence for hemispherical power asymmetry in the five-year WMAP data, Astrophys. J. **699**, 985 (2009).

[26] A. L. Erickcek, M. Kamionkowski, and S. M. Carroll, A hemispherical power asymmetry from inflation, Phys. Rev. D **78**, 123520 (2008).

[27] P. A. R. Ade *et al.*, Planck 2013 results. XXIII. Isotropy and statistics of the CMB, Astron. Astrophys. **571**, A23 (2014).

[28] F. Paci, A. Gruppuso, F. Finelli, A. De Rosa, N. Mandolesi, and P. Natoli, Hemispherical power asymmetries in the WMAP 7-year low-resolution temperature and polarization maps, Mon. Not. R. Astron. Soc. **434**, 3071 (2013).

[29] F. Schmidt and L. Hui, Cosmic Microwave Background Power Asymmetry from Non-Gaussian Modulation, Phys. Rev. Lett. **110**, 011301 (2013).

[30] Y. Akrami, Y. Fantaye, A. Shafieloo, H. K. Eriksen, F. K. Hansen, A. J. Banday, and K. M. Górski, Power asymmetry in WMAP and Planck temperature sky maps as measured by a local variance estimator, Astrophys. J. Lett. **784**, L42 (2014).

[31] P. K. Aluri and P. Jain, Large-scale anisotropy due to pre-inflationary phase of cosmic evolution, Mod. Phys. Lett. A **27**, 1250014 (2012).

[32] P. K. Rath, T. Mudholkar, P. Jain, P. K. Aluri, and S. Panda, Direction dependence of the power spectrum and its effect on the cosmic microwave background radiation, J. Cosmol. Astropart. Phys. 04 (2013) 007.

[33] P. Jain and J. P. Ralston, Anisotropy in the propagation of radio polarizations from cosmologically distant galaxies, Mod. Phys. Lett. A **14**, 417 (1999).

[34] D. Hutsemekers, Evidence for very large scale coherent orientations of quasar polarization vectors, Astron. Astrophys. **332**, 410 (1998).

[35] A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga, and A. Hamilton, Significance of the largest scale CMB fluctuations in WMAP, Phys. Rev. D **69**, 063516 (2004).

[36] J. P. Ralston and P. Jain, The Virgo alignment puzzle in propagation of radiation on cosmological scales, Int. J. Mod. Phys. D **13**, 1857 (2004).

[37] D. J. Schwarz, G. D. Starkman, D. Huterer, and C. J. Copi, Is the Low-ℓ Microwave Background Cosmic?, Phys. Rev. Lett. **93**, 221301 (2004).

[38] A. K. Singal, Large peculiar motion of the solar system from the dipole anisotropy in sky brightness due to distant radio sources, Astrophys. J. Lett. **742**, L23 (2011).

[39] P. Tiwari and P. Jain, Dipole anisotropy in integrated linearly polarized flux density in NVSS data, Mon. Not. R. Astron. Soc. **447**, 2658 (2015).

[40] C. Gordon, W. Hu, D. Huterer, and T. Crawford, Spontaneous isotropy breaking: A mechanism for CMB multipole alignments, Phys. Rev. D **72**, 103002 (2005).

[41] C. Gordon, Broken isotropy from a linear modulation of the primordial perturbations, Astrophys. J. **656**, 636 (2007).

[42] S. Prunet, J.-P. Uzan, F. Bernardeau, and T. Brunier, Constraints on mode couplings and modulation of the CMB with WMAP data, Phys. Rev. D **71**, 083508 (2005).

[43] C. L. Bennett *et al.*, Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Are there cosmic microwave background anomalies?, Astrophys. J. Suppl. Ser. **192**, 17 (2011).

[44] S. Ghosh, R. Kothari, P. Jain, and P. K. Rath, Dipole modulation of cosmic microwave background temperature

and polarization, J. Cosmol. Astropart. Phys. 01 (2016) 046.

[45] A. P. Balachandran, A. Ibort, G. Marmo, and M. Martone, Inequivalence of quantum field theories on noncommutative spacetimes: Moyal versus Wick-Voros planes, Phys. Rev. D **81**, 085017 (2010).

[46] M. Artin, *Algebra*, 2nd ed. (Prentice Hall of India, New Delhi, 2011).

[47] M. Abou-Zeid and H. Dorn, Comments on the energy-momentum tensor in non-commutative field theories, Phys. Lett. B **514**, 183 (2001).

[48] R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. **378**, 207 (2003).

[49] A. B. Hammou, M. Lagraa, and M. M. Sheikh-Jabbari, Coherent state induced star product on R^3_λ and the fuzzy sphere, Phys. Rev. D **66**, 025025 (2002).

[50] R. Kothari, S. Ghosh, P. K. Rath, G. Kashyap, and P. Jain, Imprint of inhomogeneous and anisotropic primordial power spectrum on CMB polarization, Mon. Not. R. Astron. Soc. **460**, 1577 (2016).

[51] P. K. Rath and P. Jain, Testing the dipole modulation model in CMBR, J. Cosmol. Astropart. Phys. 12 (2013) 014.

[52] P. K. Rath, P. K. Aluri, and P. Jain, Relating the inhomogeneous power spectrum to the CMB hemispherical anisotropy, Phys. Rev. D **91**, 023515 (2015).

[53] F. Casas, A. Murua, and M. Nadinic, Efficient computation of the Zassenhaus formula, Comput. Phys. Commun. **183**, 2386 (2012).

[54] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis*, 3rd ed. (Wiley, New York, 2007).