Critical behavior of (2 + 1)-dimensional QED: $1/N_f$ corrections in the Landau gauge

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The dynamical generation of a fermion mass is studied within (2 + 1)-dimensional QED with N four-component fermions in the leading and next-to-leading orders of the $1/N$ expansion. The analysis is carried out in the Landau gauge, which is supposed to insure the gauge independence of the critical fermion flavor number, N_c. It is found that the dynamical fermion mass appears for $N < N_c$, where $N_c = 3.29$, which is only about 1% larger than its value at leading order.

I. INTRODUCTION

Quantum electrodynamics in 2 + 1 dimensions (QED$_3$) has been extensively studied for more than three decades now. Originally, the interest in QED$_3$ came from its similarities to (3 + 1)-dimensional QCD and the fact that phenomena such as dynamical chiral symmetry breaking (DχSB) and mass generation may be studied systematically in such a toy model; see, e.g., Refs. [1–16]. Later, a strong interest in QED$_3$ arose in connection with planar condensed matter physics systems having relativisticlike low-energy excitations, such as some two-dimensional antiferromagnets [17] and graphene [18]. The study of a dynamically generated gap in the fermion spectrum of graphene has now become an active area of research; see, e.g., the reviews in Refs. [19,20]. In all cases, the understanding of the phase structure of QED$_3$ is a crucial prerequisite to understanding nonperturbative dynamic phenomena in more realistic particle and condensed matter physics models.

Despite the fact that a large number of investigations have been carried out to study DχSB in QED$_3$, very different results have been obtained. Without being exhaustive, let us indeed recall that, in his seminal paper [1], Pisarski solved the Schwinger-Dyson (SD) gap equation using a leading order (LO) $1/N$ expansion and found that a fermion mass is generated for all values of N, decreasing exponentially with N and vanishing only in the limit $N \to \infty$. Later, he confirmed his finding by a renormalization group analysis [4]. Support of Pisarski's result was given by Pennington and collaborators [3], who adopted a more general nonperturbative approach to solving the SD equations. On the other hand, in a more refined analysis of the gap equation at LO of the $1/N$-expansion, Appelquist et al. [2] showed that the theory exhibits a critical behavior as the number N of fermion flavors approaches $N_c = 32/\pi^2$; that is, a fermion mass is dynamically generated only for $N < N_c$. Contrary to all previous results,

an alternative nonperturbative study by Atkinson et al. [5] suggested that chiral symmetry is unbroken at a sufficiently large N. The theory has also been simulated on the lattice [6–8]. Remarkably, the conclusions of Ref. [6] are in agreement with the existence of a critical N, as predicted in the analysis of Ref. [2], while a second paper [7] finds DχSB for all N and a recent third one [8] finds no sign of DχSB at all. Even in the case where a finite N_c is found, its value is subject to uncertainty, with estimates ranging from $N_c = 1$ to $N_c = 4$; see Ref. [9] for a review. Moreover, Ref. [10] found an upper bound, $N_c < 3/2$, while, more recently, Ref. [11] found that $N_c < 4.4$ and Ref. [12] that $N_c < 9/4$. Clearly, all these disagreements reflect our poor understanding of this problem.

The purpose of this work is to include $1/N$ corrections in the LO result of Ref. [2]. Because the critical value N_c is not large, the contribution of such higher orders in the $1/N$ expansion can be essential, and their proper study may lead to a better understanding of the problem. This important issue has rarely been addressed in the past. To the best of our knowledge, the main discussions are in Refs. [13,14], where rather different results were obtained. The well-known results of Ref. [13] demonstrated a quite strong stability of the $1/N$ expansion, while the ones of Ref. [14] showed that a similar property holds only in the Landau gauge. The strong gauge dependence found in Ref. [14] is in agreement with the studies of Ref. [15] in the so-called rainbow approximation. In the following, we shall refine the analysis of Ref. [14] and perform an accurate computation of all $1/N$ corrections in the Landau gauge, with a special focus on the most complicated ones, in order to extract the value of N_c.

The last years witnessed strong progress in the study of the gauge dependence of DχSB in various models; see Ref. [21] as well as the references and discussions therein. The progress is related to the use of the Landau-Khalatnikov-Fradkin transformation [22]. In the case of

QED$_3$ in the $1/N$-expansion, the application of this transformation [16] has revealed the almost complete lack of gauge dependence for N_c. This confirms that we can limit our analysis to the case of the Landau gauge.

II. MODEL AND SCHWINGER-DYSON EQUATIONS

The Lagrangian of massless QED$_3$ with N flavors of fermions reads

$$L = \overline{\Psi}(i\hat{\partial} - e\hat{A})\Psi - \frac{1}{4}F_{\mu\nu}^2, \qquad (1)$$

where Ψ is taken to be a four-component complex spinor. In the massless case, which we are considering, the model contains infrared divergences. The latter soften when the model is analyzed in a $1/N$ expansion [23,24]. Since the theory is superrenormalizable, the mass scale is given by the dimensionful coupling constant, $a = Ne^2/8$, which is kept fixed as $N \to \infty$. In the four-component case, we can introduce the matrices γ_3 and γ_5, which anticommute with γ_0, γ_1, and γ_2. Then the massless case is invariant under the transformations: $\Psi \to \exp(i\alpha_1\gamma_3)\Psi$ and $\Psi \to \exp(i\alpha_2\gamma_5)\Psi$. Together with the identity matrix and $[\gamma_3, \gamma_5]$, we have a $U(2)$ symmetry for each spinor and the full global "chiral" (or rather flavor) symmetry is $U(2N)$. A mass term will break this symmetry to $U(N) \times U(N)$. It is the dynamical generation of such a mass that we shall consider in the following. It is also possible to include a parity non-conserving mass (see, for example, Ref. [25]), but we will not consider that possibility here.

Following Ref. [2], we now study the solution of the SD equation. The inverse fermion propagator has the form

$$S^{-1}(p) = [1 + A(p)](i\hat{p} + \Sigma(p)), \qquad (2)$$

where $A(p)$ is the wave-function renormalization and $\Sigma(p)$ is the dynamically generated parity-conserving mass, which is taken to be the same for all of the fermions. Notice that, in our definition of $\Sigma(p)$, Eq. (2), the choice of the free vertex corresponds to the so-called central Ball-Chiu vertex [26] for the "more standard" definition $\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$. With these conventions, the SD equation for the fermion propagator may be decomposed into scalar and vector components as follows:

$$\tilde{\Sigma}(p) = \frac{2a}{N}\text{Tr}\int\frac{d^3k}{(2\pi)^3}\frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\Gamma^\nu(p,k)}{[1 + A(k)](k^2 + \Sigma^2(k))}, \qquad (3a)$$

$$A(p)p^2 = -\frac{2a}{N}\text{Tr}\int\frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\Gamma^\nu(p,k)}{[1 + A(k)](k^2 + \Sigma^2(k))}, \qquad (3b)$$

where $D_{\mu\nu}(p)$ is the photon propagator in the Landau gauge:

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2[1 + \Pi(p)]}, \qquad (4)$$

$\Pi(p)$ is the polarization operator and $\Gamma^\nu(p,k)$ is the vertex function. In the following, we shall first consider Eqs. (3a) and (3b) at the LO approximation and will then study Eq. (3a) at the next-to-leading order (NLO) level.

III. LEADING ORDER

The LO approximations in the $1/N$ expansion are given by

$$A(p) = 0, \qquad \Pi(p) = a/|p|, \qquad \Gamma^\nu(p,k) = \gamma^\nu, \quad (5)$$

where the fermion mass has been neglected [27] in the calculation of $\Pi(p)$. A single diagram contributes to the gap equation (3a) at LO (see Fig. 1), and the latter reads

$$\Sigma(p) = \frac{16a}{N}\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{(k^2 + \Sigma^2(k))[(p-k)^2 + a|p-k|]}. \qquad (6)$$

Performing the angular integration in Eq. (6) yields

$$\Sigma(p) = \frac{4a}{\pi^2 N|p|}\int_0^\infty d|k|\frac{|k|\Sigma(|k|)}{k^2 + \Sigma^2(|k|)}\ln\left(\frac{|k| + |p| + a}{|k - p| + a}\right). \qquad (7)$$

The study of Eq. (7) in Ref. [2] has revealed the existence of a critical number of fermion flavors N_c such that, for $N > N_c$, $\Sigma(p) = 0$. As was argued in that reference, QED$_3$ is strongly damped for $|p| > a$, i.e., all relevant physics occur at $|p|/a < 1$. Hence, only the lowest order terms in $|p|/a$ have to be kept on the rhs of Eq. (7), with a hard cutoff at $|p| = a$. Moreover, considering an N close to N_c, the value of $\Sigma(|k|)$ can be made arbitrarily small. Thus, $k^2 + \Sigma^2(|k|)$ can be replaced by k^2 on the rhs of Eq. (7), which then further simplifies as

$$\Sigma(p) = \frac{8}{\pi^2 N}\int_0^a d|k|\frac{\Sigma(|k|)}{\text{Max}(|k|, |p|)}. \qquad (8)$$

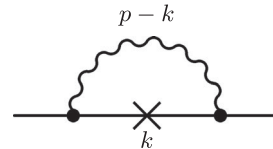Following Ref. [2], the mass function may then be parametrized as

FIG. 1. LO diagram for the dynamically generated mass $\Sigma(p)$. The crossed line denotes mass insertion.

$$\Sigma(k) = B(k^2)^{-\alpha} \qquad (9)$$

(with an arbitrary B value), where the index α has to be self-consistently determined. Substituting (9) into Eq. (8), the gap equation reads

$$1 = \frac{2\beta}{L} \quad \text{where } \beta = \frac{1}{\alpha(1/2 - \alpha)} \quad \text{and} \quad L \equiv \pi^2 N. \quad (10)$$

Solving the gap equation, the following values of α are obtained:

$$\alpha_\pm = \frac{1}{4}\left(1 \pm \sqrt{1 - \frac{32}{L}}\right), \qquad (11)$$

which reproduces the solution given by Appelquist *et al.* in Ref. [2]. Their analysis yields a critical number of fermions, $N_c = 32/\pi^2 \approx 3.24$ (i.e., $L_c = 32$), such that $\Sigma(p) = 0$ for $N > N_c$ and

$$\Sigma(0) \simeq \exp[-2\pi/(N_c/N - 1)^{1/2}] \qquad (12)$$

for $N < N_c$. Thus, DχSB occurs when α becomes complex—that is, for $N < N_c$.

As was shown in Ref. [14], the same result for $\Sigma(p)$ can be obtained in another way. Taking the limit of a large a, the linearized version of Eq. (6) has the following form:

$$\Sigma(p) = \frac{16}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2|p-k|}. \qquad (13)$$

Interestingly, the large-N limit of the photon propagator in QED$_3$ has precisely the same momentum dependence as the one in the so-called reduced QED; see Refs. [28,29]. The multiloop structure of the latter was recently explored in Refs. [30,31]. With the help of the ansatz (9), one can then see that the rhs of Eq. (13) may be calculated with the help of the standard rules of perturbation theory for massless Feynman diagrams, as in Ref. [32]; see also the recent short review in Ref. [33]. Indeed, given these rules, the computation of Eq. (13) is straightforward and reads

$$\Sigma^{(\text{LO})}(p) = \frac{8B}{N} \frac{(p^2)^{-\alpha}}{(4\pi)^{3/2}} \frac{2\beta}{\pi^{1/2}}. \qquad (14)$$

This immediately yields the gap equation (10) and, hence, the results of Eq. (11) together with the critical value $N_c = 32/\pi^2$, at which the index α becomes complex.

Similarly, such rules allow for a straightforward evaluation of the wave-function renormalization. At LO, Eq. (3b) simplifies as

$$A(p)p^2 = -\frac{2a}{N}\text{Tr}\int \frac{d^D k}{(2\pi)^D} \frac{(g_{\mu\nu} - \frac{(p-k)_\mu(p-k)_\nu}{(p-k)^2})\hat{p}\gamma^\mu\hat{k}\gamma^\nu}{k^2|p-k|}, \qquad (15)$$

where the integral has been dimensionally regularized with $D = 3 - 2\varepsilon$. Taking the trace and computing the integral on the rhs yields

$$A(p) = \frac{\Gamma(1+\varepsilon)(4\pi)^\varepsilon\mu^{2\varepsilon}}{p^{2\varepsilon}}C_1 = \frac{\bar{\mu}^{2\varepsilon}}{p^{2\varepsilon}}C_1 + \text{O}(\varepsilon), \qquad (16)$$

where the $\overline{\text{MS}}$ parameter $\bar{\mu}$ has the standard form $\bar{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$, with the Euler constant γ_E and

$$C_1 = +\frac{4}{3\pi^2 N}\left(\frac{1}{\varepsilon} + \frac{7}{3} - 2\ln 2\right). \qquad (17)$$

The corresponding anomalous scaling dimension of the fermion field then reads $\eta = \mu^2(d/d\mu^2)A(p) = 4/(3\pi^2 N)$, and it coincides with the one found in Ref. [34].

IV. NEXT-TO-LEADING ORDER

The ease with which the standard rules for computing massless Feynman diagrams allowed us to derive LO results suggests the possibility of extending these computations beyond LO. We therefore consider the NLO contributions to the dynamically generated mass and parametrize them as

$$\Sigma^{(\text{NLO})}(p) = \left(\frac{8}{N}\right)^2 B\frac{(p^2)^{-\alpha}}{(4\pi)^3}(\Sigma_A + \Sigma_1 + 2\Sigma_2 + \Sigma_3), \qquad (18)$$

where each NLO contribution is represented graphically in Fig. 2. Because we are dealing with the linearized gap equation, each contribution contains a single mass insertion. Adding these contributions to the LO result, Eq. (14), the gap equation has the following general form:

$$1 = \frac{2\beta}{L} + \frac{\pi}{L^2}[\Sigma_A + \Sigma_1 + 2\Sigma_2 + \Sigma_3]. \qquad (19)$$

After very tedious and lengthy calculations, all NLO contributions could be evaluated exactly using the rules for computing massless Feynman diagrams. For the most complicated scalar diagrams [see $I_1(\alpha)$ and $I_2(\alpha)$ below], the Gegenbauer-polynomial technique has been used to follow Ref. [35]. We now summarize our results (details of the calculations will be published elsewhere).

The contribution Σ_A, [see diagram (A) in Fig. 2], originates from the LO value of $A(p)$ and is singular. Using dimensional regularization, it reads
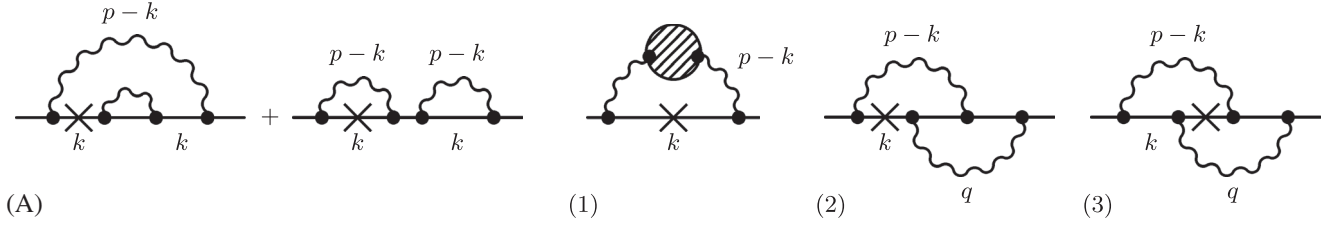
FIG. 2. NLO diagrams of the dynamically generated mass $\Sigma(p)$. The shaded blob is defined in Eq. (23).

$$\overline{\Sigma}_A = +\frac{16}{3}\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\beta\left(\frac{1}{\varepsilon} + \Psi_1 + \frac{4}{3} - \frac{\beta}{4}\right) + \mathrm{O}(\varepsilon), \quad (20)$$

where $\overline{\Sigma}_i = \pi\Sigma_i$, $(i = 1, 2, 3.A)$ and

$$\Psi_1 = \Psi(\alpha) + \Psi(1/2 - \alpha) - 2\Psi(1) + \frac{3}{1/2 - \alpha} - 2\ln 2,$$
$$(21)$$

and Ψ is the digamma function. The contribution of diagram (1) in Fig. 2 is finite and reads

$$\overline{\Sigma}_1 = -4\hat{\Pi}\beta, \qquad \hat{\Pi} = \frac{92}{9} - \pi^2, \quad (22)$$

where [30,36] the contribution of $\hat{\Pi}$ arises from the two-loop polarization operator in dimension $D = 3$, which may be graphically represented as

The contribution of diagram (2) in Fig. 2 is again singular. Dimensionally regularizing it yields

$$2\overline{\Sigma}_2 = -\frac{16}{3}\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\beta\left(\frac{1}{\varepsilon} + \Psi_1 + \frac{7}{3} + \frac{5\beta}{8}\right) - 2\hat{\Sigma}_2 + O(\varepsilon),$$
$$(24)$$

where

$$\hat{\Sigma}_2 = (1 - 4\alpha)\beta[\Psi'(\alpha) - \Psi'(1/2 - \alpha)]$$
$$-\frac{\pi}{2\alpha}\tilde{I}_1(\alpha) - \frac{\pi}{2(1/2 - \alpha)}\tilde{I}_1(\alpha + 1), \quad (25)$$

and Ψ' is the trigamma function. Notice that the singularities in $\overline{\Sigma}_A$ and $\overline{\Sigma}_2$ cancel each other out and their sum is therefore finite:

$$\overline{\Sigma}_A + 2\overline{\Sigma}_2 = -\frac{2}{3}\beta(7\beta + 8) - 2\hat{\Sigma}_2. \quad (26)$$

This cancellation corresponds to one of the logarithms, $\ln(p/\alpha)$, in Ref. [13]; the importance of such cancellations

was discussed previously, in Ref. [2]. The dimensionless integral $\tilde{I}_1(\alpha)$ appearing in Eq. (25) is defined as

$$I_1(\alpha) \equiv \frac{(p^2)^{-\alpha}}{(4\pi)^3}\tilde{I}_1(\alpha)$$
$$= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{|p - k_1|k_1^{2\alpha}(k_1 - k_2)^2(p - k_2)^2|k_2|},$$
$$(27)$$

and it obeys the following relation (it can be obtained by analogy with the ones in Ref. [32]):

$$\tilde{I}_1(\alpha + 1) = \frac{(\alpha - 1/2)^2}{\alpha^2}\tilde{I}_1(\alpha) - \frac{1}{\pi\alpha^2}[\Psi'(\alpha) - \Psi'(1/2 - \alpha)].$$
$$(28)$$

Using the results of Ref. [35], the integral $\tilde{I}_1(\alpha)$ can be represented in the form of a twofold series,

$$\tilde{I}_1(\alpha) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{B(l, n, 1, 1/2)}{(n + 1/2)\Gamma(1/2)}$$
$$\times\left[\frac{2}{n + 1/2}\left(\frac{1}{l + n + \alpha} + \frac{1}{l + n + 3/2 - \alpha}\right)\right.$$
$$\left.+ \frac{1}{(l + n + \alpha)^2} + \frac{1}{(l + n + 3/2 - \alpha)^2}\right], \quad (29)$$

where

$$B(m, n, \alpha, 1/2) = \frac{\Gamma(m + n + \alpha)\Gamma(m + \alpha - 1/2)}{m!\Gamma(m + n + 3/2)\Gamma(\alpha)\Gamma(\alpha - 1/2)}.$$
$$(30)$$

Finally, the contribution of diagram (3) in Fig. 2 is finite and reads

$$\overline{\Sigma}_3 = \hat{\Sigma}_3 + 3\beta^2,$$
$$\hat{\Sigma}_3 = (1/2 - \alpha)\pi\tilde{I}_2(1 + \alpha) + \frac{\pi}{2}\tilde{I}_2(\alpha) + (\alpha - 2)\pi\tilde{I}_3(\alpha).$$
$$(31)$$

The dimensionless integrals in Eq. (31) are defined as
$$\tilde{I}_2(\alpha) = \tilde{I}(\gamma = 1/2, \alpha) \quad \text{and} \quad \tilde{I}_3(\alpha) = \tilde{I}(\gamma = -1/2, 1 + \alpha),$$
where

$$I(\gamma,\alpha) \equiv \frac{(p^2)^{-\alpha-\gamma+1/2}}{(4\pi)^3} \tilde{I}(\gamma,\alpha)$$

$$= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{(p-k_1)^{2\gamma} k_1^2 (k_1-k_2)^{2\alpha} (p-k_2)^2 |k_2|}.$$

$$(32)$$

They satisfy the following relations:

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2 - \alpha),$$

$$\tilde{I}_3(\alpha) = \frac{2}{4\alpha-1}(\alpha\tilde{I}_2(1+\alpha) - (1/2-\alpha)\tilde{I}_2(\alpha)) - \frac{\beta^2}{\pi},$$

$$(33)$$

and thus only one of them is independent. Using the results from Ref. [35], the integral $\tilde{I}_2(\alpha)$ can be represented in the form of a threefold series:

$$\tilde{I}_2(\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B(m,n,\beta,1/2) \sum_{l=0}^{\infty} B(l,n,1,1/2)$$

$$\times C(n,m,l,\alpha), \qquad (34a)$$

$$C(n,m,l,\alpha) = \frac{1}{(m+n+\alpha)(l+n+\alpha)}$$

$$+ \frac{1}{(m+n+\alpha)(l+m+n+1)}$$

$$+ \frac{1}{(m+n+1/2)(l+m+n+\alpha)}$$

$$+ \frac{1}{(m+n+1/2)(l+n+3/2-\alpha)}$$

$$+ \frac{1}{(n+l+\alpha)(l+m+n+\alpha)}$$

$$+ \frac{1}{(l+n+3/2-\alpha)(l+n+m+\alpha)}.$$

$$(34b)$$

Combining all of the above results, the gap equation (19) may be written in an explicit form as

$$1 = \frac{2\beta}{L} + \frac{1}{L^2}\left[8S(\alpha) - \frac{5}{3}\beta^2 - \frac{16}{3}\beta - 4\hat{\Pi}\beta\right], \quad (35)$$

where

$$S(\alpha) = (\hat{\Sigma}_3(\alpha) - 2\hat{\Sigma}_2(\alpha))/8. \qquad (36)$$

At this point, we consider Eq. (35) directly at the critical point $\alpha = 1/4$, i.e., at $\beta = 16$. This yields

$$L_c^2 - 32L_c - 8(S - 64 - 8\hat{\Pi}) = 0, \qquad (37)$$

where $S = S(\alpha = 1/4)$. Solving Eq. (37), we have two standard solutions:

$$L_{c,\pm} = 16 \pm \sqrt{D}, \qquad D = 8(S - 32 - 8\hat{\Pi}). \quad (38)$$

It turns out that the "−" solution is unphysical and has to be rejected because $L_{c,-} < 0$. So, the physical solution is unique and corresponds to $L_c = L_{c,+}$. In order to provide a numerical estimate for N_c, we have used the series representations in order to evaluate the integrals: $\pi\tilde{I}_1(\alpha = 1/4) \equiv R_1$ and $\pi\tilde{I}_2(\alpha = 1/4 + i\delta) \equiv R_2 - iP_2\delta + O(\delta^2)$, where $\delta \to 0$ regulates an artificial singularity in $\pi\tilde{I}_3(\alpha = 1/4) = R_2 + P_2/4$. With 10,000 iterations for each series, we obtain the following numerical estimates:

$$R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720. \quad (39)$$

From these results, we may then obtain the numerical value of $S = R_1 - R_2/8 - 7P_2/128$, which, combined with the $\hat{\Pi}$ one, yields $L_c = 32.45$, and therefore $N_c = 3.29$. This result shows that the inclusion of the $1/N$ corrections increases the critical value of N_c by only 1.5% with respect to its LO value.

V. CONCLUSION

We have included $O(1/N^2)$ contributions to the SD equation exactly and have found that the critical value N_c increased by 1.5% with respect to the LO result. Our analysis is in nice agreement with Ref. [13] and therefore gives further evidence for the solution found by Appelquist *et al.* [2]. Our results are in support of the fact that the $1/N$ expansion of the kernel of the SD equation describes reliably the critical behavior of the theory.

In closing, let us briefly compare our study with that of Nash [13], which, to the best of our knowledge, is the only popular paper which included NLO contributions in the gap equation of QED₃. Our good agreement with Ref. [13] is nice but is also rather strange because the two analyses are done in quite different ways. While we have used the Landau gauge (in accordance with recent results [16] showing the gauge invariance of N_c in this gauge when using the Ball-Chiu vertex), Nash worked with an arbitrary gauge fixing parameter, ξ. He resummed the most important NLO terms ($\propto \beta^2$, in our definition), which, together with the LO ones, led to a gauge invariant result for N_c. This result is larger by a factor of 4/3 than the pure LO one [2]. The rest of the NLO terms ($\propto \beta$) were evaluated (mostly numerically) in the Feynman gauge, which modifies N_c another time and gives the final result of Nash: $N_c = 3.28$. Finally, we also note that Nash obtained two possible solutions (one was considered unphysical), while we obtained a unique one. For these reasons, and despite the surprising closeness of the final results, our analysis

substantially differs from that of Nash and intermediate expressions are difficult to compare [37].

We also note that, very recently, NLO corrections were computed by Gusynin and Pyatkovskiy [38] using a slightly different approach than ours; they obtained a gauge-independent value, $N_c = 2.85$. Their value is remarkably close to the one recently obtained by Herbut [39], $N_c = 2.89$, using a completely different method. In order to clear up the beautiful agreement we have with Nash's results [13] as well as the difference with the results

of Gusynin and Pyatkovskiy [38], we plan to take into account all ξ-dependent terms in our forthcoming publication.

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