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Geometric transition in the nonperturbative topological string

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We study a geometric transition in a nonperturbative topological string. We consider two cases. One is the geometric transition from the closed topological string on the local $\mathcal{B}_3$ to the closed topological string on the resolved conifold. The other is the geometric transition from the closed topological string on the local $\mathcal{B}_3$ to the open topological string on the resolved conifold with a toric A-brane. We find that, in both cases, the geometric transition can be applied for the nonperturbative topological string. We also find the corrections of the value of the Kähler parameters at which the geometric transition occurs.

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I. INTRODUCTION

Recently, in unrefined topological string theory on noncompact toric Calabi-Yau threefolds, the free energy including nonperturbative effects is proposed [1]. The nonperturbative parts can be obtained by considering the Nekrasov-Shatashvili limit [2]. We call this "nonperturbative free energy." Nonperturbative free energy is finite for any string coupling owing to the Hatsuda-Moriyama-Okuyama (HMO) cancellation mechanism [3] This free energy has been studied in various situations [4–14]. For example, it is known that this free energy is closely related to the quantization of a mirror curve for the noncompact toric Calabi-Yau threefold [15–21]. This provides us with the nonperturbative definition of the topological string. However, it is unclear whether the important properties of the perturbative topological string hold, even if we consider the nonperturbative topological string.

In this paper, we study a geometric transition [22–24] in the nonperturbative topological string. As an example, we study the geometric transition in the closed topological string on the local $\mathcal{B}_3$. We consider two cases. One is the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold in the closed topological string. The other is the geometric transition from the closed topological string on the local $\mathcal{B}_3$ to the open topological string on the resolved conifold with a toric A-brane.

We first consider the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold in the closed topological string. Then we find that, by calculating the nonperturbative free energy of the closed topological string on the local $\mathcal{B}_3$ and the resolved conifold, the geometric transition can be applied even if the nonperturbative effects are included. We also find that, in contrast to the case involving perturbative free energy, the values of the Kähler parameters in which the geometric transition occurs are corrected by the nonperturbative effects.

Next we consider the geometric transition from the closed topological string to the open topological string with a toric A-brane. Then we find that the HMO cancellation mechanism can be applied, even if there is a toric A-brane. We also find that the nonperturbative parts of this free energy have the same structure as the one in Refs. [20,25–27]. We check this statement up to $\mathcal{O}(Q_b^2)$. The Kähler parameters are corrected by the nonperturbative effects as in the above case.

This paper is organized as follows. In Sec. II, we calculate the nonperturbative free energy of the closed topological string on the local $\mathcal{B}_3$ by using the refined topological vertex formalism [28–35]. In Sec. III, we consider the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold in the closed topological string. We also consider the geometric transition in which the toric A-brane occurs. Finally, we summarize our results and discuss future work in Sec. IV.

II. FREE ENERGY FOR THE TOPOLOGICAL STRING ON THE LOCAL $\mathcal{B}_3$

In this section, we calculate the nonperturbative free energy of the closed topological string on the local $\mathcal{B}_3$. We also check the HMO cancellation mechanism for this free energy.

A. Calculation of the refined topological string

In order to calculate the nonperturbative free energy, we use the refined topological vertex formalism. The web diagram of the local $\mathcal{B}_3$ is shown in Fig. 1, where we define $Q_{1,2,b,f} \coloneqq \mathrm{e}^{-t_{1,2,b,f}}$, and $t_{1,2,b,f}$ are the Kähler parameters.

We can then write the partition function of the refined topological string on this geometry $\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q)$ as follows:

$$\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q) = \sum_{\mu_b,\tilde{\mu}_b}(-Q_b)^{|\mu_b|}(-Q_bQ_f)^{|\tilde{\mu}_b|}f^2_{\tilde{\mu}_b}(t,q)\mathcal{Z}^{(1)}_{\mu_b\tilde{\mu}_b}\mathcal{Z}^{(2)}_{\mu_b\tilde{\mu}_b}, \tag{2.1}$$

where we define

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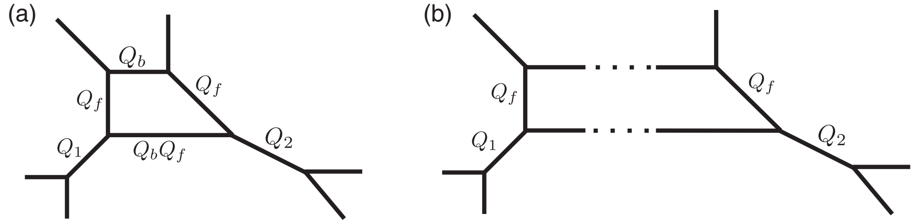

FIG. 1. (a) Web diagram of the local $\mathcal{B}_3$. (b) Its building blocks.

$$\mathcal{Z}^{(1)}_{\mu_b\tilde{\mu}_b} \coloneqq \sum_{\mu_f,\mu_1}(-Q_f)^{|\mu_f|}(-Q_1)^{|\mu_1|}\tilde{f}_{\mu_f}(t,q)C_{\emptyset\mu_f\mu_b}(t,q) \times C_{\mu_f^t\mu_1\tilde{\mu}_b}(t,q)C_{\emptyset\mu_1^t\emptyset}(q,t), \tag{2.2}$$

$$\mathcal{Z}^{(2)}_{\mu_b\tilde{\mu}_b} \coloneqq \sum_{\mu_f,\mu_2}(-Q_f)^{|\mu_f|}(-Q_2)^{|\mu_2|}\tilde{f}^{-1}_{\mu_f}(t,q)C_{\mu_f\emptyset\mu_b^t}(q,t) \times C_{\mu_2\mu_f^t\tilde{\mu}_b^t}(q,t)C_{\mu_2^t\emptyset\emptyset}(t,q). \tag{2.3}$$

(2.2) and (2.3) correspond to the building blocks on the left side and right sides in Fig. 1(b), respectively. We also define the refined topological vertex $C_{\lambda\mu\nu}(t,q)$ and the framing factors $f_\mu(t,q)$, $\tilde{f}_\mu(t,q)$ as follows:

$$C_{\lambda\mu\nu}(t,q) = t^{-\frac{\|\mu^t\|^2}{2}}q^{\frac{\|\mu\|^2+\|\nu\|^2}{2}}\tilde{Z}_\nu(t,q) \times \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})s_{\mu/\eta}(t^{-\nu^t}q^{-\rho}), \tag{2.4}$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu}(1-q^{\nu_i-j}t^{\nu_j^t-i+1})^{-1}, \tag{2.5}$$

$$f_\mu(t,q) = (-1)^{|\mu|}q^{-\frac{\|\mu\|^2}{2}}t^{\frac{\|\mu^t\|^2}{2}}, \tag{2.6}$$

$$\tilde{f}_\mu(t,q) = (-1)^{|\mu|}\left(\frac{t}{q}\right)^{\frac{|\mu|}{2}}q^{-\frac{\|\mu\|^2}{2}}t^{\frac{\|\mu^t\|^2}{2}}, \tag{2.7}$$

where the function $s_{\lambda/\eta}(x_1,x_2,\ldots)$ is the skew Schur function. By using some formulas from the appendixes, we obtain

$$\mathcal{Z}^{(1)}_{\mu_b\tilde{\mu}_b} = \tilde{Z}_{\mu_b}(t,q)\tilde{Z}_{\tilde{\mu}_b}(t,q)q^{\frac{\|\mu_b\|^2+\|\tilde{\mu}_b\|^2}{2}} \times \prod_{i,j=1}^{\infty}\frac{(1-Q_1t^{-\tilde{\mu}^t_{b,j}+i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_1Q_ft^{-\mu^t_{b,j}+i-\frac{1}{2}}q^{j-\frac{1}{2}})}{(1-Q_ft^{-\mu^t_{b,j}+i}q^{-\tilde{\mu}_{b,i}+j-1})}, \tag{2.8}$$

$$\mathcal{Z}^{(2)}_{\mu_b\tilde{\mu}_b} = \tilde{Z}_{\mu_b^t}(q,t)\tilde{Z}_{\tilde{\mu}_b^t}(q,t)t^{\frac{\|\mu_b^t\|^2+\|\tilde{\mu}_b^t\|^2}{2}} \times \prod_{i,j=1}^{\infty}\frac{(1-Q_2t^{-\tilde{\mu}^t_{b,j}+i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_2Q_ft^{-\mu^t_{b,j}+i-\frac{1}{2}}q^{j-\frac{1}{2}})}{(1-Q_ft^{-\mu^t_{b,j}+i-1}q^{-\tilde{\mu}_{b,i}+j})}. \tag{2.9}$$

In order to clarify the discussion in the next section, we normalize (2.8) and (2.9) by dividing the trivial building blocks $\mathcal{Z}^{(1)}_{\emptyset\emptyset}$ and $\mathcal{Z}^{(2)}_{\emptyset\emptyset}$. Again, by using some formulas, we obtain

$$\begin{aligned}\hat{\mathcal{Z}}^{(1)}_{\mu_b\tilde{\mu}_b} &\coloneqq \mathcal{Z}^{(1)}_{\mu_b\tilde{\mu}_b}/\mathcal{Z}^{(1)}_{\emptyset\emptyset}\\ &= \tilde{Z}_{\mu_b}(t,q)\tilde{Z}_{\tilde{\mu}_b}(t,q)q^{\frac{\|\mu_b\|^2+\|\tilde{\mu}_b\|^2}{2}}\\ &\quad\times\prod_{(i,j)\in\mu_b}\frac{1-Q_1Q_ft^{-i+\frac{1}{2}}q^{\mu_{b,i}-j+\frac{1}{2}}}{1-Q_ft^{\tilde{\mu}^t_{b,j}-i+1}q^{\mu_{b,i}-j}}\\ &\quad\times\prod_{(i,j)\in\tilde{\mu}_b}\frac{1-Q_1t^{-i+\frac{1}{2}}q^{\tilde{\mu}_{b,i}-j+\frac{1}{2}}}{1-Q_ft^{-\mu^t_{b,j}+i}q^{-\tilde{\mu}_{b,i}+j-1}},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\hat{\mathcal{Z}}^{(2)}_{\mu_b\tilde{\mu}_b} &\coloneqq \mathcal{Z}^{(2)}_{\mu_b\tilde{\mu}_b}/\mathcal{Z}^{(2)}_{\emptyset\emptyset}\\ &= \tilde{Z}_{\mu_b^t}(q,t)\tilde{Z}_{\tilde{\mu}_b^t}(q,t)t^{\frac{\|\mu_b^t\|^2+\|\tilde{\mu}_b^t\|^2}{2}}\\ &\quad\times\prod_{(i,j)\in\mu_b}\frac{1-Q_2Q_ft^{-i+\frac{1}{2}}q^{\mu_{b,i}-j+\frac{1}{2}}}{1-Q_ft^{\tilde{\mu}^t_{b,j}-i}q^{\mu_{b,i}-j+1}}\\ &\quad\times\prod_{(i,j)\in\tilde{\mu}_b}\frac{1-Q_2t^{-i+\frac{1}{2}}q^{\tilde{\mu}_{b,i}-j+\frac{1}{2}}}{1-Q_ft^{-\mu^t_{b,j}+i-1}q^{-\tilde{\mu}_{b,i}+j}}.\end{aligned} \tag{2.11}$$

Thus, the partition function $\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q)$ is as follows:

$$\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q) = \mathcal{Z}^{(1)}_{\emptyset\emptyset}\mathcal{Z}^{(2)}_{\emptyset\emptyset}\sum_{\mu_b,\tilde{\mu}_b}(-Q_b)^{|\mu_b|}(-\tilde{Q}_b)^{|\tilde{\mu}_b|} \times f^2_{\tilde{\mu}_b}(t,q)\hat{\mathcal{Z}}^{(1)}_{\mu_b\tilde{\mu}_b}\hat{\mathcal{Z}}^{(2)}_{\mu_b\tilde{\mu}_b}. \tag{2.12}$$

B. Nonperturbative free energy of the topological string

Now we define the perturbative free energy of the refined topological string $F(Q;t,q)$ and the unrefined topological string $F_{\mathrm{WS}}(Q;q)$ as follows:

$$F(Q;t,q) \coloneqq -\log[\mathcal{Z}(Q;t,q)], \qquad F_{\mathrm{WS}}(Q;q) = F(Q;q). \tag{2.13}$$

Then we define the perturbative parts of the nonperturbative free energy as

$$F_{\mathrm{WS}}(\mathrm{e}^{-2\pi t_1/h+\pi\mathrm{i}},\mathrm{e}^{-2\pi t_2/h-\pi\mathrm{i}},\mathrm{e}^{-2\pi t_b/h-\pi\mathrm{i}},\mathrm{e}^{-2\pi t_f/h};\mathrm{e}^{4\pi^2\mathrm{i}/\hbar}). \tag{2.14}$$

Note that we redefine the Kähler parameters due to the HMO cancellation.

The nonperturbative parts of the nonperturbative free energy are obtained by using the Nekrasov-Shatashvili limit of the refined topological string [2],

$$F_{\mathrm{NS}}(Q;q) \coloneqq \lim_{\epsilon_2\to 0}\epsilon_2 F(Q;t,q) \quad (q=\mathrm{e}^{\epsilon_1}, t=\mathrm{e}^{-\epsilon_2}). \tag{2.15}$$

Then the nonperturbative parts of the free energy $F_{\mathrm{M2}}(\mathbf{t};\hbar)$ are defined as follows:

$$F_{\mathrm{M2}}(\mathbf{t};\hbar) = \frac{\mathrm{i}}{2\pi}\left[\mathbf{t}\cdot\frac{\partial}{\partial\mathbf{t}}F_{\mathrm{NS}}(\mathrm{e}^{-t_i};\mathrm{e}^{\mathrm{i}\hbar}) + \hbar^2\frac{\partial}{\partial\hbar}(F_{\mathrm{NS}}(\mathrm{e}^{-t_i};\mathrm{e}^{\mathrm{i}\hbar})/h)\right], \tag{2.16}$$

where we define

$$\hbar \coloneqq 4\pi^2/g_s \tag{2.17}$$

and $\mathbf{t} \coloneqq (t_1, t_2, t_b, t_f)$.

Thus, the nonperturbative free energy of the topological string on the local $\mathcal{B}_3$ $J_{\mathrm{Local}\mathcal{B}_3}(\mathbf{t};\hbar)$ is as follows:

$$J_{\mathrm{Local}\mathcal{B}_3}(\mathbf{t};\hbar) \coloneqq F_{\mathrm{WS}}(\mathrm{e}^{-2\pi t_1/h+\pi\mathrm{i}}, \mathrm{e}^{-2\pi t_2/h-\pi\mathrm{i}}, \mathrm{e}^{-2\pi t_b/h-\pi\mathrm{i}}, \mathrm{e}^{-2\pi t_f/h}; \mathrm{e}^{4\pi^2\mathrm{i}/\hbar}) + F_{\mathrm{M2}}(\mathbf{t};\hbar). \tag{2.18}$$

In this case, $J_{\mathrm{Local}\mathcal{B}_3}(\mathbf{t};\hbar)$ is finite for any g_s or $\hbar$. For example, when we set $\hbar = 2\pi$, we obtain

$$\begin{aligned}
\lim_{\hbar\to 2\pi} J_{\mathrm{Local}\mathcal{B}_3}(\mathbf{t};\hbar)
&= -\frac{2+\pi^2+2t_1+t_1^2}{8\pi^2}\mathrm{e}^{-t_1} - \frac{2+\pi^2+2t_2+t_2^2}{8\pi^2}\mathrm{e}^{-t_2} \\
&\quad + \frac{2+\pi^2+2t_b+t_b^2}{8\pi^2}\mathrm{e}^{-t_b} + \frac{2+2t_f+t_f^2}{4\pi^2}\mathrm{e}^{-t_f} \\
&\quad - \frac{2+\pi^2+2t_1+2t_f+2t_1t_f+t_1^2+t_f^2}{8\pi^2}\mathrm{e}^{-t_1-t_f} \\
&\quad - \frac{2+\pi^2+2t_2+2t_f+2t_2t_f+t_2^2+t_f^2}{8\pi^2}\mathrm{e}^{-t_2-t_f} \\
&\quad - \frac{2+9\pi^2+2t_b-6t_f-6t_bt_f+t_b^2-3t_f^2}{8\pi^2}\mathrm{e}^{-t_b-t_f} + \dots.
\end{aligned} \tag{2.19}$$

A more general discussion for this pole cancellation appears in Ref. [3].

III. GEOMETRIC TRANSITION IN THE NONPERTURBATIVE TOPOLOGICAL STRING

In this section, we consider the geometric transition.

We first consider the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold. In order to know how to set the Kähler parameters for this geometric transition, we consider the geometric transition in the perturbative topological string at the beginning. After consideration, we consider the geometric transition in the nonperturbative topological string. Then we find that the nonperturbative free energy after the geometric transition agrees with the one on the resolved conifold which is obtained in Ref. [9]. We also find that the Kähler parameters are corrected by the nonperturbative effects.

We next consider the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold with a toric A-brane. As with the above case, we consider the geometric transition in the perturbative topological string at the beginning. After consideration, we consider the geometric transition in the nonperturbative topological string. Then we find that the nonperturbative parts of this free energy after the geometric transition have the same structure as the one in Ref. [27].

A. Geometric transition from the local $\mathcal{B}_3$ to the resolved conifold

In this subsection, we consider the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold in the closed string (see Fig. 2).

1. Perturbative free energy

To begin with, we consider how to set the Kähler parameters to special values in the refined topological string. In accordance with Refs. [36–39], we set the parameters Q_1 and Q_2 as follows:

$$Q_1 = \sqrt{\frac{t}{q}}, \qquad Q_2 = \sqrt{\frac{q}{t}}. \tag{3.1}$$

Then the factors in $\hat{\mathcal{Z}}^{(1)}_{\mu_b\tilde{\mu}_b}$

$$\prod_{(i,j)\in\tilde{\mu}_b}(1-Q_1 t^{-i+\frac{1}{2}}q^{\tilde{\mu}_{b,i}-j+\frac{1}{2}}) = \prod_{(i,j)\in\tilde{\mu}_b}(1-t^{-i+1}q^{\tilde{\mu}_{b,i}-j}) \tag{3.2}$$

become zero unless the Young diagram $\tilde{\mu}_b$ becomes empty. Then, after several cancellations, we obtain

$$\begin{aligned}
\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q) &= \prod_{i,j=1}^{\infty}(1-t^i q^{j-1})(1-t^{i-1}q^j) \\
&\quad \times \sum_{\mu_b}(-Q_b)^{|\mu_b|}\tilde{Z}_{\mu_b}(t,q) \\
&\quad \times \tilde{Z}_{\mu_b^t}(q,t) q^{\frac{\|\mu_b\|^2}{2}} t^{\frac{\|\mu_b^t\|^2}{2}}.
\end{aligned} \tag{3.3}$$

This expression agrees with the partition function of the closed topological string on the resolved conifold, except for the factors $\prod_{i,j=1}^{\infty}(1-t^iq^{j-1})(1-t^{i-1}q^j)$.[1] Thus, in an

[1]This difference is due to the normalization of the topological vertex.

unrefined case, this geometric transition occurs when we set

$$Q_1 = Q_2 = 1. \tag{3.4}$$

The same can be said of the perturbative free energy about the relation between the partition function and the free energy.

2. Nonperturbative free energy

We next consider the nonperturbative free energy. Then, by considering the HMO cancellation mechanism, the Kähler parameters of the perturbative parts are replaced as follows:

$$t_1 \to -2\pi t_1/\hbar + \pi \mathrm{i}, \tag{3.5}$$

$$t_2 \to -2\pi t_2/\hbar - \pi \mathrm{i}, \tag{3.6}$$

$$t_f \to -2\pi t_f/\hbar - \pi \mathrm{i}. \tag{3.7}$$

Thus, in order to consider the above geometric transition in the perturbative part, we set the Kähler parameters as follows:

$$-\frac{2\pi t_1}{\hbar} + \pi \mathrm{i} = 0, \tag{3.8}$$

$$-\frac{2\pi t_2}{\hbar} - \pi \mathrm{i} = 0. \tag{3.9}$$

By using the relation (2.17), we obtain

$$t_1 = \frac{2\pi^2 \mathrm{i}}{g_s}, \tag{3.10}$$

$$t_2 = -\frac{2\pi^2 \mathrm{i}}{g_s}. \tag{3.11}$$

This means that the Kähler parameters are corrected by the nonperturbative parts. Then the nonperturbative parts $F_{\mathrm{M2}}(\mathbf{t};\hbar)$ become

$$\begin{aligned} F_{\mathrm{M2}}(\mathbf{t};\hbar)|_{t_1,t_2\,\mathrm{fixed}} &= f(\hbar) + \frac{(\hbar/2)\cos[\frac{\hbar}{2}] + (1+t_b)\sin[\frac{\hbar}{2}]}{4\pi \sin^2[\frac{\hbar}{2}]} \mathrm{e}^{-t_b} \\ &+ \frac{\hbar \cos[\hbar] + (1+2t_b)\sin[\hbar]}{16\pi \sin^2[\hbar]} \mathrm{e}^{-2t_b} \\ &\cdots, \end{aligned} \tag{3.12}$$

where $f(\hbar)$ is the function which is independent of the Kähler parameters.[2] Thus, this function corresponds to the factors $\prod_{i,j=1}^{\infty}(1 - t^i q^{j-1})(1 - t^{i-1} q^j)$. This expression agrees with the one on the resolved conifold which is obtained in Ref. [9] except for $f(\hbar)$. Therefore, we conclude that the geometric transition can be applied, even if we consider the nonperturbative topological string.

[2] It would be interesting to consider the meaning of $f(\hbar)$ in terms of the constant map. We would like to thank Sanefumi Moriyama for discussing it with us.

B. Geometric transition from the local $\mathcal{B}_3$ to the resolved conifold with a toric A-brane

In this subsection, we consider the geometric transition from the closed topological string on the local $\mathcal{B}_3$ to the open topological string on the resolved conifold with a toric A-brane.

1. Perturbative free energy

We consider again the refined topological string at the beginning. In accordance with Refs. [36–39], we set the parameters Q_1 and Q_2 as follows:

$$Q_1 = \sqrt{\frac{t}{q}}, \qquad Q_2 = t\sqrt{\frac{q}{t}} \tag{3.13}$$

Then, as we said in the previous section, the Young diagram $\tilde{\mu}_b$ becomes empty. Thus, by using the expressions (2.10) and (2.11), we obtain

$$\begin{aligned} \mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q) &= \prod_{i,j=1}(1 - t^i q^j)(1 - t^i q^{j-1}) \\ &\times \sum_{\mu_b} (-Q_b)^{|\mu_b|} \tilde{Z}_{\mu_b}(t,q) \\ &\times \tilde{Z}_{\mu_b^t}(q,t) q^{\frac{\|\mu_b\|^2}{2}} t^{\frac{\|\mu_b^t\|^2}{2}} \prod_{j=1}^{\infty} \frac{1}{1 - Q_f t^{-\mu_{b,j}^t} q^j}. \end{aligned} \tag{3.14}$$

In order to check for consistency, we calculate the partition function of the open topological string on the resolved conifold with a toric A-brane. Its web diagram is shown in Fig. 3. Then we write the partition function from the web diagram as follows:

$$\mathcal{Z}_{\mathrm{open}} = \sum_{\text{all indices}} (-Q_b)^{|\mu|} C_{\emptyset\emptyset\mu}(t,q) C_{\mu'\emptyset\mu^t}(q,t) \mathrm{Tr}_{\mu''} V, \tag{3.15}$$

where V is the holonomy matrix. Since there is a single A-brane, the matrix V is the one by one matrix,

$$V = \mathrm{diag}(z). \tag{3.16}$$

Then we obtain

$$\begin{aligned} \mathcal{Z}_{\mathrm{open}} &= \sum_{\mu} (-Q_b)^{|\mu|} \tilde{Z}_{\mu}(t,q) \tilde{Z}_{\mu^t}(q,t) q^{\frac{\|\mu\|^2}{2}} t^{\frac{\|\mu^t\|^2}{2}} \\ &\times \prod_{j=1}^{\infty} \frac{1}{1 - z t^{-\mu_j^t + \frac{1}{2}} q^{j-1}}. \end{aligned} \tag{3.17}$$

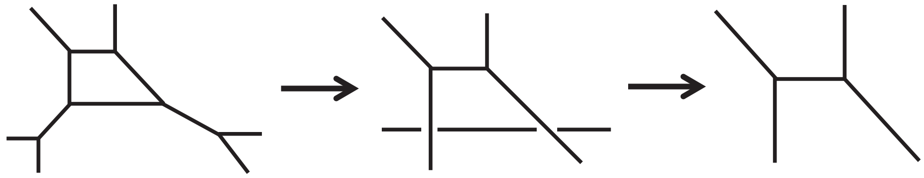

FIG. 2. The geometric transition in the closed topological string.

Then (3.14) agrees with (3.17) under the relation between z and Q_f,

$$z = t^{-\frac{1}{2}} q Q_f, \tag{3.18}$$

except for the factors $\prod_{i,j=1}(1-t^i q^j)(1-t^i q^{j-1})$. Thus, in the perturbative topological string, the geometric transition occurs when we set

$$Q_1 = 1, \qquad Q_2 = q. \tag{3.19}$$

2. Nonperturbative free energy

We now consider the nonperturbative free energy of the open topological string. With the above discussion in mind, we set the Kähler parameters for the geometric transition in the perturbative parts:

$$t_1 = \frac{2\pi^2 \mathrm{i}}{g_s}, \tag{3.20}$$

$$t_2 = -\frac{2\pi^2 \mathrm{i}}{g_s} - 2\pi \mathrm{i}. \tag{3.21}$$

Then we obtain the nonperturbative effects $F_{\mathrm{M2}}(\mathbf{t};\hbar)$:

$$\begin{aligned} F_{\mathrm{M2}}(\mathbf{t};\hbar)|_{t_1,t_2\text{fixed}} &= g(\hbar) + \frac{(\hbar/2)\cos[\frac{\hbar}{2}] + (1+t_b)\sin[\frac{\hbar}{2}]}{4\pi \sin^2[\frac{\hbar}{2}]} \mathrm{e}^{-t_b} \\ &+ \frac{\hbar\cos[\hbar] + (1+2t_b)\sin[\hbar]}{16\pi\sin^2[\hbar]}\mathrm{e}^{-2t_b} + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\hbar/2}}{2\sin[\frac{\hbar}{2}]}\mathrm{e}^{-t_f} \\ &+ \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\hbar}}{2\sin[\frac{\hbar}{2}]}\mathrm{e}^{-t_b-t_f} + \left(\frac{\mathrm{i}\mathrm{e}^{5\mathrm{i}\hbar/2}}{2\sin[\hbar]} + \frac{\mathrm{i}\mathrm{e}^{3\mathrm{i}\hbar/2}}{2\sin[\hbar]}\right)\mathrm{e}^{-t_b-2t_f} \\ &- \left(\frac{\mathrm{i}\mathrm{e}^{2\mathrm{i}\hbar}}{4\sin[\hbar]} + \frac{\mathrm{i}\mathrm{e}^{3\mathrm{i}\hbar}}{2\sin[\hbar]}\right)\mathrm{e}^{-2t_b-2t_f} + \cdots, \end{aligned} \tag{3.22}$$

where $g(\hbar)$ is the function which is independent of the Kähler parameters. Moreover, according to the relation (3.18), we set the correspondence between Q_f and z as follows:

$$-\frac{2\pi}{\hbar}t_f + \frac{2\pi^2\mathrm{i}}{\hbar} = -\frac{2\pi}{\hbar}x \Leftrightarrow t_f = x + \pi\mathrm{i}, \tag{3.23}$$

where we define $z = \mathrm{e}^{-x}$. Thus, we obtain

$$\begin{aligned} F_{\mathrm{M2}}(\mathbf{t};\hbar)|_{t_1,t_2\text{fixed}} &= g(\hbar) + \frac{(\hbar/2)\cos[\frac{\hbar}{2}] + (1+t_b)\sin[\frac{\hbar}{2}]}{4\pi \sin^2[\frac{\hbar}{2}]} \mathrm{e}^{-t_b} \\ &+ \frac{\hbar\cos[\hbar] + (1+2t_b)\sin[\hbar]}{16\pi\sin^2[\hbar]}\mathrm{e}^{-2t_b} + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\hbar/2}}{2\sin[\hbar]}(-\mathrm{e}^{-x}) \\ &+ \frac{\mathrm{i}\mathrm{e}^{\hbar}}{2\sin[\frac{\hbar}{2}]}\mathrm{e}^{-t_b}(-\mathrm{e}^{-x}) + \left(\frac{\mathrm{i}\mathrm{e}^{5\mathrm{i}\hbar/2}}{2\sin[\hbar]} + \frac{\mathrm{i}\mathrm{e}^{3\mathrm{i}\hbar/2}}{2\sin[\hbar]}\right)\mathrm{e}^{-t_b}(-\mathrm{e}^{-x})^2 \\ &- \left(\frac{\mathrm{i}\mathrm{e}^{2\mathrm{i}\hbar}}{4\sin[\hbar]} + \frac{\mathrm{i}\mathrm{e}^{3\mathrm{i}\hbar}}{2\sin[\hbar]}\right)\mathrm{e}^{-2t_b}(-\mathrm{e}^{-x})^2 + \cdots. \end{aligned} \tag{3.24}$$

Then we find that the nonperturbative parts of the open topological string have the same structure as the one in Ref. [27]. We check this up to second order of the Kähler parameters.

Thus, we obtain the nonperturbative free energy of the open topological string on the resolved conifold with a toric A-brane,

$$J_X^{\mathrm{open}}(\mathbf{t};\hbar) \coloneqq J_X(\mathbf{t};\hbar)\big|_{t_1=\frac{2\pi^2\mathrm{i}}{g_s}, t_2=-\frac{2\pi^2\mathrm{i}}{g_s}-2\pi\mathrm{i}}. \tag{3.25}$$

This free energy is finite for any g_s or $\hbar$ because of the HMO cancellation mechanism. For example, when we set $\hbar = 2\pi$, the free energy $J_X^{\mathrm{open}}(\mathbf{t};\hbar)$ becomes

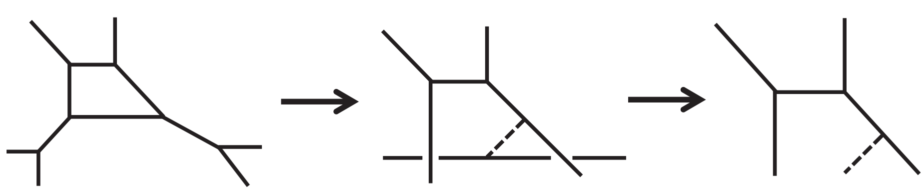

FIG. 3. The geometric transition from the closed topological string to the open topological string.

$$\begin{aligned}
&\lim_{\hbar\to 2\pi} J_X^{\rm open}(\mathbf{t};\hbar)\\
&=\frac{-9+24\pi^2+10\pi\mathrm{i}}{16\pi^2}+\frac{2\pi+\mathrm{i}(1+t_f)}{2\pi}\mathrm{e}^{-t_f}\\
&\quad+\frac{4\pi+\pi\mathrm{i}(1+2t_f)}{8\pi^2}\mathrm{e}^{-2t_f}+\frac{2+\pi^2+2t_b-t_b^2}{8\pi^2}\mathrm{e}^{-t_b}\\
&\quad-\frac{3\pi+\mathrm{i}(1+t_b+t_f)}{2\pi}\mathrm{e}^{-t_b-t_f}\\
&\quad-\frac{1+2\pi^2+2t_b(1+t_b)}{32\pi^2}\mathrm{e}^{-2t_b}+\cdots, \qquad (3.26)
\end{aligned}$$

where $t_f = x - \pi\mathrm{i}$.

IV. SUMMARY AND FUTURE WORK

In this paper, we have considered the geometric transition in the nonperturbative topological string in two cases. One is the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold. The other is the geometric transition from the local $\mathcal{B}_3$ to the resolved conifold with a toric A-brane. Then we have found that the geometric transition can be applied, even if the nonperturbative effects are included. We have also found that the Kähler parameters are corrected by the nonperturbative effects. In the open topological string, the nonperturbative free energy which we have obtained has had the same structure as the one in Ref. [27].

We have various areas of future work. First of all, considering the general formula of the nonperturbative open topological string would be interesting. Its structure would be similar to the free energy of the closed topological string which is derived in Ref. [1].

In this paper, we have considered the open topological string with a toric A-brane. Now we want to consider the open topological string in the presence of N A-branes. We would be able to use the geometric transition to obtain this free energy. In this case, we set the Kähler parameters as follows:

$$t_1 = \frac{2\pi^2\mathrm{i}}{g_s}, \qquad (4.1)$$

$$t_2 = -\frac{2\pi^2\mathrm{i}}{g_s} - 2N\pi\mathrm{i}, \qquad (4.2)$$

$$-\frac{2\pi}{\hbar}t_f + \frac{4\pi^2\mathrm{i}}{\hbar}\left(j-\frac{1}{2}\right) = -\frac{2\pi}{\hbar}x_j, \quad (j=1,2,\ldots,N), \qquad (4.3)$$

where the variables x_j correspond to the positions of the A-branes. The justification of this parameter choices would be important.

In terms of the mirror curve, we consider the mirror curve of genus 1 in this paper. Then, naively, we can derive the mirror curve of genus 0 by using the geometric transition. On the other hand, according to Ref. [17], the quantization of the mirror curve relates to the free energy of the topological string on the toric Calabi-Yau manifold associated with the mirror curve. However, there is a crucial problem. For the quantization of the mirror curve of genus 0, the expectation value of the trace class operator which is defined by the quantization of the mirror curve diverges since the spectrum of this operator is continuous. Then, by using our result and mirror symmetry [40,41], we might obtain the finite expectation value for the mirror curve of genus 0 by setting some parameters in the expectation value of the mirror curve of genus 1, whose expectation value is finite.

The above discussion about the mirror curve can be applied to the open string. Then, the free energy of the open topological string might be able to relate to the quantization of the mirror curve. It would be interesting to also consider this.

ACKNOWLEDGMENTS

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APPENDIX A: DEFINITIONS AND SOME FORMULAS

In this appendix, we summarize the definitions and the formulas which we use in this paper.

(i) The refined topological vertex is

$$\begin{aligned}
C_{\lambda\mu\nu}(t,q) &= t^{-\frac{\|\mu^t\|^2}{2}} q^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} \tilde{Z}_\nu(t,q)\\
&\quad\times\sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho}),\\
\tilde{Z}_\nu(t,q) &= \prod_{(i,j)\in\nu}(1-q^{\nu_i-j}t^{\nu_j^t-i+1})^{-1}.\\
&\left(|\mu| := \sum_{i=1}^{l(\mu)}\mu_i, \|\mu\|^2 := \sum_{i=1}^{l(\mu)}\mu_i^2\right). \qquad (A1)
\end{aligned}$$

(ii) The gluing factors are

$$\begin{aligned}
f_\mu(t,q) &= (-1)^{|\mu|} q^{-\frac{\|\mu\|^2}{2}} t^{\frac{\|\mu^t\|^2}{2}},\\
\tilde{f}_\mu(t,q) &= (-1)^{|\mu|}\left(\frac{t}{q}\right)^{\frac{|\mu|}{2}} q^{-\frac{\|\mu\|^2}{2}} t^{\frac{\|\mu^t\|^2}{2}}, \qquad (A2)
\end{aligned}$$

where $s_{\mu/\eta}(x_1, x_2, \ldots)$ is the skew Schur function. We also define the Young diagram μ as in Fig. 4. When we set $t = q$, the refined topological vertex becomes the unrefined topological vertex.

We also include some useful formulas.

(i) Some formulas pertaining to the Schur polynomial are

$$s_{\lambda/\mu}(\alpha\mathbf{x}) = \alpha^{|\lambda|-|\mu|} s_{\lambda/\mu}(\mathbf{x}), \qquad (A3)$$

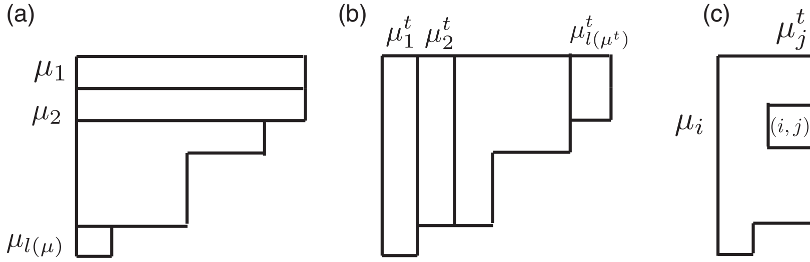

FIG. 4. The Young diagram. Definitions are for (a) μ_i, (b) μ_j^t, and (c) the coordinate (i,j).

$$\sum_\eta s_{\eta/\lambda}(\mathbf{x}) s_{\eta/\mu}(\mathbf{y}) = \prod_{i,j=1}^{\infty}(1-x_i y_j)^{-1} \sum_\tau s_{\mu/\tau}(\mathbf{x}) s_{\lambda/\tau}(\mathbf{y}), \tag{A4}$$

$$\sum_\eta s_{\eta^t/\lambda}(\mathbf{x}) s_{\eta/\mu}(\mathbf{y}) = \prod_{i,j=1}^{\infty}(1+x_i y_j) \sum_\tau s_{\mu^t/\tau}(\mathbf{x}) s_{\lambda^t/\tau^t}(\mathbf{y}), \tag{A5}$$

(ii) Normalization formulas include

$$\prod_{i,j=1}^{\infty} \frac{1-Qq^{\nu_i-j}t^{\mu_j^t-i+1}}{1-Qq^{-j}t^{-i+1}} = \prod_{(i,j)\in\nu}(1-Qq^{\nu_i-j}t^{\mu_j^t-i+1}) \prod_{(i,j)\in\mu}(1-Qq^{-\mu_i+j-1}t^{-\nu_j^t+i}), \tag{A6}$$

$$\prod_{i,j=1}^{\infty} \frac{1-Qt^{\nu_j^t-i+\frac{1}{2}}q^{-j+\frac{1}{2}}}{1-Qt^{-i+\frac{1}{2}}q^{-j+\frac{1}{2}}} = \prod_{(i,j)\in\nu}(1-Qq^{-j+\frac{1}{2}}t^{i-\frac{1}{2}}), \tag{A7}$$

$$\prod_{i,j=1}^{\infty} \frac{1-Qq^{\nu_i-j+\frac{1}{2}}t^{-i+\frac{1}{2}}}{1-Qq^{-j+\frac{1}{2}}t^{-i+\frac{1}{2}}} = \prod_{(i,j)\in\nu}(1-Qq^{j-\frac{1}{2}}t^{-i+\frac{1}{2}}). \tag{A8}$$

APPENDIX B: TOPOLOGICAL STRING ON LOCAL $\mathcal{B}_3$

1. Partition function of the refined topological string

We write the partition function of the refined topological string on the local $\mathcal{B}_3$ for certain orders explicitly,

$$\begin{aligned}
\mathcal{Z}_{\mathrm{Local}\mathcal{B}_3}(Q;t,q) = {} & \prod_{i,j=1}^{\infty} \frac{(1-Q_1 t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_2 t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_1 Q_f t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_1 Q_f t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})}{(1-Q_f t^{i-1}q^{j})(1-Q_f t^{i}q^{j-1})} \\
& \times \left\{1 - \left[\frac{(tq)^{\frac{1}{2}}(1-Q_1Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})}{(1-t)(1-q)(1-Q_f)(1-Q_f t^{-1}q)} + \frac{t^{\frac{3}{2}}q^{-\frac{1}{2}}(1-Q_1 t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2 t^{-\frac{1}{2}}q^{\frac{1}{2}})}{(1-t)(1-q)(1-Q_f)(1-Q_f tq^{-1})}Q_f\right]Q_b\right. \\
& + \left[\left(\frac{t^2q(1-Q_1Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_1Q_f t^{-\frac{3}{2}}q^{\frac{1}{2}})(1-Q_2Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2Q_f t^{-\frac{3}{2}}q^{\frac{1}{2}})}{(1-t)(1-t^2)(1-tq)(1-q)(1-Q_f)(1-Q_f t^{-1})(1-Q_f t^{-1}q)(1-Q_f t^{-2}q)}\right.\right. \\
& \left. + \frac{tq^2(1-Q_1Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_1Q_f t^{-\frac{1}{2}}q^{\frac{3}{2}})(1-Q_2Q_f t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2Q_f t^{-\frac{1}{2}}q^{\frac{3}{2}})}{(1-q)(1-q^2)(1-tq)(1-t)(1-Q_f)(1-Q_f q)(1-Q_f qt^{-1})(1-Q_f q^2t^{-1})}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^2(1-Q_1t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_1Q_ft^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2Q_ft^{-\frac{1}{2}}q^{\frac{1}{2}})}{(1-t)^2(1-q)^2(1-Q_ft)(1-Q_fq)(1-Q_ft^{-1})(1-Q_fq^{-1})}Q_f\\
&+\left(\frac{t^6q^{-1}(1-Q_1t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_1t^{-\frac{3}{2}}q^{\frac{1}{2}})(1-Q_2t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2t^{-\frac{3}{2}}q^{\frac{1}{2}})}{(1-t)(1-t^2)(1-tq)(1-q)(1-Q_f)(1-Q_ft)(1-Q_ftq^{-1})(1-Q_ft^2q^{-1})}\right.\\
&\left.+\frac{t^3q^{-2}(1-Q_1t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_1t^{-\frac{1}{2}}q^{\frac{3}{2}})(1-Q_2t^{-\frac{1}{2}}q^{\frac{1}{2}})(1-Q_2t^{-\frac{1}{2}}q^{\frac{3}{2}})}{(1-q)(1-q^2)(1-tq)(1-t)(1-Q_f)(1-Q_fq^{-1})(1-Q_ftq^{-1})(1-Q_ftq^{-2})}\right)Q_f^2\Bigg]Q_b^2+\cdots\Bigg\}. \qquad \text{(B1)}
\end{aligned}
$$

In this paper, we use this expression.

2. Free energy of the topological string

a. Free energy of the unrefined topological string

The free energy of the unrefined closed topological string on the local $\mathcal{B}_3$ is as follows:

$$
\begin{aligned}
F_{\mathrm{WS}}(Q;q)=&\left(\sum_{m=1}^{\infty}\frac{Q_1^m+Q_2^m+(Q_1Q_m)^m+(Q_2Q_m)^m-2Q_f^m}{m(q^{\frac{m}{2}}-q^{-\frac{m}{2}})^2}\right)+\left(\frac{1}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}+\frac{3}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_f+\frac{5}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_f^2\right.\\
&-\frac{2}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f-\frac{4}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f^2-\frac{2}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f-\frac{4}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f^2+\frac{1}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_2Q_f\\
&\left.+\frac{3}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_2Q_f^2\right)Q_b+\left(\frac{1}{2(q-q^{-1})^2}-\left(\frac{6}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}-\frac{3}{2(q-q^{-1})^2}\right)Q_f^2+\frac{5}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f^2\right.\\
&\left.-\frac{2}{2(q-q^{-1})^2}Q_2^2Q_f^2+\frac{5}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f^2-\frac{2}{2(q-q^{-1})^2}Q_1^2Q_f^2-\frac{4}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_2Q_f^2\right)Q_b^2+\cdots. \qquad \text{(B2)}
\end{aligned}
$$

b. Free energy of the topological string in the NS limit

The NS limit for the free energy of the closed topological string on the local $\mathcal{B}_3$ is as follows:

$$
\begin{aligned}
F_{\mathrm{NS}}(Q;q)=&\left(\sum_{m=1}^{\infty}\frac{Q_1^m+Q_2^m+(Q_1Q_m)^m+(Q_2Q_m)^m}{m^2(q^{\frac{m}{2}}-q^{-\frac{m}{2}})}-\sum_{m=1}^{\infty}\frac{Q_f^m(q^m-q^{-m})}{m^2(q^{\frac{m}{2}}-q^{-\frac{m}{2}})^2}\right)\\
&+\left(\frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}+\frac{q^{\frac{3}{2}}-q^{-\frac{3}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_f+\frac{q^{\frac{5}{2}}-q^{-\frac{5}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_f^2-\frac{q-q^{-1}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f-\frac{q-q^{-1}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f\right.\\
&\left.-\frac{q^2-q^{-2}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f^2-\frac{q^2-q^{-2}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f^2+\frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}Q_1Q_2Q_f+\frac{q^{\frac{3}{2}}-q^{-\frac{3}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_2Q_f^2\right)Q_b\\
&+\left(\frac{1}{4(q-q^{-1})}+\left(\frac{q^2-q^{-2}}{(q-q^{-1})^2}+\frac{7(q^3-q^{-3})}{4(q-q^{-1})^2}+\frac{q^4-q^{-4}}{(q-q^{-1})^2}\right)Q_f^2+\frac{q^{\frac{5}{2}}-q^{-\frac{5}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_f^2\right.\\
&+\frac{q^{\frac{5}{2}}-q^{-\frac{5}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_2Q_f^2-\frac{q^2-q^{-2}}{4(q-q^{-1})^2}Q_1^2Q_f^2-\frac{q^2-q^{-2}}{4(q-q^{-1})^2}Q_2^2Q_f^2-\frac{q^2-q^{-2}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}Q_1Q_2Q_f^2\\
&\left.+\frac{1}{4(q-q^{-1})}Q_1^2Q_2^2Q_f^2\right)Q_b^2+\ldots. \qquad \text{(B3)}
\end{aligned}
$$

APPENDIX C: PROOF OF THE GEOMETRIC TRANSITION FROM THE LOCAL $\mathcal{B}_3$ TO THE RESOLVED CONIFOLD

In this section, we show that the nonperturbative parts of the free energy of the local $\mathcal{B}_3$ become the free energy of the resolved conifold after setting the Kähler parameters to (3.8) and (3.9).

We first consider some derivatives. We define

$$W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q) \coloneqq \prod_{(i,j)\in\mu_b} \frac{1-Q_1Q_f t^{-i+\frac{1}{2}} q^{\mu_{b,i}-j+\frac{1}{2}}}{1-Q_f t^{\tilde{\mu}^t_{b,j}-i+1} q^{\mu_{b,i}-j}} \times \prod_{(i,j)\in\tilde{\mu}_b} \frac{1-Q_1 t^{-i+\frac{1}{2}} q^{\tilde{\mu}_{b,i}-j+\frac{1}{2}}}{1-Q_f t^{-\mu^t_{b,j}+i} q^{-\tilde{\mu}_{b,i}+j-1}}, \quad \text{(C1)}$$

$$W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q) \coloneqq \prod_{(i,j)\in\mu_b} \frac{1-Q_2Q_f t^{-i+\frac{1}{2}} q^{\mu_{b,i}-j+\frac{1}{2}}}{1-Q_f t^{\tilde{\mu}^t_{b,j}-i} q^{\mu_{b,i}-j+1}} \times \prod_{(i,j)\in\tilde{\mu}_b} \frac{1-Q_2 t^{-i+\frac{1}{2}} q^{\tilde{\mu}_{b,i}-j+\frac{1}{2}}}{1-Q_f t^{-\mu^t_{b,j}+i-1} q^{-\tilde{\mu}_{b,i}+j}}. \quad \text{(C2)}$$

These expressions are factors of $\hat{\mathcal{Z}}^{(1)}_{\mu_b\tilde{\mu}_b}$ and $\hat{\mathcal{Z}}^{(2)}_{\mu_b\tilde{\mu}_b}$. The Nekrasov-Shatashvili limit does not affect $W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q)$, $W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q)$ since they do not have the pole. Then, by performing the derivatives, we obtain

$$[\partial_{t_1}[\lim_{\epsilon_2\to 0} W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q)W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q)]]|_{t_1,\ t_2\ \text{fixed}} = \sum_{\mu_b} \frac{Q_f q^{\mu_{b,i}-j}}{1-Q_f q^{\mu_{b,i}-j}} + \frac{1}{1-Q_f q^{-1}} \times \prod_{(i,j)\in\tilde{\mu}_b,\tilde{\mu}_{b,i}-j\neq 0} \frac{1-q^{\tilde{\mu}_{b,i}-j}}{1-Q_f q^{-\tilde{\mu}_{b,i}+j-1}} \prod_{(i,j)\in\tilde{\mu}_b} \frac{1-q^{\tilde{\mu}_{b,i}-j+1}}{1-Q_f q^{-\tilde{\mu}_{b,i}+j}}, \quad \text{(C3)}$$

$$[\partial_{t_2}[\lim_{\epsilon_2\to 0} W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q)W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q)]]|_{t_1,\ t_2\ \text{fixed}} = \sum_{\mu_b} \frac{Q_f q^{\mu_{b,i}-j+1}}{1-Q_f q^{\mu_{b,i}-j+1}}, \quad \text{(C4)}$$

$$[\partial_{t_f}[\lim_{\epsilon_2\to 0} W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q)W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q)]]|_{t_1,\ t_2\ \text{fixed}} = 0, \quad \text{(C5)}$$

$$[\hbar\partial_{\hbar}[\lim_{\epsilon_2\to 0} W^{(1)}_{\mu_b\tilde{\mu}_b}(Q;t,q)W^{(2)}_{\mu_b\tilde{\mu}_b}(Q;t,q)]]|_{t_1,\ t_2\ \text{fixed}} = \hbar\Bigg\{-\frac{\mathrm{i}}{2}\sum_{\mu_b} \frac{Q_f q^{\mu_{b,i}-j}}{1-Q_f q^{\mu_{b,i}-j}} + \frac{\mathrm{i}}{2}\sum_{\mu_b} \frac{Q_f q^{\mu_{b,i}-j+1}}{1-Q_f q^{\mu_{b,i}-j+1}} - \frac{\mathrm{i}}{2}\frac{1}{1-Q_f q^{-1}} \prod_{(i,j)\in\tilde{\mu}_b,\tilde{\mu}_{b,i}-j\neq 0} \frac{1-q^{\tilde{\mu}_{b,i}-j}}{1-Q_f q^{-\tilde{\mu}_{b,i}+j-1}} \times \prod_{(i,j)\in\tilde{\mu}_b} \frac{1-q^{\tilde{\mu}_{b,i}-j+1}}{1-Q_f q^{-\tilde{\mu}_{b,i}+j}}\Bigg\}. \quad \text{(C6)}$$

Then, we can show that the contributions in $F_{\rm M2}(\mathbf{t},\hbar)$ which come from the above terms cancel out each other. Thus, we obtain

$$F_{\rm M2}(\mathbf{t};\hbar) = \frac{\mathrm{i}}{2\pi}\left[t_b\frac{\partial}{\partial t_b}\tilde{F}_{\rm NS}(\mathrm{e}^{-t_b};\mathrm{e}^{\mathrm{i}\hbar}) + \hbar^2\frac{\partial}{\partial\hbar}(\tilde{F}_{\rm NS}(\mathrm{e}^{-t_b};\mathrm{e}^{\mathrm{i}\hbar})/h)\right]\Bigg| + f(\hbar), \quad \text{(C7)}$$

where we define

$$\tilde{F}_{\rm NS}(\mathrm{e}^{-t_b};\mathrm{e}^{\mathrm{i}\hbar}) = \lim_{\epsilon_2\to 0}\epsilon_2 \log\left[\sum_{\mu_b}(-Q_b)^{|\mu_b|}\tilde{Z}_{\mu_b}(t,q)\tilde{Z}_{\mu^t_b} \times (q,t) q^{\frac{\|\mu_b\|^2}{2}} t^{\frac{\|\mu^t_b\|^2}{2}}\right]. \quad \text{(C8)}$$

This expression agrees with the nonperturbative parts of the free energy of the resolved conifold, except for $f(\hbar)$.

For $\mathcal{Z}^{(1)}_{\emptyset\emptyset}$ and $\mathcal{Z}^{(2)}_{\emptyset\emptyset}$, we can easily show that these contributions become parts of $f(\hbar)$ by using the following formula:

$$\prod_{i,j=1}^{\infty}(1-Qt^{i-\frac{1}{2}}q^{j-\frac{1}{2}}) = \exp\left[-\sum_{m=1}^{\infty}\frac{1}{m}\frac{Q^m}{(t^{m/2}-t^{-m/2})(q^{m/2}-q^{-m/2})}\right]. \quad \text{(C9)}$$

APPENDIX D: BUBBLING CALABI-YAU

In this section, we show that the partition function of the refined open topological string on the conifold with N toric A-branes agrees with the one of the refined closed topological string on the local $\mathcal{B}_3$ under some correspondence. This phenomenon is known as the bubbling Calabi-Yau [36,38].

Let us consider the following web diagram. We write the partition function of the refined open topological string by using the refined topological vertex formalism,

$$\mathcal{Z}_{\rm open} = \sum_{\mu} C_{\emptyset\mu\emptyset}(q,t)\mathrm{Tr}_{\mu^t}V. \quad \text{(D1)}$$

By using the formula

$$\mathrm{Tr}_{\mu}V = s_{\mu}(x), \qquad V = \mathrm{diag}[z_1,z_2,\ldots,z_N], \quad \text{(D2)}$$

we obtain

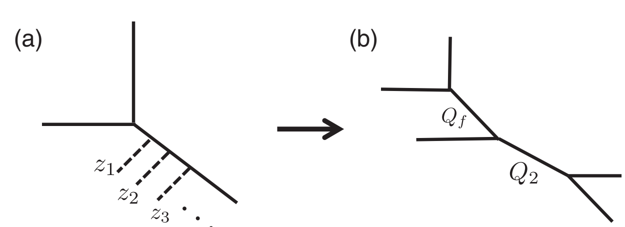

FIG. 5. The web diagram. The figure (a) is the web diagram with the toric A-branes. This web diagram becomes the web diagram (b) via the geometric transition. This phenomenon is known as the bubbling Calabi-Yau.

$$\mathcal{Z}_{\text{open}} = \prod_{j=1}^{\infty}\prod_{i=1}^{N}\frac{1}{1 - z_i t^{\frac{1}{2}} q^{j-1}}. \tag{D3}$$

We next consider the web diagram in Fig. 5(b). The partition function of this diagram is

$$\mathcal{Z}_{\text{closed}} := \sum_{\mu_f,\mu_2}(-Q_f)^{|\mu_f|}(-Q_2)^{|\mu_2|}\tilde{f}_{\mu_f}^{-1}(t,q)C_{\mu_f\emptyset\emptyset}(q,t) \times C_{\mu_2\mu_f^t\emptyset}(q,t)C_{\mu_2^t\emptyset\emptyset}(t,q). \tag{D4}$$

After some calculation, we obtain

$$\mathcal{Z}_{\text{closed}} = \prod_{i,j=1}\frac{(1 - Q_2 t^{i-\frac{1}{2}} q^{j-\frac{1}{2}})(1 - Q_2 Q_f t^{i-\frac{1}{2}} q^{j-\frac{1}{2}})}{(1 - Q_f t^{i-1} q^j)}. \tag{D5}$$

We then set

$$Q_2 = t^N\sqrt{\frac{q}{t}}, \tag{D6}$$

$$Q_f q t^{i-\frac{3}{2}} = z_i (i = 1, 2, \ldots, N). \tag{D7}$$

The partition function (D3) agrees with (D5) except for the factors $\prod_{i,j=1}(1 - t^{i+N-1}q^j)$. This difference is due to the difference of normalization between the refined Chern-Simons theory and the refined topological vertex. This is the refinement of the bubbling Calabi-Yau.

The web diagram we consider in Sec. III corresponds to the case of $N = 1$.

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