PHYSICAL REVIEW D **94,** 054036 (2016)

Pion scalar form factor with correct mass and width of scalar mesons $f_0(500)$ and $f_0(980)$

Stanislav Dubnicka,[1] Anna Zuzana Dubnickova,[2] Robert Kamiński,[3] and Andrej Liptaj[1]

[1]*Institute of Physics, Slovak Academy of Sciences, 84511 Bratislava, Slovak Republik*
[2]*Department of Theoretical Physics, Comenius University, 84248 Bratislava, Slovak Republik*
[3]*Henryk Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, 31-342 Krakow, Poland*

(Received 4 February 2016; revised manuscript received 5 September 2016; published 26 September 2016)

Construction of the pion scalar isoscalar form factor $\Gamma_\pi(t)$ in the elastic region, with an emphasis on the values of the S-wave isoscalar $\pi\pi$ scattering length a_0^0 and the quadratic pion scalar radius $\langle r^2\rangle_s^\pi$ to be in conformity with predictions of the chiral perturbation theory, is presented. It is based on a precise S-wave isoscalar $\pi\pi$ scattering phase shift generated by dispersive analysis of experimental data with and imposed crossing symmetry condition. The final result for values of the $f_0(500)$ scalar meson mass and width is $m_\sigma = (487 \pm 31)$ MeV; $\Gamma_\sigma = (542 \pm 60)$ MeV and for values of the $f_0(980)$ scalar meson mass and width is $m_{f(980)} = (988 \pm 78)$ MeV; $\Gamma_{f(980)} = (97 \pm 29)$ MeV. The $f_0(500)$ scalar meson parameters are compatible with the results from dispersive analyses of the BERN and MADRID/CRACOW groups to be considered now as the most reliable values of the $f_0(500)$ scalar meson parameters, though in presented analysis another, unusual way has been applied. The $f_0(980)$ meson parameters agree well with values given by the Particle Data Group.

DOI: 10.1103/PhysRevD.94.054036

I. INTRODUCTION

The lightest hadronic resonance with vacuum quantum numbers 0^{++}, $f_0(500)$, is the most controversial particle from the whole spectrum of existing scalar mesons [1]. From the first identification of this particle in 1974 a lot of work has been done and many papers concerned with this scalar meson have been published up to now. However, only recently a clarification of this controversial situation with $f_0(500)$ has been achieved in the papers of the BERN [2] and of the MADRID/CRACOW group [3], which are now considered to be the most reliable determinations of the $f_0(500)$ scalar meson parameters.

In the paper [4] another method for a determination of the $f_0(500)$ scalar meson parameters by means of the pion scalar form factor (FF) analysis in the elastic region has been elaborated. However, there was no pretension on the precision of the obtained results as the published inaccurate, and in some region even contradicting, experimental data on the S-wave isoscalar $\pi\pi$-scattering phase shift have been exploited. So, the obtained mass and width of the $f_0(500)$ scalar meson in [4] can be maximally considered as an indication for an existence of $f_0(500)$, however to be far away from the true parameters of this particle. The same can be said for the $f_0(980)$ meson whose parameters have been determined to be far from those given by the Particle Data Group [1].

In this paper analysis of the pion scalar FF at the elastic region is carried out with true S-wave isoscalar $\pi\pi$ scattering phase shift data with theoretical errors, which have been generated by the Garcia-Martin-Kamiński-Pela'ez-Yndurain (MADRID/CRACOW group) [3] Roy-like equations. Moreover, a more simple method of a calculation of the integral under consideration, in comparison with that in [4], is applied for finding an explicit form of the pion scalar form factor at the elastic region.

Nevertheless, in a fitting procedure of such S-wave isoscalar $\pi\pi$ scattering phase shift data with five free parameters the obtained value of the S-wave isoscalar $\pi\pi$ scattering length a_0^0 and the quadratic pion scalar radius, following from the explicit form of the obtained in such a way the pion scalar isoscalar FF, is not in agreement with predictions of the chiral perturbation theory (χPT) [5], which is revealed to be a precision theory of the low energy hadron physics. Therefore we have repeated the analysis with a fixed value $a_0^0 = 0.220 \pm 0.005$ of the S-wave isoscalar $\pi\pi$ scattering length and also by a strict requirement of the quadratic pion scalar radius to be $\langle r^2\rangle_s^\pi = (0.63\text{–}0.65)\ \mathrm{fm}^2$ in accordance with predictions of the χPT.

As a result we have found that no more minimum of χ^2/ndf is achieved with five free parameters, but only with seven parameters, which have resulted in appearance of two additional complex conjugate and very near to the real axis zeros on the first sheet of the Riemann surface in t-variable. In this way the result of the paper [6] is reproduced, in a completely different way.

In the next section all known properties of the pion scalar FF $\Gamma_\pi(t)$ are summarized and an explicit form of the pion scalar FF phase representation with one subtraction is presented, which leads to a quadratic pion scalar radius as the rapidly convergent integral also through the phase δ_Γ of the pion scalar FF.

Section III contains a detailed derivation of the $\tan\delta_\Gamma$ parametrization in order to demonstrate that its further application is not due to its simplicity, but it is a rigorous consequence of the analyticity, unitarity and the reality condition of the pion scalar FF.

In Sec. IV the true S-wave isoscalar $\pi\pi$ scattering phase shift data with theoretical errors to be generated by the MADRID/CRACOW group Roy-like equations without any restriction on the values of a_0^0 and $\langle r^2\rangle_s^\pi$ are analyzed.

In Sec. V a more simple calculation of the corresponding integrals than in the previous analysis [4] are carried out.

Section VI is devoted to the repeated analysis of the true S-wave isoscalar $\pi\pi$ scattering phase shift data, however with strict restriction on the values of the S-wave isoscalar $\pi\pi$ scattering length a_0^0 and the quadratic pion scalar radius $\langle r^2\rangle_s^\pi$, which follow from the χPT. Here also the resultant behavior of the pion scalar FF in the elastic region is presented graphically and compared with the pion scalar FF from [7] to be drawn by a dashed line.

Conclusions are given in the last section.

II. PION SCALAR FORM FACTOR AND ITS PHASE REPRESENTATION

The pion scalar FF $\Gamma_\pi(t)$ is defined by the parametrization of the matrix element of the scalar quark density

$$\langle\pi^i(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|\pi^j(p_1)\rangle = \delta^{ij}\Gamma_\pi(t), \tag{1}$$

where $t=(p_2-p_1)^2$ and $\hat{m}=\frac{1}{2}(m_u+m_d)$.

It possesses all known properties of the pion electromagnetic FF $F_\pi(t)$ like

(i) analyticity in the t-plane besides cuts on the positive real axis from two-pion threshold $t=4m_\pi^2$ to $+\infty$;

(ii) elastic unitarity condition $\mathrm{Im}\Gamma_\pi(t)=\Gamma_\pi(t)e^{-i\delta_0^0}\sin\delta_0^0$, where $\delta_0^0(t)$ is the S-wave isoscalar $\pi\pi$ scattering phase shift;

(iii) asymptotic behavior $\Gamma_\pi(t)_{|t|\to\infty}\sim\frac{1}{t}$;

(iv) reality condition $\Gamma_\pi^*(t)=\Gamma_\pi(t^*)$;

(v) normalization, however, now to the pion sigma term value $\Gamma_\pi(0)=(0.99\pm0.02)m_\pi^2$ to be predicted by the χPT [8].

The analyticity of the pion scalar FF $\Gamma_\pi(t)$ in the t-plane together with its asymptotic behavior allow one to derive (through the Cauchy formula) a dispersion relation with one subtraction at the pion sigma term value $\Gamma_\pi(0)=m_\pi^2$ (further we take $m_\pi=1$), which in combination with the elastic pion scalar FF unitarity condition leads to the pion scalar FF phase representation with one subtraction,

$$\Gamma_\pi(t)=P_n(t)\exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^\infty\frac{\delta_\Gamma(t')}{t'(t'-t)}dt'\right], \tag{2}$$

to be the starting point for our further investigations.

The one-subtracted pion scalar FF phase representation (2) ensures that the dominant contribution to $\Gamma_\pi(t)$ gives the low-energy part of the integral and the same integral is convergent independently if the pion scalar FF phase takes asymptotically nonzero real value or it is zero at ∞.

The rapidly convergent representation for the quadratic pion scalar radius,

$$\langle r^2\rangle_s^\pi=\frac{6}{\pi}\int_{4m_\pi^2}^\infty\frac{\delta_\Gamma(t')}{t'^2}dt', \tag{3}$$

follows directly from the relation (2).

III. PARAMETRIZATION OF THE PION SCALAR FORM FACTOR PHASE δ_Γ

The pion scalar FF $\Gamma_\pi(t)$ on the positive real axis in the t-plane for $t>4m_\pi^2$ is a complex function and its phase δ_Γ is determined by behaviors of the $\mathrm{Im}\Gamma_\pi(t)$ and the $\mathrm{Re}\Gamma_\pi(t)$ through the relation

$$\tan\delta_\Gamma(t)=\frac{\mathrm{Im}\Gamma_\pi(t)}{\mathrm{Re}\Gamma_\pi(t)}. \tag{4}$$

But the pion scalar FF $\Gamma_\pi(t)$ is an analytic function in the whole complex t-plane besides branch points on the positive real axis, where the lowest one at $t=4m_\pi^2$, corresponding to the opening of two-pion channel, is a square-root type. The latter can be demonstrated by the analytic continuation of $\Gamma_\pi(t)$ through upper and lower boundaries of the two-pion cut to the second Riemann sheet, utilizing the pion scalar FF elastic unitarity condition, and obtaining the identical functional expressions. Then by an application of the relation for the absolute value of the pion c.m. three-momentum

$$q=[(t-4)/4]^{1/2}, \qquad m_\pi=1, \tag{5}$$

the two-sheeted Riemann surface of $\Gamma_\pi(t)$, generated by the branch point $t=4m_\pi^2$, is mapped into one q-plane and the elastic cut $4m_\pi^2<t<16m_\pi^2$ disappears. If we take into account results of the phenomenological analysis of the $\pi\pi$ reactions [9] that final states containing more than two particles start playing a significant role only well above $t=4m_K^2$, where the inelastic two-body channel $\pi\pi\to K\bar{K}$ opens, the latter elastic region can be extended up to $t\approx1\ \mathrm{GeV}^2$.

Noticing the conformal mapping (5) in more detail, the first sheet in the t-variable, containing only branch points and zeros of $\Gamma_\pi(t)$, is mapped into the upper half of the q-plane, whereby the branch point $t=4m_\pi^2$ and the normalization point $t=0$ are mapped into $q=0$ and $q=+i$, respectively, and the real axis from $-\infty$ up to $t=4m_\pi^2$, on which $\Gamma_\pi(t)$ is a real function, is mapped into the positive imaginary axis of the q-plane.

The second Riemann sheet in the t-variable, containing branch points, zeros and also complex conjugate pairs of poles, which control the shape of $\Gamma_\pi(t)$, is mapped into the lower half of the q-plane.

If we restrict ourselves just to the elastic region and neglect contributions to $\Gamma_\pi(t)$ of all branch points beyond 1 GeV^2, then there are only zeros in the q-plane and poles exclusively in the lower-half plane, which can be taken into account as roots of the polynomials in the numerator and the denominator of $\Gamma_\pi(t)$. As a result $\Gamma_\pi(t)$ can be represented in the form of the following rational function:

$$\Gamma_\pi(t) = \frac{\sum_{n=0}^{M} a_n q^n}{\sum_{r=0}^{N} b_r q^r}. \tag{6}$$

Multiplying the numerator and denominator by the complex conjugate denominator Eq. (6) is changed to the form

$$\Gamma_\pi(t) = \frac{\sum_{s=0}^{M+N} c_s q^s}{(\sum_{r=0}^{N} b_r q^r)(\sum_{r=0}^{N} b_r q^r)^*}. \tag{7}$$

The reality condition $\Gamma_\pi(t^*) = \Gamma_\pi^*(t)$ results in the reality of (7) on the positive imaginary axis of the q-plane. One can see immediately that the expression

$$\Gamma_\pi(t) = \frac{(c_0 + c_2 q^2 + c_4 q^4 + \cdots) + i(c_1 q + c_3 q^3 + c_5 q^5 \cdots)}{(\sum_{r=0}^{N} b_r q^r)(\sum_{r=0}^{N} b_r q^r)^*} \tag{8}$$

fulfils the latter claim, which leads through (4) to the following parametrization of the pion scalar FF phase $\delta_\Gamma(t)$,

$$\tan\delta_\Gamma(t) = \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \cdots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \cdots}, \tag{9}$$

with all coefficients to be real.

Now, utilizing the equality between the phase of $\Gamma_\pi(t)$ and the S-wave isoscalar $\pi\pi$ scattering phase shift $\delta_0^0(t)$,

$$\delta_\Gamma(t) \equiv \delta_0^0(t), \tag{10}$$

following directly from the elastic unitarity condition of the pion scalar FF, one obtains from (9) very effective parametrization of $\delta_0^0(t)$ at the elastic region

$$\delta_0^0(t) = \arctan\frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \cdots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \cdots}, \tag{11}$$

where the parameter A_1 is identical with the S-wave isoscalar $\pi\pi$-scattering length a_0^0.

IV. TRUE S-WAVE ISOSCALAR $\pi\pi$ SCATTERING PHASE SHIFT DATA

In order to find an explicit form of the correct pion scalar FF at the elastic region by means of the relation (2) one is in need of the true S-wave isoscalar $\pi\pi$ scattering phase shift data.

These have been obtained in dispersive data analysis of the MADRID/CRACOW group [10] in which amplitudes for the S and P waves were fitted simultaneously to experimental data and to dispersion relations constrained by the crossing symmetry condition with one subtraction (so-called GKPY equations).

In another dispersive data analysis of the BERN group [11], very elegant from a mathematical point of view, analytical solutions for the S and P amplitudes between the $\pi\pi$ threshold and 800 MeV were presented. The authors used Roy's equations [12], which need two subtractions one of which is a linear function of squared energy. It leads to larger uncertainties of the output amplitudes (i.e. phase shifts in the elastic region) than those from the GKPY equations.

Therefore, in order to minimize, in our analysis, the uncertainties of the calculated form factor, we decided to use results of the GKPY equations presented in [10]. The authors of this analysis parametrized all important amplitudes (i.e. for the S, P, D and F waves), in the phenomenological region below 1420 MeV where available are data for the phase shifts and inelasticities. Parametrizations were model independent i.e. purely mathematical without any physical bias. In case of quite complex scalar isoscalar amplitude, the whole region below 1420 MeV was divided into three parts (below 850 MeV, up to $K\bar{K}$ threshold and above) and was described by three parametrizations matched smoothly at 850 MeV and at the $K\bar{K}$ threshold (i.e. values of the phase shifts and their first derivatives).

In the first step of analysis in [10], such constructed amplitudes for all S-F partial waves were fitted separately to the corresponding experimental data sets. Obtained amplitudes were subsequently used in the GKPY equations as input. In the second step the output amplitudes for the S and P waves from these equations were used in a simultaneous fit to the experimental data and to GKPY equations. In the minimization procedure the χ^2 function, which was used in the first step, was supplemented by squared differences between input and output amplitudes for the S and P waves (strictly saying between their real parts) with appropriate weights (see [10]).

In results, those fits delivered a set of all important $\pi\pi$ partial wave amplitudes well describing experimental data including S and P wave amplitudes fulfilling also crossing symmetry. It is also important to notice that although only these two amplitudes were directly fitted to the GKPY equations, the amplitudes for all other waves were also fitted to them in an indirect way. As was proved in [10], the phase shifts calculated in this way have much smaller errors than those obtained in experimental analyses [15] and then those constrained by the twice subtracted Roy's equations. At, for

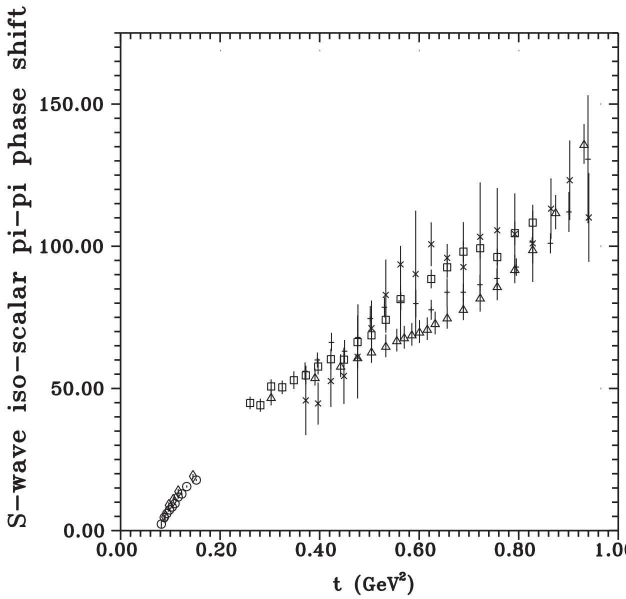

FIG. 1. Existing unprecise data on S-wave isoscalar $\pi\pi$ scattering phase shift [15].

example 800 MeV, uncertainties of the experimental data (including also systematic differences between various data sets) are about 6 times larger that those from GKPY dispersive analysis [13].

In the preliminary work [4] determination of parameters A_1–A_5 and position of the σ and $f_0(980)$ poles has been performed using fits to dispersed experimental data presented in Fig. 1. It resulted in positions of these poles considerably different from presented in the Particle Data tables [1]. In this paper we use precise phase shifts from [10] fully representing (in elastic region) amplitudes fulfilling crossing symmetry, in order to obtain a true behavior of the δ_0^0 phase shift and consequently also more reliable values of $f_0(500)$ and $f_0(980)$ parameters.

In the analysis of the output phase shifts we used 72 points at every 10 MeV between 280 and 990 MeV. They are presented in Fig. 2.

These data have been analyzed by the relation (11) up to the moment, when the minimum of χ^2/ndf was achieved. The latter has been found, as in [4], with first five nonzero coefficients A_i, which, however, now take the following values:

$$A_1 = 0.2219 \pm 0.0029$$
$$A_2 = -0.0764 \pm 0.0423$$
$$A_3 = 0.1390 \pm 0.0251$$
$$A_4 = -0.0062 \pm 0.0053$$
$$A_5 = -0.0135 \pm 0.0020$$

and the roots of the corresponding polynomials in the numerator and denominator of equivalent form

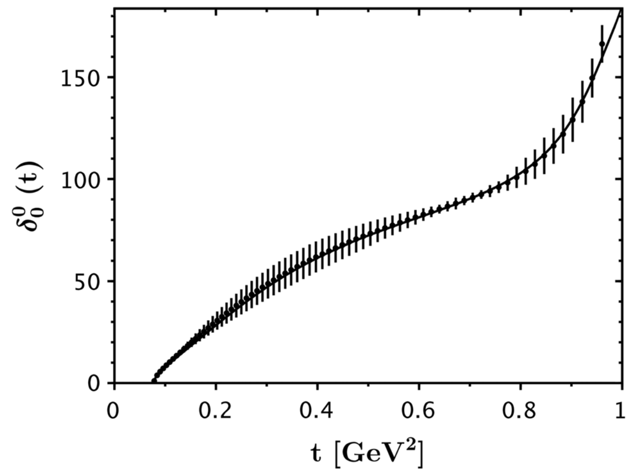

FIG. 2. The data on $\delta_0^0(t)$ from [10] with theoretical errors to be generated by parametrization of the GKPY equations for the S-wave isoscalar $\pi\pi$ scattering amplitude in [10]. The solid line represents our fit with (11).

$$\delta_0^0(t) = \frac{1}{2i} \ln \left[\frac{(1 + A_2 q^2 + A_4 q^4) + i(A_1 q + A_3 q^3 + A_5 q^5)}{(1 + A_2 q^2 + A_4 q^4) - i(A_1 q + A_3 q^3 + A_5 q^5)} \right] \tag{12}$$

to (11) are

$$q_1 = 0.00 \quad - i2.0430 \pm 0.2029$$
$$q_2 = 3.3827 \pm 0.0115 \quad + i0.1744 \pm 0.0340$$
$$q_3 = -3.3827 \pm 0.0115 \quad + i0.1744 \pm 0.0340$$
$$q_4 = 1.41470 \pm 0.0579 \quad + i1.0749 \pm 0.0162$$
$$q_5 = -1.4147 \pm 0.0579 \quad + i1.0749 \pm 0.0162$$

$$q_1^* = -q_1$$
$$q_2^* = -q_3$$
$$q_3^* = -q_2$$
$$q_4^* = -q_5$$
$$q_5^* = -q_4.$$

V. SIMPLE CALCULATION OF THE CORRESPONDING INTEGRALS

The substitution of $\delta_0^0(t)$ of the form (11) into (2), with the numerical values of the coefficients $A_1, \ldots A_5$, leads to the expression which does not allow one to calculate the corresponding integral explicitly. Therefore we have used the equivalent form (12) to (11).

Then

$$\Gamma_\pi(t) = P_n(t)\exp\left[\frac{(q^2+1)}{2\pi i} \times \int_{-\infty}^{\infty} \frac{q' \ln\frac{(1+A_2q'^2+A_4q'^4)+i(A_1q'+A_3q'^3+A_5q'^5)}{(1+A_2q'^2+A_4q'^4)-i(A_1q'+A_3q'^3+A_5q'^5)}}{(q'^2+1)(q'^2-q^2)} dq'\right], \tag{13}$$

and the integral

$$I = \int_{-\infty}^{\infty} \frac{q' \ln\frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq'$$

$$q^2 < 0 \text{ i.e. } q = i\sqrt{\frac{4-t}{4}} \equiv ib \tag{14}$$

now can be calculated in the framework of the theory of residua explicitly.

In order to carry it out practically, it is convenient to decompose the integral into a sum of two integrals,

$$I = I_1 + I_2 = \int_{-\infty}^{\infty} \frac{q' \ln\frac{(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq' + \int_{-\infty}^{\infty} \frac{q' \ln\frac{(q'-q_1)}{(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq', \tag{15}$$

according to singularities to be placed in the upper or lower half q-plane, as it is sketched in Fig. 3.

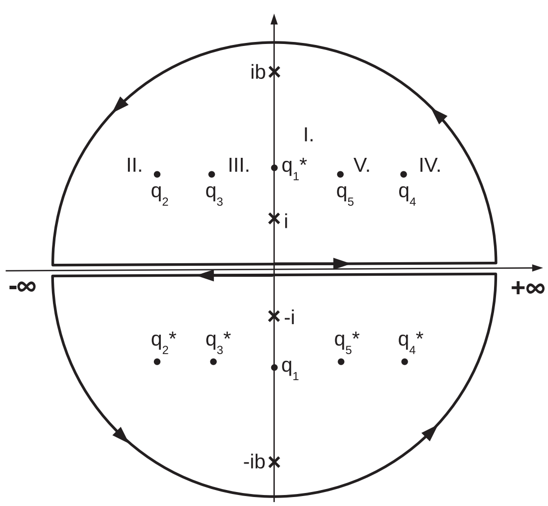

FIG. 3. Poles (times) and branch points (filled circle) of the integrands $\phi_1(q')$ and $\phi_2(q')$ with contours of integrations in the upper and the lower half q-planes, respectively.

Then the explicit form of

$$I = \frac{2\pi i}{(q^2+1)} ln\left(\frac{(q-q_1^*)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \times \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)}\right) \tag{16}$$

is obtained in the straightforward way, if in the case of the first integral

$$\oint \phi_1(q')dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n \tag{17}$$

the contour of integration is closed in the lower half q-plane and in the second integral

$$\oint \phi_2(q')dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n \tag{18}$$

the contour of integration is closed in the upper half q-plane (see Fig. 3).

In a such way one avoids complicated calculations of the cut contributions to be carried out in [4], which are automatically generated by branch points under logarithms.

The substitution of (16) into (13) leads to the explicit form of the pion scalar FF,

$$\Gamma_\pi(t) = P_n(t)\frac{(q-q_1^*)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \times \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)}, \tag{19}$$

where $P_n(t)$ is any polynomial normalized at $t=0$ to one, however, it has not violated the asymptotic behavior of the pion scalar FF.

In a similar way by means of the theory of residua one finds from (3) also an explicit form of the quadratic pion scalar radius

$$\langle r^2\rangle_s^\pi = \frac{3}{4}\left[\frac{i}{i-q_2^*}+\frac{i}{i-q_3^*}+\frac{i}{i-q_4^*}+\frac{i}{i-q_5^*}-\frac{i}{i-q_1}\right], \tag{20}$$

which gives the value $\langle r^2\rangle_s^\pi = 0.77\ \mathrm{fm}^2$.

All positions of the poles are in pion mass = 139.57 MeV and the errors correspond to maximal deviations of all A_i parameters from their central values.

The pole $q = q_3^*$ on the second Riemann sheet in the t-variable corresponds to the $f_0(500)$ meson resonance, now with the mass and the width, $m_\sigma = (459 \pm 29)$ MeV and $\Gamma_\sigma = (517 \pm 77)$ MeV, respectively, which are compatible with the parameters obtained in [2,3]. The pole $q = q_2^*$ represents $f_0(980)$ with parameters $m_{f_{980}} = (985 \pm 82)$ MeV and $\Gamma_{f_{980}} = (93 \pm 34)$ MeV, respectively.

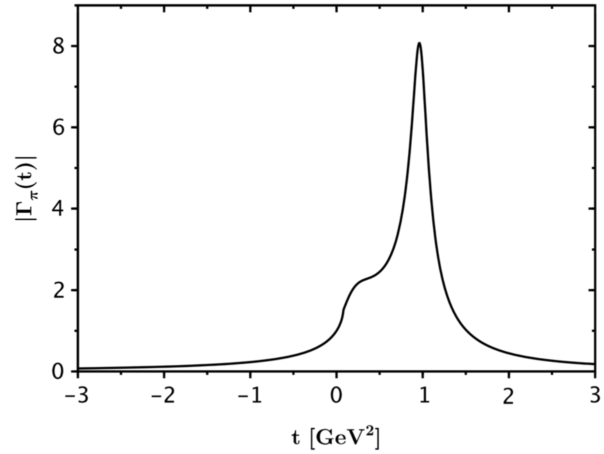

FIG. 4. Behavior of the pion scalar form factor (19) with one zero and four poles in the region $-3\ \mathrm{GeV}^2 < t < 3\ \mathrm{GeV}^2$. Results correspond to fit to the output phase shifts from [10] by five free parameters. Physically may be interpreted region only below 990 MeV ($t \approx 0.98\ \mathrm{GeV}^2$), which corresponds to the elastic region.

A behavior of the $\Gamma_\pi(t)$ (19) at the interval $-3\ \mathrm{GeV}^2 < t < 3\ \mathrm{GeV}^2$ is presented in Fig. 4.

Clearly seen are contributions of both the $f_0(500)$ and $f_0(980)$ poles and normalization to 1 at $t = 0$. Of course, as construction of the form factor was based on amplitudes (i.e. phase shifts) only from the elastic region, physical interpretation of the energy distribution of this form factor is also limited to this region.

There is a question if the pion scalar FF in Fig. 4 is the true FF. One cannot believe it as in its construction the S-wave isoscalar $\pi\pi$-scattering length value a_0^0 has been found not to be in conformity with the prediction of χPT and the same can be said also about the obtained value $\langle r^2\rangle_s^\pi = 0.77\ \mathrm{fm}^2$ of the quadratic pion scalar radius.

Therefore, in order to find a true pion scalar FF behavior in the elastic region, in the next section an analysis is repeated, however, now with fixed a_0^0 and $\langle r^2\rangle_s^\pi$ at the values following from χPT.

VI. REPEATED ANALYSIS OF TRUE $\delta_0^0(t)$ WITH STRICT RESTRICTION ON THE VALUES OF a_0^0 AND $\langle r^2\rangle_s^\pi$

The precision theory of the low energy hadron physics [5], the χPT, predicts the values $a_0^0 = 0.220 \pm 0.005$ and $\langle r^2\rangle_s^\pi = (0.63\text{–}0.65)\ \mathrm{fm}^2$.

If in (11) the coefficient A_1 is fixed at the value $A_1 = a_0^0 = 0.2200$ and the value of $\langle r^2\rangle_s^\pi$ in a fitting procedure of the data on δ_0^0 is required to be $\langle r^2\rangle_s^\pi = (0.63\text{–}0.65)\ \mathrm{fm}^2$, then no more the minimum of χ^2/ndf is achieved with five free parameters, but only with six free parameters $A_2, \ldots A_7$ and the result is

$$
\begin{aligned}
A_1 &= 0.2200\\
A_2 &= -0.1447 \pm 0.0096\\
A_3 &= 0.0955 \pm 0.0081\\
A_4 &= 0.0015 \pm 0.0017\\
A_5 &= -0.0148 \pm 0.0012\\
A_6 &= 0.0001 \pm 0.0001\\
A_7 &= 0.0004 \pm 0.0001
\end{aligned}
$$

$$
\begin{aligned}
q_1 &= 0.00 \quad - i2.3111 \pm 0.0581\\
q_2 &= 3.3946 \pm 0.0070 \quad + i0.1804 \pm 0.0102\\
q_3 &= -3.3946 \pm 0.0070 \quad + i0.1804 \pm 0.0102\\
q_4 &= 1.5286 \pm 0.0257 \quad + i1.1093 \pm 0.0074\\
q_5 &= -1.5286 \pm 0.0257 \quad + i1.1093 \pm 0.0074\\
q_6 &= 4.9338 \pm 0.0723 \quad - i0.0033 \pm 0.0446\\
q_7 &= -4.9338 \pm 0.0723 \quad - i0.0033 \pm 0.0446
\end{aligned}
$$

$$
\begin{aligned}
q_1^* &= -q_1\\
q_2^* &= -q_3\\
q_3^* &= -q_2\\
q_4^* &= -q_5\\
q_5^* &= -q_4\\
q_6^* &= -q_7\\
q_7^* &= -q_6.
\end{aligned}
$$

Then the explicit form of the pion scalar FF is

$$
\Gamma_\pi(t) = P_n(t)\frac{(q-q_1^*)(q-q_6^*)(q-q_7^*)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \times \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)(i-q_6^*)(i-q_7^*)} \qquad (21)
$$

and the result for the quadratic pion scalar radius reads

$$
\langle r^2\rangle_s^\pi = \frac{3}{4}\left[\frac{i}{i-q_2^*}+\frac{i}{i-q_3^*}+\frac{i}{i-q_4^*}+\frac{i}{i-q_5^*} - \frac{i}{i-q_1}-\frac{i}{i-q_6}-\frac{i}{i-q_7}\right], \qquad (22)
$$

which gives the value $\langle r^2\rangle_s^\pi = 0.63\ \mathrm{fm}^2$.

The pole $q = q_3^*$ on the second Riemann sheet in the t-variable corresponds to the $f_0(500)$ meson resonance, now with the mass and the width, $m_\sigma = (487 \pm 31)$ MeV and $\Gamma_\sigma = (542 \pm 60)$ MeV, respectively, which are compatible with the parameters obtained in [2,3]. The pole

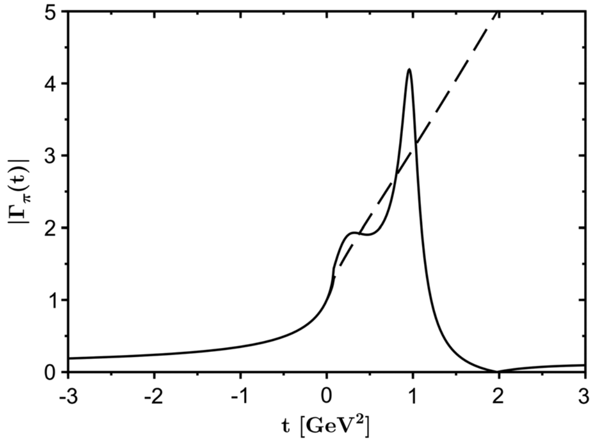

FIG. 5. Behavior of the pion scalar form factor (21) with three zeros and four poles in the region $-3\ \mathrm{GeV}^2 < t < 3\ \mathrm{GeV}^2$. Results correspond to fit to the output phase shifts from [10] by six free parameters $A_2, \ldots A_7$ if the parameter A_1 is fixed at the value of $a_0^0 = 0.22$. Physically may be interpreted region only below 990 MeV ($t \approx 0.98\ \mathrm{GeV}^2$). The dashed line represents the pion scalar FF from [7].

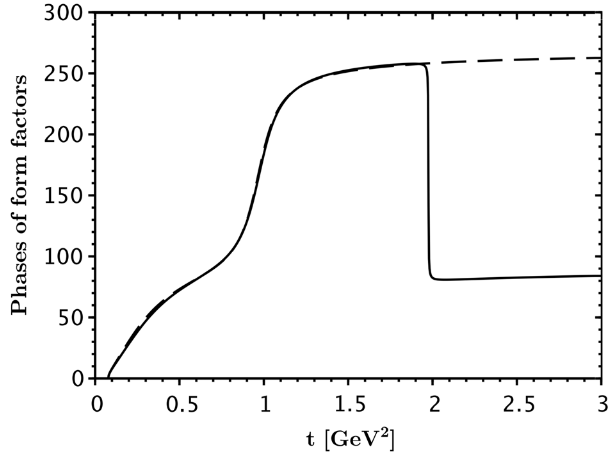

FIG. 6. Behavior of phases of the pion scalar form factors in Figs. 4 and 5. The phase of the pion scalar FF in Fig. 5, in the construction of which the values of a_0^0 and $\langle r^2\rangle_s^\pi$ are strictly required to be in conformity with the predictions of χPT is represented by a full line. The dashed line corresponds to the FF in Fig. 4, the construction of which is carried out without any restrictions.

$q = q_2^*$ represents $f_0(980)$ with parameters $m_{f_{980}} = (988 \pm 78)$ MeV and $\Gamma_{f_{980}} = (97 \pm 29)$ MeV, respectively.

The true behavior of the pion scalar FF is graphically presented in Fig. 5, where also a comparison with the pion scalar FF from [7] is carried out by the dashed line.

The phases of the FFs in Figs. 4 and 5 are compared in Fig. 6 by dashed and full lines, respectively. One can see from Fig. 6 that strong requirement of the values of a_0^0 and $\langle r^2\rangle_s^\pi$ in construction of the true pion scalar FF to be in conformity with the predictions of the χPT, leads to appearance of two conjugate zeros of the FF [see (21)] and the phase of such FF falls down beyond 2 GeV^2. So, finally the results of the papers [6] and [14] are reproduced, however, by means of the completely different way.

VII. CONCLUSIONS

The construction of the true pion scalar isoscalar FF $\Gamma_\pi(t)$ in the elastic region, with an emphasis on the values of the S-wave isoscalar $\pi\pi$ scattering length a_0^0 and the quadratic pion scalar radius $\langle r^2\rangle_s^\pi$ to be in conformity with predictions of the chiral perturbation theory, is presented.

It has been based on true S-wave isoscalar $\pi\pi$ scattering phase shift data at the elastic region with theoretical errors, which have been generated in dispersive analysis of existing experimental points with an imposed crossing symmetry condition by the Garcia-Martin-Kamiński-Pela'ez-Yndurain (MADRID/CRACOW group) [10] Roy-like equations.

For an explicit form fully solvable mathematical scheme has been exploited, however, now a more simple calculation of the corresponding integrals than in the previous analysis [4] has been found, avoiding rather complicated calculations of the cut contributions. As a result the pion scalar FF takes the form of a rational function in the absolute value of the pion c.m. three-momentum q-variable with three zeros and four poles in the lower half plane, which corresponds to the second Riemann sheet in the t-variable.

Positions of two lowest scalar mesons, $f_0(500)$ and $f_0(980)$, have been quite precisely determined from the four poles in (21). Together with three zeros, one on the imaginary axis and two conjugate according to the imaginary axis, in the absolute value of the pion c.m. three-momentum q-variable, they completely describe energy dependence of the pion scalar form factor in the full elastic region as it is demonstrated in Fig. 5.

ACKNOWLEDGMENTS

The support of the Slovak Grant Agency for Sciences VEGA under Grant No. 2/0158/13 and of the Slovak Research and Development Agency under Contract No. APVV-0463-12 is acknowledged. This work has been also partially supported by the Polish Science Center (NCN) Grant No. Dec-2013/09/B/ST2/04382.

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