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Analytic three-loop static potential

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We present analytic results for the three-loop static potential of two heavy quarks. The analytic calculation of the missing ingredients is outlined, and results for the singlet and octet potential are provided.

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I. INTRODUCTION

The static potential between two heavy quarks belongs to the fundamental quantities of QCD. In lowest order, it is described by the Coulomb potential adapted to QCD. Such an approach was already used more than 40 years ago [1] to describe the bound state of heavy quarks. Shortly afterward, the one-loop corrections were computed [2,3], and the two-loop terms were added toward the end of the 1990s [4–6]. Light quark mass effects at two loops can be found in Ref. [7]. About eight years ago, the three-loop corrections were computed by two groups in Refs. [8–10]. However, in contrast to the lower-order expressions, the three-loop results could only be presented in numerical form. In fact, in Refs. [8,9], three coefficients in the expansion of the master integrals around $d = 4$, where d is the space-time dimension, could only be evaluated numerically (see also below). The evaluation of one of them is described in detail in Ref. [11] (in a broader context), and the remaining two coefficients are considered in Sec. II of this paper. We are thus in the position to present analytic results at three loops. The corresponding expressions can be found in Sec. III.

A generalization of the three-loop singlet potential has been considered in Ref. [12]. It is still assumed that the heavy color sources form a singlet state; however, the color representation is kept general.

The new results can also be used to present analytic expressions for the so-called octet potential which describes the situation where the quark and antiquark do not form a color-singlet but a color-octet state. Two- and (numerical) three-loop results have been obtained in Refs. [13,14], and [15], respectively. Analytic results for the octet potential are presented in Sec. IV.

In order to fix the notation, we write the momentum space potential in the form

$$V^{[c]}(|\vec{q}|) = -4\pi C^{[c]} \frac{\alpha_s(|\vec{q}|)}{\vec{q}^2}\left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1^{[c]} + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2^{[c]} + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^3 \left(a_3^{[c]} + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^2}\right) + \ldots\right], \tag{1}$$

with $C^{[1]} = C_F$ for the color-singlet and $C^{[8]} = C_F - C_A/2$ for the color-octet case. Here, $C_A = N_c$ and $C_F = (N_c^2 - 1)/(2N_c)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $\mathrm{SU}(N_c)$ color gauge group, respectively. The strong coupling α_s is defined in the $\overline{\mathrm{MS}}$ scheme, and for the renormalization scale, we choose $\mu = |\vec{q}|$ in order to suppress the corresponding logarithms. The general results, both in momentum and coordinate space, can, e.g., be found in Appendix A of Ref. [15].

The logarithmic term in Eq. (1) has its origin in an infrared divergence which is present for the first time at three loops as has been pointed out in Ref. [16]. The corresponding pole has been subtracted minimally. Its presence can be understood in the context of methods of regions and potential nonrelativistic QCD [17–21] where $V^{[c]}$ appears as a matching coefficient. Thus, the infrared divergence cancels against ultraviolet divergences of the ultrasoft contributions. The latter have been studied in Refs. [20,22,23]. For the resummation of leading and next-to-leading ultrasoft logarithms, we refer to Refs. [24–26].

The three-loop coefficient a_3 only has a moderate numerical value (see, e.g., the discussion in Ref. [9]) and has thus only a relatively small influence on phenomenological quantities. This is in contrast to the two-loop coefficient which is of the same order of magnitude as a_1. However, since the static potential is a matching coefficient, it is hence not a physical quantity. In fact, a_3 is scheme dependent, and only the combination with all other building blocks leads to meaningful quantities.

For later convenience, we decompose the three-loop corrections according to the number of closed fermion loops

$$a_3^{[c]} = a_3^{[c],(3)} n_l^3 + a_3^{[c],(2)} n_l^2 + a_3^{[c],(1)} n_l + a_3^{[c],(0)}, \quad (2)$$

where n_l is the number of light (massless) quarks. We furthermore consider the difference between the singlet and octet contributions and write ($i = 0, 1, 2, 3$)

$$a_3^{[8],(i)} = a_3^{[1],(i)} + \delta a_3^{[8],(i)}. \quad (3)$$

In Sec. IV, we provide analytical results for $\delta a_3^{[8],(i)}$.

The three-loop coefficient of the color-singlet potential, $a_3^{[1]}$, has entered a number of physical applications as a building block (see also Ref. [27] for a recent review on applications of nonrelativistic QCD to high-energy processes). To name a few of them, we want to mention the next-to-next-to-next-to-leading-order corrections to the leptonic decay width of the $\Upsilon(1S)$ meson [28] and the top quark threshold production in electron positron colliders [29]. Furthermore, a_3 has entered analyses to determine precise values for the charm and bottom quark masses [30–33] and the strong coupling constant [34].

II. CALCULATION OF I_{11} AND I_{16}

The calculation of $a_3^{[1]}$ as performed in Ref. [9] requires the evaluation of 41 master integrals which can be subdivided into three different classes. There are ten integrals which do not have any static line [i.e., a propagator of the form $1/(-k_0 \pm i0)$, see also Fig. 1]. They are known since long. Furthermore, we have 14 integrals with a massless one-loop insertion. They can easily be integrated in terms of Γ functions using standard techniques. The corresponding results have been presented in Ref. [35]. Results for 16 more complicated integrals can be found in Ref. [36] as expansions in $\epsilon = (4-d)/2$ to the necessary order except for two integrals [I_{11} and I_{16} of Ref. [36]; see also Figs. 1(a) and 1(b)]. Their $\mathcal{O}(\epsilon)$ terms enter $a_3^{[1]}$; however, they were only known numerically. The evaluation of these coefficients will be described in the remainder of this section. For completeness, we want to mention that the third numerical ingredient required in Ref. [9] comes from the finite diagram in Fig. 1(c) (the 41st master integral) which has been computed in a parallel article [11].

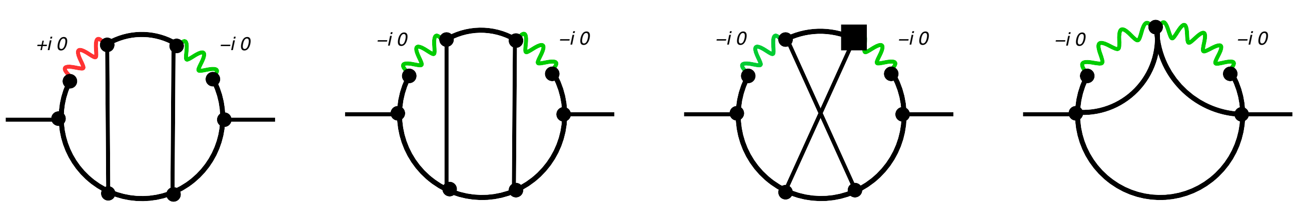
Let us also mention that techniques which have been used to compute master integrals in Ref. [10] can be found in Ref. [37]; see also Ref. [38] for a status report of the approach used in Ref. [10].

The method which is used to compute I_{11} and I_{16} is based on the dimensional recurrence relation and analyticity with respect to space-time dimensionality d (the so-called "DRA method") and was developed in Ref. [39]. In Ref. [40], this method was applied for the first time to the case with more than one master integral in a sector. Some integrals taken from families of integrals for the three-loop static quark potential and denoted in Ref. [40] by I_{14} and I_{15} [see Fig. 1(d)] have been calculated. Note that I_{14} and I_{15} are the only nontrivial integrals entering the right-hand side of the dimensional recurrence relation for I_{16}. Therefore, in principle, the results of Ref. [40] make the calculation of I_{16} straightforward.

However, the numerical issues related to the calculation of contributions to the inhomogeneous terms proportional to I_{14} and I_{15} in the right-hand side of dimensional recurrence relations for I_{16} are quite involved. The most complicated part of this contribution has the form

$$T(\nu) = \sum_{k=0}^{\infty} v^T(\nu + k) \sum_{n=k}^{\infty} \left(\prod_{l=k}^{n} \mathrm{M}(\nu + l) \right) u(\nu + n), \quad (4)$$

where $\nu = d/2$ and $v^T(x)$, $\mathrm{M}(x)$, and $u(x)$ are a row vector, a 2×2 matrix, and a column vector, respectively. Their components are rational functions of the variable x. In order



(a) I_{11} (b) I_{16} (c) I_{18} (d) I_{14}

FIG. 1. (a)–(c): Master integrals entering $a_3^{[c]}$ which were only known numerically. Solid lines denote relativistic scalar propagators, and wavy lines refer to static propagators. For the latter, the causality prescription is given explicitly where $\pm i0$ indicates a propagator of the form $1/(-k_0 \pm i0)$ with k_0 being the zeroth component of the momentum flowing through the corresponding line. The square in I_{18} indicates a convenient choice for the numerator which is specified in Ref. [11]. I_{18} is finite, and only the $\mathcal{O}(\epsilon^0)$ term is needed. For I_{11} and I_{16}, also the $\mathcal{O}(\epsilon^1)$ terms enter $a_3^{[c]}$. (d): Master integral which is needed for the computation of the integrals in (b) and (c). The integral I_{15} belongs to the same integral family as I_{14} but has an additional dot on the lower line.

to calculate the sums in Eq. (4) without nested loops, we apply the standard trick of the DRA method; see Ref. [41]. Namely, let us denote

$$F(k) = \sum_{n=k}^{\infty} \mathrm{P}(k,n) u(\nu+n), \tag{5}$$

where $\mathrm{P}(k,n) = \prod_{l=k}^{n} \mathrm{M}(\nu+l)$. Then,

$$T(\nu) = \sum_{k=0}^{\infty} v^T(\nu+k) F(k). \tag{6}$$

Using Eq. (5), the function $F(k)$ can be calculated for given k in one loop if one takes into account the recurrence relation $\mathrm{P}(k,n+1) = \mathrm{P}(k,n)\,\mathrm{M}(\nu+n+1)$. Now, we note that $F(k)$ satisfies the recurrence relation

$$F(k+1) = \mathrm{M}^{-1}(\nu+k)F(k) - u(\nu+k). \tag{7}$$

Therefore, in order to calculate consecutive terms of the sum in Eq. (6), we need to use Eq. (5) only once and then use the recurrence relation (7). However, the price we have to pay is much higher than for scalar sums. This is connected with the multiplication by the inverse matrix $\mathrm{M}^{-1}(\nu+k)$. For $x \to \infty$, the elements of $\mathrm{M}(x)$ are of order unity, while its determinant tends to $1/1024$. Due to this fact, the multiplication by M^{-1} involves large cancellations which result in rapid precision loss. For example, using a precision of 7000 digits in the initial expression, we obtain only about 370 digits in the final result.

Besides, it appears that the sum over n in the definition of $F(k)$ converges very slowly, with the summand behaving as $n^{-\alpha}$ ($\alpha > 1$) at large n. So, in order to obtain the high-precision numerical result suitable for using PSLQ [42], one has to apply the matrix analog of the convergence acceleration algorithm described in Ref. [43]. In particular, one needs to know the exponent α of the powerlike decay. This appears to be possible thanks to Ref. [44], where a method for finding the asymptotic behavior of the solutions of recurrence relations was developed. Once we dealt with these numerical issues, we obtained the result[1]

$$\begin{aligned} I_{16} = & -\frac{56\pi^4}{135\epsilon} - \left(\frac{112\pi^4}{135} + \frac{16\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3}\right) + \left(\frac{968\zeta(5)}{3} - 16\pi^4 l_2 + \frac{136\zeta(3)^2}{3} + \frac{400\pi^2\zeta(3)}{9} - \frac{838\pi^6}{2835} + \frac{1792\pi^4}{135}\right)\epsilon \\ & + \left(\frac{6144 s_6 l_2}{7} - \frac{6144 s_{7a}}{7} + \frac{15360 s_{7b}}{7} + 1536\alpha_4\zeta(3) + 1024\pi^2\alpha_5 - 256\pi^2\alpha_4 - \frac{64}{9}\pi^4 l_2^3 \right. \\ & - 2976\zeta(5) l_2^2 - 64\pi^2\zeta(3) l_2^2 - \frac{112}{3}\pi^4 l_2^2 - \frac{7680\zeta(3)^2 l_2}{7} - \frac{544\pi^6 l_2}{315} + 128\pi^4 l_2 + \frac{306202\zeta(7)}{21} - \frac{12182\pi^2\zeta(5)}{7} \\ & \left. + \frac{64\zeta(5)}{3} - \frac{1168\zeta(3)^2}{3} - \frac{11828\pi^4\zeta(3)}{945} + \frac{1664\pi^2\zeta(3)}{9} + \frac{1376\pi^6}{135} - \frac{12544\pi^4}{135} + 768 s_6\right)\epsilon^2 + O(\epsilon^3), \end{aligned} \tag{8}$$

where $\zeta(n)$ is Riemann's zeta function evaluated at n and

$$\begin{aligned} l_2 &= \log(2), \\ \alpha_n &= \mathrm{Li}_n(1/2) + \frac{(-\log 2)^n}{n!}, \\ s_6 &= \zeta(-5,-1) + \zeta(6), \\ s_{7a} &= \zeta(-5,1,1) + \zeta(-6,1) + \zeta(-5,2) + \zeta(-7), \\ s_{7b} &= \zeta(7) + \zeta(5,2) + \zeta(-6,-1) + \zeta(5,-1,-1). \end{aligned} \tag{9}$$

$\zeta(m_1,\ldots,m_k)$ are multiple zeta values given by

$$\zeta(m_1,\ldots,m_k) = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \tag{10}$$

In order to apply the DRA method to I_{11}, one has to take into account that the dimensional recurrence relation for I_{11} contains now two nontrivial integrals denoted in Ref. [40] by I_9 and I_{10}. So, in a first step, one has to apply the DRA method to these two integrals. Fortunately, they can be calculated along the same lines as I_{14} and I_{15} from which they differ only by the $\pm i0$ prescription in one of the linear denominators. In particular, the summing factor has the same form as in Ref. [40] [see Eq. (4.14) of that paper]. Plugging the results for I_9 and I_{10} in the dimensional recurrence relation for I_{11} and applying the DRA method, we obtain[2]

[1]See Fig. 1(b) for a graphical definition and Eq. (4.1) of Ref. [40] for normalization factors.

[2]See Fig. 1(a) for a graphical definition and Eq. (4.1) of Ref. [40] for normalization factors.

$$
\begin{aligned}
I_{11} = {}& \frac{64\pi^4}{135\epsilon} + \left(\frac{128\pi^4}{135} + \frac{32\pi^2\zeta(3)}{9} - \frac{8\zeta(5)}{3}\right) + \left(16\pi^4 l_2 + \frac{968\zeta(5)}{3} + \frac{136\zeta(3)^2}{3} - \frac{800\pi^2\zeta(3)}{9} + \frac{548\pi^6}{2835} - \frac{2048\pi^4}{135}\right)\epsilon \\
& + \left(\frac{6144 s_6 l_2}{7} - \frac{6144 s_{7a}}{7} + \frac{15360 s_{7b}}{7} + 1536\alpha_4\zeta(3) - 2048\pi^2\alpha_5 + 512\pi^2\alpha_4 - \frac{64}{9}\pi^4 l_2^3 - 2976\zeta(5) l_2^2 \right. \\
& - 64\pi^2\zeta(3) l_2^2 + \frac{80}{3}\pi^4 l_2^2 - \frac{7680\zeta(3)^2 l_2}{7} - \frac{208\pi^6 l_2}{315} - 128\pi^4 l_2 + \frac{306202\zeta(7)}{21} + \frac{1482\pi^2\zeta(5)}{7} + \frac{64\zeta(5)}{3} \\
& \left. - \frac{1168\zeta(3)^2}{3} - \frac{70208\pi^4\zeta(3)}{945} - \frac{3328\pi^2\zeta(3)}{9} - \frac{1504\pi^6}{135} + \frac{14336\pi^4}{135} + 768 s_6\right)\epsilon^2 + O(\epsilon^3). \qquad (11)
\end{aligned}
$$

Note that the $\mathcal{O}(\epsilon^2)$ terms of I_{16} and I_{11} in Eqs. (8) and (11) are not needed for $a_3^{[c]}$. We nevertheless provide these results to demonstrate the powerfulness of the DRA method.

In principle, the DRA method is also applicable to the calculation of I_{18}. However, the difficulties related to the slow convergence of certain matrix sums and the corresponding precision loss appear to be overwhelming. For this reason, the method of differential equations has been applied to I_{18}; see Ref. [11].

III. SINGLET POTENTIAL

In this section, we present analytic expressions for $a_3^{[1]}$. One- and two-loop results using the same notation can be found in Ref. [15]. Analytic results for the coefficients of n_l^3 and n_l^2 have already been presented in Ref. [8]. Here, they are repeated for completeness:

$$
\begin{aligned}
a_3^{[1],(3)} &= -\left(\frac{20}{9}\right)^3 T_F^3, \\
a_3^{[1],(2)} &= \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2. \qquad (12)
\end{aligned}
$$

Let us now turn to the n_l^1 and n_l^0 term. Expressed in terms of the eigenvalues of the Casimir operators and higher-order group invariants d_F^{abcd} and d_A^{abcd} (see, e.g., Ref. [45]), we obtain for the linear-n_l term the analytic result

$$
\begin{aligned}
a_3^{[1],(1)} = {}& \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left\{\pi^2\left(\frac{1264}{9} - \frac{976\zeta(3)}{3} + l_2(64 + 672\zeta(3))\right) + \pi^4\left(-\frac{184}{3} + \frac{32 l_2}{3} - 32 l_2^2\right) + \frac{10\pi^6}{3}\right\} \\
& + T_F\left\{C_F^2\left(\frac{286}{9} + \frac{296\zeta(3)}{3} - 160\zeta(5)\right) + C_A C_F\left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right)\right. \\
& + C_A^2\left[-\frac{58747}{486} + \pi^2\left(\frac{17}{27} - 32\alpha_4 + l_2\left(-\frac{4}{3} - 14\zeta(3)\right) - \frac{19\zeta(3)}{3}\right) - 356\zeta(3)\right. \\
& \left.\left. + \pi^4\left(-\frac{157}{54} - \frac{5 l_2}{9} + l_2^2\right) + \frac{1091\zeta(5)}{6} + \frac{57(\zeta(3))^2}{2} + \frac{761\pi^6}{2520} - 48 s_6\right]\right\}, \qquad (13)
\end{aligned}
$$

and the gluonic part is given by

$$
\begin{aligned}
a_3^{[1],(0)} = {}& \frac{d_F^{abcd} d_A^{abcd}}{N_A}\left\{\pi^2\left[\frac{7432}{9} - 4736\alpha_4 + l_2\left(\frac{14752}{3} - 3472\zeta(3)\right) - \frac{6616\zeta(3)}{3}\right]\right. \\
& \left. + \pi^4\left(-156 + \frac{560 l_2}{3} + \frac{496 l_2^2}{3}\right) + \frac{1511\pi^6}{45}\right\} + C_A^3\left\{\frac{385645}{2916} + \pi^2\left[-\frac{953}{54} + \frac{584\alpha_4}{3} + \frac{175\zeta(3)}{2}\right.\right. \\
& \left.\left. + l_2\left(-\frac{922}{9} + \frac{217\zeta(3)}{3}\right)\right] + \frac{584\zeta(3)}{3} + \pi^4\left(\frac{1349}{270} - \frac{20 l_2}{9} - \frac{40 l_2^2}{9}\right) - \frac{1927\zeta(5)}{6} - \frac{143(\zeta(3))^2}{2} - \frac{4621\pi^6}{3024} + 144 s_6\right\}. \\
& \qquad (14)
\end{aligned}
$$

The numerical evaluation of the analytic results is in full agreement (including all digits) with Refs, [8–10].

It is interesting to note that the contributions proportional to $d_F^{abcd}d_F^{abcd}$ and $d_F^{abcd}d_A^{abcd}$ only involve π^2, π^4, and π^6 terms. Note that these color structures appear for the first time at three-loop order. On the other hand, the other color structures basically involve all constants one expects up to transcendentality weight 6. Note, however, that the constant s_6 is only present in the most non-Abelian parts, i.e., $T_F C_A^2$ and C_A^3. Let us also mention that log(2) terms are present to first, second, and fourth powers, but there are no cubic terms.

In a next step, we specify to $\mathrm{SU}(N_c)$ and replace the color factors by

$$C_A = N_c, \qquad C_F = \frac{N_c^2-1}{2N_c}, \qquad T_F = \frac{1}{2}, \qquad N_A = N_c^2 - 1,$$
$$\frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96N_c^2}, \qquad \frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48}. \tag{15}$$

This leads to

$$\begin{aligned}
a_3^{[1],(1)} &= \frac{66133}{648} + \pi^2\left(-\frac{79}{9} + l_2(-4 - 42\zeta(3)) + \frac{61\zeta(3)}{3}\right) - \frac{272\zeta(3)}{3} + \pi^4\left(\frac{23}{6} - \frac{2l_2}{3} + 2l_2^2\right) + 20\zeta(5) - \frac{5\pi^6}{24} \\
&\quad + \frac{1}{N_c^2}\left\{\frac{143}{36} + \pi^2\left[\frac{79}{3} - 61\zeta(3) + l_2(12 + 126\zeta(3))\right] + \frac{37\zeta(3)}{3} + \pi^4\left(-\frac{23}{2} + 2l_2 - 6l_2^2\right) - 20\zeta(5) + \frac{5\pi^6}{8}\right\} \\
&\quad + N_c^2\left\{-\frac{323615}{1944} + \pi^2\left(\frac{16}{9} - 16\alpha_4 - \frac{59\zeta(3)}{9}\right) - \frac{299\zeta(3)}{3} + \pi^4\left(-\frac{113}{54} - \frac{l_2}{6} + \frac{l_2^2}{6}\right)\right. \\
&\quad \left. + \frac{1091\zeta(5)}{12} + \frac{57(\zeta(3))^2}{4} + \frac{13\pi^6}{70} - 24s_6\right\}, \\
a_3^{[1],(0)} &= N_c\left\{\pi^2\left[\frac{929}{9} - 592\alpha_4 + l_2\left(\frac{1844}{3} - 434\zeta(3)\right) - \frac{827\zeta(3)}{3}\right] + \pi^4\left(-\frac{39}{2} + \frac{70l_2}{3} + \frac{62l_2^2}{3}\right) + \frac{1511\pi^6}{360}\right\} \\
&\quad + N_c^3\left\{\frac{385645}{2916} + \pi^2\left(-\frac{4}{9} + 96\alpha_4 + \frac{374\zeta(3)}{9}\right) + \frac{584\zeta(3)}{3} + \pi^4\left(\frac{943}{540} + \frac{5l_2}{3} - l_2^2\right)\right. \\
&\quad \left. - \frac{1927\zeta(5)}{6} - \frac{143(\zeta(3))^2}{2} - \frac{29\pi^6}{35} + 144s_6\right\}.
\end{aligned} \tag{16}$$

Finally, for $N_c = 3$, we have

$$\begin{aligned}
a_3^{[1],(1)} &= -\frac{452213}{324} + \pi^2\left[\frac{274}{27} - \frac{409\zeta(3)}{9} - 144\alpha_4 + l_2\left(-\frac{8}{3} - 28\zeta(3)\right)\right] - \frac{26630\zeta(3)}{27} \\
&\quad + \pi^4\left(-\frac{293}{18} - \frac{35l_2}{18} + \frac{17l_2^2}{6}\right) + \frac{30097\zeta(5)}{36} + \frac{1931\pi^6}{1260} + \frac{513(\zeta(3))^2}{4} - 216s_6,
\end{aligned} \tag{17}$$

$$\begin{aligned}
a_3^{[1],(0)} &= \frac{385645}{108} + \pi^2\left[\frac{893}{3} + 816\alpha_4 + l_2(1844 - 1302\zeta(3)) + 295\zeta(3)\right] + 5256\zeta(3) \\
&\quad + \pi^4\left(-\frac{227}{20} + 115l_2 + 35l_2^2\right) - \frac{17343\zeta(5)}{2} - \frac{1643\pi^6}{168} - \frac{3861(\zeta(3))^2}{2} + 3888s_6,
\end{aligned} \tag{18}$$

which in numerical form is given by

$$a_3^{[1]} = 13432.5648565 - 3289.9052968n_l + 185.9900266n_l^2 - 1.3717421n_l^3. \tag{19}$$

IV. OCTET POTENTIAL

In this section, we proceed similarly to the previous one and present results for $\delta a_3^{[8],(i)}$ defined in Eq. (3). We discuss the results in terms of C_A, C_F, etc., in Appendix A and present in this section expressions in terms of N_c. We have $\delta a_3^{[8],(i)} = 0$ for $i = 2$ and $i = 3$, and for the linear-n_l and n_l-independent terms, we get

$$\begin{aligned}
\delta a_3^{[8],(1)} &= \pi^2\left[-\frac{11}{3} - 31\zeta(3) + l_2(4 + 42\zeta(3))\right] + \pi^4\left(-\frac{7}{6} + \frac{2l_2}{3} - 2l_2^2\right) + \frac{5\pi^6}{24} \\
&\quad + N_c^2\left[\pi^2\left(\frac{8}{9} + 48\alpha_4 + 25\zeta(3)\right) + \pi^4\left(\frac{2}{3} + \frac{2l_2}{3}\right) - \frac{13\pi^6}{20}\right], \\
\delta a_3^{[8],(0)} &= N_c^3\left\{\pi^2\left[\frac{139}{9} + 304\alpha_4 + 15\zeta(3) + l_2\left(-\frac{1844}{3} + 434\zeta(3)\right)\right]\right. \\
&\quad \left. + \pi^4\left(\frac{295}{6} - 30l_2 - \frac{62l_2^2}{3}\right) - \frac{1187\pi^6}{360}\right\},
\end{aligned} \tag{20}$$

which for $N_c = 3$ leads to

$$\begin{aligned}
\delta a_3^{[8],(1)} &= -\frac{677\pi^6}{120} + \pi^4\left(\frac{29}{6} + \frac{20l_2}{3} - 2l_2^2\right) + \pi^2\left[\frac{13}{3} + 432\alpha_4 + 194\zeta(3) + l_2(4 + 42\zeta(3))\right], \\
\delta a_3^{[8],(0)} &= \pi^2[417 + 8208\alpha_4 + 405\zeta(3) + l_2(-16596 + 11718\zeta(3))] + \pi^4\left(\frac{2655}{2} - 810l_2 - 558l_2^2\right) - \frac{3561\pi^6}{40}.
\end{aligned} \tag{21}$$

It is interesting to note that $\delta a_3^{[8],(0)}$ and $\delta a_3^{[8],(1)}$ have an overall factor π^2 which was predicted in Ref. [15] on the basis of the involved master integrals. Although they could not be computed analytically, it was possible to show that there is an overall factor π^2, a feature which is also observed at two-loop order in QCD [13,14] and in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories [46].

In numerical form, we obtain for the complete three-loop coefficient

$$\delta a_3^{[8]} = -2634.7351731 + 367.9626044 n_l. \tag{22}$$

V. CONCLUSIONS

The interaction of a slowly moving heavy quark-antiquark pair can be described with the help of a static potential, a concept which is familiar from ordinary quantum mechanics. Its perturbative part is obtained from the exchange of soft gluons which are conveniently considered in the framework of nonrelativistic QCD. Numerical results for the three-loop potential, which have entered a number of physical observables, were obtained eight years ago by two independent groups [8–10]. The obtained precision has been sufficient for all physical applications where a_3 entered as a building block. However, from the aesthetic point of view, it is important to obtain analytic results for higher-order quantum corrections. This has been achieved in this paper. We have obtained analytic results for the three-loop corrections to the singlet and octet potentials which are presented in Secs. III and IV, respectively.

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APPENDIX: $\delta a_2^{[8]}$ AND $\delta a_3^{[8]}$ IN TERMS OF COLOR INVARIANTS

In this Appendix, we present results for $\delta a_3^{[8],(1)}$ and $\delta a_3^{[8],(0)}$ in terms of C_A, C_F, T_F, N_A, d_F^{abcd}, and d_A^{abcd}. Let us mention that the representation given in Eq. (1) is only valid for SU(N_c). Thus, in the following, we present results for $C^{[8]}\delta a_3^{[8],(i)}$ $(i = 0,\ 1)$ with $C^{[8]} = C_F - C_A/2$. For completeness, we also present the two-loop expression; at one-loop order, we have $\delta a_1^{[8]} = 0$. Our results read

$$C^{[8]}\delta a_2^{[8]} = \left(\frac{\pi^4}{12} - \pi^2\right)\left(C_A^3 - 48\frac{d_F^{abcd}d_A^{abcd}}{N_A}\right),$$

$$C^{[8]}\delta a_3^{[8],(1)} = C_A\frac{d_F^{abcd}d_F^{abcd}}{N_A}\left[\pi^2\left(\frac{88}{9} - \frac{32l_2}{3} + \frac{248\zeta(3)}{3} - 112\zeta(3)l_2\right) + \pi^4\left(\frac{28}{9} - \frac{16l_2}{9} + \frac{16l_2^2}{3}\right) - \frac{5\pi^6}{9}\right]$$
$$+ \frac{d_F^{abcd}d_A^{abcd}}{N_A}\left[\pi^2\left(\frac{4}{3} - 192\alpha_4 - \frac{16l_2}{3} - \frac{176\zeta(3)}{3} - 56l_2\zeta(3)\right) + \pi^4\left(-\frac{10}{9} - \frac{32l_2}{9} + \frac{8l_2^2}{3}\right) + \frac{209\pi^6}{90}\right]$$
$$+ C_A^3T_F\left[\pi^2\left(-\frac{7}{27} + 8\alpha_4 + \frac{4l_2}{9} + \frac{13\zeta(3)}{18} + \frac{14l_2\zeta(3)}{3}\right) + \pi^4\left(-\frac{1}{54} + \frac{5l_2}{27} - \frac{2l_2^2}{9}\right) - \frac{23\pi^6}{270}\right],$$

$$C^{[8]}\delta a_3^{[8],(0)} = C_A\frac{d_F^{abcd}d_A^{abcd}}{N_A}\left[\pi^2\left(-\frac{2356}{9} + 3520\alpha_4 - \frac{7376l_2}{3} + 1420\zeta(3) + 1736\zeta(3)l_2\right) + \pi^4\left(66 - \frac{200l_2}{3} - \frac{248l_2^2}{3}\right)\right.$$
$$\left.- \frac{511\pi^6}{18}\right] + \frac{d_A^{abcd}d_A^{abcd}}{N_A}\left[\pi^2\left(\frac{50}{3} - \frac{1184\alpha_4}{3} + \frac{3688l_2}{9} - \frac{370\zeta(3)}{3} - \frac{868l_2\zeta(3)}{3}\right)\right.$$
$$\left.+ \pi^4\left(-\frac{197}{9} + \frac{140l_2}{9} + \frac{124l_2^2}{9}\right) + \frac{1871\pi^6}{540}\right] + C_A^4\left[\pi^2\left(\frac{257}{54} - \frac{512\alpha_4}{9} + \frac{922l_2}{27} - \frac{220\zeta(3)}{9}\right.\right.$$
$$\left.\left.- \frac{217l_2\zeta(3)}{9}\right) + \pi^4\left(-\frac{25}{54} + \frac{20l_2}{27} + \frac{31l_2^2}{27}\right) + \frac{2897\pi^6}{6480}\right], \tag{A1}$$

with

$$\frac{d_A^{abcd}d_A^{abcd}}{N_A} = \frac{N_c^2(N_c^2 + 36)}{24}. \tag{A2}$$

Numerical results of Eq. (A1) are given in Ref. [47]. All color factors have been computed with the help of the program `color` [45].

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