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Dibaryons with two strange quarks and one heavy flavor in a constituent quark model

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We investigate the symmetry property and the stability of dibaryons containing two strange quarks and one heavy flavor with isospin $I = \frac{1}{2}$. We construct the wave function of the dibaryon in two ways. First, we directly construct the color and spin state of the dibaryon starting from the four possible $SU(3)$ flavor states. Second, we consider the states composed of five light quarks and then construct the wave function of the dibaryon by adding one heavy quark. The stability of the dibaryon against the strong decay into two baryons is discussed by using the variational method in a constituent quark model with a confining and hyperfine potential. We find that, for all configurations with spin $S = 0$, 1, 2, the ground states of the dibaryons are the sum of two baryons, and there is no compact bound state that is stable against the strong decay.

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I. INTRODUCTION

Investigating the stability of multiquark hadrons has been pursued in various models after Jaffe suggested the possible existence of such particles in QCD [1–3]. The observation of many charmoniumlike states, such as $X(3872)$, $Z_c(3900)$, and $Z^+(4430)$, and of heavy pentaquark states [4] revived great interest in the studies of multiquark hadrons and/or of molecular bound states containing heavy quark hadrons. Additionally, a new particle called $X(5568)$ was recently observed by the D0 Collaboration in the $B_s^0\pi^\pm$ invariant mass spectrum with 5.1σ significance [5]. $X(5568)$ may be the first observed tetraquark which has four different flavors: up, down, strange, and bottom. If all the flavors are different, for any typical two-body interaction, one can always find the most attractive combination so that one has the advantage to form a bound multiquark state [6].

The stability of the dibaryons with heavy quarks was studied already in several models: those based on chromomagnetic models [6,7] and chiral constituent quark model [8,9]. Furthermore, Huang, Ping, and Wang studied H-like dibaryon states containing heavy quarks instead of strange quarks within the framework of the quark delocalization color screening model [10]. Dibaryons within the diquark models with heavy quarks are also considered [11].

Most of the models studying the possible existence of dibaryons are looking at the most attractive color-spin interaction channel [1–3]. For example, for the H dibaryon, the attraction in the color-spin interaction is larger than those coming from two Λ's, which is the most attractive two-baryon channel that the dibaryon can decay. However, it should be noted that whether such an attraction really leads to a stable compact dibaryon states is determined by whether the attraction is strong enough to overcome the extra repulsion coming from bringing all the quarks together into a compact configuration. As we will discuss later, the magnitude of each effect depends on the masses of the quarks involved that can be systematically studied only within a complete model that consistently treats the kinetic terms and the interaction terms within one framework.

In this work, to investigate the subtle interplay between the two competing effects, in a simple but consistent model, we will study the stability of the $uudssQ$ dibaryon using the variational method in a constituent quark model. This is a generalization of the H dibaryon to include one heavy quark so that it contains the most attractive color-spin interaction channel but, at the same time, reduced kinetic energy from combining six quarks in a compact configuration. In particular, we focus our attention on $I = \frac{1}{2}$, because the states with the lowest isospin are the most attractive bound for a given quark system [6].

Moreover, we will demonstrate how to consistently construct the color-spin flavor wave functions that contain the $uudssQ$ quarks. There are two ways of constructing the wave function of a dibaryon with one heavy quark. First, we can directly construct the color and spin wave function of the dibaryon starting from the four possible $SU(3)$ flavor states. Or we can consider the color and spin state of q^5 and then construct the wave function of the dibaryon by adding one heavy quark. We show the two approaches lead to identical wave functions, showing the consistency of our approach. Technically, the second approach is more convenient to obtain the wave function compared to using the first approach, because the former utilizes Clebsch-Gordan coefficients of S_6 while the second approach uses that of S_5.

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This paper is organized as follows. We first present the Hamiltonian and calculate the masses of baryons to determine the fitting parameters of the model in Sec. II. In Sec. III, we explain why we choose the dibaryon with one heavy flavor in terms of the relation between the hyperfine potential and the stability condition. In Sec. IV, we construct the spatial wave function of the dibaryon. In Sec. V, we classify q^5 with $SU(3)$ flavor symmetry and construct the color and spin wave function of the dibaryon using the first method. In Sec. VI, we construct the color and spin wave function of the dibaryon using the second method. In Sec. VII, we calculate the wave function of the dibaryon and show that their results are the same in both methods. In Sec. VIII, we represent the numerical results obtained from the variational method, and finally we summarize the results in Sec. IX. The Appendixes include some details of the calculations.

II. HAMILTONIAN

We take a nonrelativistic Hamiltonian with the confinement and hyperfine potential given by

$$H = \sum_{i=1}^{6}\left(m_i + \frac{\mathbf{p}_i^2}{2m_i}\right) - \frac{3}{16}\sum_{i<j}^{6}\lambda_i^c\lambda_j^c(V_{ij}^C + V_{ij}^{SS}), \quad (1)$$

where m_i's are the quark masses, $\lambda_i^c/2$ are the color operator of the ith quark for the color $SU(3)$, and V_{ij}^C and V_{ij}^{SS} are the confinement and hyperfine potential, respectively:

$$V_{ij}^C = -\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a_0} - D. \quad (2)$$

The hyperfine term which effectively splits the multiplets of baryon with respect to spin is expressed as

$$V_{ij}^{SS} = \frac{\hbar^2 c^2 \kappa'}{m_i m_j c^4}\frac{1}{(r_{0ij})^2 r_{ij}} e^{-(r_{ij})^2/(r_{0ij})^2}\sigma_i \cdot \sigma_j. \quad (3)$$

Here, r_{ij} is the distance between interquarks, $|\mathbf{r}_i - \mathbf{r}_j|$, and r_{0ij} are chosen to depend on the masses of interquarks given by

$$r_{0ij} = 1/\left(\alpha + \beta\frac{m_i m_j}{m_i + m_j}\right). \quad (4)$$

We choose to keep the isospin symmetry by requiring that $m_u = m_d$. In the Hamiltonian, the parameters have been chosen so that the fitted masses of baryons are comparable with those of experiments.

When we calculate the expectation value of the potential terms for a baryon with certain symmetry, it is convenient to introduce the following three Jacobian coordinates. Then it reduces our problem to the two-body system in the center of mass frame.

(i) *Coordinate I.—*

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2),$$
$$\boldsymbol{x}_2 = \frac{m_1 + m_2}{\sqrt{2m_1^2 + 2m_2^2 + 2m_1 m_2}} \times \left(\frac{m_1}{m_1 + m_2}\mathbf{r}_1 + \frac{m_2}{m_1 + m_2}\mathbf{r}_2 - \mathbf{r}_3\right). \quad (5)$$

(ii) *Coordinate II.—*

$$\boldsymbol{y}_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_3),$$
$$\boldsymbol{y}_2 = \frac{m_2 + m_3}{\sqrt{2m_2^2 + 2m_3^2 + 2m_2 m_3}} \times \left(\frac{m_2}{m_2 + m_3}\mathbf{r}_2 + \frac{m_3}{m_2 + m_3}\mathbf{r}_3 - \mathbf{r}_1\right). \quad (6)$$

(iii) *Coordinate III.—*

$$\boldsymbol{z}_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_3 - \mathbf{r}_1),$$
$$\boldsymbol{z}_2 = \frac{m_1 + m_3}{\sqrt{2m_1^2 + 2m_3^2 + 2m_1 m_3}} \times \left(\frac{m_1}{m_1 + m_3}\mathbf{r}_1 + \frac{m_3}{m_1 + m_3}\mathbf{r}_3 - \mathbf{r}_2\right). \quad (7)$$

There exist orthogonal transformations between these coordinate systems, which we will make use of when calculating the matrix elements. As for the spatial wave function, we will introduce the following form using the first coordinate system:

$$R = \exp\left[-a(\boldsymbol{x}_1)^2 - b(\boldsymbol{x}_2)^2\right]. \quad (8)$$

By using this simple Gaussian function, we calculate the baryon masses containing a charm or bottom quark. The fitting parameters in Hamiltonian are given in Table I. These parameters were obtained by making a best fit to the baryon octet and decuplet masses. The variational parameters and the masses obtained with these parameters are

TABLE I. Parameters fitted to the experimental baryon masses using the variational method with a single Gaussian. The respective units are given in the bottom row.

κ	κ'	a_0	D	α	β	m_q	m_s	m_c	m_b
0.59	0.5	5.386	0.960	2.6	0.552	0.343	0.632	1.93	5.3
		GeV^{-2}	GeV	$(\mathrm{fm})^{-1}$		GeV	GeV	GeV	GeV

TABLE II. The masses of baryons obtained from the variational method and the experimental data in GeV units. The fourth and fifth rows show the variational parameters of the spatial wave function with units in fm^{-2}.

(I,S)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{3}{2})$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{3}{2})$
	N, P	Δ	Λ	Σ	Σ^*	Ξ	Ξ^*
Mass	0.977	1.23	1.12	1.2	1.38	1.324	1.52
Exp	0.938	1.232	1.115	1.189	1.382	1.315	1.532
a	2.5	1.8	3.3	2.2	1.8	3.4	2.9
b	2.5	1.8	2.8	3.4	2.3	3.2	2.1
(I,S)	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{3}{2})$	$(0,\frac{1}{2})$	$(0,\frac{3}{2})$
	Λ_c	Σ_c	Σ_c^*	Ξ_c	Ξ_c^*	Ω_c	Ω_c^*
Mass	2.285	2.45	2.526	2.476	2.649	2.687	2.763
Exp	2.286	2.453	2.518	2.468	2.646	2.695	2.766
a	3.4	2.1	1.9	3.7	2.4	3.6	3.2
b	3.8	3.9	3.3	4.6	4.1	5.8	4.8
(I,S)	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{3}{2})$	$(0,\frac{1}{2})$	$(0,\frac{3}{2})$
	Λ_b	Σ_b	Σ_b^*	Ξ_b	Ξ_b^*	Ω_b	Ω_b^*
Mass	5.608	5.809	5.839	5.787	5.95	6.019	6.053
Exp	5.619	5.811	5.832	5.792	5.949	6.048	
a	3.5	2.1	2.0	3.8	2.5	3.5	3.4
b	4.4	4.1	3.8	5.5	5.0	6.6	6.1

given in Table II. The method to calculate the color, flavor, and spin basis was explained in Ref. [12].

III. COLOR-SPIN INTERACTION AND THE STABILITY CONDITION

It is well known that color-spin interaction is an important factor in investigating the stability of a multiquark system. In $SU(3)$ flavor symmetry, there is a simple formula [13] from which one can easily calculate the expectation value of the hyperfine potential:

$$H_{SS} = -\sum_{i<j}^{N} \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j = N(N-10) + \frac{4}{3} S(S+1) + 2C_C + 4C_F, \quad (9)$$

where $C_F = \frac{1}{4}\lambda^F \lambda^F$. For the flavor singlet H dibaryon, $H_{SS} = -24$, and for Λ, $H_{SS} = -8$ so for $\Lambda\Lambda$, $H_{SS} = -16$. Hence, the H dibaryon is more attractive than the $\Lambda\Lambda$ system in terms of color-spin interaction. This is the basis for a possible stable H dibaryon.

At the same time, it is interesting to point out that we can split the dibaryon into five quarks and one quark system. By using Eq. (9), we can calculate the expectation value of the hyperfine potential of five quarks in the H dibaryon. We represent the expectation values of the hyperfine potential for the H dibaryon with a flavor singlet and $\Lambda\Lambda$ in Table III.

TABLE III. The expectation value of $-\sum_{i<j}\langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$ for the H dibaryon with a flavor singlet (F^1) and $\Lambda\Lambda$.

$-\sum_{i<j}\langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$	$i<j=1$–5	$i=1-5,\ j=6$
H dibaryon, F^1	-16	-8

$-\sum_{i<j}\langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$	$i<j=1$–3	$i=4,\ j=5$	$i=4-5,\ j=6$
$\Lambda\Lambda$	-8	-8	0

In that case, the flavor and color state of five quarks are antitriplet and $C_C = C_F = \frac{4}{3}$ for the antitriplet state, so it leads to $H_{SS} = -16$, which is the same as $H_{SS}^{\Lambda\Lambda}$. So it shows that the interaction between the sixth quark and the other quarks gives a more attractive effect than the $\Lambda\Lambda$ system and agrees with our recent work [14]. As we shall see later, we note that the same color-spin matrix element is obtained in the most attractive flavor color-spin channel of our configuration ($uudssQ$) with $S=0$ and $I=1/2$. For this case, we can also choose Q to be the sixth quark with the color-spin matrix element as given in Table III. Unfortunately, the H dibaryon is not stable in our model when we consider the Hamiltonian, as the repulsion coming from the kinetic energy and confinement potential dominates over the attraction coming from the hyperfine potential as we bring six quarks to a compact configuration.

If we replace the sixth quark with a heavy quark, then the situation becomes more subtle. In the infinite heavy quark mass limit, the contribution from the sixth quark becomes zero because the hyperfine potential has a $1/m_Q$ factor. And in that case, the expectation value of the hyperfine potential is the same as that of Λ and the diquark system. However, the heavy quark mass is not infinite, so the $1/m_Q$ factor weakens the attractive effect, but it will also reduce the kinetic energy if it does not change the interquark distances. So we can consider the dibaryon with heavy flavor, and it may lead to a better chance to form the stable state than the H dibaryon. It should be noted that the antitriplet flavor state is the most attractive color-spin interaction when we consider five quarks only.

IV. SPATIAL FUNCTION

In order to construct an antisymmetric wave function of the dibaryon, we choose the spatial function to be symmetric such that the rest of the wave function represented by color $\otimes$ flavor $\otimes$ spin should be antisymmetric. Here, we calculate in the flavor $SU(3)$ breaking case and fix the position of each quark on $u(1)u(2)d(3)s(4)s(5)Q(6)$. So our wave function should have the specific symmetry property which is antisymmetric among 1, 2, and 3 and at the same time antisymmetric between 4 and 5. And among various Jacobi coordinates, we choose the baryon-baryon configuration because it is convenient to investigate the strong decay mode:

$$\sum_{i=1}^{6}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2-\frac{1}{2}M\dot{\mathbf{r}}_{CM}^2=\sum_{i=1}^{5}\frac{1}{2}M_i\dot{\mathbf{x}}_i^2,$$

$$\text{where } M=\sum_{i=1}^{6}m_i,$$
$$M_1=M_2=m, \qquad M_3=m_s,$$
$$M_4=\frac{3m_s m_Q}{2m_s+m_Q},$$
$$M_5=\frac{2m(5m_s^2+2m_s m_Q+2m_Q^2)}{(3m+2m_s+m_Q)(2m_s+m_Q)},$$
$$\mathbf{r}_{CM}=\frac{1}{M}\sum_{i=1}^{6}m_i\mathbf{r}_i. \tag{10}$$

The Jacobian coordinates are given by

$$\boldsymbol{x_1}=\frac{1}{\sqrt{2}}(\mathbf{r}_1-\mathbf{r}_2),$$
$$\boldsymbol{x_2}=\sqrt{\frac{2}{3}}\left(\frac{1}{2}\mathbf{r}_1+\frac{1}{2}\mathbf{r}_2-\mathbf{r}_3\right),$$
$$\boldsymbol{x_3}=\frac{1}{\sqrt{2}}(\mathbf{r}_4-\mathbf{r}_5),$$
$$\boldsymbol{x_4}=\sqrt{\frac{2}{3}}\left(\frac{1}{2}\mathbf{r}_4+\frac{1}{2}\mathbf{r}_5-\mathbf{r}_6\right),$$
$$\boldsymbol{x_5}=\frac{\sqrt{3}(2m_s+m_Q)}{\sqrt{10m_s^2+4m_s m_Q+4m_Q^2}}\left(\frac{1}{3}\mathbf{r}_1+\frac{1}{3}\mathbf{r}_2+\frac{1}{3}\mathbf{r}_3-\frac{m_s}{2m_s+m_Q}\mathbf{r}_4-\frac{m_s}{2m_s+m_Q}\mathbf{r}_5-\frac{m_Q}{2m_s+m_Q}\mathbf{r}_6\right). \tag{11}$$

Then, we can construct the spatial wave function of the dibaryon in a single Gaussian form that can accommodate the required symmetry property:

$$R=\exp[-a(x_1^2+x_2^2)-bx_3^2-cx_4^2-dx_5^2], \tag{12}$$

where a, b, c, and d are the variational parameters. The spatial function in Eq. (12) is symmetric among 1, 2, and 3 and at the same time symmetric between 4 and 5. We will denote this symmetry property of the spatial function by [123][45]6. Considering the dibaryon to be formed by bringing together a baryon composed of particle [123] and a baryon composed of [45]6, one notes that the additional kinetic term will involve coordinate $\boldsymbol{x_5}$ with mass M_5. Hence, for fixed $\boldsymbol{x_5}$, the additional kinetic term becomes smaller when m_Q increases but only becomes zero when more than one quark becomes heavy as can be seen in Eq. (10). This additional kinetic term has to be smaller than the additional attraction coming from the color-spin interaction for the dibaryon to form a stable compact state.

V. CLASSIFICATION OF q^5Q WITH $SU(3)$ FLAVOR SYMMETRY

In this section, we directly construct the color and spin wave function of the dibaryon from the four possible $SU(3)$ flavor states.

A. Flavor state of q^5

Here, we classify the flavor states in terms of $SU(3)_F$ symmetry and will break the flavor symmetry later. Since the spatial wave function is symmetric, we have to construct the flavor, color, and spin wave function to be antisymmetric. Under the general group $SU(18)_{CFS}$, a totally antisymmetric multiplet of $[1^5]_{FCS}$ can be decomposed as

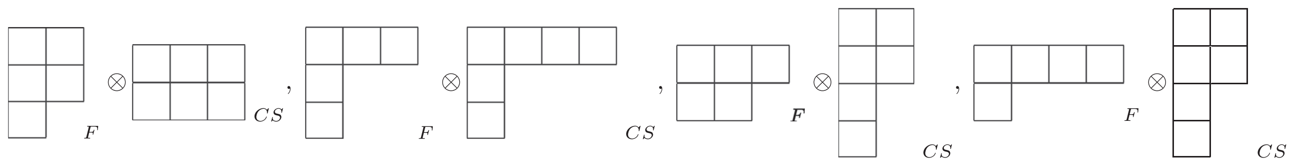
$$[1^5]_{FCS}=([\bar{3}]_F,[420]_{CS})\oplus([6]_F,[336]_{CS})\oplus([\overline{15}]_F,[210]_{CS})\oplus([24]_F,[84]_{CS})\oplus([21]_F,[\bar{6}]_{CS}). \tag{13}$$

In this article, we consider only $I=\frac{1}{2}$, so that we exclude the $[21]_F$ flavor state. Hence, there are four possible flavor states as follows:

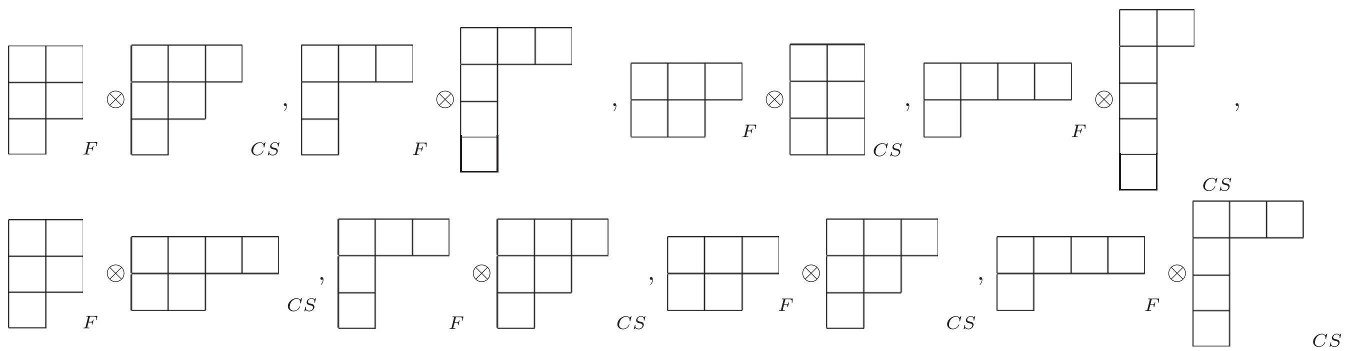
$$[\bar{3}]_F=\ydiagram{2,2,1},\quad [6]_F=\ydiagram{3,1,1},\quad [\overline{15}]_F=\ydiagram{3,2},$$
$$[24]_F=\ydiagram{4,1}.$$

For each flavor state, we can determine the possible Young tableau of the color and spin state of q^5Q. According to the group theory, for a given Young tableau, the fully antisymmetric state can be constructed by multiplying the Young tableau by its conjugate of the Young tableau, where the conjugate representation of a given Young tableau can be obtained by exchanging the row and column in the Young tableau. Additionally, the Young tableau of the color and spin state that can contribute to the final state depends on the spin of the dibaryon. For a given spin state, the possible color and spin state can be obtained by taking the direct product of the color singlet dibaryon configuration to the spin state and taking the conjugate, with the addition of the sixth quark, of the fixed flavor state. We represent the possible flavor, color, and spin state of q^5Q with $I=\frac{1}{2}$ for each spin states as follows.

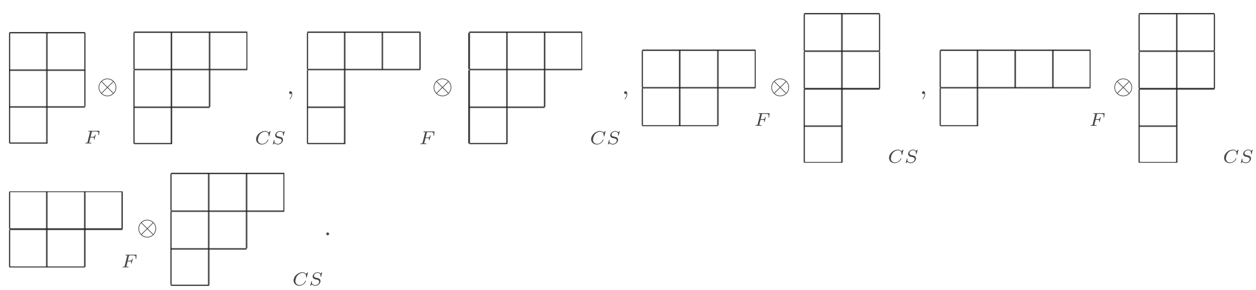
(i) $S = 0$.—Four states:



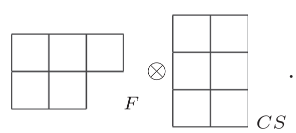
(ii) $S = 1$.—Eight states:



(iii) $S = 2$.—Five states:



(iv) $S = 3$.—One state:



After $SU(3)_F$ breaking, there are only two flavor bases:

$$|F_1\rangle = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}, \ \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}, \ \begin{array}{|c|}\hline 6 \\ \hline\end{array}\right), \qquad |F_2\rangle = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, \ \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}, \ \begin{array}{|c|}\hline 6 \\ \hline\end{array}\right). \tag{14}$$

B. Flavor, color, and spin state of q^5Q

Here, we fix u and d quarks to be 1, 2, and 3 and two s quarks to be 4 and 5. Since there is a flavor symmetry between strange quarks, the color and spin wave function should be antisymmetric between 4 and 5. The details to construct a color and spin wave function which has the specific symmetry property was explained in Ref. [14].

For example, in the case of $[\bar{3}]_F$ and the $S = 0$ state, the Young tableau of the color and spin state should be [3,3]. Furthermore, the color and spin state have to be antisymmetric between 4 and 5 because of their flavor symmetry. Hence, by using the Young-Yamanouchi representation and permutation property [15], we can construct the flavor, color, and spin wave function which has the required symmetry.

(i) $S = 0$.—

$$\phi_1^{S=0} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 & 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 & 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 & 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 & 6 \end{ytableau}_{CS} \right),$$

$$\phi_2^{S=0} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 4 & 6 \\ 2 \\ 5 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 \\ 5 \end{ytableau}_{CS} \right),$$

$$\phi_3^{S=0} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \\ 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \\ 6 \end{ytableau}_{CS} \right),$$

$$\phi_4^{S=0} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \begin{ytableau} 1 & 3 \\ 2 & 6 \\ 4 \\ 5 \end{ytableau}_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \begin{ytableau} 1 & 2 \\ 3 & 6 \\ 4 \\ 5 \end{ytableau}_{CS}. \quad (15)$$

(ii) $S = 1$.—

$$\phi_1^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \\ 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{ytableau}_{CS} \right),$$

$$\phi_2^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 5 \\ 2 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 4 \\ 2 \\ 5 \\ 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}} F_2 \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 5 \\ 3 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 4 \\ 3 \\ 5 \\ 6 \end{ytableau}_{CS} \right),$$

$$\phi_3^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{ytableau}_{CS} \right),$$

$$\phi_4^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \begin{ytableau} 1 & 3 \\ 2 \\ 4 \\ 5 \\ 6 \end{ytableau}_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \begin{ytableau} 1 & 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau}_{CS}, \quad (16)$$

$$\phi_5^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 4 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 & 4 & 6 \\ 2 & 5 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 4 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 & 5 \end{ytableau}_{CS} \right),$$

$$\phi_6^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 4 \\ 2 & 6 \\ 5 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 5 \\ 3 & 6 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 \\ 5 \end{ytableau}_{CS} \right),$$

$$\phi_7^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 & 6 \\ 2 & 5 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 & 6 \\ 2 & 4 \\ 5 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 & 6 \\ 3 & 5 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 & 6 \\ 3 & 4 \\ 5 \end{ytableau}_{CS} \right),$$

$$\phi_8^{S=1} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \begin{ytableau} 1 & 3 & 6 \\ 2 \\ 4 \\ 5 \end{ytableau}_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \begin{ytableau} 1 & 2 & 6 \\ 3 \\ 4 \\ 5 \end{ytableau}_{CS}.$$

(iii) $S = 2$.—

$$\phi_1^{S=2} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \\ 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{ytableau}_{CS} \right),$$

$$\phi_2^{S=2} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 3 & 4 \\ 2 & 6 \\ 5 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 5 \\ 3 & 6 \\ 4 \end{ytableau}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 \\ 5 \end{ytableau}_{CS} \right),$$

$$\phi_3^{S=2} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \\ 6 \end{ytableau}_{CS} \right) - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \\ 6 \end{ytableau}_{CS} - \frac{\sqrt{3}}{2} \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \\ 6 \end{ytableau}_{CS} \right),$$

$$\phi_4^{S=2} = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \ytableaushort{13,26,4,5}_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \ytableaushort{12,36,4,5}_{CS},$$

$$\begin{aligned}\phi_5^{S=2} = {}& \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \ytableaushort{136,25,4}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \ytableaushort{136,24,5}_{CS} \right) \\ & - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \ytableaushort{126,35,4}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \ytableaushort{126,34,5}_{CS} \right). \end{aligned} \tag{17}$$

(iv) $S = 3$.—

$$\begin{aligned}\phi_1^{S=3} = {}& \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2} \ytableaushort{13,25,46}_{CS} - \frac{\sqrt{3}}{2} \ytableaushort{13,24,56}_{CS} \right) \\ & - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2} \ytableaushort{12,35,46}_{CS} - \frac{\sqrt{3}}{2} \ytableaushort{12,34,56}_{CS} \right). \end{aligned}$$

Using the Clebsch-Gordon coefficients which are presented in the Appendixes, we can obtain the flavor, color, and spin state of the dibaryon.

VI. CLASSIFICATION OF q^5 WITH $SU(3)$ FLAVOR SYMMETRY

A. Flavor and spin state of q^5

We can construct the wave function of the dibaryon in another way. Here, we consider the state of five light quarks first, and then we will add a heavy quark later. The totally antisymmetric multiplet of $[1^5]_{CFS}$ can be decomposed as

$$\begin{aligned}[1^5]_{CFS} = {}& ([\bar{3}]_C, [420]_{FS}) \oplus ([6]_C, [336]_{FS}) \\ & \oplus ([\overline{15}]_C, [210]_{FS}) \oplus ([24]_C, [84]_{FS}) \\ & \oplus ([21]_C, [\bar{6}]_{FS}). \end{aligned} \tag{18}$$

By using the Young tableau, we can find that the multiplets in the right-hand side of Eq. (18) are fully antisymmetric:

$$([\bar{3}]_C, [420]_{FS}) = \ydiagram{2,2,1} \otimes \ydiagram{3,2},$$

$$([6]_C, [336]_{FS}) = \ydiagram{3,1,1} \otimes \ydiagram{3,1,1},$$

$$([\bar{15}]_C, [210]_{FS}) = \ydiagram{3,2} \otimes \ydiagram{2,2,1},$$

$$([24]_C, [84]_{FS}) = \ydiagram{4,1} \otimes \ydiagram{2,1,1,1},$$

$$([21]_C, [6]_{FS}) = \ydiagram{5} \otimes \ydiagram{1,1,1,1,1}.$$

Since the dibaryon is a color singlet, the color state of q^5 should be antitriplet. So flavor and spin state of q^5 should be $[420]_{FS}$. The $[420]_{FS}$ multiplet can be decomposed as

$$\begin{aligned}[420]_{FS} = {}& ([\bar{3}]_F, [2]_S) \oplus ([6]_F, [2]_S) \oplus ([\overline{15}]_F, [2]_S) \\ & \oplus ([21]_F, [2]_S) \oplus ([24]_F, [2]_S) \oplus ([\bar{3}]_F, [4]_S) \\ & \oplus ([6]_F, [4]_S) \oplus ([\overline{15}]_F, [4]_S) \oplus ([24]_F, [4]_S) \\ & \oplus ([\overline{15}]_F, [6]_S). \end{aligned} \tag{19}$$

The corresponding Young tableaus of each state are given as follows:

$$([\bar{3}]_F,[2]_S) = \ydiagram{2,2,1} \otimes \ydiagram{3,2},$$

$$([6]_F,[2]_S) = \ydiagram{3,1,1} \otimes \ydiagram{3,2},$$

$$([\bar{15}]_F,[2]_S) = \ydiagram{3,2} \otimes \ydiagram{3,2},$$

$$([24]_F,[2]_S) = \ydiagram{4,1} \otimes \ydiagram{3,2},$$

$$([\bar{3}]_F,[4]_S) = \ydiagram{2,2,1} \otimes \ydiagram{4,1},$$

$$([6]_F,[4]_S) = \ydiagram{3,1,1} \otimes \ydiagram{4,1},$$

$$([\bar{15}]_F,[4]_S) = \ydiagram{3,2} \otimes \ydiagram{4,1},$$

$$([24]_F,[4]_S) = \ydiagram{4,1} \otimes \ydiagram{4,1},$$

$$([\bar{15}]_F,[6]_S) = \ydiagram{3,2} \otimes \ydiagram{5}.$$

In this article, we consider only $I = \frac{1}{2}$, so we exclude the $[21]_F$ case. Therefore, there are four flavor states for each $S = \frac{1}{2}$ and $S = \frac{3}{2}$ and one state for $S = \frac{5}{2}$. Hence, as we can see in Fig. 1, we can determine the number of possible states of q^5Q when adding one heavy quark. For the dibaryons, there can be four spin states $S = 0, 1, 2, 3$. From the above decomposition, for the $S = 0$ case, there are four possible states. For $S = 1$, there are eight possible states. For $S = 2$, there are five possible states. For $S = 3$, there is only one flavor that is $[\bar{15}]$.

B. Flavor, color, and spin state of q^5

After $SU(3)_F$ breaking, there are only two flavor bases for q^5:

$$|F_1\rangle = \left(\ytableau{1 & 2 \\ 3}, \ytableau{4 & 5} \right), \qquad |F_2\rangle = \left(\ytableau{1 & 3 \\ 2}, \ytableau{4 & 5} \right). \quad (20)$$

We can construct the wave function of q^5 using the same method as in Sec. V B.

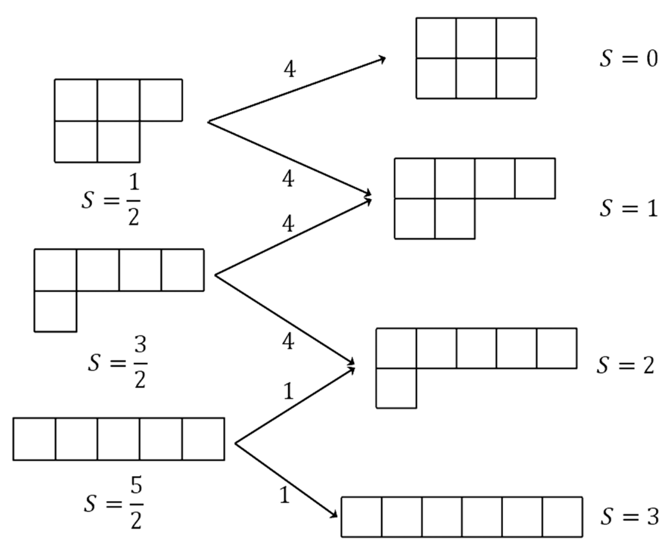


FIG. 1. Spin Young tableau of q^5 and q^5Q.

(i) $F = [\bar{3}]$.—The Young tableau of the color and spin state is [3, 2]:

$$\psi_1 = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2}\ytableaushort{135,24} - \frac{\sqrt{3}}{2}\ytableaushort{134,25}\right)_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2}\ytableaushort{125,34} - \frac{\sqrt{3}}{2}\ytableaushort{124,35}\right)_{CS}. \quad (21)$$

(ii) $F = [6]$.—The Young tableau of the color and spin state is [3, 1, 1]:

$$\psi_2 = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}}\ytableaushort{135,2,4} - \frac{\sqrt{5}}{\sqrt{8}}\ytableaushort{134,2,5}\right)_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}}\ytableaushort{125,3,4} - \frac{\sqrt{5}}{\sqrt{8}}\ytableaushort{124,3,5}\right)_{CS}. \quad (22)$$

(iii) $F = [\overline{15}]$.—The Young tableau of the color and spin state is [2, 2, 1]:

$$\psi_3 = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \left(\frac{1}{2}\ytableaushort{13,25,4} - \frac{\sqrt{3}}{2}\ytableaushort{13,24,5}\right)_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \left(\frac{1}{2}\ytableaushort{12,35,4} - \frac{\sqrt{3}}{2}\ytableaushort{12,34,5}\right)_{CS}. \quad (23)$$

(iv) $F = [24]$.—The Young tableau of the color and spin state is [2, 1, 1, 1]:

$$\psi_4 = \frac{1}{\sqrt{2}}|F_1\rangle \otimes \ytableaushort{13,2,4,5}_{CS} - \frac{1}{\sqrt{2}}|F_2\rangle \otimes \ytableaushort{12,3,4,5}_{CS}. \quad (24)$$

VII. FLAVOR, COLOR, AND SPIN STATE OF q^5Q

In the Appendixes, we represent the color and spin state of q^5 and Clebsch-Gordan coefficients, from which we can construct the color and spin state of q^5Q using the following basis functions.

A. Flavor basis function

Since the isospin of the dibaryon is $\frac{1}{2}$, there are two flavor basis functions:

$$|F_1\rangle = \left(\ytableaushort{12,3}, \ytableaushort{45}, \ytableaushort{6}\right), \qquad |F_2\rangle = \left(\ytableaushort{13,2}, \ytableaushort{45}, \ytableaushort{6}\right). \quad (25)$$

B. Color basis function

Color singlet.—There are five basis functions with Young tableau [2, 2, 2]:

$$|C_1\rangle = \ytableaushort{12,34,56}, \quad |C_2\rangle = \ytableaushort{13,24,56}, \quad |C_3\rangle = \ytableaushort{12,35,46}, \quad |C_4\rangle = \ytableaushort{13,25,46}, \quad |C_5\rangle = \ytableaushort{14,25,36}. \quad (26)$$

C. Spin basis function

(i) $S = 0$.—There are five basis functions with Young tableau [3, 3]:

$$|S_1^0\rangle = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array}, \quad |S_2^0\rangle = \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array}, \quad |S_3^0\rangle = \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&6\\ \hline\end{array},$$

$$|S_4^0\rangle = \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&6\\ \hline\end{array}, \quad |S_5^0\rangle = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array}.$$

(ii) $S = 1$.—There are nine basis functions with Young tableau [4, 2]:

$$|S_1^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6\\ \cline{1-2}\end{array}, \quad |S_2^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\ \hline 4&6\\ \cline{1-2}\end{array}, \quad |S_3^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&6\\ \cline{1-2}\end{array},$$

$$|S_4^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&6\\ \cline{1-2}\end{array}, \quad |S_5^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4&5\\ \cline{1-2}\end{array}, \quad |S_6^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5\\ \cline{1-2}\end{array},$$

$$|S_7^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&5\\ \cline{1-2}\end{array}, \quad |S_8^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3&4\\ \cline{1-2}\end{array}, \quad |S_9^1\rangle = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4\\ \cline{1-2}\end{array}.$$

(iii) $S = 2$.—There are five basis functions with Young tableau [5, 1]:

$$|S_1^2\rangle = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 6\\ \cline{1-1}\end{array} \quad |S_2^2\rangle = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\ \hline 5\\ \cline{1-1}\end{array}$$

$$|S_3^2\rangle = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&6\\ \hline 4\\ \cline{1-1}\end{array} \quad |S_4^2\rangle = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\ \hline 3\\ \cline{1-1}\end{array},$$

$$|S_5^2\rangle = \begin{array}{|c|c|c|c|c|}\hline 1&3&4&5&6\\ \hline 2\\ \cline{1-1}\end{array}.$$

(iv) $S = 3$.—There is one basis function with Young tableau [6]:

$$|S^3\rangle = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\ \hline\end{array}.$$

D. Flavor, color, and spin state of q^5Q

Here, we can construct the wave function of q^5Q from the state of q^5. Since the color state of q^5 is $[\bar{3}]$, there is only one way to construct the color singlet state by adding one heavy flavor. There is no change in the color basis function from q^5 to q^5Q. However, we should treat the spin state transformation carefully. The spin basis function of q^5 for $S = \frac{1}{2}$ can be either $S = 0$ or $S = 1$ of the q^5Q state. We represent the spin basis transformation for each case in the Appendixes.

(i) $S = 0$.—Four possible states:
$\psi^5_{1,S=\frac{1}{2}} \to \psi^6_{1,S=0}$, $\psi^5_{2,S=\frac{1}{2}} \to \psi^6_{2,S=0}$,
$\psi^5_{3,S=\frac{1}{2}} \to \psi^6_{3,S=0}$, $\psi^5_{4,S=\frac{1}{2}} \to \psi^6_{4,S=0}$.

(ii) $S = 1$.—Eight possible states:
$\psi^5_{1,S=\frac{1}{2}} \to \psi^6_{1,S=1}$, $\psi^5_{2,S=\frac{1}{2}} \to \psi^6_{2,S=1}$,
$\psi^5_{3,S=\frac{1}{2}} \to \psi^6_{3,S=1}$, $\psi^5_{4,S=\frac{1}{2}} \to \psi^6_{4,S=1}$,
$\psi^5_{1,S=\frac{3}{2}} \to \psi^6_{5,S=1}$, $\psi^5_{2,S=\frac{3}{2}} \to \psi^6_{6,S=1}$,
$\psi^5_{3,S=\frac{3}{2}} \to \psi^6_{7,S=1}$, $\psi^5_{4,S=\frac{3}{2}} \to \psi^6_{8,S=1}$.

(iii) $S = 2$.—Five possible states:
$\psi^5_{1,S=\frac{3}{2}} \to \psi^6_{1,S=2}$, $\psi^5_{2,S=\frac{3}{2}} \to \psi^6_{2,S=2}$,
$\psi^5_{3,S=\frac{3}{2}} \to \psi^6_{3,S=2}$, $\psi^5_{4,S=\frac{3}{2}} \to \psi^6_{4,S=2}$,
$\psi^5_{3,S=\frac{5}{2}} \to \psi^6_{5,S=2}$.

(iv) $S = 3$.—One possible state:
$\psi^5_{3,S=\frac{5}{2}} \to \psi^6_{1,S=3}$.

We find that there is an orthogonal transformation between $\{\psi_i\}$ and $\{\phi_i\}$.

(i) $S = 0$:

$$\phi_1 = \psi_1, \quad \phi_2 = -\psi_2, \quad \phi_3 = \psi_3, \quad \phi_4 = \psi_4.$$

(ii) $S = 1$:

$$\frac{2}{\sqrt{5}}\phi_1 - \frac{1}{\sqrt{5}}\phi_5 = \psi_1, \qquad \frac{1}{\sqrt{5}}\phi_1 + \frac{2}{\sqrt{5}}\phi_5 = \psi_5,$$
$$-\frac{\sqrt{5}}{3}\phi_2 + \frac{2}{3}\phi_6 = \psi_2, \qquad \frac{2}{3}\phi_2 + \frac{\sqrt{5}}{3}\phi_6 = \psi_6,$$
$$\frac{\sqrt{5}}{3}\phi_3 + \frac{2}{3}\phi_7 = \psi_3, \qquad \frac{2}{3}\phi_3 - \frac{\sqrt{5}}{3}\phi_7 = \psi_7,$$
$$-\frac{2}{3}\phi_4 - \frac{\sqrt{5}}{3}\phi_8 = \psi_4, \qquad -\frac{\sqrt{5}}{3}\phi_4 + \frac{2}{3}\phi_8 = \psi_8.$$

(iii) $S = 2$:

$$\phi_1 = -\psi_1, \quad \phi_2 = \psi_2, \quad \phi_4 = -\psi_4,$$
$$-\frac{4}{5}\phi_3 + \frac{3}{5}\phi_5 = \psi_3, \qquad -\frac{3}{5}\phi_3 - \frac{4}{5}\phi_5 = \psi_5.$$

(iv) $S = 3$:

$$\phi_1 = \psi_1.$$

Since we calculate the expectation values using these wave functions, the results from both methods are the same as expected.

E. Baryon-baryon configuration of q^5Q

Considering the decay channel, we construct the baryon-baryon wave function of q^5Q. Among the five color basis functions, $|C_5\rangle$ is the color singlet for 1, 2, and 3 and at the same time the color singlet for 4, 5, and 6. Therefore, we have to construct the wave function only by using $|C_5\rangle$ as the color state if we want to investigate the decay channel through the uud baryon and the ssQ baryon. As done in Sec. VI B, we can construct the color, flavor, and spin state of q^5Q which has the specific symmetry property for $S = 0$, 1, 2:

$$\psi^{BB}_{S=0} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(\frac{1}{2}|S^0_2\rangle + \frac{\sqrt{3}}{2}|S^0_4\rangle\right) + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(\frac{1}{2}|S^0_3\rangle + \frac{\sqrt{3}}{2}|S^0_5\rangle\right), \quad (27)$$

$$\psi^{BB}_{S=1} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(A_1|S^1_3\rangle + A_2\left(\frac{1}{2}|S^1_6\rangle + \frac{\sqrt{3}}{2}|S^1_8\rangle\right)\right) + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(A_1|S^1_4\rangle + A_2\left(\frac{1}{2}|S^1_7\rangle + \frac{\sqrt{3}}{2}|S^1_9\rangle\right)\right), \quad (28)$$

$$\psi^{BB}_{S=2} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S^2_4\rangle) + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S^2_5\rangle. \quad (29)$$

In the above expressions, A_1 and A_2 are undetermined constants. However, to construct the $S = 1$ dibaryon, the spin of the ssQ baryon can be either $\frac{1}{2}$ or $\frac{3}{2}$. When the spin of the ssQ baryon is $\frac{3}{2}$, it should have the symmetry property $\{456\}$. Using this symmetry property, we can decide A_1 and A_2. Once this state is determined, we can obtain the other state which consists of the ssQ baryon ($S = \frac{1}{2}$) by using the orthogonality of the wave function.

(i) $uud(S = \frac{1}{2}) + ssQ(S = \frac{1}{2})$:

$$\psi^{BB}_{S=1} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(-\frac{2\sqrt{2}}{3}|S^1_3\rangle + \frac{1}{6}|S^1_6\rangle + \frac{\sqrt{3}}{6}|S^1_8\rangle\right) + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(-\frac{2\sqrt{2}}{3}|S^1_4\rangle + \frac{1}{6}|S^1_7\rangle + \frac{\sqrt{3}}{6}|S^1_9\rangle\right). \quad (30)$$

(ii) $uud(S = \frac{1}{2}) + ssQ(S = \frac{3}{2})$:

$$\psi^{BB}_{S=1} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(\frac{1}{3}|S^1_3\rangle + \frac{\sqrt{2}}{3}|S^1_6\rangle + \frac{\sqrt{6}}{3}|S^1_8\rangle\right) + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(\frac{1}{3}|S^1_4\rangle + \frac{\sqrt{2}}{3}|S^1_7\rangle + \frac{\sqrt{6}}{3}|S^1_9\rangle\right). \quad (31)$$

We cannot construct the baryon $\{123\}$-baryon $\{456\}$ wave function with $S = 3$, because there is no uud baryon with $I = \frac{1}{2}$ and $S = \frac{3}{2}$.

VIII. NUMERICAL RESULTS

In this section, we present our numerical results. Before using the variational method, we can estimate the stability condition only by using the simple color-spin interaction formula given as $-\sum_{i<j}^{N} 1/(m_i m_j)\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$ without taking into account the r dependence of the hyperfine potential. In the upper part of Table IV, we show the matrix elements without the masses, which reflects the attraction

TABLE IV. The expectation values of $-\sum_{i<j}^{N}\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$ and $-\sum_{i<j}^{N} 1/(m_i m_j)\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$ (unit GeV^{-2}) for each antitriplet flavor state and corresponding decay channel.

$-\sum_{i<j}^{N}\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$				
$uudssQ$	$\psi_{1,S=0}$	$\psi_{1,S=1}$	$\psi_{5,S=1}$	$\psi_{1,S=2}$
	-24	$-\frac{40}{3}$	$-\frac{76}{3}$	-4
Decay channel	$N\Omega_Q$	$N\Omega_Q$	$N\Omega_Q$	$N\Omega^*_Q$
	-16	-16	-16	0
Decay channel	$\Lambda\Xi_Q$	$\Lambda\Xi_Q$	$\Lambda\Xi_Q$	$\Lambda\Xi^*_Q$
	-16	-16	-16	0
$-\sum_{i<j}^{N} 1/(m_i m_j)\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$				
$uudssc$	$\psi_{1,S=0}$	$\psi_{1,S=1}$	$\psi_{5,S=1}$	$\psi_{1,S=2}$
	-98.1	-87.1	-91.2	-64.9
Decay channel	$N\Omega_c$	$N\Omega_c$	$N\Omega_c$	$N\Omega^*_c$
	-70.1	-70.1	-70.1	-57
Decay channel	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi^*_c$
	-104.9	-104.9	-104.9	-49.5
$-\sum_{i<j}^{N} 1/(m_i m_j)\langle\lambda^c_i\lambda^c_j\sigma_i\cdot\sigma_j\rangle$				
$uudssb$	$\psi_{1,S=0}$	$\psi_{1,S=1}$	$\psi_{5,S=1}$	$\psi_{1,S=2}$
	-92.9	-88.8	-80.7	-71.2
Decay channel	$N\Omega_b$	$N\Omega_b$	$N\Omega_b$	$N\Omega^*_b$
	-64.5	-64.5	-64.5	-59.7
Decay channel	$\Lambda\Xi_b$	$\Lambda\Xi_b$	$\Lambda\Xi_b$	$\Lambda\Xi^*_b$
	-104.9	-104.9	-104.9	-53.4

TABLE V. Variational parameters fitted to the ground state of the dibaryon for $S = 0$, 1, 2 states using the variational method. The units for the variational parameters are fm^{-2}, and the mass unit is GeV.

$uudssc$	a	b	c	d	Mass	Decay mode
$S = 0$	2.5	3.5	5.7	0	3.666	$N\Omega_c$
$S = 1$	2.5	3.5	5.7	0	3.666	$N\Omega_c$
$S = 2$	2.5	3.2	4.8	0	3.737	$N\Omega_c^*$

$uudssb$	a	b	c	d	Mass	Decay mode
$S = 0$	2.5	3.5	6.5	0	6.997	$N\Omega_b$
$S = 1$	2.5	3.5	6.5	0	6.997	$N\Omega_b$
$S = 2$	2.5	3.3	6.1	0	7.028	$N\Omega_b^*$

in the $SU(4)$ flavor symmetric limit. As one can see here, the $\psi_{5,S=1}$ is the most attractive channel. The lower part of Table IV is the realistic case where all the relevant constituent quark masses are included. One notes that, for $S = 0$, 1, $\Lambda\Xi_c$ and $\Lambda\Xi_b$ systems are now more attractive than the corresponding flavor antitriplet dibaryon states, but for $S = 2$ the dibaryon states are more attractive. Therefore, we can expect to have a compact stable dibaryon state for the $S = 2$ configuration when considering the color-spin interaction only.

We now analyze the numerical results obtained from the variational method. Varying the parameters of the spatial function, we find the ground state of the dibaryon. The parameters and masses are in Table V. Here, we represent the color, flavor, and spin wave function of the ground state dibaryon obtained from the variational method for $S = 0$, 1, 2, 3:

$$\begin{aligned}\psi^{\text{ground}}_{S=0} &= -0.5\psi^6_1 + 0.71\psi^6_2 - 0.17\psi^6_3 - 0.47\psi^6_4 \\ &= 0.35|C_5\rangle \otimes |F_1\rangle \otimes |S^0_2\rangle \\ &\quad + 0.35|C_5\rangle \otimes |F_2\rangle \otimes |S^0_3\rangle \\ &\quad + 0.61|C_5\rangle \otimes |F_1\rangle \otimes |S^0_4\rangle \\ &\quad + 0.61|C_5\rangle \otimes |F_2\rangle \otimes |S^0_5\rangle, \end{aligned} \tag{32}$$

$$\begin{aligned}\psi^{\text{ground}}_{S=1} &= -0.17\psi^6_1 + 0.24\psi^6_2 - 0.06\psi^6_3 - 0.16\psi^6_4 \\ &\quad + 0.67\psi^6_5 + 0.42\psi^6_6 - 0.5\psi^6_7 + 0.14\psi^6_8 \\ &= -0.67|C_5\rangle \otimes |F_1\rangle \otimes |S^1_3\rangle - 0.67|C_5\rangle \otimes |F_2\rangle \otimes |S^1_4\rangle \\ &\quad + 0.12|C_5\rangle \otimes |F_1\rangle \otimes |S^1_6\rangle + 0.12|C_5\rangle \otimes |F_2\rangle \otimes |S^1_7\rangle \\ &\quad + 0.2|C_5\rangle \otimes |F_1\rangle \otimes |S^1_8\rangle + 0.2|C_5\rangle \otimes |F_2\rangle \otimes |S^1_9\rangle, \end{aligned} \tag{33}$$

$$\begin{aligned}\psi^{\text{ground}}_{S=2} &= -0.71\psi^6_1 - 0.45\psi^6_2 + 0.53\psi^6_3 - 0.15\psi^6_4 \\ &= 0.71|C_5\rangle \otimes |F_1\rangle \otimes |S^2_4\rangle \\ &\quad + 0.71|C_5\rangle \otimes |F_2\rangle \otimes |S^2_5\rangle, \end{aligned} \tag{34}$$

$$\begin{aligned}\psi^{\text{ground}}_{S=3} &= \psi^6_1 = \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S^3_1\rangle \\ &\quad - \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S^3_1\rangle \\ &\quad - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S^3_1\rangle \\ &\quad + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S^3_1\rangle. \end{aligned} \tag{35}$$

As we can see, the color wave function of the ground state for $S = 0$, 1, 2 is $|C_5\rangle$, and the coefficients are almost the same as in Sec. VII E. Therefore, the ground state of q^5Q for $S = 0$, 1, 2 is the sum of two baryon states. However, for $S = 3$, the color state is not $|C_5\rangle$, because there is no baryon state with $I = 1/2$ and $S = 3/2$ for uud. Therefore, the ground state for $S = 3$ is not the sum of two baryon states. Also, as can be seen in Table V, $d = 0$ for the ground state other than the $S = 3$. This parameter corresponds to a well-separated baryon-baryon configuration. As $uudssQ$ has the $\{123\}\{45\}6$ symmetry property, even with a single Gaussian trial wave function given in Eq. (12), one can express the wave function for a well-separated baryon(123)-baryon(456) configuration. Therefore, the mass quoted in the first two tables of Table V are just the mass sum of the decaying baryons.

In Table VI, we show the result for the dibaryon state with $S = 3$. It should be first noted that, while the dibaryon ground state can be written as a sum of two baryon states for $S = 0$, 1, 2 states, such decomposition is not possible for the $S = 3$ state. This is so because while, for the $S = 0$, 1, 2 states, the required symmetry $\{123\}\{45\}6$ can be satisfied by the combining the symmetry property of the first decaying baryon $\{123\}$ and the second baryon $\{45\}6$, which also happens to be the lowest decaying mode, one cannot decompose the symmetry into a single combination of two baryon states if the dibaryon state is in the $S = 3$, $I = 1/2$ configuration. That is why we obtain a positive binding energy for the dibaryon in this case.

We can still study the stability as we change the heavy quark mass. In fact, we can find that $E_B/(m_{\Sigma^*} + m_{\Xi^*_Q})$ becomes smaller by changing $Q = c \rightarrow b$, which implies that the dibaryon mass actually becomes relatively lower when Q becomes heavier. It would be interesting if we

TABLE VI. Variational parameters fitted to the ground state of the $S = 3$ dibaryon state using the variational method. The binding energy E_B is taken to be the difference between the mass of the dibaryon and the masses of the lowest two baryon decay modes. The units for the variational parameters are fm^{-2}, and the mass and binding energy units are GeV.

$S = 3$	a	b	c	d	Mass	E_B	Decay mode
$uudssc$	1.6	3.3	2.8	2.4	4.404	0.375	$\Sigma^*\Xi_c^*$
$uudssb$	1.6	3.3	4.5	2.5	7.803	0.473	$\Sigma^*\Xi_b^*$

compute the dibaryon mass for the $uudsss$ to study the systematics with different quark masses. However, in that case, there will be three identical s quarks in our system of study which will require all the s quarks to be in the antisymmetric configuration. That will modify the color-spin part of the wave function and thus potentially give different matrix elements for same-spin states as compared to our case. While such a configuration is interesting, we will leave such a configuration for future study.

Additionally, we have to notice that there are two cases to make the $S = 1$ dibaryon from the two baryon states. The first one is two $S = 1/2$ baryons, and the second is one $S = 1/2$ baryon and one $S = 3/2$ baryon. The ground state in Eq. (33) corresponds to the first case. We can also calculate the excited state for $S = 1$ by using variational parameters obtained from the ground state. We find that the flavor color-spin wave function for the first excited state is given as

$$\begin{aligned}\psi^{\text{excited}}_{S=1} = {}& 0.24|C_5\rangle \otimes |F_1\rangle \otimes |S^1_3\rangle + 0.24|C_5\rangle \otimes |F_2\rangle \otimes |S^1_4\rangle \\ & + 0.33|C_5\rangle \otimes |F_1\rangle \otimes |S^1_6\rangle + 0.33|C_5\rangle \otimes |F_2\rangle \otimes |S^1_7\rangle \\ & + 0.58|C_5\rangle \otimes |F_1\rangle \otimes |S^1_8\rangle + 0.58|C_5\rangle \otimes |F_2\rangle \otimes |S^1_9\rangle. \end{aligned} \tag{36}$$

This form turns out to be the same as the second case for $S = 1$ given in Sec. VII E, which represents the flavor color-spin state of the $N\Omega^*_c$ system.

IX. SUMMARY

In this work, we investigate the symmetry property and the stability of the dibaryon containing two strange quarks and one heavy flavor with $I = \frac{1}{2}$. To obtain the wave function of the dibaryon with the required symmetry property, we utilize two methods; the first one is to construct the color and spin wave function of the dibaryon from each flavor state directly, and the second one is to form the color and spin state of q^5 first and then construct the wave function of the dibaryon by adding one heavy quark. We verify that their results are the same.

In the $SU(3)$ flavor limit, by using Eq. (9), we expect the flavor antitriplet state to be the most attractive channel when five light quarks are considered. Additionally, if we increase the mass of the sixth quark, then its additional kinetic term will become smaller so that it may lead to a better chance to form the stable and compact dibaryon state. Hence, we calculate the mass of the dibaryon with two strange quarks and one heavy flavor.

We first estimate the stability condition by using only the color-spin interaction without the r dependence. In the $SU(4)$ flavor symmetric limit, all the antitriplet flavor states of the dibaryon with $S = 0, 1, 2$ are more attractive than the decay channel. However, when we consider the relevant constituent quark masses obtained from baryon fitting results, only the antitriplet state for $S = 2$ is more attractive.

Finally, we calculate the masses of the dibaryons in a nonrelativistic Hamiltonian by using the variational method. We find that the ground state of the dibaryon for $S = 0, 1, 2$ is the well-separated two baryon states. As for $S = 3$, with our single Gaussian form, the mass of the ground state is found to be more repulsive than the decay channel. Hence, we conclude that there are no stable and compact $uudssQ$ dibaryon states with $I = \frac{1}{2}$ within the given potential. It should be noted that the spatial wave function that we used as shown in Sec. II is limited to symmetric configurations. However, it is important to improve the spatial part to include possible correlations between quarks as was taken into account in Ref. [16] through the orbital mixed symmetry [4, 2]. We will leave such improvements as an important future work. If we consider the dibaryon containing more than one heavy quark, then their additional kinetic terms will decrease further than the dibaryon with one heavy quark so that it may lead to a better chance to form the stable dibaryon.

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APPENDIX A: COLOR AND SPIN STATE OF q^5

Here, we represent the color and spin basis function of q^5.

1. Color basis function

$$|C_1\rangle = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \quad |C_2\rangle = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \quad |C_3\rangle = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad |C_4\rangle = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array},$$

$$|C_5\rangle = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}. \tag{A1}$$

2. Spin basis function

(i) $S = 1/2$.—Five basis functions with Young tableau [3, 3]:

$$|S^{\frac{1}{2}}_1\rangle = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \quad |S^{\frac{1}{2}}_2\rangle = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad |S^{\frac{1}{2}}_3\rangle = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array},$$

$$|S^{\frac{1}{2}}_4\rangle = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \quad |S^{\frac{1}{2}}_5\rangle = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

(ii) $S = 3/2$.—Four basis functions with Young tableau [4, 2]:

$$|S_1^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1&2&3&4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array}, \qquad |S_2^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1&2&3&5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}, \qquad |S_3^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1&2&4&5 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array},$$

$$|S_4^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1&3&4&5 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}.$$

(iii) $S = 5/2$.—One basis function with Young tableau [6]:

$$|S_1^{\frac{5}{2}}\rangle = \begin{array}{|c|c|c|c|c|} \hline 1&2&3&4&5 \\ \hline \end{array}.$$

APPENDIX B: CS COUPLING OF q^5Q

In this section, we present the color ⊗ spin basis. The Clebsch-Gordon coefficients of combining the color and spin basis are calculated by using K matrix [15,17].

1. $S=0$

$$\begin{array}{|c|c|c|} \hline 1&2&3 \\ \hline 4&5&6 \\ \hline \end{array}_{CS} = \frac{1}{2}|C_1\rangle \otimes |S_2^0\rangle + \frac{1}{2}|C_2\rangle \otimes |S_3^0\rangle + \frac{1}{2}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2}|C_4\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|c|} \hline 1&2&4 \\ \hline 3&5&6 \\ \hline \end{array}_{CS} = \frac{1}{2}|C_1\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^0\rangle - \frac{1}{2}|C_5\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|c|} \hline 1&3&4 \\ \hline 2&5&6 \\ \hline \end{array}_{CS} = \frac{1}{2}|C_2\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^0\rangle + \frac{1}{2}|C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|c|} \hline 1&2&5 \\ \hline 3&4&6 \\ \hline \end{array}_{CS} = \frac{1}{2}|C_3\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_3^0\rangle + \frac{1}{2}|C_5\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|c|} \hline 1&3&5 \\ \hline 2&4&6 \\ \hline \end{array}_{CS} = \frac{1}{2}|C_4\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_2^0\rangle - \frac{1}{2}|C_5\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\right._{CS} = \frac{1}{\sqrt{2}}|C_1\rangle\otimes|S_4^0\rangle + \frac{1}{\sqrt{2}}|C_2\rangle\otimes|S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\ \hline 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\right._{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S_2^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle\otimes|S_2^0\rangle - \frac{1}{\sqrt{10}}|C_2\rangle\otimes|S_3^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle\otimes|S_3^0\rangle + \frac{1}{\sqrt{10}}|C_3\rangle\otimes|S_4^0\rangle + \frac{1}{\sqrt{10}}|C_4\rangle\otimes|S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\right._{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S_1^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle\otimes|S_1^0\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_2^0\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_3^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_3^0\rangle - \frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_4^0\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_5^0\rangle - \frac{1}{\sqrt{10}}|C_5\rangle\otimes|S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\right._{CS} = -\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S_1^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle\otimes|S_1^0\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_2^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_2^0\rangle + \frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_3^0\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_4^0\rangle + \frac{1}{\sqrt{10}}|C_5\rangle\otimes|S_4^0\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\right._{CS} = -\frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle\otimes|S_2^0\rangle - \frac{1}{\sqrt{5}}|C_3\rangle\otimes|S_2^0\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle\otimes|S_3^0\rangle - \frac{1}{\sqrt{5}}|C_4\rangle\otimes|S_3^0\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_4^0\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_5^0\rangle,$$

$$\left|\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3\\ \cline{1-1} 5\\ \cline{1-1}\end{array}\right._{CS} = -\frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle\otimes|S_1^0\rangle + \frac{1}{\sqrt{5}}|C_3\rangle\otimes|S_1^0\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_2^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_3^0\rangle - \frac{1}{\sqrt{5}}|C_5\rangle\otimes|S_3^0\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_4^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_5^0\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2\\ \cline{1-1} 5\\ \cline{1-1}\end{array}_{CS} = -\frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle\otimes|S_1^0\rangle+\frac{1}{\sqrt{5}}|C_4\rangle\otimes|S_1^0\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_2^0\rangle+\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S_2^0\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_3^0\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_4^0\rangle+\frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle\otimes|S_4^0\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1}\end{array}_{CS} = \frac{1}{2}|C_1\rangle\otimes|S_1^0\rangle-\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_2^0\rangle+\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_3^0\rangle-\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_4^0\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_5^0\rangle$$
$$+\frac{1}{2}|C_5\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1}\end{array}_{CS} = \frac{1}{2}|C_2\rangle\otimes|S_1^0\rangle+\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_2^0\rangle+\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_3^0\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_4^0\rangle-\frac{1}{2}|C_5\rangle\otimes|S_4^0\rangle$$
$$+\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&4&5&6\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2}|C_2\rangle\otimes|S_2^0\rangle+\frac{1}{3}|C_1\rangle\otimes|S_3^0\rangle-\frac{1}{2}|C_4\rangle\otimes|S_4^0\rangle+\frac{1}{2}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline 5\\ \cline{1-1} 6\\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_1^0\rangle+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_2^0\rangle-\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_3^0\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_3^0\rangle-\frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_4^0\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_4^0\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_1^0\rangle - \frac{1}{3}|C_3\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_2^0\rangle - \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_3^0\rangle$$
$$+\frac{\sqrt{2}}{3}|C_4\rangle \otimes |S_3^0\rangle - \frac{1}{3}|C_5\rangle \otimes |S_3^0 - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_4^0\rangle\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_5^0\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_1^0\rangle - \frac{1}{3}|C_4\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_2^0\rangle + \frac{\sqrt{2}}{3}|C_4\rangle \otimes |S_2^0\rangle + \frac{1}{3}|C_5\rangle \otimes |S_2^0\rangle$$
$$-\frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_3^0\rangle + \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_3^0 + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_4^0\rangle\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}_{CS} = \frac{2}{3}|C_5\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_2^0\rangle + \frac{1}{3}|C_4\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_3^0\rangle - \frac{1}{3}|C_3\rangle \otimes |S_3^0\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_1^0\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_2^0\rangle + \frac{1}{3}|C_3\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_3^0\rangle$$
$$-\frac{1}{3}|C_4\rangle \otimes |S_3^0\rangle + \frac{1}{3\sqrt{2}}|C_5\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_5^0\rangle + \frac{1}{\sqrt{6}}|C_5\rangle \otimes |S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_1^0\rangle+\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S_1^0\rangle-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_2^0\rangle-\frac{1}{3}|C_4\rangle\otimes|S_2^0\rangle-\frac{1}{3\sqrt{2}}|C_5\rangle\otimes|S_2^0\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_3^0\rangle-\frac{1}{3}|C_3\rangle\otimes|S_3^0\rangle+\frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_4^0\rangle-\frac{1}{\sqrt{6}}|C_5\rangle\otimes|S_4^0\rangle+\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}_{CS} = -\frac{\sqrt{2}}{3}|C_5\rangle\otimes|S_1^0\rangle+\frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_2^0\rangle-\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S_2^0\rangle-\frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_3^0\rangle+\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S_3^0\rangle$$
$$-\frac{1}{\sqrt{6}}|C_4\rangle\otimes|S_4^0\rangle+\frac{1}{\sqrt{6}}|C_3\rangle\otimes|S_5^0\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle\otimes|S_1^0\rangle-\frac{\sqrt{2}}{\sqrt{15}}|C_4\rangle\otimes|S_2^0\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_3\rangle\otimes|S_3^0\rangle-\frac{\sqrt{3}}{\sqrt{10}}|C_2\rangle\otimes|S_4^0\rangle+\frac{\sqrt{3}}{\sqrt{10}}|C_1\rangle\otimes|S_5^0\rangle.$$

2. $S=1$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_3^1\rangle-\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_4^1\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_4^1\rangle+\frac{1}{\sqrt{5}}|C_1\rangle\otimes|S_6^1\rangle$$
$$+\frac{1}{\sqrt{5}}|C_2\rangle\otimes|S_7^1\rangle+\frac{1}{\sqrt{5}}|C_3\rangle\otimes|S_8^1\rangle+\frac{1}{\sqrt{5}}|C_4\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_2^1\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_2^1\rangle-\frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S_3^1\rangle+\frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S_4^1\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_4^1\rangle$$
$$+\frac{1}{\sqrt{5}}|C_1\rangle\otimes|S_5^1\rangle+\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S_6^1\rangle-\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S_7^1\rangle-\frac{1}{\sqrt{10}}|C_3\rangle\otimes|S_8^1\rangle+\frac{1}{\sqrt{10}}|C_4\rangle\otimes|S_9^1\rangle$$
$$-\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2} 6\\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S^1_2\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S^1_2\rangle+\frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S^1_3\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S^1_3\rangle+\frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S^1_4\rangle$$
$$+\frac{1}{\sqrt{5}}|C_2\rangle\otimes|S^1_5\rangle-\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S^1_6\rangle-\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S^1_7\rangle+\frac{1}{\sqrt{10}}|C_4\rangle\otimes|S^1_8\rangle+\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S^1_8\rangle$$
$$+\frac{1}{\sqrt{10}}|C_3\rangle\otimes|S^1_9\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2} 6\\ \cline{1-1}\end{array}_{CS} = \frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S^1_1\rangle+\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S^1_2\rangle-\frac{1}{4\sqrt{5}}|C_3\rangle\otimes|S^1_3\rangle+\frac{1}{4\sqrt{5}}|C_4\rangle\otimes|S^1_4\rangle+\frac{1}{2\sqrt{10}}|C_5\rangle\otimes|S^1_4\rangle$$
$$+\frac{1}{\sqrt{5}}|C_3\rangle\otimes|S^1_5\rangle-\frac{1}{\sqrt{10}}|C_3\rangle\otimes|S^1_6\rangle+\frac{1}{\sqrt{10}}|C_4\rangle\otimes|S^1_7\rangle+\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S^1_7\rangle-\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S^1_8\rangle$$
$$+\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S^1_9\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2} 6\\ \cline{1-1}\end{array}_{CS} = \frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S^1_1\rangle+\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S^1_2\rangle+\frac{1}{4\sqrt{5}}|C_4\rangle\otimes|S^1_3\rangle-\frac{1}{2\sqrt{10}}|C_5\rangle\otimes|S^1_3\rangle+\frac{1}{4\sqrt{5}}|C_3\rangle\otimes|S^1_4\rangle$$
$$+\frac{1}{\sqrt{5}}|C_4\rangle\otimes|S^1_5\rangle+\frac{1}{\sqrt{10}}|C_4\rangle\otimes|S^1_6\rangle-\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S^1_6\rangle+\frac{1}{\sqrt{10}}|C_3\rangle\otimes|S^1_7\rangle+\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S^1_8\rangle$$
$$+\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S^1_9\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&6\\ \cline{1-2} 5\\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S^1_3\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S^1_3\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S^1_4\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S^1_4\rangle+\frac{1}{\sqrt{15}}|C_1\rangle\otimes|S^1_6\rangle$$
$$+\frac{2}{3\sqrt{5}}|C_3\rangle\otimes|S^1_6\rangle+\frac{1}{\sqrt{15}}|C_2\rangle\otimes|S^1_6\rangle+\frac{2}{3\sqrt{5}}|C_4\rangle\otimes|S^1_7\rangle-\frac{1}{\sqrt{15}}|C_3\rangle\otimes|S^1_8\rangle-\frac{1}{\sqrt{15}}|C_4\rangle\otimes|S^1_9\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&6\\ \cline{1-2} 5\\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S^1_2\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S^1_2\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle\otimes|S^1_3\rangle-\frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle\otimes|S^1_4\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle\otimes|S^1_4\rangle$$
$$+\frac{1}{\sqrt{15}}|C_1\rangle\otimes|S^1_5\rangle-\frac{2}{3\sqrt{5}}|C_3\rangle\otimes|S^1_5\rangle+\frac{1}{\sqrt{30}}|C_1\rangle\otimes|S^1_6\rangle-\frac{1}{\sqrt{30}}|C_2\rangle\otimes|S^1_7\rangle+\frac{2}{3\sqrt{5}}|C_5\rangle\otimes|S^1_7\rangle$$
$$+\frac{1}{\sqrt{30}}|C_3\rangle\otimes|S^1_8\rangle-\frac{1}{\sqrt{30}}|C_4\rangle\otimes|S^1_9\rangle+\frac{1}{\sqrt{15}}|C_5\rangle\otimes|S^1_9\rangle,$$

$$\begin{aligned}\left.\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\right._{CS} = {}& \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S^1_2\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S^1_2\rangle - \frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle\otimes|S^1_3\rangle - \frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle\otimes|S^1_4\rangle \\ & + \frac{1}{\sqrt{15}}|C_2\rangle\otimes|S^1_5\rangle - \frac{2}{3\sqrt{5}}|C_4\rangle\otimes|S^1_5\rangle - \frac{1}{\sqrt{30}}|C_2\rangle\otimes|S^1_6\rangle - \frac{2}{3\sqrt{5}}|C_5\rangle\otimes|S^1_6\rangle - \frac{1}{\sqrt{30}}|C_1\rangle\otimes|S^1_7\rangle \\ & - \frac{1}{\sqrt{30}}|C_4\rangle\otimes|S^1_8\rangle - \frac{1}{\sqrt{15}}|C_5\rangle\otimes|S^1_8\rangle - \frac{1}{\sqrt{30}}|C_3\rangle\otimes|S^1_9\rangle,\end{aligned}$$

$$\begin{aligned}\left.\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\right._{CS} = {}& \frac{\sqrt{5}}{3\sqrt{2}}|C_3\rangle\otimes|S^1_1\rangle + \frac{1}{6\sqrt{2}}|C_1\rangle\otimes|S^1_2\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S^1_2\rangle - \frac{1}{12}|C_1\rangle\otimes|S^1_3\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S^1_3\rangle \\ & + \frac{1}{12}|C_2\rangle\otimes|S^1_4\rangle - \frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S^1_4\rangle + \frac{1}{2\sqrt{6}}|C_5\rangle\otimes|S^1_4\rangle - \frac{1}{3}|C_1\rangle\otimes|S^1_5\rangle + \frac{1}{3\sqrt{2}}|C_1\rangle\otimes|S^1_6\rangle \\ & - \frac{1}{3\sqrt{2}}|C_2\rangle\otimes|S^1_7\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S^1_8\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S^1_9\rangle - \frac{1}{3}|C_5\rangle\otimes|S^1_9\rangle,\end{aligned}$$

$$\begin{aligned}\left.\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\right._{CS} = {}& \frac{\sqrt{5}}{3\sqrt{2}}|C_4\rangle\otimes|S^1_1\rangle + \frac{1}{6\sqrt{2}}|C_2\rangle\otimes|S^1_2\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S^1_2\rangle + \frac{1}{12}|C_2\rangle\otimes|S^1_3\rangle - \frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S^1_3\rangle \\ & - \frac{1}{2\sqrt{6}}|C_5\rangle\otimes|S^1_3\rangle + \frac{1}{12}|C_1\rangle\otimes|S^1_4\rangle - \frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S^1_4\rangle - \frac{1}{3}|C_2\rangle\otimes|S^1_5\rangle - \frac{1}{3\sqrt{2}}|C_2\rangle\otimes|S^1_6\rangle \\ & - \frac{1}{3\sqrt{2}}|C_1\rangle\otimes|S^1_7\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S^1_8\rangle + \frac{1}{3}|C_5\rangle\otimes|S^1_8\rangle - \frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S^1_9\rangle,\end{aligned}$$

$$\begin{aligned}\left.\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array}\right._{CS} = {}& \frac{\sqrt{5}}{3\sqrt{2}}|C_5\rangle\otimes|S^1_1\rangle - \frac{1}{\sqrt{6}}|C_5\rangle\otimes|S^1_2\rangle - \frac{1}{6\sqrt{2}}|C_2\rangle\otimes|S^1_3\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S^1_3\rangle + \frac{1}{6\sqrt{2}}|C_1\rangle\otimes|S^1_4\rangle \\ & + \frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S^1_4\rangle + \frac{1}{3}|C_2\rangle\otimes|S^1_6\rangle - \frac{1}{3}|C_1\rangle\otimes|S^1_7\rangle + \frac{1}{3}|C_4\rangle\otimes|S^1_8\rangle - \frac{1}{3}|C_3\rangle\otimes|S^1_9\rangle,\end{aligned}$$

$$\begin{aligned}\left.\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\right._{CS} = {}& \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S^1_1\rangle - \frac{5}{6\sqrt{6}}|C_3\rangle\otimes|S^1_2\rangle + \frac{5}{12\sqrt{3}}|C_3\rangle\otimes|S^1_3\rangle - \frac{5}{12\sqrt{3}}|C_4\rangle\otimes|S^1_4\rangle - \frac{5}{6\sqrt{6}}|C_5\rangle\otimes|S^1_4\rangle \\ & - \frac{1}{3\sqrt{3}}|C_3\rangle\otimes|S^1_5\rangle + \frac{1}{3\sqrt{6}}|C_3\rangle\otimes|S^1_6\rangle - \frac{1}{3\sqrt{6}}|C_4\rangle\otimes|S^1_7\rangle - \frac{1}{3\sqrt{3}}|C_5\rangle\otimes|S^1_7\rangle - \frac{1}{\sqrt{6}}|C_1\rangle\otimes|S^1_8\rangle \\ & + \frac{1}{\sqrt{6}}|C_2\rangle\otimes|S^1_9\rangle,\end{aligned}$$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array}_{CS} = \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_4\rangle\otimes|S_2^1\rangle - \frac{5}{12\sqrt{3}}|C_4\rangle\otimes|S_3^1\rangle + \frac{5}{6\sqrt{6}}|C_5\rangle\otimes|S_3^1\rangle - \frac{5}{12\sqrt{3}}|C_3\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{3\sqrt{3}}|C_4\rangle\otimes|S_5^1\rangle - \frac{1}{3\sqrt{6}}|C_4\rangle\otimes|S_6^1\rangle + \frac{1}{3\sqrt{3}}|C_5\rangle\otimes|S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_3\rangle\otimes|S_7^1\rangle + \frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_8^1\rangle$$
$$+\frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}_{CS} = -\frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle\otimes|S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_1\rangle\otimes|S_2^1\rangle + \frac{5}{18\sqrt{2}}|C_3\rangle\otimes|S_2^1\rangle + \frac{5}{12\sqrt{3}}|C_1\rangle\otimes|S_3^1\rangle + \frac{5}{18}|C_3\rangle\otimes|S_3^1\rangle$$
$$-\frac{5}{12\sqrt{3}}|C_2\rangle\otimes|S_4^1\rangle - \frac{5}{18}|C_4\rangle\otimes|S_4^1\rangle + \frac{5}{18\sqrt{2}}|C_5\rangle\otimes|S_4^1\rangle - \frac{1}{3\sqrt{3}}|C_1\rangle\otimes|S_5^1\rangle - \frac{2}{9}|C_3\rangle\otimes|S_5^1\rangle$$
$$+\frac{1}{3\sqrt{6}}|C_1\rangle\otimes|S_6^1\rangle - \frac{2\sqrt{2}}{9}|C_3\rangle\otimes|S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_2\rangle\otimes|S_7^1\rangle + \frac{2\sqrt{2}}{9}|C_4\rangle\otimes|S_7^1\rangle - \frac{2}{9}|C_5\rangle\otimes|S_7^1\rangle$$
$$-\frac{1}{3\sqrt{6}}|C_3\rangle\otimes|S_8^1\rangle + \frac{1}{3\sqrt{6}}|C_4\rangle\otimes|S_9^1\rangle + \frac{1}{3\sqrt{3}}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array}_{CS} = -\frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle\otimes|S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_2\rangle\otimes|S_2^1\rangle + \frac{5}{18\sqrt{2}}|C_4\rangle\otimes|S_2^1\rangle - \frac{5}{12\sqrt{3}}|C_2\rangle\otimes|S_3^1\rangle - \frac{5}{18}|C_4\rangle\otimes|S_3^1\rangle$$
$$-\frac{5}{18\sqrt{2}}|C_5\rangle\otimes|S_3^1\rangle - \frac{5}{12\sqrt{3}}|C_1\rangle\otimes|S_4^1\rangle - \frac{5}{18}|C_3\rangle\otimes|S_4^1\rangle - \frac{1}{3\sqrt{3}}|C_2\rangle\otimes|S_5^1\rangle - \frac{2}{9}|C_4\rangle\otimes|S_5^1\rangle$$
$$-\frac{1}{3\sqrt{6}}|C_2\rangle\otimes|S_6^1\rangle + \frac{2\sqrt{2}}{9}|C_4\rangle\otimes|S_6^1\rangle + \frac{2}{9}|C_5\rangle\otimes|S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_1\rangle\otimes|S_7^1\rangle + \frac{2\sqrt{2}}{9}|C_3\rangle\otimes|S_7^1\rangle$$
$$+\frac{1}{3\sqrt{6}}|C_4\rangle\otimes|S_8^1\rangle - \frac{1}{3\sqrt{3}}|C_5\rangle\otimes|S_8^1\rangle + \frac{1}{3\sqrt{6}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}_{CS} = -\frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle\otimes|S_1^1\rangle - \frac{5}{9\sqrt{2}}|C_5\rangle\otimes|S_2^1\rangle + \frac{5}{6\sqrt{6}}|C_2\rangle\otimes|S_3^1\rangle - \frac{5}{18\sqrt{2}}|C_4\rangle\otimes|S_3^1\rangle - \frac{5}{6\sqrt{6}}|C_1\rangle\otimes|S_4^1\rangle$$
$$+\frac{5}{18\sqrt{2}}|C_3\rangle\otimes|S_4^1\rangle + \frac{4}{9}|C_5\rangle\otimes|S_5^1\rangle + \frac{1}{3\sqrt{3}}|C_2\rangle\otimes|S_6^1\rangle + \frac{2}{9}|C_4\rangle\otimes|S_6^1\rangle - \frac{1}{3\sqrt{3}}|C_1\rangle\otimes|S_7^1\rangle$$
$$-\frac{2}{9}|C_3\rangle\otimes|S_7^1\rangle - \frac{1}{3\sqrt{3}}|C_4\rangle\otimes|S_8^1\rangle + \frac{1}{3\sqrt{3}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_3^1\rangle - \frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S_3^1\rangle + \frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_4^1\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S_4^1\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_6^1\rangle$$
$$-\frac{1}{3}|C_3\rangle\otimes|S_6^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_7^1\rangle - \frac{1}{3}|C_4\rangle\otimes|S_7^1\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_8^1\rangle + \frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_2^1\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S_2^1\rangle + \frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_3^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_4^1\rangle - \frac{1}{3\sqrt{2}}|C_5\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_5^1\rangle + \frac{1}{3}|C_3\rangle\otimes|S_5^1\rangle - \frac{1}{2\sqrt{6}}|C_1\rangle\otimes|S_6^1\rangle + \frac{1}{2\sqrt{6}}|C_2\rangle\otimes|S_7^1\rangle - \frac{1}{3}|C_5\rangle\otimes|S_7^1\rangle$$
$$-\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_8^1\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_9^1\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_2^1\rangle + \frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S_2^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_3^1\rangle + \frac{1}{3\sqrt{2}}|C_5\rangle\otimes|S_3^1\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_5^1\rangle + \frac{1}{3}|C_4\rangle\otimes|S_5^1\rangle + \frac{1}{2\sqrt{6}}|C_2\rangle\otimes|S_6^1\rangle + \frac{1}{3}|C_5\rangle\otimes|S_6^1\rangle + \frac{1}{2\sqrt{6}}|C_1\rangle\otimes|S_7^1\rangle$$
$$+\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_8^1\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_8^1\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = \frac{\sqrt{2}}{3}|C_3\rangle\otimes|S_1^1\rangle + \frac{1}{3\sqrt{10}}|C_1\rangle\otimes|S_2^1\rangle + \frac{1}{\sqrt{30}}|C_3\rangle\otimes|S_2^1\rangle - \frac{1}{6\sqrt{5}}|C_1\rangle\otimes|S_3^1\rangle - \frac{1}{\sqrt{15}}|C_3\rangle\otimes|S_3^1\rangle$$
$$+\frac{1}{6\sqrt{5}}|C_2\rangle\otimes|S_4^1\rangle - \frac{1}{\sqrt{15}}|C_4\rangle\otimes|S_4^1\rangle + \frac{1}{\sqrt{30}}|C_5\rangle\otimes|S_4^1\rangle + \frac{\sqrt{5}}{6}|C_1\rangle\otimes|S_5^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_1\rangle\otimes|S_6^1\rangle$$
$$+\frac{\sqrt{5}}{6\sqrt{2}}|C_2\rangle\otimes|S_7^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S_9^1\rangle + \frac{\sqrt{5}}{6}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline \multicolumn{1}{|c|}{2}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{6}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}_{CS} = \frac{\sqrt{2}}{3}|C_4\rangle\otimes|S_1^1\rangle+\frac{1}{3\sqrt{10}}|C_2\rangle\otimes|S_2^1\rangle+\frac{1}{\sqrt{30}}|C_4\rangle\otimes|S_2^1\rangle+\frac{1}{6\sqrt{5}}|C_2\rangle\otimes|S_3^1\rangle-\frac{1}{\sqrt{15}}|C_4\rangle\otimes|S_3^1\rangle$$
$$-\frac{1}{\sqrt{30}}|C_5\rangle\otimes|S_3^1\rangle+\frac{1}{6\sqrt{5}}|C_1\rangle\otimes|S_4^1\rangle-\frac{1}{\sqrt{15}}|C_3\rangle\otimes|S_4^1\rangle+\frac{\sqrt{5}}{6}|C_2\rangle\otimes|S_5^1\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_2\rangle\otimes|S_6^1\rangle$$
$$+\frac{\sqrt{5}}{6\sqrt{2}}|C_1\rangle\otimes|S_7^1\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S_8^1\rangle-\frac{\sqrt{5}}{6}|C_5\rangle\otimes|S_8^1\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&4&5\\\hline \multicolumn{1}{|c|}{2}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{3}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{6}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}_{CS} = \frac{\sqrt{2}}{3}|C_5\rangle\otimes|S_1^1\rangle-\frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle\otimes|S_2^1\rangle-\frac{1}{3\sqrt{10}}|C_2\rangle\otimes|S_3^1\rangle-\frac{1}{\sqrt{30}}|C_4\rangle\otimes|S_3^1\rangle+\frac{1}{3\sqrt{10}}|C_1\rangle\otimes|S_4^1\rangle$$
$$+\frac{1}{\sqrt{30}}|C_3\rangle\otimes|S_4^1\rangle-\frac{\sqrt{5}}{6}|C_2\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{6}|C_1\rangle\otimes|S_7^1\rangle-\frac{\sqrt{5}}{6}|C_4\rangle\otimes|S_8^1\rangle+\frac{\sqrt{5}}{6}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&6\\\hline \multicolumn{1}{|c|}{3}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{5}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}_{CS} = -\frac{2}{3\sqrt{3}}|C_3\rangle\otimes|S_1^1\rangle+\frac{4}{3\sqrt{15}}|C_1\rangle\otimes|S_2^1\rangle+\frac{2}{9\sqrt{5}}|C_3\rangle\otimes|S_2^1\rangle-\frac{2\sqrt{2}}{3\sqrt{15}}|C_1\rangle\otimes|S_3^1\rangle+\frac{2\sqrt{2}}{9\sqrt{5}}|C_3\rangle\otimes|S_3^1\rangle$$
$$+\frac{2\sqrt{2}}{3\sqrt{15}}|C_2\rangle\otimes|S_4^1\rangle-\frac{2\sqrt{2}}{9\sqrt{5}}|C_4\rangle\otimes|S_4^1\rangle+\frac{2}{9\sqrt{5}}|C_5\rangle\otimes|S_4^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle\otimes|S_5^1\rangle-\frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle\otimes|S_5^1\rangle$$
$$-\frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S_6^1\rangle-\frac{\sqrt{5}}{9}|C_3\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S_7^1\rangle+\frac{\sqrt{5}}{9}|C_4\rangle\otimes|S_7^1\rangle-\frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle\otimes|S_7^1\rangle$$
$$+\frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S_8^1\rangle-\frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S_9^1\rangle-\frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}_{CS} = -\frac{2}{3\sqrt{3}}|C_4\rangle\otimes|S_1^1\rangle+\frac{4}{3\sqrt{15}}|C_2\rangle\otimes|S_2^1\rangle+\frac{2}{9\sqrt{5}}|C_4\rangle\otimes|S_2^1\rangle+\frac{2\sqrt{2}}{3\sqrt{15}}|C_2\rangle\otimes|S_3^1\rangle-\frac{2\sqrt{2}}{9\sqrt{5}}|C_4\rangle\otimes|S_3^1\rangle$$
$$-\frac{2}{9\sqrt{5}}|C_5\rangle\otimes|S_3^1\rangle+\frac{2\sqrt{2}}{3\sqrt{15}}|C_1\rangle\otimes|S_4^1\rangle-\frac{2\sqrt{2}}{9\sqrt{5}}|C_3\rangle\otimes|S_4^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle\otimes|S_5^1\rangle-\frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle\otimes|S_5^1\rangle$$
$$+\frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{9}|C_4\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S_7^1\rangle+\frac{\sqrt{5}}{9}|C_3\rangle\otimes|S_7^1\rangle$$
$$-\frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S_8^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle\otimes|S_8^1\rangle-\frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 5\\\cline{1-1}\end{array}_{CS} = -\frac{2}{3\sqrt{3}}|C_5\rangle\otimes|S_1^1\rangle-\frac{4}{9\sqrt{5}}|C_5\rangle\otimes|S_2^1\rangle-\frac{4}{3\sqrt{15}}|C_2\rangle\otimes|S_3^1\rangle-\frac{2}{9\sqrt{5}}|C_4\rangle\otimes|S_3^1\rangle+\frac{4}{3\sqrt{15}}|C_1\rangle\otimes|S_4^1\rangle$$
$$+\frac{2}{9\sqrt{5}}|C_3\rangle\otimes|S_4^1\rangle+\frac{\sqrt{10}}{9}|C_5\rangle\otimes|S_5^1\rangle-\frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle\otimes|S_6^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle\otimes|S_7^1\rangle$$
$$-\frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle\otimes|S_7^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle\otimes|S_8^1\rangle-\frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&5&6\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}_{CS} = \frac{2}{3\sqrt{3}}|C_5\rangle\otimes|S_2^1\rangle-\frac{2}{3\sqrt{3}}|C_4\rangle\otimes|S_3^1\rangle+\frac{2}{3\sqrt{3}}|C_3\rangle\otimes|S_4^1\rangle-\frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle\otimes|S_5^1\rangle+\frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle\otimes|S_6^1\rangle$$
$$-\frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle\otimes|S_7^1\rangle+\frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_8^1\rangle-\frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_1^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle\otimes|S_2^1\rangle-\frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S_3^1\rangle+\frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S_4^1\rangle+\frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle\otimes|S_4^1\rangle$$
$$-\frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S_5^1\rangle+\frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle\otimes|S_6^1\rangle-\frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle\otimes|S_7^1\rangle-\frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle\otimes|S_7^1\rangle-\frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S_8^1\rangle$$
$$+\frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S_9^1\rangle,$$

$$
\begin{array}{|c|c|}\hline 1&3\\ \hline 2&4\\ \hline 5&6\\ \hline \end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_2\rangle\otimes|S^1_1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle\otimes|S^1_2\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S^1_3\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S^1_4\rangle
$$
$$
-\frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S^1_5\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle\otimes|S^1_6\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle\otimes|S^1_6\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle\otimes|S^1_7\rangle + \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S^1_8\rangle
$$
$$
+\frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S^1_9\rangle,
$$

$$
\begin{array}{|c|c|}\hline 1&2\\ \hline 3&5\\ \hline 4&6\\ \hline \end{array}_{CS} = \frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle\otimes|S^1_1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle\otimes|S^1_2\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle\otimes|S^1_2\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S^1_3\rangle - \frac{\sqrt{5}}{9}|C_3\rangle\otimes|S^1_3\rangle
$$
$$
+\frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S^1_4\rangle + \frac{\sqrt{5}}{9}|C_4\rangle\otimes|S^1_4\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle\otimes|S^1_4\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S^1_5\rangle - \frac{\sqrt{5}}{9}|C_3\rangle\otimes|S^1_5\rangle
$$
$$
+\frac{\sqrt{5}}{6\sqrt{6}}|C_1\rangle\otimes|S^1_6\rangle - \frac{\sqrt{10}}{9}|C_3\rangle\otimes|S^1_6\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_2\rangle\otimes|S^1_7\rangle + \frac{\sqrt{10}}{9}|C_4\rangle\otimes|S^1_5\rangle - \frac{\sqrt{5}}{9}|C_5\rangle\otimes|S^1_7\rangle
$$
$$
-\frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle\otimes|S^1_8\rangle + \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle\otimes|S^1_9\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle\otimes|S^1_9\rangle,
$$

$$
\begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4&6\\ \hline \end{array}_{CS} = \frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle\otimes|S^1_1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle\otimes|S^1_2\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle\otimes|S^1_2\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{9}|C_4\rangle\otimes|S^1_3\rangle
$$
$$
+\frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S^1_4\rangle + \frac{\sqrt{5}}{9}|C_3\rangle\otimes|S^1_4\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S^1_5\rangle - \frac{\sqrt{5}}{9}|C_4\rangle\otimes|S^1_5\rangle
$$
$$
-\frac{\sqrt{5}}{6\sqrt{6}}|C_2\rangle\otimes|S^1_6\rangle + \frac{\sqrt{10}}{9}|C_4\rangle\otimes|S^1_6\rangle + \frac{\sqrt{5}}{9}|C_5\rangle\otimes|S^1_6\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_1\rangle\otimes|S^1_7\rangle + \frac{\sqrt{10}}{9}|C_3\rangle\otimes|S^1_7\rangle
$$
$$
+\frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle\otimes|S^1_8\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle\otimes|S^1_8\rangle + \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle\otimes|S^1_9\rangle,
$$

$$
\begin{array}{|c|c|}\hline 1&4\\ \hline 2&5\\ \hline 3&6\\ \hline \end{array}_{CS} = \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle\otimes|S^1_1\rangle + \frac{\sqrt{10}}{9}|C_5\rangle\otimes|S^1_2\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle\otimes|S^1_3\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle\otimes|S^1_4\rangle
$$
$$
-\frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle\otimes|S^1_4\rangle + \frac{2\sqrt{5}}{9}|C_5\rangle\otimes|S^1_5\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle\otimes|S^1_6\rangle + \frac{\sqrt{5}}{9}|C_4\rangle\otimes|S^1_6\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle\otimes|S^1_7\rangle
$$
$$
-\frac{\sqrt{5}}{9}|C_3\rangle\otimes|S^1_7\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle\otimes|S^1_8\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle\otimes|S^1_9\rangle,
$$

$$\left.\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right._{CS} = \frac{\sqrt{5}}{3\sqrt{3}}|C_3\rangle\otimes|S^1_1\rangle - \frac{2}{3\sqrt{3}}|C_1\rangle\otimes|S^1_2\rangle - \frac{1}{9}|C_3\rangle\otimes|S^1_2\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle\otimes|S^1_3\rangle - \frac{\sqrt{2}}{9}|C_3\rangle\otimes|S^1_3\rangle$$
$$-\frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle\otimes|S^1_4\rangle + \frac{\sqrt{2}}{9}|C_4\rangle\otimes|S^1_4\rangle - \frac{1}{9}|C_5\rangle\otimes|S^1_4\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle\otimes|S^1_5\rangle - \frac{\sqrt{2}}{9}|C_3\rangle\otimes|S^1_5\rangle$$
$$-\frac{1}{3\sqrt{3}}|C_1\rangle\otimes|S^1_6\rangle - \frac{2}{9}|C_3\rangle\otimes|S^1_6\rangle + \frac{1}{3\sqrt{3}}|C_2\rangle\otimes|S^1_7\rangle + \frac{2}{9}|C_4\rangle\otimes|S^1_7\rangle - \frac{\sqrt{2}}{9}|C_5\rangle\otimes|S^1_7\rangle$$
$$+\frac{1}{3\sqrt{3}}|C_3\rangle\otimes|S^1_8\rangle - \frac{1}{3\sqrt{3}}|C_4\rangle\otimes|S^1_9\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle\otimes|S^1_9\rangle,$$

$$\left.\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right._{CS} = \frac{\sqrt{5}}{3\sqrt{3}}|C_4\rangle\otimes|S^1_1\rangle - \frac{2}{3\sqrt{3}}|C_2\rangle\otimes|S^1_2\rangle - \frac{1}{9}|C_4\rangle\otimes|S^1_2\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle\otimes|S^1_3\rangle + \frac{\sqrt{2}}{9}|C_4\rangle\otimes|S^1_3\rangle$$
$$+\frac{1}{9}|C_5\rangle\otimes|S^1_3\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle\otimes|S^1_4\rangle + \frac{\sqrt{2}}{9}|C_3\rangle\otimes|S^1_4\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle\otimes|S^1_5\rangle - \frac{\sqrt{2}}{9}|C_4\rangle\otimes|S^1_5\rangle$$
$$+\frac{1}{3\sqrt{3}}|C_2\rangle\otimes|S^1_6\rangle + \frac{2}{9}|C_4\rangle\otimes|S^1_6\rangle + \frac{\sqrt{2}}{9}|C_5\rangle\otimes|S^1_6\rangle + \frac{1}{3\sqrt{3}}|C_1\rangle\otimes|S^1_7\rangle + \frac{2}{9}|C_3\rangle\otimes|S^1_7\rangle$$
$$-\frac{1}{3\sqrt{3}}|C_4\rangle\otimes|S^1_8\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle\otimes|S^1_8\rangle - \frac{1}{3\sqrt{3}}|C_3\rangle\otimes|S^1_9\rangle,$$

$$\left.\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right._{CS} = \frac{\sqrt{5}}{3\sqrt{3}}|C_5\rangle\otimes|S^1_1\rangle + \frac{2}{9}|C_5\rangle\otimes|S^1_2\rangle + \frac{2}{3\sqrt{3}}|C_2\rangle\otimes|S^1_3\rangle + \frac{1}{9}|C_4\rangle\otimes|S^1_3\rangle - \frac{2}{3\sqrt{3}}|C_1\rangle\otimes|S^1_4\rangle$$
$$-\frac{1}{9}|C_3\rangle\otimes|S^1_4\rangle + \frac{2\sqrt{2}}{9}|C_5\rangle\otimes|S^1_5\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle\otimes|S^1_6\rangle + \frac{\sqrt{2}}{9}|C_4\rangle\otimes|S^1_6\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle\otimes|S^1_7\rangle$$
$$-\frac{\sqrt{2}}{9}|C_3\rangle\otimes|S^1_7\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle\otimes|S^1_8\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle\otimes|S^1_9\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}_{CS} = -\frac{\sqrt{5}}{3\sqrt{3}}|C_5\rangle \otimes |S^1_2\rangle + \frac{\sqrt{5}}{3\sqrt{3}}|C_4\rangle \otimes |S^1_3\rangle - \frac{\sqrt{5}}{3\sqrt{3}}|C_3\rangle \otimes |S^1_4\rangle - \frac{2\sqrt{2}}{3\sqrt{15}}|C_5\rangle \otimes |S^1_5\rangle + \frac{2\sqrt{2}}{3\sqrt{15}}|C_4\rangle \otimes |S^1_6\rangle$$

$$-\frac{2\sqrt{2}}{3\sqrt{15}}|C_3\rangle \otimes |S^1_7\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S^1_8\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S^1_9\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}_{CS} = \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S^1_5\rangle - \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S^1_6\rangle + \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S^1_7\rangle + \frac{1}{\sqrt{5}}|C_2\rangle \otimes |S^1_8\rangle - \frac{1}{\sqrt{5}}|C_1\rangle \otimes |S^1_9\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} \end{array}_{CS} = \frac{1}{\sqrt{2}}|C_1\rangle \otimes |S^1_8\rangle + \frac{1}{\sqrt{2}}|C_2\rangle \otimes |S^1_9\rangle$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} \end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S^1_6\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S^1_6\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S^1_7\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S^1_7\rangle + \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S^1_8\rangle$$

$$+\frac{1}{\sqrt{10}}|C_4\rangle \otimes |S^1_9\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} \end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S^1_5\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S^1_5\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S^1_6\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S^1_7\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S^1_7\rangle$$

$$-\frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S^1_8\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S^1_9\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S^1_9\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} \end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_2\rangle \otimes |S^1_5\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S^1_5\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S^1_6\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S^1_6\rangle + \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S^1_7\rangle$$

$$+\frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S^1_8\rangle + \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S^1_8\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S^1_9\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5\\\cline{1-2}\end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle\otimes|S_3^1\rangle-\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S_4^1\rangle-\frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle\otimes|S_4^1\rangle-\frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_6^1\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_7^1\rangle-\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_8^1\rangle-\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5\\\cline{1-2}\end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_1\rangle\otimes|S_2^1\rangle+\frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle\otimes|S_2^1\rangle-\frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_3^1\rangle+\frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_4^1\rangle-\frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_5^1\rangle-\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_6^1\rangle+\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_7^1\rangle+\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_8^1\rangle-\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_9^1\rangle$$
$$+\frac{1}{2\sqrt{5}}|C_5\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array}_{CS} = -\frac{1}{\sqrt{10}}|C_2\rangle\otimes|S_2^1\rangle+\frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle\otimes|S_2^1\rangle+\frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_3^1\rangle+\frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_3^1\rangle+\frac{1}{2\sqrt{5}}|C_1\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_2\rangle\otimes|S_5^1\rangle+\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_6^1\rangle+\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_7^1\rangle-\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_8^1\rangle-\frac{1}{2\sqrt{5}}|C_5\rangle\otimes|S_8^1\rangle$$
$$-\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4\\\cline{1-2}\end{array}_{CS} = \frac{1}{\sqrt{2}}|C_1\rangle\otimes|S_1^1\rangle+\frac{1}{\sqrt{10}}|C_3\rangle\otimes|S_2^1\rangle-\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_3^1\rangle+\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_4^1\rangle+\frac{1}{\sqrt{10}}|C_5\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_5^1\rangle+\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_6^1\rangle-\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_7^1\rangle-\frac{1}{2\sqrt{5}}|C_5\rangle\otimes|S_7^1\rangle+\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_8^1\rangle$$
$$-\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_9^1\rangle,$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array}_{CS} = \frac{1}{\sqrt{2}}|C_2\rangle\otimes|S_1^1\rangle+\frac{1}{\sqrt{10}}|C_4\rangle\otimes|S_2^1\rangle+\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_3^1\rangle-\frac{1}{\sqrt{10}}|C_5\rangle\otimes|S_3^1\rangle+\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_4^1\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_5^1\rangle-\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_6^1\rangle+\frac{1}{2\sqrt{5}}|C_5\rangle\otimes|S_6^1\rangle-\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_7^1\rangle-\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_8^1\rangle$$
$$-\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_9^1\rangle.$$

3. $S=2$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}_{CS} = \frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_4^2\rangle+\frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle\otimes|S_4^2\rangle+\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_5^2\rangle+\frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = \frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_3^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle\otimes|S_3^2\rangle + \frac{1}{4}|C_1\rangle\otimes|S_4^2\rangle - \frac{1}{4}|C_2\rangle\otimes|S_5^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = \frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_3^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle\otimes|S_3^2\rangle - \frac{1}{4}|C_2\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle\otimes|S_4^2\rangle - \frac{1}{4}|C_1\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{\sqrt{5}}{2\sqrt{2}}|C_1\rangle\otimes|S_2^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_3^2\rangle + \frac{1}{4}|C_3\rangle\otimes|S_4^2\rangle - \frac{1}{4}|C_4\rangle\otimes|S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}_{CS} = -\frac{\sqrt{5}}{2\sqrt{2}}|C_2\rangle\otimes|S_2^2\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_3^2\rangle - \frac{1}{4}|C_4\rangle\otimes|S_4^2\rangle + \frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_4^2\rangle - \frac{1}{4}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_4^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_4^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_3^2\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_3^2\rangle + \frac{\sqrt{3}}{4}|C_1\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{4}|C_2\rangle\otimes|S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_3^2\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_3^2\rangle - \frac{\sqrt{3}}{4}|C_2\rangle\otimes|S_4^2\rangle + \frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{4}|C_1\rangle\otimes|S_5^2\rangle,$$

$$\left.\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\right._{CS} = \frac{1}{\sqrt{2}}|C_3\rangle\otimes|S_2^2\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_3^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_3^2\rangle - \frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S_4^2\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_4^2\rangle$$
$$+\frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S_5^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_5^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\left.\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\right._{CS} = \frac{1}{\sqrt{2}}|C_4\rangle\otimes|S_2^2\rangle + \frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_3^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_3^2\rangle + \frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle$$
$$-\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_4^2\rangle + \frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S_5^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\left.\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array}\right._{CS} = \frac{1}{\sqrt{2}}|C_5\rangle\otimes|S_2^2\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_3^2\rangle - \frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_4^2\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_5^2\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\left.\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\right._{CS} = -\frac{4}{5}|C_1\rangle\otimes|S_1^2\rangle - \frac{3\sqrt{3}}{10\sqrt{2}}|C_1\rangle\otimes|S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_3^2\rangle - \frac{\sqrt{3}}{4\sqrt{5}}|C_3\rangle\otimes|S_4^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_4\rangle\otimes|S_5^2\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\left.\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\right._{CS} = -\frac{4}{5}|C_2\rangle\otimes|S_1^2\rangle - \frac{3\sqrt{3}}{10\sqrt{2}}|C_2\rangle\otimes|S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_3^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_4^2\rangle$$
$$+\frac{\sqrt{3}}{4\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}_{CS} = -\frac{4}{5}|C_3\rangle\otimes|S_1^2\rangle+\frac{\sqrt{3}}{5\sqrt{2}}|C_3\rangle\otimes|S_2^2\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_3^2\rangle-\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_3^2\rangle-\frac{\sqrt{3}}{4\sqrt{5}}|C_1\rangle\otimes|S_4^2\rangle$$
$$-\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_4^2\rangle+\frac{\sqrt{3}}{4\sqrt{5}}|C_2\rangle\otimes|S_5^2\rangle+\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_5^2\rangle-\frac{1}{2\sqrt{10}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array}_{CS} = -\frac{4}{5}|C_4\rangle\otimes|S_1^2\rangle+\frac{\sqrt{3}}{5\sqrt{2}}|C_4\rangle\otimes|S_2^2\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_3^2\rangle-\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_3^2\rangle+\frac{\sqrt{3}}{4\sqrt{5}}|C_2\rangle\otimes|S_4^2\rangle$$
$$+\frac{1}{2\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle+\frac{1}{2\sqrt{10}}|C_5\rangle\otimes|S_4^2\rangle+\frac{\sqrt{3}}{4\sqrt{5}}|C_1\rangle\otimes|S_5^2\rangle+\frac{1}{2\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}_{CS} = -\frac{4}{5}|C_5\rangle\otimes|S_1^2\rangle+\frac{\sqrt{3}}{5\sqrt{2}}|C_5\rangle\otimes|S_2^2\rangle+\frac{1}{\sqrt{10}}|C_5\rangle\otimes|S_3^2\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_4^2\rangle+\frac{1}{2\sqrt{10}}|C_4\rangle\otimes|S_4^2\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_5^2\rangle-\frac{1}{2\sqrt{10}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = -\frac{3}{5}|C_1\rangle\otimes|S_1^2\rangle+\frac{\sqrt{6}}{5}|C_1\rangle\otimes|S_2^2\rangle-\frac{\sqrt{2}}{\sqrt{15}}|C_3\rangle\otimes|S_3^2\rangle+\frac{1}{\sqrt{15}}|C_3\rangle\otimes|S_4^2\rangle-\frac{1}{\sqrt{15}}|C_4\rangle\otimes|S_5^2\rangle$$
$$-\frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1} 6\\\cline{1-1}\end{array}_{CS} = -\frac{3}{5}|C_2\rangle\otimes|S_1^2\rangle+\frac{\sqrt{6}}{5}|C_2\rangle\otimes|S_2^2\rangle-\frac{\sqrt{2}}{\sqrt{15}}|C_4\rangle\otimes|S_3^2\rangle-\frac{1}{\sqrt{15}}|C_4\rangle\otimes|S_4^2\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle\otimes|S_4^2\rangle$$
$$-\frac{1}{\sqrt{15}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3&5\\ \hline 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array}_{CS} = -\frac{3}{5}|C_3\rangle\otimes|S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_3\rangle\otimes|S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_3^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_3^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle\otimes|S_4^2\rangle$$

$$+\frac{2}{3\sqrt{5}}|C_3\rangle\otimes|S_4^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle\otimes|S_5^2\rangle - \frac{2}{3\sqrt{5}}|C_4\rangle\otimes|S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array}_{CS} = -\frac{3}{5}|C_4\rangle\otimes|S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_4\rangle\otimes|S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_3^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_3^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle\otimes|S_4^2\rangle$$

$$-\frac{2}{3\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_5\rangle\otimes|S_4^2\rangle - \frac{1}{\sqrt{15}}|C_1\rangle\otimes|S_5^2\rangle - \frac{2}{3\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&4\\ \hline 2&5\\ \hline 3\\ \cline{1-1} 6\\ \cline{1-1}\end{array}_{CS} = -\frac{3}{5}|C_5\rangle\otimes|S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_5\rangle\otimes|S_2^2\rangle - \frac{2\sqrt{2}}{3\sqrt{5}}|C_5\rangle\otimes|S_3^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle$$

$$-\frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3&6\\ \hline 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{3}}|C_3\rangle\otimes|S_2^2\rangle - \frac{2}{\sqrt{15}}|C_1\rangle\otimes|S_3^2\rangle - \frac{1}{3\sqrt{5}}|C_3\rangle\otimes|S_3^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_4^2\rangle$$

$$-\frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_5^2\rangle - \frac{1}{3\sqrt{5}}|C_5\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}_{CS} = \frac{1}{\sqrt{3}}|C_4\rangle\otimes|S_2^2\rangle - \frac{2}{\sqrt{15}}|C_2\rangle\otimes|S_3^2\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle\otimes|S_3^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_4^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle$$

$$+\frac{1}{3\sqrt{5}}|C_5\rangle\otimes|S_4^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}_{CS} = \frac{1}{\sqrt{3}}|C_5\rangle\otimes|S_2^2\rangle + \frac{2}{3\sqrt{5}}|C_5\rangle\otimes|S_3^2\rangle + \frac{2}{\sqrt{15}}|C_2\rangle\otimes|S_4^2\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle\otimes|S_4^2\rangle - \frac{2}{\sqrt{15}}|C_1\rangle\otimes|S_5^2\rangle$$

$$-\frac{1}{3\sqrt{5}}|C_3\rangle\otimes|S_5^2\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}_{CS} = -\frac{1}{\sqrt{3}}|C_5\rangle\otimes|S_3^2\rangle + \frac{1}{\sqrt{3}}|C_4\rangle\otimes|S_4^2\rangle - \frac{1}{\sqrt{3}}|C_3\rangle\otimes|S_5^2\rangle.$$

4. $S = 3$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}_{CS} = |C_1\rangle\otimes|S_1^3\rangle, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}_{CS} = |C_2\rangle\otimes|S_1^3\rangle, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} = |C_3\rangle\otimes|S_1^3\rangle, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} = |C_4\rangle\otimes|S_1^3\rangle, \tag{B1}$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}_{CS} = |C_5\rangle\otimes|S_1^3\rangle.$$

APPENDIX C: CS COUPLING OF q^5

1. $S=\frac{1}{2}$

$$\left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right]_{CS} = \frac{1}{2}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{2}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\right]_{CS} = \frac{1}{2}|C_1\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle-\frac{1}{2}|C_5\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\right]_{CS} = \frac{1}{2}|C_2\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2}|C_5\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\right]_{CS} = \frac{1}{2}|C_3\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2}|C_5\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right]_{CS} = \frac{1}{2}|C_4\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2}|C_5\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}\right]_{CS} = \frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{\sqrt{5}}|C_3\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{\sqrt{5}}|C_4\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle$$
$$-\frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\left[\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}\right]_{CS} = \frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{\sqrt{5}}|C_3\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{\sqrt{5}}|C_5\rangle\otimes|S_3^{\frac{1}{2}}\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle\otimes|S_1^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{5}}|C_4\rangle\otimes|S_1^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{5}}|C_5\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle$$

$$-\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle\otimes|S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2}|C_1\rangle\otimes|S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle$$

$$-\frac{1}{2}|C_5\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2}|C_2\rangle\otimes|S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle + \frac{1}{2}|C_5\rangle\otimes|S_4^{\frac{1}{2}}\rangle$$

$$-\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}_{CS} = \frac{1}{2}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{2}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle + \frac{1}{2}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle - \frac{1}{2}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_3^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_4^{\frac{1}{2}}\rangle$$

$$+\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_2^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_3^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_4^{\frac{1}{2}}\rangle$$

$$+\frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{3}|C_3\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{\sqrt{2}}{3}|C_3\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle$$
$$+\frac{\sqrt{2}}{3}|C_4\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{3}|C_5\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{3}|C_4\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{\sqrt{2}}{3}|C_4\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{3}|C_5\rangle\otimes|S_2^{\frac{1}{2}}\rangle$$
$$-\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{\sqrt{2}}{3}|C_3\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}_{CS} = \frac{2}{3}|C_5\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{3}|C_4\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{3}|C_3\rangle\otimes|S_3^{\frac{1}{2}}\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_1\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_2^{\frac{1}{2}}\rangle+\frac{1}{3}|C_3\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_3^{\frac{1}{2}}\rangle$$
$$-\frac{1}{3}|C_4\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{3\sqrt{2}}|C_5\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_5^{\frac{1}{2}}\rangle+\frac{1}{\sqrt{6}}|C_5\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{6}}|C_2\rangle\otimes|S_1^{\frac{1}{2}}\rangle+\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|S_1^{\frac{1}{2}}\rangle-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{3}|C_4\rangle\otimes|S_2^{\frac{1}{2}}\rangle-\frac{1}{3\sqrt{2}}|C_5\rangle\otimes|S_2^{\frac{1}{2}}\rangle$$
$$-\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|S_3^{\frac{1}{2}}\rangle-\frac{1}{3}|C_3\rangle\otimes|S_3^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_4\rangle\otimes|S_4^{\frac{1}{2}}\rangle-\frac{1}{\sqrt{6}}|C_5\rangle\otimes|S_4^{\frac{1}{2}}\rangle+\frac{1}{2\sqrt{3}}|C_3\rangle\otimes|S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}_{CS} = -\frac{\sqrt{2}}{3}|C_5\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle$$

$$-\frac{1}{\sqrt{6}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{6}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}_{CS} = \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_2\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_1\rangle \otimes |S_5^{\frac{1}{2}}\rangle.$$

2. $S = \frac{3}{2}$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}_{CS} = -\frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle \otimes |S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}_{CS} = -\frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{1}{4}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{1}{4}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle \otimes |S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}_{CS} = \frac{1}{4}|C_1\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{1}{4}|C_2\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle \otimes |S_3^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}_{CS} = \frac{\sqrt{5}}{2\sqrt{2}}|C_1\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{1}{4}|C_3\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{1}{4}|C_4\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}_{CS} = \frac{\sqrt{5}}{2\sqrt{2}}|C_2\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{1}{4}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{1}{4}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_3^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{5}\\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_3\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{3}}{4}|C_1\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{4}|C_2\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{5}\\ \cline{1-1}\end{array}_{CS} = \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_4\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{4}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{1}{2\sqrt{2}}|C_5\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{4}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{2}}|C_3\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S_3^{\frac{3}{2}}\rangle$$
$$+\frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{2}}|C_4\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{1}{4\sqrt{5}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle$$
$$-\frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{1}{4\sqrt{5}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|c|}\hline 1&4&5\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{2}}|C_5\rangle\otimes|S_1^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{1}{2\sqrt{10}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle$$
$$+\frac{1}{2\sqrt{10}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline \multicolumn{1}{|c|}{5}\\ \cline{1-1}\end{array}_{CS} = -\frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_3\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{4\sqrt{3}}|C_3\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_4\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_5\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$
\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}_{CS} = -\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_4\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{2\sqrt{6}}|C_5\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,
$$

$$
\begin{aligned}
\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}_{CS} = {} & \frac{1}{\sqrt{6}}|C_3\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle\otimes|S_3^{\frac{3}{2}}\rangle \\
& -\frac{\sqrt{5}}{6}|C_3\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{6}|C_4\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle\otimes|S_4^{\frac{3}{2}}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}_{CS} = {} & \frac{1}{\sqrt{6}}|C_4\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle \\
& +\frac{\sqrt{5}}{6}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{6}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}_{CS} = {} & \frac{1}{\sqrt{6}}|C_5\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{3\sqrt{2}}|C_5\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle \\
& +\frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}_{CS} = {} & -\frac{1}{\sqrt{3}}|C_3\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{2}{\sqrt{15}}|C_1\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{1}{3\sqrt{5}}|C_3\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_3^{\frac{3}{2}}\rangle \\
& +\frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{1}{3\sqrt{5}}|C_5\rangle\otimes|S_4^{\frac{3}{2}}\rangle,
\end{aligned}
$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1} 4&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{3}}|C_4\rangle\otimes|S_1^{\frac{3}{2}}\rangle+\frac{2}{\sqrt{15}}|C_2\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{1}{3\sqrt{5}}|C_4\rangle\otimes|S_2^{\frac{3}{2}}\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle$$

$$-\frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{1}{3\sqrt{5}}|C_5\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle-\frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1} 3&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = -\frac{1}{\sqrt{3}}|C_5\rangle\otimes|S_1^{\frac{3}{2}}\rangle-\frac{2}{3\sqrt{5}}|C_5\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{2}{\sqrt{15}}|C_2\rangle\otimes|S_3^{\frac{3}{2}}\rangle-\frac{1}{3\sqrt{5}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle$$

$$+\frac{2}{\sqrt{15}}|C_1\rangle\otimes|S_4^{\frac{3}{2}}\rangle+\frac{1}{3\sqrt{5}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle,$$

$$\begin{array}{|c|c|}\hline 1&5\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1} 3&\multicolumn{1}{c}{}\\\cline{1-1} 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = \frac{1}{\sqrt{3}}|C_5\rangle\otimes|S_2^{\frac{3}{2}}\rangle-\frac{1}{\sqrt{3}}|C_4\rangle\otimes|S_3^{\frac{3}{2}}\rangle+\frac{1}{\sqrt{3}}|C_3\rangle\otimes|S_4^{\frac{3}{2}}\rangle.$$

3. $S=\frac{5}{2}$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = |C_1\rangle\otimes|S_1^{\frac{5}{2}}\rangle,\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = |C_2\rangle\otimes|S_1^{\frac{5}{2}}\rangle,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = |C_4\rangle\otimes|S_1^{\frac{5}{2}}\rangle,$$
$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = |C_4\rangle\otimes|S_1^{\frac{5}{2}}\rangle,\quad \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}_{CS} = |C_5\rangle\otimes|S_1^{\frac{5}{2}}\rangle. \tag{C1}$$

APPENDIX D: SPIN BASIS TRANSFORMATION: $q^5 \rightarrow q^5Q$

(i) $S = 0$:

$$|S_1^{\frac{1}{2}}\rangle \rightarrow |S_1^0\rangle, \qquad |S_2^{\frac{1}{2}}\rangle \rightarrow |S_2^0\rangle, \qquad |S_3^{\frac{1}{2}}\rangle \rightarrow |S_3^0\rangle, \qquad |S_4^{\frac{1}{2}}\rangle \rightarrow |S_4^0\rangle, \qquad |S_5^{\frac{1}{2}}\rangle \rightarrow |S_5^0\rangle.$$

(ii) $S = 1$:

$$|S_1^{\frac{1}{2}}\rangle \rightarrow |S_5^1\rangle, \qquad |S_2^{\frac{1}{2}}\rangle \rightarrow |S_6^1\rangle, \qquad |S_3^{\frac{1}{2}}\rangle \rightarrow |S_7^1\rangle, \qquad |S_4^{\frac{1}{2}}\rangle \rightarrow |S_8^1\rangle, \qquad |S_5^{\frac{1}{2}}\rangle \rightarrow |S_9^1\rangle,$$
$$|S_1^{\frac{3}{2}}\rangle \rightarrow |S_1^1\rangle, \qquad |S_2^{\frac{3}{2}}\rangle \rightarrow |S_2^1\rangle, \qquad |S_3^{\frac{3}{2}}\rangle \rightarrow |S_3^1\rangle, \qquad |S_4^{\frac{3}{2}}\rangle \rightarrow |S_4^1\rangle.$$

(iii) $S = 2$:

$$|S_1^{\frac{3}{2}}\rangle \rightarrow |S_2^2\rangle, \qquad |S_2^{\frac{3}{2}}\rangle \rightarrow |S_3^2\rangle, \qquad |S_3^{\frac{3}{2}}\rangle \rightarrow |S_4^2\rangle, \qquad |S_4^{\frac{3}{2}}\rangle \rightarrow |S_5^2\rangle, \qquad |S_1^{\frac{5}{2}}\rangle \rightarrow |S_1^2\rangle.$$

(iv) $S = 3$:

$$|S_1^{\frac{5}{2}}\rangle \rightarrow |S_1^3\rangle.$$

APPENDIX E: FLAVOR, COLOR, AND SPIN STATE OF q^5Q

$$\begin{aligned}
\psi_{1,S=0} = &-\frac{\sqrt{3}}{4\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{4\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{\sqrt{3}}{4\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{4\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
&+ \frac{\sqrt{3}}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{1}{8}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{4\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle \\
&+ \frac{1}{8}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{8}|C_1\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle + \frac{1}{8}|C_3\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle - \frac{\sqrt{3}}{8}|C_2\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle \\
&- \frac{1}{8}|C_4\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle - \frac{1}{4\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle + \frac{1}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle - \frac{\sqrt{3}}{8}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle \\
&- \frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + \frac{1}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle - \frac{\sqrt{3}}{8}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle + \frac{1}{8}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle \\
&- \frac{\sqrt{3}}{8}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle - \frac{1}{8}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle + \frac{\sqrt{3}}{8}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle - \frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle,
\end{aligned}$$

$$\begin{aligned}
\psi_{2,S=0} = &-\frac{\sqrt{3}}{4}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{4}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{\sqrt{3}}{4}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{4}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
&+ \frac{1}{4}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{1}{4}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{4}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{4}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle,
\end{aligned}$$

$$\begin{aligned}\psi_{3,S=0} = &-\frac{1}{4\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^0\rangle+\frac{1}{12\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^0\rangle+\frac{1}{4\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^0\rangle-\frac{1}{12\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^0\rangle\\ &-\frac{1}{8\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle+\frac{7}{24}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle-\frac{1}{12\sqrt{2}}|C_5\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle-\frac{1}{8\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^0\rangle\\ &+\frac{7}{24}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^0\rangle-\frac{1}{8\sqrt{3}}|C_1\rangle\otimes|F_1\rangle\otimes|S_3^0\rangle+\frac{7}{24}|C_3\rangle\otimes|F_1\rangle\otimes|S_3^0\rangle+\frac{1}{8\sqrt{3}}|C_2\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle\\ &-\frac{7}{24}|C_4\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle-\frac{1}{12\sqrt{2}}|C_5\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle-\frac{3}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_4^0\rangle+\frac{1}{8\sqrt{3}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^0\rangle\\ &-\frac{1}{4\sqrt{6}}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^0\rangle-\frac{3}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_4^0\rangle+\frac{1}{8\sqrt{3}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^0\rangle-\frac{3}{8}|C_1\rangle\otimes|F_1\rangle\otimes|S_5^0\rangle\\ &+\frac{1}{8\sqrt{3}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^0\rangle+\frac{3}{8}|C_2\rangle\otimes|F_2\rangle\otimes|S_5^0\rangle-\frac{1}{8\sqrt{3}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^0\rangle-\frac{1}{4\sqrt{6}}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^0\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{4,S=0} = &-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^0\rangle+\frac{1}{6}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^0\rangle+\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^0\rangle-\frac{1}{6}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^0\rangle\\ &-\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle-\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle-\frac{1}{6}|C_5\rangle\otimes|F_1\rangle\otimes|S_2^0\rangle-\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^0\rangle\\ &-\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^0\rangle-\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|F_1\rangle\otimes|S_3^0\rangle-\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_3^0\rangle+\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle\\ &+\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle-\frac{1}{6}|C_5\rangle\otimes|F_2\rangle\otimes|S_3^0\rangle+\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^0\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^0\rangle\\ &+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^0\rangle+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^0\rangle-\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^0\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^0\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{1,S=1} = &-\frac{\sqrt{3}}{4\sqrt{2}}|C_2\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{4\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{\sqrt{3}}{4\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle-\frac{1}{4\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle\\ &+\frac{\sqrt{3}}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle+\frac{1}{8}|C_4\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{4\sqrt{2}}|C_5\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle+\frac{\sqrt{3}}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle\\ &+\frac{1}{8}|C_3\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle+\frac{\sqrt{3}}{8}|C_1\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle+\frac{1}{8}|C_3\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle-\frac{\sqrt{3}}{8}|C_2\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle\\ &-\frac{1}{8}|C_4\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle-\frac{1}{4\sqrt{2}}|C_5\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle+\frac{1}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle-\frac{\sqrt{3}}{8}|C_4\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle\\ &-\frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle+\frac{1}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_8^1\rangle-\frac{\sqrt{3}}{8}|C_3\rangle\otimes|F_2\rangle\otimes|S_8^1\rangle+\frac{1}{8}|C_1\rangle\otimes|F_1\rangle\otimes|S_9^1\rangle\\ &-\frac{\sqrt{3}}{8}|C_3\rangle\otimes|F_1\rangle\otimes|S_9^1\rangle-\frac{1}{8}|C_2\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle+\frac{\sqrt{3}}{8}|C_4\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle-\frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{2,S=1} = &-\frac{\sqrt{3}}{4}|C_2\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{4}|C_4\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{\sqrt{3}}{4}|C_1\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle-\frac{1}{4}|C_3\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle\\ &+\frac{1}{4}|C_5\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle+\frac{1}{4}|C_5\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle+\frac{\sqrt{3}}{4}|C_5\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle+\frac{\sqrt{3}}{4}|C_5\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{3,S=1} = &-\frac{1}{4\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{12\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{4\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle-\frac{1}{12\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle\\ &-\frac{1}{8\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle+\frac{7}{24}|C_4\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{12\sqrt{2}}|C_5\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{8\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle\\ &+\frac{7}{24}|C_3\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle-\frac{1}{8\sqrt{3}}|C_1\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle+\frac{7}{24}|C_3\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle+\frac{1}{8\sqrt{3}}|C_2\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle\\ &-\frac{7}{24}|C_4\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle-\frac{1}{12\sqrt{2}}|C_5\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle-\frac{3}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle+\frac{1}{8\sqrt{3}}|C_4\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle\\ &-\frac{1}{4\sqrt{6}}|C_5\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle-\frac{3}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_8^1\rangle+\frac{1}{8\sqrt{3}}|C_3\rangle\otimes|F_2\rangle\otimes|S_8^1\rangle-\frac{3}{8}|C_1\rangle\otimes|F_1\rangle\otimes|S_9^1\rangle\\ &+\frac{1}{8\sqrt{3}}|C_3\rangle\otimes|F_1\rangle\otimes|S_9^1\rangle+\frac{3}{8}|C_2\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle-\frac{1}{8\sqrt{3}}|C_4\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle-\frac{1}{4\sqrt{6}}|C_5\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{4,S=1} = &-\frac{1}{2\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{6}|C_4\rangle\otimes|F_1\rangle\otimes|S_5^1\rangle+\frac{1}{2\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle-\frac{1}{6}|C_3\rangle\otimes|F_2\rangle\otimes|S_5^1\rangle\\ &-\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{6}|C_5\rangle\otimes|F_1\rangle\otimes|S_6^1\rangle-\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle\\ &-\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_6^1\rangle-\frac{1}{2\sqrt{6}}|C_1\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle-\frac{1}{3\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_7^1\rangle+\frac{1}{2\sqrt{6}}|C_2\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle\\ &+\frac{1}{3\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle-\frac{1}{6}|C_5\rangle\otimes|F_2\rangle\otimes|S_7^1\rangle+\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|F_1\rangle\otimes|S_8^1\rangle\\ &+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|F_2\rangle\otimes|S_8^1\rangle+\frac{1}{2\sqrt{6}}|C_3\rangle\otimes|F_1\rangle\otimes|S_9^1\rangle-\frac{1}{2\sqrt{6}}|C_4\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle-\frac{1}{2\sqrt{3}}|C_5\rangle\otimes|F_2\rangle\otimes|S_9^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{5,S=1} = &\frac{\sqrt{5}}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^1\rangle-\frac{\sqrt{5}}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^1\rangle+\frac{\sqrt{3}}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle-\frac{1}{4}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle\\ &-\frac{\sqrt{3}}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle+\frac{1}{4}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle+\frac{1}{8\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle\\ &-\frac{1}{2}|C_5\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle\\ &+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle+\frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{8\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{2}|C_5\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{6,S=1} = &\frac{\sqrt{3}}{4\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^1\rangle-\frac{\sqrt{3}}{4\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^1\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle-\frac{1}{4\sqrt{10}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle\\ &+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle+\frac{1}{4\sqrt{10}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle-\frac{3}{8\sqrt{5}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle\\ &-\frac{1}{\sqrt{10}}|C_5\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle-\frac{3}{8\sqrt{5}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle\\ &-\frac{3}{8\sqrt{5}}|C_3\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle-\frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle+\frac{3}{8\sqrt{5}}|C_4\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{\sqrt{10}}|C_5\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{7,S=1} = &\frac{3}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^1\rangle+\frac{1}{4\sqrt{3}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^1\rangle-\frac{3}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^1\rangle-\frac{1}{4\sqrt{3}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^1\rangle\\&+\frac{\sqrt{5}}{8\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle-\frac{\sqrt{5}}{6}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle-\frac{\sqrt{5}}{8\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle+\frac{\sqrt{5}}{6}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle\\&+\frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle-\frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle+\frac{\sqrt{5}}{6}|C_5\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle+\frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle\\&-\frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle+\frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle-\frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle-\frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle\\&+\frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle+\frac{\sqrt{5}}{6}|C_5\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle,\end{aligned}\tag{E1}$$

$$\begin{aligned}\psi_{8,S=1} = &-\frac{1}{\sqrt{6}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^1\rangle+\frac{1}{\sqrt{6}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^1\rangle+\frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle+\frac{1}{3\sqrt{10}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^1\rangle\\&-\frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle-\frac{1}{3\sqrt{10}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^1\rangle+\frac{1}{\sqrt{15}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle-\frac{1}{3\sqrt{5}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle\\&-\frac{1}{3\sqrt{10}}|C_5\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle+\frac{1}{\sqrt{15}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle-\frac{1}{3\sqrt{5}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle+\frac{1}{\sqrt{15}}|C_1\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle\\&-\frac{1}{3\sqrt{5}}|C_3\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle-\frac{1}{\sqrt{15}}|C_2\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle+\frac{1}{3\sqrt{5}}|C_4\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{3\sqrt{10}}|C_5\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{1,S=2} = &\frac{\sqrt{5}}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^2\rangle-\frac{\sqrt{5}}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^2\rangle+\frac{\sqrt{3}}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle-\frac{1}{4}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle\\&-\frac{\sqrt{3}}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle+\frac{1}{4}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle+\frac{1}{8\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle\\&-\frac{1}{2}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle\\&+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle+\frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle-\frac{1}{8\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle-\frac{1}{2}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{2,S=2} = &\frac{\sqrt{3}}{4\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^2\rangle-\frac{\sqrt{3}}{4\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^2\rangle-\frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle-\frac{1}{4\sqrt{10}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle\\&+\frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle+\frac{1}{4\sqrt{10}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle-\frac{3}{8\sqrt{5}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle\\&-\frac{1}{\sqrt{10}}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle-\frac{3}{8\sqrt{5}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle+\frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle\\&-\frac{3}{8\sqrt{5}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle-\frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle+\frac{3}{8\sqrt{5}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle-\frac{1}{\sqrt{10}}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle,\end{aligned}$$

$$\begin{aligned}\psi_{3,S=2} = &\frac{3}{8}|C_2\rangle\otimes|F_1\rangle\otimes|S_2^2\rangle + \frac{1}{4\sqrt{3}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^2\rangle - \frac{3}{8}|C_1\rangle\otimes|F_2\rangle\otimes|S_2^2\rangle - \frac{1}{4\sqrt{3}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^2\rangle \\ &+ \frac{\sqrt{5}}{8\sqrt{3}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle - \frac{\sqrt{5}}{6}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle - \frac{\sqrt{5}}{8\sqrt{3}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle + \frac{\sqrt{5}}{6}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle \\ &+ \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle + \frac{\sqrt{5}}{6}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle \\ &- \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle - \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle \\ &+ \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle + \frac{\sqrt{5}}{6}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle,\end{aligned} \tag{E2}$$

$$\begin{aligned}\psi_{4,S=2} = &-\frac{1}{\sqrt{6}}|C_4\rangle\otimes|F_1\rangle\otimes|S_2^2\rangle + \frac{1}{\sqrt{6}}|C_3\rangle\otimes|F_2\rangle\otimes|S_2^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle + \frac{1}{3\sqrt{10}}|C_4\rangle\otimes|F_1\rangle\otimes|S_3^2\rangle \\ &- \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle - \frac{1}{3\sqrt{10}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^2\rangle + \frac{1}{\sqrt{15}}|C_2\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle \\ &- \frac{1}{3\sqrt{10}}|C_5\rangle\otimes|F_1\rangle\otimes|S_4^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle - \frac{1}{3\sqrt{5}}|C_3\rangle\otimes|F_2\rangle\otimes|S_4^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle \\ &- \frac{1}{3\sqrt{5}}|C_3\rangle\otimes|F_1\rangle\otimes|S_5^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle - \frac{1}{3\sqrt{10}}|C_5\rangle\otimes|F_2\rangle\otimes|S_5^2\rangle,\end{aligned}$$

$$\psi_{5,S=2} = -\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^2\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^2\rangle,$$

$$\psi_{1,S=3} = -\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle\otimes|F_1\rangle\otimes|S_1^3\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle\otimes|F_1\rangle\otimes|S_1^3\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_1^3\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_1^3\rangle.$$

APPENDIX F: MATRIX ELEMENTS OF $\lambda_i\lambda_j$ AND $\lambda_i\lambda_j\sigma_i\cdot\sigma_j$

We represent the matrix elements of $\lambda_i\lambda_j = \langle\psi|\lambda_i\lambda_j|\psi\rangle$ and $\lambda_i\lambda_j\sigma_i\cdot\sigma_j = \langle\psi|\lambda_i\lambda_j\sigma_i\cdot\sigma_j|\psi\rangle$ only for $S = 0, 1, 2, 3$.

1. $S = 0$

$$\lambda_1\lambda_2 = \begin{pmatrix} -\frac{7}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{5}{3} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ -\frac{1}{6} & \frac{1}{3\sqrt{2}} & -\frac{13}{18} & -\frac{\sqrt{2}}{9} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{\sqrt{2}}{9} & -\frac{10}{9} \end{pmatrix}, \quad \lambda_1\lambda_4 = \begin{pmatrix} -\frac{11}{12} & -\frac{1}{2\sqrt{2}} & -\frac{1}{12} & \frac{\sqrt{2}}{3} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{6\sqrt{2}} & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{1}{6\sqrt{2}} & -\frac{41}{36} & \frac{2\sqrt{2}}{9} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2\sqrt{2}}{9} & -\frac{10}{9} \end{pmatrix}, \quad \lambda_4\lambda_5 = \begin{pmatrix} -\frac{5}{3} & 0 & 1 & -\sqrt{2} \\ 0 & -\frac{8}{3} & 0 & 0 \\ 1 & 0 & -\frac{5}{3} & -\sqrt{2} \\ -\sqrt{2} & 0 & -\sqrt{2} & -\frac{2}{3} \end{pmatrix},$$

$$\lambda_1\lambda_6 = \begin{pmatrix} -\frac{7}{6} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{3\sqrt{2}} & -\frac{29}{18} & -\frac{2\sqrt{2}}{9} \\ 0 & -\frac{2}{3} & -\frac{2\sqrt{2}}{9} & -\frac{8}{9} \end{pmatrix}, \quad \lambda_5\lambda_6 = \begin{pmatrix} -\frac{11}{12} & \frac{3}{2\sqrt{2}} & -\frac{3}{4} & 0 \\ \frac{3}{2\sqrt{2}} & -\frac{7}{6} & \frac{1}{2\sqrt{2}} & 1 \\ -\frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{2}}{3} \\ 0 & 1 & \frac{\sqrt{2}}{3} & -\frac{4}{3} \end{pmatrix}. \tag{F1}$$

$$\lambda_1\lambda_2\sigma_1\sigma_2 = \begin{pmatrix} \frac{5}{6} & -\frac{5}{3\sqrt{2}} & \frac{1}{18} & \frac{\sqrt{2}}{9} \\ -\frac{5}{3\sqrt{2}} & 1 & -\frac{5}{9\sqrt{2}} & -\frac{10}{9} \\ \frac{1}{18} & -\frac{5}{9\sqrt{2}} & \frac{37}{54} & \frac{\sqrt{2}}{27} \\ \frac{\sqrt{2}}{9} & -\frac{10}{9} & \frac{\sqrt{2}}{27} & \frac{22}{27} \end{pmatrix}, \qquad \lambda_1\lambda_4\sigma_1\sigma_4 = \begin{pmatrix} \frac{11}{4} & \frac{5}{6\sqrt{2}} & \frac{1}{36} & -\frac{\sqrt{2}}{9} \\ \frac{5}{6\sqrt{2}} & \frac{23}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} \\ \frac{1}{36} & \frac{5}{18\sqrt{2}} & \frac{17}{108} & -\frac{2\sqrt{2}}{27} \\ -\frac{\sqrt{2}}{9} & \frac{5}{9} & -\frac{2\sqrt{2}}{27} & -\frac{50}{27} \end{pmatrix},$$

$$\lambda_4\lambda_5\sigma_4\sigma_5 = \begin{pmatrix} -3 & 0 & -\frac{1}{3} & \frac{\sqrt{2}}{3} \\ 0 & -\frac{8}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -3 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{10}{3} \end{pmatrix}, \qquad \lambda_1\lambda_6\sigma_1\sigma_6 = \begin{pmatrix} \frac{5}{6} & \frac{5}{3\sqrt{2}} & -\frac{1}{6} & 0 \\ \frac{5}{3\sqrt{2}} & \frac{31}{9} & \frac{5}{9\sqrt{2}} & \frac{10}{9} \\ -\frac{1}{6} & \frac{5}{9\sqrt{2}} & -\frac{43}{54} & -\frac{22\sqrt{2}}{27} \\ 0 & \frac{10}{9} & -\frac{22\sqrt{2}}{27} & -\frac{4}{27} \end{pmatrix},$$

$$\lambda_5\lambda_6\sigma_5\sigma_6 = \begin{pmatrix} \frac{11}{4} & -\frac{5}{2\sqrt{2}} & \frac{1}{4} & 0 \\ -\frac{5}{2\sqrt{2}} & \frac{17}{6} & -\frac{5}{6\sqrt{2}} & -\frac{5}{3} \\ \frac{1}{4} & -\frac{5}{6\sqrt{2}} & -\frac{101}{36} & \frac{11\sqrt{2}}{9} \\ 0 & -\frac{5}{3} & \frac{11\sqrt{2}}{9} & \frac{20}{9} \end{pmatrix}. \tag{F2}$$

2. $S = 1$

$$\lambda_1\lambda_2 = \begin{pmatrix} -\frac{7}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{5}{3} & \frac{1}{3\sqrt{2}} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3\sqrt{2}} & -\frac{13}{18} & -\frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{\sqrt{2}}{9} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{3} & -\sqrt{\frac{2}{5}} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & -\frac{16}{15} & \frac{\sqrt{2}}{3} & -\frac{2}{15} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{2}}{3} & -\frac{11}{9} & \frac{\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{2}{5}}}{3} & -\frac{2}{15} & \frac{\sqrt{2}}{9} & -\frac{32}{45} \end{pmatrix},$$

$$\lambda_1\lambda_4 = \begin{pmatrix} -\frac{11}{12} & -\frac{1}{2\sqrt{2}} & -\frac{1}{12} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{6\sqrt{2}} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{6\sqrt{2}} & -\frac{41}{36} & \frac{2\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2\sqrt{2}}{9} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{13}{24} & \frac{3}{4\sqrt{10}} & -\frac{5\sqrt{5}}{24} & \frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & \frac{3}{4\sqrt{10}} & -\frac{19}{20} & -\frac{7}{12\sqrt{2}} & \frac{4}{15} \\ 0 & 0 & 0 & 0 & -\frac{5\sqrt{5}}{24} & -\frac{7}{12\sqrt{2}} & -\frac{67}{72} & \frac{\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{3} & \frac{4}{15} & \frac{\sqrt{2}}{9} & -\frac{56}{45} \end{pmatrix},$$

$$\lambda_4\lambda_5 = \begin{pmatrix} -\frac{5}{3} & 0 & 1 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{5}{3} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{29}{12} & \frac{3}{2\sqrt{10}} & \frac{\sqrt{5}}{4} & -\sqrt{\frac{2}{5}} \\ 0 & 0 & 0 & 0 & \frac{3}{2\sqrt{10}} & -\frac{53}{30} & \frac{3}{2\sqrt{2}} & -\frac{6}{5} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{4} & \frac{3}{2\sqrt{2}} & -\frac{17}{12} & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & -\frac{6}{5} & -\sqrt{2} & -\frac{16}{15} \end{pmatrix},$$

$$\lambda_1\lambda_6 = \begin{pmatrix} -\frac{7}{6} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3\sqrt{2}} & -\frac{29}{18} & -\frac{2\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & -\frac{2\sqrt{2}}{9} & -\frac{8}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{11}{12} & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{13}{10} & -\frac{1}{6\sqrt{2}} & -\frac{4}{15} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{4} & -\frac{1}{6\sqrt{2}} & -\frac{37}{36} & -\frac{4\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{15} & -\frac{4\sqrt{2}}{9} & -\frac{64}{45} \end{pmatrix},$$

$$\lambda_5\lambda_6 = \begin{pmatrix} -\frac{11}{12} & \frac{3}{2\sqrt{2}} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2\sqrt{2}} & -\frac{7}{6} & \frac{1}{2\sqrt{2}} & 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{2}}{3} & -\frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{31}{24} & -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{3\sqrt{5}}{8} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{5}{2}}}{4} & -\frac{43}{60} & \frac{1}{4\sqrt{2}} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3\sqrt{5}}{8} & \frac{1}{4\sqrt{2}} & -\frac{9}{8} & \frac{2\sqrt{2}}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{2\sqrt{2}}{3} & -\frac{8}{15} \end{pmatrix}. \tag{F3}$$

$$\lambda_1\lambda_2\sigma_1\sigma_2 = \begin{pmatrix} \frac{5}{6} & -\frac{5}{3\sqrt{2}} & \frac{1}{18} & \frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ -\frac{5}{3\sqrt{2}} & 1 & -\frac{5}{9\sqrt{2}} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & -\frac{5}{9\sqrt{2}} & \frac{37}{54} & \frac{\sqrt{2}}{27} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{9} & -\frac{10}{9} & \frac{\sqrt{2}}{27} & \frac{22}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{6} & \frac{7}{3\sqrt{10}} & -\frac{11\sqrt{5}}{18} & \frac{\sqrt{\frac{2}{5}}}{9} \\ 0 & 0 & 0 & 0 & \frac{7}{3\sqrt{10}} & \frac{3}{5} & -\frac{7}{9\sqrt{2}} & \frac{34}{45} \\ 0 & 0 & 0 & 0 & -\frac{11\sqrt{5}}{18} & -\frac{7}{9\sqrt{2}} & \frac{19}{54} & -\frac{\sqrt{2}}{27} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{9} & \frac{34}{45} & -\frac{\sqrt{2}}{27} & -\frac{16}{135} \end{pmatrix},$$

$$\lambda_1\lambda_4\sigma_1\sigma_4 = \begin{pmatrix} \frac{11}{4} & \frac{5}{6\sqrt{2}} & \frac{1}{36} & -\frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ \frac{5}{6\sqrt{2}} & \frac{23}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{5}{18\sqrt{2}} & \frac{17}{108} & -\frac{2\sqrt{2}}{27} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{9} & \frac{5}{9} & -\frac{2\sqrt{2}}{27} & -\frac{50}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{15}{8} & -\frac{13}{12\sqrt{10}} & \frac{23\sqrt{5}}{72} & -\frac{\sqrt{\frac{2}{5}}}{9} \\ 0 & 0 & 0 & 0 & -\frac{13}{12\sqrt{10}} & \frac{31}{36} & \frac{17}{36\sqrt{2}} & -\frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{23\sqrt{5}}{72} & \frac{17}{36\sqrt{2}} & -\frac{71}{216} & -\frac{\sqrt{2}}{27} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{2}{5}}}{9} & -\frac{4}{9} & -\frac{\sqrt{2}}{27} & -\frac{56}{27} \end{pmatrix},$$

$$\lambda_4\lambda_5\sigma_4\sigma_5 = \begin{pmatrix} -3 & 0 & -\frac{1}{3} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -3 & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{10}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{11}{4} & -\frac{1}{2\sqrt{10}} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{10}} & -\frac{89}{30} & -\frac{1}{2\sqrt{2}} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{12} & -\frac{1}{2\sqrt{2}} & -\frac{37}{12} & \frac{\sqrt{2}}{3} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{3} & \frac{2}{5} & \frac{\sqrt{2}}{3} & -\frac{16}{5} \end{pmatrix},$$

$$\lambda_1\lambda_6\sigma_1\sigma_6 = \begin{pmatrix} -\frac{5}{18} & -\frac{5}{9\sqrt{2}} & \frac{1}{18} & 0 & -\frac{16}{9} & -\frac{2\sqrt{10}}{9} & -\frac{4\sqrt{5}}{9} & 0 \\ -\frac{5}{9\sqrt{2}} & -\frac{31}{27} & -\frac{5}{27\sqrt{2}} & -\frac{10}{27} & -\frac{2\sqrt{2}}{3} & \frac{56}{27\sqrt{5}} & \frac{2\sqrt{10}}{27} & \frac{32}{27\sqrt{5}} \\ \frac{1}{18} & -\frac{5}{27\sqrt{2}} & \frac{43}{162} & \frac{22\sqrt{2}}{81} & \frac{4}{9} & -\frac{22\sqrt{\frac{2}{5}}}{27} & -\frac{88\sqrt{5}}{81} & -\frac{32\sqrt{\frac{2}{5}}}{81} \\ 0 & -\frac{10}{27} & \frac{22\sqrt{2}}{81} & \frac{4}{81} & 0 & \frac{56}{27\sqrt{5}} & -\frac{8\sqrt{10}}{81} & -\frac{464}{81\sqrt{5}} \\ -\frac{16}{9} & -\frac{2\sqrt{2}}{3} & \frac{4}{9} & 0 & \frac{95}{36} & -\frac{5\sqrt{\frac{5}{2}}}{6} & \frac{7\sqrt{5}}{36} & 0 \\ -\frac{2\sqrt{10}}{9} & \frac{56}{27\sqrt{5}} & -\frac{22\sqrt{\frac{2}{5}}}{27} & \frac{56}{27\sqrt{5}} & -\frac{5\sqrt{\frac{5}{2}}}{6} & \frac{5}{54} & \frac{47}{54\sqrt{2}} & -\frac{20}{27} \\ -\frac{4\sqrt{5}}{9} & \frac{2\sqrt{10}}{27} & -\frac{88\sqrt{5}}{81} & -\frac{8\sqrt{10}}{81} & \frac{7\sqrt{5}}{36} & \frac{47}{54\sqrt{2}} & \frac{307}{324} & \frac{28\sqrt{2}}{81} \\ 0 & \frac{32}{27\sqrt{5}} & -\frac{32\sqrt{\frac{2}{5}}}{81} & -\frac{464}{81\sqrt{5}} & 0 & -\frac{20}{27} & \frac{28\sqrt{2}}{81} & \frac{80}{81} \end{pmatrix},$$

$$\lambda_5\lambda_6\sigma_5\sigma_6 = \begin{pmatrix} -\frac{11}{12} & \frac{5}{6\sqrt{2}} & -\frac{1}{12} & 0 & -\frac{4}{3} & \frac{\sqrt{10}}{3} & \frac{2\sqrt{5}}{3} & 0 \\ \frac{5}{6\sqrt{2}} & -\frac{17}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} & \sqrt{2} & \frac{32}{9\sqrt{5}} & -\frac{\sqrt{10}}{9} & -\frac{16}{9\sqrt{5}} \\ -\frac{1}{12} & \frac{5}{18\sqrt{2}} & \frac{101}{108} & -\frac{11\sqrt{2}}{27} & -\frac{2}{3} & \frac{11\sqrt{\frac{2}{5}}}{9} & \frac{8\sqrt{5}}{27} & \frac{16\sqrt{\frac{2}{5}}}{27} \\ 0 & \frac{5}{9} & -\frac{11\sqrt{2}}{27} & -\frac{20}{27} & 0 & -\frac{28}{9\sqrt{5}} & \frac{4\sqrt{10}}{27} & -\frac{128}{27\sqrt{5}} \\ -\frac{4}{3} & \sqrt{2} & -\frac{2}{3} & 0 & \frac{65}{24} & \frac{5\sqrt{\frac{5}{2}}}{4} & -\frac{7\sqrt{5}}{24} & 0 \\ \frac{\sqrt{10}}{3} & \frac{32}{9\sqrt{5}} & \frac{11\sqrt{\frac{2}{5}}}{9} & -\frac{28}{9\sqrt{5}} & \frac{5\sqrt{\frac{5}{2}}}{4} & -\frac{5}{36} & -\frac{47}{36\sqrt{2}} & \frac{10}{9} \\ \frac{2\sqrt{5}}{3} & -\frac{\sqrt{10}}{9} & \frac{8\sqrt{5}}{27} & \frac{4\sqrt{10}}{27} & -\frac{7\sqrt{5}}{24} & -\frac{47}{36\sqrt{2}} & \frac{557}{216} & -\frac{14\sqrt{2}}{27} \\ 0 & -\frac{16}{9\sqrt{5}} & \frac{16\sqrt{\frac{2}{5}}}{27} & -\frac{128}{27\sqrt{5}} & 0 & \frac{10}{9} & -\frac{14\sqrt{2}}{27} & -\frac{40}{27} \end{pmatrix}. \quad \text{(F4)}$$

3. $S=2$

$$\lambda_1\lambda_2 = \begin{pmatrix} -\frac{5}{3} & -\sqrt{\frac{2}{5}} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ -\sqrt{\frac{2}{5}} & -\frac{16}{15} & \frac{\sqrt{2}}{3} & -\frac{2}{15} & 0 \\ \frac{\sqrt{5}}{3} & \frac{\sqrt{2}}{3} & -\frac{11}{9} & \frac{\sqrt{2}}{9} & 0 \\ -\frac{\sqrt{\frac{2}{5}}}{3} & -\frac{2}{15} & \frac{\sqrt{2}}{9} & -\frac{32}{45} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix},$$

$$\lambda_1\lambda_4 = \begin{pmatrix} -\frac{13}{24} & \frac{3}{4\sqrt{10}} & -\frac{5\sqrt{5}}{24} & \frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ \frac{3}{4\sqrt{10}} & -\frac{19}{20} & -\frac{7}{12\sqrt{2}} & \frac{4}{15} & 0 \\ -\frac{5\sqrt{5}}{24} & -\frac{7}{12\sqrt{2}} & -\frac{67}{72} & \frac{\sqrt{2}}{9} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{3} & \frac{4}{15} & \frac{\sqrt{2}}{9} & -\frac{56}{45} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\lambda_4\lambda_5 = \begin{pmatrix} -\frac{29}{12} & \frac{3}{2\sqrt{10}} & \frac{\sqrt{5}}{4} & -\sqrt{\frac{2}{5}} & 0 \\ \frac{3}{2\sqrt{10}} & -\frac{53}{30} & \frac{3}{2\sqrt{2}} & -\frac{6}{5} & 0 \\ \frac{\sqrt{5}}{4} & \frac{3}{2\sqrt{2}} & -\frac{17}{12} & -\sqrt{2} & 0 \\ -\sqrt{\frac{2}{5}} & -\frac{6}{5} & -\sqrt{2} & -\frac{16}{15} & 0 \\ 0 & 0 & 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

$$\lambda_1\lambda_6 = \begin{pmatrix} -\frac{11}{12} & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{\sqrt{5}}{4} & 0 & 0 \\ \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{13}{10} & -\frac{1}{6\sqrt{2}} & -\frac{4}{15} & 0 \\ -\frac{\sqrt{5}}{4} & -\frac{1}{6\sqrt{2}} & -\frac{37}{36} & -\frac{4\sqrt{2}}{9} & 0 \\ 0 & -\frac{4}{15} & -\frac{4\sqrt{2}}{9} & -\frac{64}{45} & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$\lambda_5\lambda_6 = \begin{pmatrix} -\frac{31}{24} & -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{3\sqrt{5}}{8} & 0 & 0 \\ -\frac{3\sqrt{\frac{5}{2}}}{4} & -\frac{43}{60} & \frac{1}{4\sqrt{2}} & \frac{2}{5} & 0 \\ \frac{3\sqrt{5}}{8} & \frac{1}{4\sqrt{2}} & -\frac{9}{8} & \frac{2\sqrt{2}}{3} & 0 \\ 0 & \frac{2}{5} & \frac{2\sqrt{2}}{3} & -\frac{8}{15} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{F5}$$

$$\lambda_1\lambda_2\sigma_1\sigma_2 = \begin{pmatrix} \frac{7}{6} & \frac{7}{3\sqrt{10}} & -\frac{11\sqrt{5}}{18} & \frac{\sqrt{\frac{2}{5}}}{9} & 0 \\ \frac{7}{3\sqrt{10}} & \frac{3}{5} & -\frac{7}{9\sqrt{2}} & \frac{34}{45} & 0 \\ -\frac{11\sqrt{5}}{18} & -\frac{7}{9\sqrt{2}} & \frac{19}{54} & -\frac{\sqrt{2}}{27} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{9} & \frac{34}{45} & -\frac{\sqrt{2}}{27} & -\frac{16}{135} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix},$$

$$\lambda_1\lambda_4\sigma_1\sigma_4 = \begin{pmatrix} \frac{15}{8} & -\frac{13}{12\sqrt{10}} & \frac{23\sqrt{5}}{72} & -\frac{\sqrt{\frac{2}{5}}}{9} & 0 \\ -\frac{13}{12\sqrt{10}} & \frac{31}{36} & \frac{17}{36\sqrt{2}} & -\frac{4}{9} & 0 \\ \frac{23\sqrt{5}}{72} & \frac{17}{36\sqrt{2}} & -\frac{71}{216} & -\frac{\sqrt{2}}{27} & 0 \\ -\frac{\sqrt{\frac{2}{5}}}{9} & -\frac{4}{9} & -\frac{\sqrt{2}}{27} & -\frac{56}{27} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\lambda_4\lambda_5\sigma_4\sigma_5 = \begin{pmatrix} -\frac{11}{4} & -\frac{1}{2\sqrt{10}} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ -\frac{1}{2\sqrt{10}} & -\frac{89}{30} & -\frac{1}{2\sqrt{2}} & \frac{2}{5} & 0 \\ -\frac{\sqrt{5}}{12} & -\frac{1}{2\sqrt{2}} & -\frac{37}{12} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{3} & \frac{2}{5} & \frac{\sqrt{2}}{3} & -\frac{16}{5} & 0 \\ 0 & 0 & 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

$$\lambda_1\lambda_6\sigma_1\sigma_6 = \begin{pmatrix} -\frac{19}{12} & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{7}{12\sqrt{5}} & 0 & \frac{4}{3\sqrt{5}} \\ \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{18} & -\frac{47}{90\sqrt{2}} & \frac{4}{9} & \frac{8\sqrt{2}}{15} \\ -\frac{7}{12\sqrt{5}} & -\frac{47}{90\sqrt{2}} & -\frac{307}{540} & -\frac{28\sqrt{2}}{135} & -\frac{92}{45} \\ 0 & \frac{4}{9} & -\frac{28\sqrt{2}}{135} & -\frac{16}{27} & -\frac{32\sqrt{2}}{45} \\ \frac{4}{3\sqrt{5}} & \frac{8\sqrt{2}}{15} & -\frac{92}{45} & -\frac{32\sqrt{2}}{45} & \frac{14}{5} \end{pmatrix},$$

$$\lambda_5\lambda_6\sigma_5\sigma_6 = \begin{pmatrix} -\frac{13}{8} & -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{7}{8\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{1}{12} & \frac{47}{60\sqrt{2}} & -\frac{2}{3} & -\frac{4\sqrt{2}}{5} \\ \frac{7}{8\sqrt{5}} & \frac{47}{60\sqrt{2}} & -\frac{557}{360} & \frac{14\sqrt{2}}{45} & -\frac{26}{15} \\ 0 & -\frac{2}{3} & \frac{14\sqrt{2}}{45} & \frac{8}{9} & \frac{16\sqrt{2}}{15} \\ -\frac{2}{\sqrt{5}} & -\frac{4\sqrt{2}}{5} & -\frac{26}{15} & \frac{16\sqrt{2}}{15} & -\frac{7}{15} \end{pmatrix}. \tag{F6}$$

4. $S = 3$

$$\lambda_1\lambda_2 = -\frac{2}{3}, \qquad \lambda_1\lambda_4 = -1, \qquad \lambda_4\lambda_5 = -\frac{8}{3},$$
$$\lambda_1\lambda_6 = -2, \qquad \lambda_5\lambda_6 = \frac{1}{3}. \tag{F7}$$

$$\lambda_1\lambda_2\sigma_1\sigma_2 = -\frac{2}{3}, \quad \lambda_1\lambda_4\sigma_1\sigma_4 = -1, \quad \lambda_4\lambda_5\sigma_4\sigma_5 = -\frac{8}{3},$$
$$\lambda_1\lambda_6\sigma_1\sigma_6 = -2, \quad \lambda_5\lambda_6\sigma_5\sigma_6 = \frac{1}{3}. \tag{F8}$$

For each spin, $\sum_{i<j}^{6} \lambda_i\lambda_j = -16 I_n$, where I_n is the $n \times n$ identity matrix.

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