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Pulsar radiation in post-Maxwellian vacuum nonlinear electrodynamics

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The effects of nonlinear vacuum electrodynamics are most clearly pronounced in a strong electromagnetic field close to Schwinger limit. Electromagnetic fields of such intensity can be obtained in laboratory conditions only on very few extreme laser facilities and during a short time interval. At the same time, the astrophysical compact objects with a strong electromagnetic field such as pulsars and magnetars are the best suited to study the effects of nonlinear vacuum electrodynamics. We present analytical calculations for pulsar proper radiation in parametrized post-Maxwellian nonlinear vacuum electrodynamics. Based on the obtained solutions, the effect of nonlinear vacuum corrections to pulsar spin down is being investigated. The analysis of torque functions show that the nonlinear vacuum electrodynamics corrections to the electromagnetic radiation for some pulsars may be comparable to the energy loss by gravitational radiation.

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I. INTRODUCTION

Investigation of the pulsar's spin down is extremely significant for modern astrophysics. It gives estimations on the pulsar's luminosity variation and on the pulsar's age. An analysis of pulsar spin down for a close binary system PSR 1913_{+16} at the first time indirectly confirmed gravitational wave radiation [1].

Pulsar spin down is generally caused by many factors such as its interaction with the surrounding magnetosphere [2], energy dissipation by telluric currents in the near-surface region of the pulsar's crust [3], time dependence of the pulsar's magnetic dipole moment caused by processes under its surface, the presence of a magnetic quadruple moment and other higher multipole components in a magnetic field [4], and gravitational wave radiation.

Nevertheless, the main reason of the pulsar spin down is its electromagnetic radiation [5,6]. This radiation is typically described in magnetic dipole approximation [7,8] of Maxwell electrodynamics. However, pulsars [9] have a considerable magnetic field, $B \sim 10^9 \div 10^{14}$ G, for which Maxwell electrodynamics should be replaced by nonlinear vacuum electrodynamics [10–13]. As it is shown in [14–18], according to nonlinear vacuum electrodynamics the electromagnetic processes do not proceed in the way predicted by Maxwell electrodynamics (see also [19–23]). Hence, it becomes important to calculate pulsar electromagnetic radiation corrections according to the laws of nonlinear vacuum electrodynamics. The necessity for such corrections in the pulsar dynamics was also mentioned in [23,24] and the first estimates in order of magnitude were obtained in [18,25,26]. The primary goal of the present paper is to obtain exact expressions for vacuum nonlinear electrodynamics corrections to dipole radiation of the pulsar in the post-Maxwellian limit. The parer organized as follows. The main nonlinear vacuum electrodynamics theories and their properties are presented in Sec. II. Section III gives the calculation of electromagnetic fields for the rotating pulsar and in Sec. IV pulsar spin-down corrections are evaluated.

II. NONLINEAR VACUUM ELECTRODYNAMIC THEORIES

Nonlinear vacuum electrodynamics corrections become substantial in a strong magnetic field of a pulsar. The nature of these corrections depends on the choice of the theoretical model. The modern field theory considers two most promising generalizations of vacuum nonlinear electrodynamics, namely, the Heisenberg-Euler [10] and Born-Infeld [11] theories, although other models have also been analyzed [27].

The Heisenberg-Euler electrodynamics takes into account one-loop vacuum polarization corrections to classical electrodynamics arising from QED. The nonlinear Born-Infeld electrodynamics was constructed from the phenomenological assumptions leading to a finite energy of a pointlike charged particle. As it became clear later, the Lagrangian of this theory can also be obtained as an asymptotic case in superstring theory [28]. The influence of nonlinear corrections in both vacuum electrodynamic models becomes substantial only in sufficiently strong fields comparable with the so-called critical or Schwinger field

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$B_c = m^2c^2/e\hbar = 4.41 \times 10^{13}$ G. When electromagnetic fields E and B are much lower than the critical field B_c, one can speak about post-Maxwellian approximation of nonlinear vacuum electrodynamics. In this approximation, the nonlinear corrections to the Lagrange function originating from Born-Infeld and Heisenberg-Euler theories can be represented in a similar form [29] by using the general parametrized expression

$$L = \frac{1}{32\pi}\{2J_2 + \xi[(\eta_1 - 2\eta_2)J_2^2 + 4\eta_2 J_4]\} - \frac{1}{c}A_k j^k, \quad (1)$$

where $\xi = 1/B_c^2 = 0.5 \times 10^{-27}\ \mathrm{G}^{-2}$, $J_2 = F_{ik}F^{ki} = 2(\mathbf{E}^2 - \mathbf{B}^2)$, and $J_4 = F_{ik}F^{kl}F_{lm}F^{mi} = 2(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\,\mathbf{B})^2$—which are invariants of the electromagnetic field tensor, and the post-Maxwellian parameters η_1 and η_2 depend on the choice of theoretical model.

The post-Maxwellian parameters η_1 and η_2 in the case of Heisenberg-Euler electrodynamics are related to the fine structure constant α [10,30]:

$$\eta_1 = \frac{\alpha}{45\pi} = 5.1 \times 10^{-5}, \qquad \eta_2 = \frac{7\alpha}{180\pi} = 9.0 \times 10^{-5}.$$

For Born-Infeld electrodynamics, these parameters are equal to each other and can be expressed through the field induction $1/a$ typical for this theory:

$$\eta_1 = \eta_2 = \frac{a^2 B_c^2}{4} = 4.9 \times 10^{-6}, \qquad a = 10^{-16}\ \mathrm{G}^{-1}.$$

The electromagnetic field equations for the post-Maxwellian vacuum electrodynamics with the Lagrangian (1) are equivalent [31] to the equations of Maxwell electrodynamics in continuous media

$$\mathrm{rot}\ \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \mathrm{div}\mathbf{D} = 4\pi\rho, \quad (2)$$

$$\mathrm{rot}\ \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\mathbf{B} = 0,$$

with specific nonlinear constitutive equations [32]:

$$\mathbf{D} = 4\pi\frac{\partial L}{\partial \mathbf{E}} = \mathbf{E} + 2\xi\{\eta_1(\mathbf{E}^2 - \mathbf{B}^2)\mathbf{E} + 2\eta_2(\mathbf{B}\,\mathbf{E})\mathbf{B}\}, \quad (3)$$

$$\mathbf{H} = -4\pi\frac{\partial L}{\partial \mathbf{B}} = \mathbf{B} + 2\xi\{\eta_1(\mathbf{E}^2 - \mathbf{B}^2)\mathbf{B} - 2\eta_2(\mathbf{B}\,\mathbf{E})\mathbf{E}\}.$$

As it follows from the constitutive relations, the influence of nonlinear terms becomes substantial in magnetic fields close to B_c. At the same time, to retain the Lagrange function (1) validity we will only focus on the case $B/B_c < 1$ which takes place for many pulsars.

To calculate the pulsar radiation intensity we should take into account not only changes in the electromagnetic field equations, but also consider the alterations in the electromagnetic field energy-momentum tensor and the energy flux density. In accordance with [33], in parametrized post-Maxwellian electrodynamics (1) the energy-momentum tensor has the form

$$T^{ik} = \frac{1}{4\pi}\Big\{(1 + \xi\eta_1 J_2)F^{ik}_{(2)} - \frac{g^{ik}}{8}[2J_2 + \xi(\eta_1 + 2\eta_2)J_2^2 - 4\eta_2\xi J_4]\Big\},$$

where $F^{ik}_{(2)} = g^{ni}F_{nm}F^{mk}$ is the second power of the electromagnetic field tensor and g^{ik} is the metric. Using the energy-momentum tensor one can readily derive the expression for the energy flux density vector $\mathbf{S}$ (Poynting vector):

$$S^\mu = cT^{0\mu} = \frac{c}{4\pi}(1 + \xi\eta_1 J_2)F^{0\mu}_{(2)} = \frac{c}{4\pi}\{1 + 2\xi\eta_1(\mathbf{E}^2 - \mathbf{B}^2)\}[\mathbf{E}\,\mathbf{B}],$$

where the index $\mu = 1, 2, 3$ denotes the spatial components of the vector and $[\mathbf{E}\,\mathbf{B}]$ refers to the cross product.

It should be noted that the Pointing vector in the post-Maxwellian approximation has the correction $\xi\eta_1(\mathbf{E}^2 - \mathbf{B}^2)[\mathbf{E}\,\mathbf{B}]$, which is nonzero in the general case. However, for the radiation intensity calculations this correction turns out to be inessential as it leads to subleading terms in Pointing vector decreasing faster than $1/r^2$ with increasing the distance r from the pulsar to the observer. This property allows us to use the expression for the Pointing vector of Maxwell electrodynamics $\mathbf{S} \simeq c[\mathbf{E}\,\mathbf{B}]/4\pi$ in our further calculations.

Now we can determine nonlinear vacuum electrodynamics corrections to the pulsar electromagnetic radiation.

III. NONLINEAR VACUUM ELECTRODYNAMICS CORRECTIONS TO THE PULSAR RADIATION

Let us consider a pulsar with the magnetic dipole moment $\mathbf{m}$ and the radius R_s placed at the coordinate origin. The magnetic dipole moment is declined at the angle θ_0 to the z axis and rotates uniformly with the angular velocity ω around this axis. So the time variation of this vector has a form $\mathbf{m}(t) = |\mathbf{m}|\{\sin\theta_0\cos\omega t, \sin\theta_0\sin\omega t, \cos\theta_0\}$.

Since for the most pulsars the linear velocity of the surface points is much lower than the speed of light c, the restriction on the angular velocity takes place $\omega \ll c/R_s$. At the same time we will suppose that the pulsar magnetic field $\mathbf{B}$ is sufficiently weak and that $B^2/B_c^2 < 1$. This assumption allows us to use the post-Maxwellian approximation for the Lagrange function (1) and the constitutive equations (2). Moreover, the corrections of nonlinear electrodynamics in the constitutive equations turn out to be much smaller than the terms related to the Maxwell electrodynamics, which makes it possible to calculate the pulsar electromagnetic field by the successive approximation method. Let us represent the pulsar electromagnetic field as a sum of the Maxwell electrodynamics field $\mathbf{E}_M$, $\mathbf{B}_M$ and small corrections of nonlinear vacuum electrodynamics $\mathbf{E}_N$, $\mathbf{B}_N$.

$$\mathbf{E} = \mathbf{E}_M + \mathbf{E}_N, \qquad \mathbf{B} = \mathbf{B}_M + \mathbf{B}_N.$$

In all the following relations we will assume that $E_N, B_N \ll E_M, B_M$ and the calculations will be performed up to quantities linear in $\mathbf{E}_N$, $\mathbf{B}_N$.

In a number of papers it was shown that electromagnetic wave beams, propagating through the pulsar magnetic field, will bend [34,35] and the propagation velocity for these beams will depend on the wave polarization [36,37]. However, in the solution of the considering problem these effects give a second-order infinitesimal correction, and we neglect them.

Introducing electromagnetic field potentials in accordance with the usual expressions $\mathbf{B}_N = \mathrm{rot}\mathbf{A}_N$, $\mathbf{E}_N = -\mathrm{grad}\varphi_N - \partial\mathbf{A}_N/c\partial t$, we reduce system (2), considering (3) and the Lorentz gauge, to equations for electromagnetic field potentials:

$$\begin{aligned}\Box\varphi_N &= 2\xi\mathrm{div}\{\eta_1(\mathbf{E}_M^2-\mathbf{B}_M^2)\mathbf{E}_M + 2\eta_2(\mathbf{B}_M\mathbf{E}_M)\mathbf{B}_M\},\\ \Box\mathbf{A}_N &= 2\xi\mathrm{rot}\{\eta_1(\mathbf{E}_M^2-\mathbf{B}_M^2)\mathbf{B}_M - 2\eta_2(\mathbf{B}_M\mathbf{E}_M)\mathbf{E}_M\}\\ &\quad -\frac{2\xi}{c}\frac{\partial}{\partial t}\{\eta_1(\mathbf{E}_M^2-\mathbf{B}_M^2)\mathbf{E}_M + 2\eta_2(\mathbf{B}_M\mathbf{E}_M)\mathbf{B}_M\}. \end{aligned} \tag{4}$$

The right-hand sides of Eq. (4) depend on components of the pulsar's electromagnetic field in Maxwell electrodynamics. For our calculations it suffices to take a classical simple model of the magnetic dipole field [7,38] for $\mathbf{E}_M$ and $\mathbf{B}_M$, although a more general description that takes into account higher multipole components [4,39] is possible. In Maxwell's electrodynamics the field of a uniformly rotating magnetic dipole may be represented in the form [40]

$$\begin{aligned}\mathbf{B}_M(\mathbf{r},\tau) &= \frac{3(\mathbf{m}(\tau)\mathbf{r})\mathbf{r} - r^2\mathbf{m}(\tau)}{r^5} - \frac{\dot{\mathbf{m}}(\tau)}{cr^2} + \frac{3(\dot{\mathbf{m}}(\tau)\mathbf{r})\mathbf{r}}{cr^4}\\ &\quad + \frac{(\ddot{\mathbf{m}}(\tau)\mathbf{r})\mathbf{r} - r^2\ddot{\mathbf{m}}(\tau)}{c^2r^3}, \qquad (5)\\ \mathbf{E}_M(\mathbf{r},\tau) &= \frac{[\mathbf{r},\dot{\mathbf{m}}(\tau)]}{cr^3} + \frac{[\mathbf{r},\ddot{\mathbf{m}}(\tau)]}{c^2r^2},\end{aligned}$$

where the point above the vector means a derivative with respect to time delay $\tau = t - r/c$.

Substituting expressions (5) for $\mathbf{B}_M$ and $\mathbf{E}_M$ to the right-hand sides of Eq. (4), we obtain equations for the electromagnetic field potentials φ_N, $\mathbf{A}_N$ caused by the nonlinearity of vacuum electrodynamics. These equations are linear inhomogeneous hyperbolic types of differential equations, which can be integrated by integration methods described in detail in the literature [41,42]. The procedure of solutions deriving appears to be rather cumbersome because of the complexity of the right-hand side, and therefore we only present the final results in the form of solutions for the fields $\mathbf{E}_N$ and $\mathbf{B}_N$. It should be noted that radiation calculations need to preserve in general solutions of Eq. (4) only summands decreasing not faster than $\sim 1/r$ in the wave zone. It is convenient to represent the expressions for the vectors $\mathbf{E}_N$ and $\mathbf{B}_N$ as a sum in which the index of each term indicates the frequency with which it varies:

$$\mathbf{E}_N = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3, \qquad \mathbf{B}_N = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3.$$

In such notation, the component $\mathbf{E}_1$ corresponds to the radiation field at the pulsar rotation frequency ω, the component $\mathbf{E}_2$ relates to the radiation at the double frequency 2ω, and $\mathbf{E}_3$ describes radiation at the frequency 3ω.

Explicit expressions for the pulsar radiation electric field calculated at the observation point with the radius-vector $\mathbf{r} = \{x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta\}$ located at the distance $r \gg c/\omega$ from the pulsar have a form

$$\begin{aligned}(\mathbf{E}_1)_x &= -\frac{U\cos\theta\sin\theta_0}{30r}\{[2(7\eta_1-3\eta_2-(3\eta_1-\eta_2)\sin^2\theta_0)\\ &\quad +(2\sin^2\theta_0-1)(3\eta_1-\eta_2)\sin^2\theta]\sin(\Psi+\varphi)\\ &\quad -(2\sin^2\theta_0-1)(3\eta_1-\eta_2)\sin^2\theta\sin(\Psi-\varphi)\},\\ (\mathbf{E}_1)_y &= \frac{U\cos\theta\sin\theta_0}{30r}\{[2(7\eta_1-3\eta_2-(3\eta_1-\eta_2)\sin^2\theta_0)\\ &\quad +(2\sin^2\theta_0-1)(3\eta_1-\eta_2)\sin^2\theta]\cos(\Psi+\varphi)\\ &\quad -(2\sin^2\theta_0-1)(3\eta_1-\eta_2)\sin^2\theta\cos(\Psi-\varphi)\},\\ (\mathbf{E}_1)_z &= \frac{U\sin\theta\sin\theta_0}{15r}\sin\Psi\\ &\quad \times\{7\eta_1-3\eta_2-(3\eta_1-\eta_2)\sin^2\theta_0\\ &\quad +(2\sin^2\theta_0-1)(3\eta_1-\eta_2)\sin^2\theta\},\\ (\mathbf{E}_2)_x &= \frac{2U\cos\theta_0\sin^2\theta_0\sin\theta}{105r}\\ &\quad \times\{[51\eta_1\sin^2\theta-7\eta_2\sin^2\theta-66\eta_1+14\eta_2]\sin(2\Psi+\varphi)\\ &\quad +(15\eta_1-7\eta_2)\sin^2\theta\sin(2\Psi-\varphi)\},\\ (\mathbf{E}_2)_y &= -\frac{2U\cos\theta_0\sin^2\theta_0\sin\theta}{105r}\\ &\quad \times\{[51\eta_1\sin^2\theta-7\eta_2\sin^2\theta-66\eta_1+14\eta_2]\cos(2\Psi+\varphi)\\ &\quad -(15\eta_1-7\eta_2)\sin^2\theta\cos(2\Psi-\varphi)\},\\ (\mathbf{E}_2)_z &= \frac{4U(33\eta_1-7\eta_2)}{105r}\cos\theta\sin^2\theta\cos\theta_0\sin^2\theta_0\sin(2\Psi),\\ (\mathbf{E}_3)_x &= -\frac{81U\eta_1}{140r}\cos\theta\sin^2\theta\sin^3\theta_0\sin(3\Psi+\varphi),\\ (\mathbf{E}_3)_y &= \frac{81U\eta_1}{140r}\cos\theta\sin^2\theta\sin^3\theta_0\cos(3\Psi+2\varphi),\\ (\mathbf{E}_3)_z &= \frac{81U\eta_1}{140r}\sin^3\theta\sin^3\theta_0\sin(3\Psi),\end{aligned} \tag{6}$$

where for brevity we use the notations $k = \omega/c$, $\Psi = \omega t - kr - \varphi$, and $U = \xi m^3 k^4/R_s^4$. Because of the cumbersomeness of the expressions, we will not present here the explicit form of the magnetic field vector $\mathbf{B}_\alpha$ (where $\alpha = 1, 2, 3$); however, it can be easily found in the wave zone from relations $\mathbf{B}_\alpha = [\mathbf{n}\mathbf{E}_\alpha]$, where $\mathbf{n} = \mathbf{r}/r$ is the unit vector in the direction to the observer.

As it follows from (6), vacuum electrodynamics nonlinearity leads to radiation at frequencies multiple to the pulsar rotation frequency. In the post-Maxwellian approximation this radiation occurs at the doubled and tripled rotation frequency. Allowance for higher-order infinitesimal corrections of nonlinear vacuum electrodynamics such as post-post-Maxwellian corrections [43], also leads to a high order harmonics appearance in the pulsar radiation spectrum, but their amplitude is much lower than the harmonics amplitude in the post-Maxwellian approximation. Now we can use obtained solutions (6) to calculate the angular distribution of pulsar radiation and the total radiation intensity.

The angular distribution of pulsar radiation (the intensity of radiation at the solid angle $d\Omega$ in the direction of observation given by the unit vector $\mathbf{n}$) can be found by using the Pointing vector $\mathbf{S}$: $dI(t)/d\Omega = r^2(\mathbf{S}\,\mathbf{n})$. As it was shown earlier, in order to calculate the Pointing vector in the wave zone one can take the Maxwell electrodynamics expression $\mathbf{S} = c[\mathbf{E}\,\mathbf{B}]/4\pi$. Using the derived expressions (6) for the pulsar electromagnetic field, we can write the angular radiation distribution within the first order of smallness:

$$\frac{dI(t)}{d\Omega} = \left(\frac{dI}{d\Omega}\right)_M + \frac{dI_0}{d\Omega} + \sum_{l=1}^{4} \frac{dI_l}{d\Omega}\cos(l\Psi), \tag{7}$$

where the notation $\Psi = \omega t - kr - \varphi$ is applied.

The summand $(dI/d\Omega)_M$ corresponds to radiation in Maxwell electrodynamics and the other terms refer to the corrections of vacuum nonlinear electrodynamics. Their indices denote the frequency at which the corresponding radiation intensity component varies. And the expressions themselves for each of the angular distribution of the pulsar radiation component have a form:

$$\begin{aligned}
\left(\frac{dI}{d\Omega}\right)_M &= \frac{Q\sin^2\theta_0}{4\pi}\left[1 - \frac{1}{2}\sin^2\theta(1+\cos(2\Psi))\right],\\
\frac{dI_0}{d\Omega} &= \frac{AQ\sin^2\theta_0}{\pi}\left[\frac{\eta_2 - 3\eta_1}{20}\sin^2\theta\sin^2\theta_0 + \frac{5\eta_1 - 2\eta_2}{30}\sin^2\theta + \frac{3\eta_1 - \eta_2}{30}\sin^2\theta_0 + \frac{3\eta_2 - 7\eta_1}{30}\right],\\
\frac{dI_1}{d\Omega} &= \frac{2AQ\cos\theta\cos\theta_0\sin\theta\sin^3\theta_0}{35\pi}\left[3\eta_1\sin^2\theta - 11\eta_1 + \frac{7\eta_2}{3}\right],\\
\frac{dI_2}{d\Omega} &= \frac{AQ\sin^2\theta\sin^2\theta_0}{4\pi}\left[\frac{81\eta_1}{140}\sin^2\theta\sin^2\theta_0 - \frac{67\eta_1}{70}\sin^2\theta_0 - \frac{\eta_2}{15}\sin^2\theta_0 + \frac{2(2\eta_1 - \eta_2)}{15}\right],\\
\frac{dI_3}{d\Omega} &= \frac{6\eta_1 AQ}{35\pi}\cos\theta\cos\theta_0\sin^3\theta\sin^3\theta_0,\\
\frac{dI_4}{d\Omega} &= \frac{81\eta_1 AQ}{560\pi}\sin^4\theta\sin^4\theta_0,
\end{aligned} \tag{8}$$

where the notations $A = (kR_sB_p/B_c)^2$ and $Q = \omega^4 B_p^2 R_s^6/c^3$ are introduced. Here, for convenience, the modulus of the pulsar magnetic dipole moment m was expressed in terms of the characteristic magnetic field on its surface B_p: $m = B_p R_s^3$.

The expressions (7) and (8) allow us to calculate the total energy radiated by the pulsar in all directions:

$$I_{\rm el} = \int \frac{dI}{d\Omega} d\Omega = \frac{2Q}{3}\sin^2\theta_0\left[1 - \frac{11\eta_1 - 5\eta_2}{15}\left(\frac{B_p}{B_c}kR_s\right)^2\right]. \tag{9}$$

Thus, the nonlinear vacuum electrodynamics action induces higher harmonics in the pulsar radiation spectrum, with frequencies multiple of the rotation frequency. Although the intensity of these harmonics (9) is small in comparison with the pulsar radiation in Maxwell electrodynamics, nevertheless their corrections should be taken into account in an analysis of the pulsar spin down.

IV. PULSAR SPIN-DOWN CORRECTIONS

For an estimation of the vacuum nonlinear electrodynamics influence on the pulsar spin down let us suppose a pulsar rotating in vacuum. Similarly, as in [38,44] we will consider that the pulsar's angular velocity change occurs only due to the electromagnetic and gravitational wave radiation. So the pulsar kinetic energy $\mathcal{E}_k$ variation rate will be caused by the electromagnetic $I_{\rm em}$ and gravitational $I_{\rm gr}$ radiation intensities:

$$\frac{d\mathcal{E}_k}{dt} = -I_{\rm el} - I_{\rm gr}. \tag{10}$$

Assuming a rigid-body rotation of the pulsar and the relativistic corrections associated with the rotation to be small, one can express the kinetic energy through the inertia moment J and the angular velocity ω so that $\mathcal{E}_k = J\omega^2/2$. We obtained the pulsar electromagnetic radiation intensity $I_{\rm el}$ in (9) and the gravitational radiation intensity may be estimated by the well-known quadrupole formula [33]:

$$I_{\rm gr} = \frac{32}{5}\frac{G}{c^5}J^2\epsilon^2\omega^6,$$

where G is the gravitational constant and ϵ is equatorial ellipticity. The law of pulsar rotation energy variation (10) leads to the spin-down equation:

$$\frac{d\omega}{dt} = (K_m + K_{3\rm ned})\omega^3 + (K_{5\rm ned} + K_{\rm gw})\omega^5, \quad (11)$$

where K_m is the Maxwell electrodynamics torque function, $K_{3\rm ned}$ and $K_{5\rm ned}$ are the torque functions originating from vacuum nonlinear electrodynamics, and $K_{\rm gw}$ is the gravitational radiation torque. Each of the these functions has the following form:

$$K_m = -\frac{2}{3c^3}\frac{B_p^2}{J}R_s^6\sin^2\theta_0, \quad K_{3\rm ned} = \frac{8\eta_1}{15}\left(\frac{B_p}{B_c}\right)^2 K_m,$$
$$K_{5\rm ned} = \left[\frac{117\eta_1 + 25\eta_2}{75} + \frac{2(3\eta_1 - 5\eta_2)}{15}\sin^2\theta_0\right]\left(\frac{R_s B_p}{cB_c}\right)^2 K_m,$$
$$K_{\rm gw} = -\frac{32}{5}\frac{GJ\varepsilon^2}{c^5}. \quad (12)$$

As it follows from Eq. (11), the influence of vacuum nonlinear electrodynamics in the post-Maxwellian approximation is manifested as additional corrections to the Maxwell electrodynamics torque function and to the pulsar gravitational radiation torque. These corrections $K_{3\rm ned}$ and $K_{5\rm ned}$ depend on the value of post-Maxwellian parameters η_1 and η_2.

For Born-Infeld and Heisenberg-Euler models $K_{3\rm ned} > 0$ and $K_{5\rm ned} > 0$, and therefore these corrections induce the pulsar spin-down increase. At the same time, expression (12) does not exclude the existence of a hypothetical theoretical model of a vacuum nonlinear electrodynamics for which $K_{5\rm ned} < 0$. In such a theory nonlinear corrections can decrease the rate of the energy loss by the pulsar.

To provide explicit estimates for the values of the torque functions we adopt typical pulsar parameters [9]: the radius $R_s = 30$ km, the inertia moment $J = 10^{45}\ \mathrm{g\cdot cm^2}$, and we also suppose that $\theta_0 = \pi/2$. When choosing the pulsar's rotation period and magnetic field induction, one should take into consideration the earlier adopted restrictions $\omega R_s/c \ll 1$ and $B_p \ll B_c$. These constraints are satisfied for a wide range of pulsars, for example, J007 + 7303 whose rotation period is $P_0 \sim 0.3$ s, and $B_p \sim 10^{13}$ G or for J1012-5830 with the rotation period $P_0 \sim 2.1$ s and the magnetic field induction on the surface $B_p \sim 0.9 \times 10^{12}$ G.

The values of post-Maxwellian parameters in the Heisenberg-Euler and Born-Infeld theories have the same order of smallness, and therefore for the initial estimation we will use the parameters of only one of them, for instance, the Heisenberg-Euler theory. For the above-mentioned pulsar characteristics the ratio between the nonlinear electrodynamics and Maxwell electrodynamics torque is sufficiently small: $K_{3\rm ned}/K_m \sim 10^{-6}$. As it was expected, the main influence on the pulsar spin down is exerted by the Maxwell electrodynamics. At the same time the ratio between the gravitational radiation and vacuum nonlinear electrodynamics torque functions strongly depends on the pulsar equatorial ellipticity ε. For the maximum ellipticity estimation $\varepsilon \sim 10^{-4}$ [38] and for the angular velocity $\omega \sim 100\ \mathrm{s}^{-1}$, we will obtain $K_{5\rm ned}/K_{\rm gw} \sim 3.7 \times 10^{-4}$, and $\omega^2 K_{\rm gw}/K_{3\rm ned} \sim 0.7$ In the case of pulsars with a smaller ellipticity $\varepsilon \sim 10^{-6}$ pulsar spin down caused by gravitational radiation decreases: $K_{5\rm ned}/K_{\rm gw} \sim 3.5$ and $\omega^2 K_{\rm gw}/K_{3\rm ned} \sim 7 \times 10^{-5}$ and becomes much smaller than the vacuum nonlinear electrodynamics effect.

V. CONCLUSION

In this paper, we have studied the proper electromagnetic radiation of a pulsar in the post-Maxwellian approximation of vacuum nonlinear electrodynamics.

Obtained analytical expressions for the electromagnetic field of a slowly rotating pulsar ($kR_s \ll 1$) show the presence of harmonics with frequencies multiple to the rotation frequency in the radiation spectrum. The field of each harmonic (6) depends on the parameters of the post-Maxwellian electrodynamic model η_1, η_2 and is largely determined by the magnetic field induction B_p on the pulsar surface and its rotation frequency ω.

The expressions of the intensity angular distribution (7) and total radiation intensity (9) were obtained for each of the harmonics. In spite of the fact that the influence of nonlinear vacuum electrodynamics has the character of small corrections to the intensity of the main pulsar radiation in Maxwell electrodynamics, the magnitude of these corrections may turn out to be comparable with the other mechanisms of pulsar energy loss, for instance, with the gravitational radiation. To analyze this possibility we studied the pulsar spin down caused by both electromagnetic and gravitational radiation. The influence of nonlinear vacuum electrodynamics in the post-Maxwellian approximation shows up simultaneously in the form of corrections $K_{3\rm ned}$ and $K_{5\rm ned}$ to torque functions of each of the pulsar spin-down mechanisms. The correction $K_{3\rm ned}$ to the Maxwell electrodynamic torque function K_m exceeds in magnitude the correction $K_{5\rm ned}$. In the case of pulsars with the characteristic rotation frequency $\omega < 100\ \mathrm{s}^{-1}$ we infer $\omega^2 K_{5\rm ned} \ll K_{3\rm ned}$. The relation between $K_{\rm gw}$ and $K_{5\rm ned}$

depends substantially on the pulsar magnetic field induction and its ellipticity. For a characteristic field $B_p \sim 10^{13}$ G on the pulsar surface and the ellipticity $\varepsilon \sim 10^{-4}$ the gravitational radiation makes a more considerable contribution to the pulsar spin down than the vacuum nonlinear electrodynamics: $K_{5\mathrm{ned}}/K_{\mathrm{gw}} \sim 10^{-4}$. At the same time, for the pulsars with lower equatorial ellipticity $\varepsilon \sim 10^{-6}$ the nonlinear vacuum electrodynamics torque $K_{5\mathrm{ned}}$ exceeds the gravitational torque $K_{5\mathrm{ned}}/K_{\mathrm{gw}} \sim 3.5$ and exerts great influence on the pulsar spin down. Therefore, for weakly elliptic pulsars the spin down due to the vacuum nonlinear electrodynamics corrections can play a more important role than the spin down due to gravitational radiation.

[1] R. A. Hulse and J. H. Taylor, Astrophys. J. **195**, L51 (1975).
[2] J. Pétri, Mon. Not. R. Astron. Soc. **433**, 986 (2013).
[3] T. M. Tauris and S. Konar, Astron. Astrophys. **376**, 543 (2001).
[4] J. Pétri, Mon. Not. R. Astron. Soc. **450**, 714 (2015).
[5] J. Pétri, Mon. Not. R. Astron. Soc. **439**, 1071 (2014).
[6] J. Pétri, Mon. Not. R. Astron. Soc. **447**, 3170 (2015).
[7] J. P. Ostriker and J. E. Gunn, Astrophys. J. **157**, 1395 (1969).
[8] P. A. Vshivtseva, V. I. Denisov, and I. P. Denisova, Dokl. Phys. **47**, 798 (2002).
[9] R. N. Manchester and J. H. Taylor, *Pulsars* (W. H. Freeman, San Francisco, 1977).
[10] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[11] M. Born and L. Infeld, Proc. R. Soc. A **144**, 425 (1934).
[12] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. **166**, 1 (2000).
[13] R. Battesti and C. Rizzo, Rep. Prog. Phys. **76**, 016401 (2013).
[14] W.-y. Tsai, Phys. Rev. D **10**, 2699 (1974).
[15] W.-y. Tsai and T. Erber, Phys. Rev. D **12**, 1132 (1975).
[16] A. E. Shabad, Ann. Phys. (N.Y.) **90**, 166 (1975).
[17] V. I. Denisov and I. P. Denisova, Dokl. Phys. **46**, 377 (2001).
[18] V. A. Sokolov, Moscow Univ. Phys. Bull. (Engl. Transl.) **64**, 349 (2009).
[19] D. B. Melrose and S. J. Stoneham, Il Nuovo Cimento **32A**, 435 (1976).
[20] A. E. Shabad and V. V. Usov, Astrophys. Space Sci. **102**, 327 (1984).
[21] F. Karbstein, Phys. Rev. D **88**, 085033 (2013).
[22] V. I. Denisov, V. A. Sokolov, and M. I. Vasili'ev, Phys. Rev. D **90**, 023011 (2014).
[23] S. L. Adler, Ann. Phys. (N.Y.) **67**, 599 (1971).
[24] M. G. Baring and A. K. Harding, Astrophys. J. Lett. **507**, L55 (1998).
[25] J. S. Heyl and L. Hernquist, J. Phys. A **30**, 6475 (1997).
[26] J. Pétri, Mon. Not. R. Astron. Soc. **451**, 3581 (2015).
[27] G. Boillat, J. Math. Phys. (N.Y.) **11**, 941 (1970).
[28] S. Cecotti and S. Ferrara, Phys. Lett. B **187**, 335 (1987).
[29] V. I. Denisov, Theor. Math. Phys. **132**, 1071 (2002).
[30] V. A. Sokolov and M. I. Vasil'ev, Moscow Univ. Phys. Bull. (Engl. Transl.) **67**, 418 (2012).
[31] V. B. Berestetskii, L. P. Pitaevskii, and E. M. Lifshitz, *Quantum Electrodynamics* (Elsevier, Oxford, UK, 1982).
[32] V. I. Denisov and I. P. Denisova, Opt. Spectrosc. **90**, 282 (2001).
[33] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, UK, 1971).
[34] V. I. Denisov, I. V. Krivchenkov, and I. P. Denisova, J. Exp. Theor. Phys. **95**, 194 (2002).
[35] J. Y. Kim and T. Lee, J. Cosmol. Astropart. Phys. 11 (2011) 017.
[36] J. Y. Kim and T. Leem, J. Cosmol. Astropart. Phys. 10 (2011) 056.
[37] V. I. Denisov and S. I. Svertilov, Phys. Rev. D **71**, 063002 (2005).
[38] C. Palomba, Astron. Astrophys. **354**, 163 (2000).
[39] J. H. Krolik, Astrophys. J. **373**, L69 (1991).
[40] V. I. Denisov, I. P. Denisova, and V. A. Sokolov, Theor. Math. Phys. **172**, 1321 (2012).
[41] K. V. Zhukovsky, Moscow Univ. Phys. Bull. (Engl. Transl.) **70**, 93 (2015).
[42] V. I. Denisov and I. P. Denisova, Opt. Spectrosc. **90**, 928 (2001).
[43] I. P. Denisova, I. V. Krivchenkov, P. A. Vshivtseva, and A. A. Zubrilo, Gen. Relativ. Gravit. **36**, 889 (2004).
[44] V. V. Usov, Nature (London) **357**, 472 (1992).