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Renormalization out of equilibrium in a superrenormalizable theory

Mathias Garny[1] and Urko Reinosa[2]
[1]*CERN Theory Division, CH-1211 Genève 23, Switzerland*
[2]*Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France*
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We discuss the renormalization of the initial value problem in nonequilibrium quantum field theory within a simple, yet instructive, example and show how to obtain a renormalized time evolution for the two-point functions of a scalar field and its conjugate momentum at all times. The scheme we propose is applicable to systems that are initially far from equilibrium and compatible with nonsecular approximation schemes which capture thermalization. It is based on Kadanoff-Baym equations for non-Gaussian initial states, complemented by usual vacuum counterterms. We explicitly demonstrate how various cutoff-dependent effects peculiar to nonequilibrium systems, including time-dependent divergences or initial-time singularities, are avoided by taking an initial non-Gaussian three-point vacuum correlation into account.

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I. INTRODUCTION

Quantum field theory out of equilibrium has received a lot of attention in recent years, especially within the framework of the Kadanoff-Baym equations [1]. Coupled to a resummation such as the one provided by the two-particle irreducible (2PI) effective action [2], these equations elude the secularity problem and yield a controlled time evolution even at late times [3,4]. Despite this major progress and the related growing number of applications [5] including condensed matter, early universe cosmology or heavy-ion collisions, a consistent formulation of the initial value problem in the case of (super)renormalizable theories, that deals with the elimination of ultraviolet divergences at all times, is still to be constructed.

Many interesting approaches have been devoted to understanding and tackling the problem, from studies in the Hartree approximation or in perturbation theory [6–10] which however do not capture thermalization or are not free of secular terms, to approaches based on appropriately chosen external sources [11,12] or on the use of information about the time evolution prior to the initialization time [13] which depart however in spirit from the strict initial value problem, and restrict the control over the initial state.

In this work, we present a consistent formulation dealing with both secular terms and UV divergences, in which the only ingredient is a proper description of the initial state. Our main result is that nonequilibrium initial states encompassing (a particular subset of) non-Gaussian vacuum correlations, together with the usual vacuum counterterms, ensure a manifestly finite evolution. Furthermore, we find that initial correlations play a role in the elimination of divergences across all time scales.

We should mention that, in various physically relevant limits, renormalization is not needed or already under control. This is the case for the kinetic regime which becomes relevant when the system is dilute enough or the classical/statistical regime which applies in the case of high mode occupancies. In this work however, we would like not to rely on those assumptions. Treating the quantum case properly can be useful for various reasons. First of all, this may lead to some constraints on the initial state, which are not obvious in the classical case. Also, there can be physical situations where the strongly-occupied or dilute limits are not applicable, or the system changes the behavior in the course of its time-evolution (e.g. preheating after inflation [14]; cf. [15] for a review and potential observational consequences). Finally, if one wants to understand response on short time scales (e.g. related to the Zeno-effect [16]) or loss of memory of the initial state (e.g. for a finite period of inflationary expansion [17]; cf. [18] for a discussion of related signatures in present and future cosmological data), a proper implementation of the initial state is important, to avoid artifacts. We also note that renormalization in the classical limit is well understood for the time-evolution of thermal correlation functions [19] but not for genuine out-of-equilibrium correlators where there exist claims that the classical limit is not renormalizable [20], at least not in the standard sense. The approach developed here avoids this difficulty and is applicable away from the limiting cases discussed above.

It is important to stress that, when it comes to discussing the renormalization out-of-equilibrium, few analytical results are known, especially in any framework that deals simultaneously with the secularity problem (and therefore goes beyond perturbation theory). For this reason, in this work we focus on a specific model/approximation that allows us to exhibit the features which render

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renormalization out of equilibrium difficult, while admitting an analytical treatment of divergences. The specificity of our model/approximation should then be balanced with the benefit of having a complete analytical understanding of renormalization out-of-equilibrium in at least one (nontrivial) case. The knowledge gained from this simple but instructive example will be later put to work to discuss other models/approximations.

The paper is organized as follows. In Sec. II, we summarize the relevant equations for the out-of-equilibrium problem and specify our model and approximation. In Sec. III, we discuss the renormalization of the model in equilibrium. In Secs. IV, we consider the out-of-equilibrium case, extract analytically the most divergent contributions to the two-point correlators and discuss their renormalization in terms of initial three-point correlations. We also compare our results to the case of a Gaussian initial state where these correlations are absent. In Sec. V, we discuss yet another peculiar feature of the Gaussian initial state and how it is cured by the three-point initial correlations.

II. NONEQUILIBRIUM EVOLUTION EQUATIONS

In what follows, we consider a theory involving two real scalar fields φ and χ in $d+1$ dimensions with trilinear coupling,

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{\lambda}{2}\varphi\chi^2. \quad (1)$$

Out of equilibrium, the information on the system is contained in its time-dependent correlation functions. Tracking the one- and two-point functions provides already a wealth of information, from which various quantities of interest can be constructed, including effective occupation and energy densities, spectral properties, and power spectra [21–23]. In general the one-point function $\langle\varphi(x)\rangle$ can be space-time dependent, and the two-point functions depend on two space-time points. They can be parametrized by the so-called statistical and spectral functions,

$$\begin{aligned} F(x,y) &= \frac{1}{2}\langle\{\varphi(x),\varphi(y)\}\rangle - \langle\varphi(x)\rangle\langle\varphi(y)\rangle, \\ \rho(x,y) &= i\langle[\varphi(x),\varphi(y)]\rangle, \end{aligned} \quad (2)$$

with analogous definitions for χ. We are interested in obtaining a properly renormalized nonequilibrium evolution for these correlators, from which the observables mentioned above can be read off. For example, the energy density involves the correlator of the canonical momentum, given by $\partial_{x^0}\partial_{y^0}F$ [22]. For simplicity we assume in the following[1] $\langle\varphi\rangle = \langle\chi\rangle = 0$ and concentrate on the time-evolution of the two-point functions.

[1]In the theory considered here, $\langle\varphi\rangle = 0$ might be achieved by means of an external source.

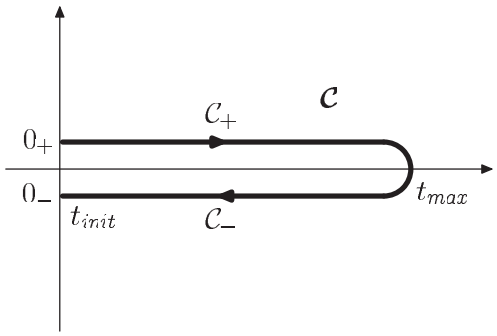

FIG. 1. Closed time path $\mathcal{C}$ used for the computation of expectation values. The time path starts and ends at the initial time $t_{\rm init} \equiv 0$, and $t_{\rm max}$ should be chosen larger than the largest time for which correlation functions are computed. The boundaries of the time path are denoted by 0_+ and 0_-, respectively.

The nonequilibrium setup can be characterized by an initial state (at time $t = 0$, say) described by a density matrix ρ. The time-evolution of any observable $\hat{O}$ can be obtained from the closed-time path representation of $\langle\hat{O}\rangle \equiv \mathrm{Tr}(\rho\hat{O})$ [1]. Let us briefly review the generating functional Z for expectation values. We consider external sources $J_a(x)$ and $K_{a,b}(x,y)$ with indices $a,b = \varphi,\chi$, and use a compact notation

$$\begin{aligned} J_a\phi^a &= \int_{\mathcal{C}} d^{d+1}x J_a(x)\phi^a(x), \\ \phi^a K_{ab}\phi^b &= \int_{\mathcal{C}} d^{d+1}x d^{d+1}y \phi^a(x)K_{ab}(x,y)\phi^b(x), \end{aligned} \quad (3)$$

where $\phi^a = \varphi,\chi$ labels the two fields and summation over a, b is implied. The time arguments are integrated over the closed time path $\mathcal{C}$ shown in Fig. 1. The generating functional for expectation values of products of field operators that are time-ordered along the path $\mathcal{C}$ is defined as

$$\begin{aligned} Z[J,K] &= \mathrm{Tr}(\rho T_{\mathcal{C}} e^{iJ_a\phi^a + \frac{i}{2}\phi^a K_{a,b}\phi^b}) \\ &= \int \mathcal{D}\phi_+ \int \mathcal{D}\phi_- \langle\phi_+|\rho|\phi_-\rangle \\ &\quad \times \langle\phi_-|T_{\mathcal{C}} e^{iJ_a\phi^a + \frac{i}{2}\phi^a K_{ab}\phi^b}|\phi_+\rangle. \end{aligned} \quad (4)$$

In the second step we have evaluated the trace in the basis of eigenstates of the field operators at $t = 0$, $\phi^a(0,\vec{x})|\phi\rangle = \phi^a(\vec{x})|\phi\rangle$, and used the completeness relation $\int\mathcal{D}|\phi\rangle\langle\phi| = 1$. The matrix element in the third line can be represented by the path integral [1,24]

$$\langle\phi_-|e^{iJ_a\phi^a + \frac{i}{2}\phi^a K_{ab}\phi^b}|\phi_+\rangle = \int_{\phi_+}^{\phi_-}\mathcal{D}\phi e^{iS + iJ_a\phi^a + \frac{i}{2}\phi^a K_{ab}\phi^b}, \quad (5)$$

over field configurations $\phi^a(t,\vec{x})$ with time argument associated to the closed time path $\mathcal{C}$, with boundary conditions $\phi^a(0_\pm,\vec{x}) = \phi^a_\pm(\vec{x})$.

The matrix element of the density matrix in the second line of Eq. (4) encodes the information about the initial state at $t = 0$. In the following it is important to describe the initial state accurately. For that, it is useful to Taylor expand the matrix element in the form

$$\langle \phi_+|\rho|\phi_-\rangle = \exp(i\mathcal{F}[\phi]), \tag{6}$$

with

$$\mathcal{F}[\phi] \equiv \sum_{n\geq 0}\sum_{a_i=\varphi,\chi}\sum_{\epsilon_i=\pm}\frac{1}{n_\varphi! n_\chi!}\int d^dx_1\cdots d^dx_n \times \alpha^{\epsilon_1\ldots\epsilon_n}_{n,a_1\ldots a_n}(\vec{x}_1,\ldots,\vec{x}_n)\phi^{a_1}_{\epsilon_1}(\vec{x}_1)\ldots\phi^{a_n}_{\epsilon_n}(\vec{x}_n), \tag{7}$$

where the coefficients α_n are analogous to cumulants, generalized to a field-theoretical setting [24], with labels for fields ($a_i = \varphi,\chi$) and bra/ket states ($\epsilon_i = \pm$). Here n_φ is the number of φ fields contained in $a_1,\ldots,a_n$ and $n_\chi = n - n_\varphi$. Terms with $n \leq 2$ encode the initial conditions for the one- and two-point functions, and non-Gaussian initial correlations are parametrized by terms with $n \geq 3$. After inserting (5) and (6) into (4), one obtains

$$Z[J,K] = \int\mathcal{D}\phi_+\int\mathcal{D}\phi_-\int_{\phi_+}^{\phi_-}\mathcal{D}\phi e^{iS+i\mathcal{F}+iJ_a\phi^a+\frac{i}{2}\phi^aK_{ab}\phi^b}. \tag{8}$$

Note that the information on the initial state entering via $\mathcal{F}[\phi]$ can be seen as an additional contribution adding to the classical action $S[\phi]$. The contributions to $\mathcal{F}[\phi]$ in the Taylor expansion (7) for $n \leq 2$ can be absorbed into a redefinition of the external sources J and K, respectively. Following [25], the non-Gaussian initial correlations with $n \geq 3$ can be incorporated in a diagrammatic expansion of the generating functional as effective n-point vertices, called *α-vertices* in what follows. For example, the vertex encoding the initial three-point correlation of one φ and two χ fields has the form

$$i\alpha_{3,\chi\chi\varphi}(x_1,x_2,x_3) \equiv \frac{i\delta^3\mathcal{F}[\phi]}{\delta\chi(x_1)\delta\chi(x_2)\delta\varphi(x_3)} = \sum_{\epsilon_i=\pm} i\alpha^{\epsilon_1,\epsilon_2,\epsilon_3}_{3,\chi\chi\varphi}(\vec{x}_1,\vec{x}_2,\vec{x}_3) \times \delta_{\epsilon_1}(x_1^0)\delta_{\epsilon_2}(x_2^0)\delta_{\epsilon_3}(x_3^0), \tag{9}$$

where $\delta_\pm(t) \equiv \delta_\mathcal{C}(t - 0_\pm)$ and $\delta_\mathcal{C}$ is the delta function on the closed time path. In addition, the usual local $\chi\chi\varphi$-vertex derived from the Lagrangian is in this notation given by

$$i\frac{\delta^3 S[\phi]}{\delta\chi(x_1)\delta\chi(x_2)\delta\varphi(x_3)} = -i\lambda\delta_\mathcal{C}(x_1 - x_2)\delta_\mathcal{C}(x_2 - x_3), \tag{10}$$

where $\delta_\mathcal{C}(x - y) \equiv \delta_\mathcal{C}(x^0 - y^0)\delta(\vec{x} - \vec{y})$. In general, the α_n-vertices connect n legs of φ/χ fields, specified by the labels a_i, and involve n delta functions supported at $t = 0_\pm$, the upper and lower boundary of the closed time path $\mathcal{C}$, respectively. Taking the additional vertices arising from the initial correlations into account, the generating functional can be constructed based on the usual Feynman rules in position space, except that each time variable is integrated over the closed time path.

By performing a double Legendre transformation of $Z[J,K]$ with respect to both sources one obtains the 2PI effective action Γ, which depends in general on the full one- two-point correlation functions of the φ and χ fields [2]. As mentioned above, we consider a setup with vanishing one-point functions. In this case the 2PI effective action can be parametrized as

$$\Gamma[G,D] = \frac{i}{2}\mathrm{Tr}\log G^{-1} + \frac{i}{2}\mathrm{Tr}\log D^{-1} + \frac{i}{2}\mathrm{Tr}(G_0^{-1}G) + \frac{i}{2}\mathrm{Tr}(D_0^{-1}D) + \Gamma_2[D,G], \tag{11}$$

where

$$G(x,y) = \frac{1}{Z}\frac{\delta Z}{\delta K_{\varphi\varphi}(y,x)} = \langle T_\mathcal{C}\varphi(x)\varphi(y)\rangle = F(x,y) - \frac{i}{2}\mathrm{sgn}_\mathcal{C}(x^0 - y^0)\rho(x,y), \tag{12}$$

where $\mathrm{sgn}_\mathcal{C}(x^0 - y^0)$ is the signum function on the closed-time path, and $G_0^{-1}(x,y) = i(\Box + m_\varphi^2)\delta_\mathcal{C}(x - y)$. The two-point functions D and D_0 for χ are defined analogously. Furthermore $\Gamma_2[D,G]$ is given by the sum of all 2PI diagrams with full propagators G and D for the φ and χ fields, respectively. The equation of motion for the two-point function G is given by the stationarity condition $\delta\Gamma/\delta G = 0$, which takes the form of a Schwinger-Dyson equation

$$G^{-1}(x,y) = G_0^{-1}(x,y) - \Pi(x,y), \tag{13}$$

where $\Pi(x,y) = 2i\delta\Gamma_2/\delta G(y,x)$ is the full self-energy for φ. Out of equilibrium the two-point functions depend on both arguments and not only on the difference $x - y$. By multiplying from the right with G and integrating one obtains the equation of motion

$$i(\Box + m_\varphi^2)G(x,y) = \delta_\mathcal{C}(x-y) + \int_\mathcal{C} d^{d+1}z\Pi(x,z)G(x,y). \tag{14}$$

The self-energy can be decomposed as [25]

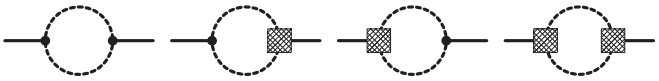

FIG. 2. One-loop contributions to the φ self-energy $\Pi(x,y)$. The box represents the initial three-point correlation $i\alpha_3$. The first and second diagrams contribute to $\Pi_{F/\rho}$ and $\Pi^{\lambda\alpha}_{F/\rho}$ respectively, while the other two are $\propto \delta_{s/a}(x_0)$ and not needed in what follows.

$$\Pi(x,y) = \Pi_F(x,y) - \frac{i}{2}\mathrm{sgn}_{\mathcal{C}}(x^0-y^0)\Pi_\rho(x,y) + i\Pi^{\lambda\alpha}_F(x,\vec{y})\delta_s(y^0) + \frac{1}{2}\Pi^{\lambda\alpha}_\rho(x,\vec{y})\delta_a(y^0), \quad (15)$$

where $\delta_{s/a}(t) = (\delta(t-0_+) \pm \delta(t-0_-))/2$ and we omitted terms $\propto \delta_{s/a}(x^0)$ which will not be needed. The self-energies $\Pi^{\lambda\alpha}_{F/\rho}$ vanish for Gaussian initial states, and contain an α-vertex attached to the right leg in a diagrammatic expansion (cf. Fig. 2). The integration over the time path on the right-hand side of (14) can be expressed in terms of integrals over the real-time axis by inserting the decompositions (12) and (15). In addition, for a spatially homogeneous state, it is convenient to use the momentum representation $F_p(x^0,y^0) = \int d^dx e^{i\vec{p}\cdot(\vec{x}-\vec{y})}F(x,y)$, and similar for ρ and Π. This yields the Kadanoff-Baym equations [1,25]

$$\begin{aligned}(\partial_t^2 + \nu_p^2 + \delta m_\varphi^2)F_p(t,t') &= \int_0^{t'} dt''\Pi_{F,p}(t,t'')\rho_p(t'',t') \\ &\quad - \int_0^{t} dt''\Pi_{\rho,p}(t,t'')F_p(t'',t') \\ &\quad + \Pi^{\lambda\alpha}_{F,p}(t)F_p(0,t') \\ &\quad + \frac{1}{4}\Pi^{\lambda\alpha}_{\rho,p}(t)\rho_p(0,t'), \\ (\partial_t^2 + \nu_p^2 + \delta m_\varphi^2)\rho_p(t,t') &= \int_t^{t'} dt''\Pi_{\rho,p}(t,t'')\rho_p(t'',t'),\end{aligned} \quad (16)$$

that describe the time-evolution starting from a general initial state at $t=0$. Here we defined $\nu_p^2 \equiv p^2 + m_\varphi^2$ and included also a mass counterterm. The initial state enters via the initial conditions $F_p(0,0)$, $\partial_t F_p(0,0) = \partial_{t'}F_p(0,0)$ and $\partial_t\partial_{t'}F_p(0,0)$ as well as the non-Gaussian initial correlations α. The latter contribute to the self-energies in a diagrammatic expansion of the self-energies via the α-vertices. The corresponding initial conditions for ρ_p are fixed by the equal time commutation relations (ETCR) to be $\rho_p|_{t=t'} = \partial_t\partial_{t'}\rho_p|_{t=t'} = 0$ and $\partial_t\rho_p|_{t=t'} = -\partial_{t'}\rho_p|_{t=t'} = 1$.

We take an initial three-point correlation into account, and approximate the self-energies at one-loop with a d-dimensional loop cutoff Λ (cf. Fig. 2),

$$\begin{aligned}\Pi_{F,p}(t,t') &= -\frac{\lambda^2}{2}\int_\Lambda \frac{d^dq}{(2\pi)^d}\Big(D_{F,q}(t,t')D_{F,p-q}(t,t') - \frac{1}{4}D_{\rho,q}(t,t')D_{\rho,p-q}(t,t')\Big), \\ \Pi_{\rho,p}(t,t') &= -\lambda^2\int_\Lambda \frac{d^dq}{(2\pi)^d}D_{F,q}(t,t')D_{\rho,p-q}(t,t'), \\ \Pi^{\lambda\alpha}_{F,p}(t) &= -\frac{\lambda}{2}\int_\Lambda \frac{d^dq}{(2\pi)^d}\Big(D_{F,q}(t,0)D_{F,p-q}(t,0)i\alpha_3^{sss} - \frac{1}{4}D_{\rho,q}(t,0)D_{\rho,p-q}(t,0)i\alpha_3^{aas}\Big), \\ \Pi^{\lambda\alpha}_{\rho,p}(t) &= -\lambda\int_\Lambda \frac{d^dq}{(2\pi)^d}D_{F,q}(t,0)D_{\rho,p-q}(t,0)i\alpha_3^{saa},\end{aligned} \quad (17)$$

where D denotes the correlator of the field χ and with initial correlations $\alpha_3^{ijk} \equiv \sum_{\epsilon_i} P^i_{\epsilon_1}P^j_{\epsilon_2}P^k_{\epsilon_3}\alpha^{\epsilon_1\epsilon_2\epsilon_3}_{3,\chi\chi\varphi}(q,p-q,-p)$, with $P^s_\pm \equiv 1$, $P^a_\pm \equiv \pm 1$ transformed into spatial Fourier modes. Including higher initial correlations is possible but would affect only the finite and thus the physical part of the time evolution.

While our findings can be generalized, we assume for simplicity that χ behaves as a thermal bath that is kept close to equilibrium by some further (unspecified) interactions. Concretely, to compute the self-energies we use the free equilibrium expressions $D_{F,q}(t,t') = (1+2n_q)\cos(\omega_q(t-t'))/(2\omega_q)$ and $D_{\rho,q}(t,t') = \sin(\omega_q(t-t'))/\omega_q$, where $n_q = 1/(e^{\omega_q/T}-1)$. This corresponds to neglecting the backreaction of the thermal bath to which the field φ is coupled. In this approximation, the spectral function is given by the equilibrium solution $\rho_p(t,t') = \rho_p^{eq}(t-t')$, while the statistical propagator $F_p(t,t')$ approaches the thermal solution for late times $t,t' \to \infty$ [23]. Furthermore, we analyze the superrenormalizable case $d<5$, which admits a nontrivial continuum limit. As mentioned before, we choose this setting because it allows us to trace all potential divergences in a nonequilibrium system analytically. For concreteness we set $d=4$ in the following because it is the highest dimension for which the cubic scalar interaction is still superrenormalizable. All results can be easily adapted to lower dimensions, for which the UV sensitivity is reduced compared to $d=4$.

III. EQUILIBRIUM PROPERTIES

Before discussing the renormalization of $F_p(t,t')$, we briefly discuss the renormalization of the spectral function by usual vacuum counterterms. We focus for simplicity on $\rho^{\mathrm{vac}}_{p=0}$ for $m_\chi = 0$, but the relevant UV properties extend to nonzero p, T or m_χ. The Fourier transform $\rho^{\mathrm{vac}}_{p=0}(\omega) = \int \frac{d\omega}{2\pi}\rho^{\mathrm{vac}}_{p=0}(t)e^{-i\omega t}$ is obtained from the vacuum Euclidean propagator $G_{\mathrm{vac}}(ip_5) = 1/(p_5^2 + m_\varphi^2 + \delta m_\varphi^2 + \Pi_{\mathrm{vac}}(ip_5))$ as $\rho^{\mathrm{vac}}_{p=0}(\omega) = -2i\mathrm{Im}G_{\mathrm{vac}}(ip_5 \to \omega + i\epsilon)$. At one-loop, we find

$$\Pi_{\rm vac}(ip_5) = -\frac{\lambda^2}{64\pi^2}\left[\Lambda - \frac{p_5}{2}\,\mathrm{Arctan}\left(\frac{2\Lambda}{p_5}\right)\right]. \quad (18)$$

There is a linear divergence which is absorbed by a mass counterterm $\delta m_\varphi^2 = (\lambda^2/64\pi^2)\Lambda$. Then the Euclidean propagator admits the continuum limit $G^\infty_{\rm vac}(ip_5) = 1/(p_5^2 + m_\varphi^2 + \gamma|p_5|)$, where we introduced $\gamma \equiv \lambda^2/(256\pi)$. The spectral function in the continuum limit reads

$$\rho^{\rm vac}_{p=0}(\omega) = \frac{-2i\gamma\omega}{(\omega^2 - m_\varphi^2)^2 + \gamma^2\omega^2}, \quad (19)$$

and obeys $\int \frac{d\omega}{2\pi}\omega\rho^{\rm vac}_{p=0}(\omega) = -i$ in agreement with the ETCR. We note also that the spectral function behaves like $1/\omega^3$ at large $|\omega|$, a property that we shall use in the next section. This property extends to $T > 0$ since the thermal contribution to the one-loop self-energy behaves like $\lambda^2 T^3/\omega_n^2$ at large external Matsubara frequency ω_n.

For the discussion below, it is finally important to realize that even though $G_{\rm vac}(ip_5)$ admits a continuum limit, the UV behavior of its Fourier transform $G_{\rm vac}(\tau)$ needs to be further analyzed. In particular, $\partial_\tau^2 G_{\rm vac}(\tau)|_{\tau=0}$ contains divergences. One is a trivial contact term which appears in the relation $\partial_\tau^2 G_{\rm vac}(\tau)|_{\tau=0} = -\delta(\tau=0) + \partial_t\partial_{t'}F^{\rm vac}_p(t,t')|_{t=t'}$. After this contact term has been subtracted, we are left with

$$\begin{aligned}\partial_t\partial_{t'}F^{\rm vac}_{p=0}|_{t=t'} &= \int_{-\infty}^{\infty}\frac{dp_5}{2\pi}(1 - p_5^2 G_{\rm vac}(ip_5)) \\ &= \int_{-\infty}^{\infty}\frac{dp_5}{2\pi}\frac{m_\varphi^2 + \frac{\lambda^2}{64\pi^2}\frac{p_5}{2}\mathrm{Arctan}(\frac{2\Lambda}{p_5})}{p_5^2 + m_\varphi^2 + \frac{\lambda^2}{64\pi^2}\frac{p_5}{2}\mathrm{Arctan}(\frac{2\Lambda}{p_5})}\end{aligned} \quad (20)$$

which produces a divergent integral for $\Lambda \to \infty$. Since the divergence originates from the "Arctan" function in the numerator and the p_5^2-term in the denominator, we have that, as $\Lambda \to \infty$ (and after performing the change of variables $p_5 = \Lambda\tilde{p}_5$)

$$\begin{aligned}\partial_t\partial_{t'}F^{\rm vac}_{p=0}|_{t=t'} &\sim \frac{\lambda^2}{128\pi^2}\int_{-\infty}^{\infty}\frac{d\tilde{p}_5}{2\pi}\frac{\tilde{p}_5\mathrm{Arctan}(\frac{2}{\tilde{p}_5})}{\tilde{p}_5^2 + \frac{m_\varphi^2}{\Lambda^2}} \\ &\sim \frac{\gamma}{\pi}\ln\frac{\Lambda}{m_\varphi},\end{aligned} \quad (21)$$

where γ was introduced above Eq. (19). This leading behavior at large Λ is not modified at nonzero T, p or m_χ. Moreover, this correlator contributes to the energy density [23], and in equilibrium the divergence can be removed by a cosmological constant counterterm.

IV. RENORMALIZATION OUT OF EQUILIBRIUM

To study the behavior of $F_p(t,t')$ for $\Lambda \to \infty$, we note that a formal analytical solution for $F_p(t,t')$ is given by $F_p(t,t') = F_p^{\rm hom}(t,t') + F_p^{inh,G}(t,t') + F_p^{inh,nG}(t,t')$ with

$$\begin{aligned}F_p^{\rm hom} &= \rho_p(t)\rho_p(t')\partial_t\partial_{t'}F_p(0,0) + \sigma_p(t)\sigma_p(t')F_p(0,0) \\ &\quad + (\sigma_p(t)\rho_p(t') + \sigma_p(t')\rho_p(t))\partial_t F_p(0,0), \\ F_p^{inh,G} &= \int_0^t du\int_0^{t'} dv\rho_p(t,u)\Pi_{F,p}(u,v)\rho_p(v,t'), \\ F_p^{inh,nG} &= \frac{1}{4}\int_0^t du\rho_p(t,u)\Pi^{\lambda\alpha}_{\rho,p}(u)\rho_p(0,t') \\ &\quad + \frac{1}{4}\int_0^{t'} dv\rho_p(t,0)\Pi^{\lambda\alpha}_{\rho,p}(v)\rho_p(v,t'),\end{aligned} \quad (22)$$

where we introduced

$$\begin{aligned}&\rho_p(t) \equiv \rho_p(t,0), \\ &\sigma_p(t) \equiv -\partial_{t'}\rho_p(t,0) + \int_0^t \rho_p(t,u)\Pi^{\lambda\alpha}_{F,p}(u).\end{aligned} \quad (23)$$

For a Gaussian initial state, $F_p^{inh,nG}$ vanishes identically and $\sigma_p(t)^{\rm Gauss} = -\partial_{t'}\rho_p(t,0)$, in accordance with [23].

We first investigate the contribution $F_p^{inh,G}(t,t')$ in (22) which is independent of the initial conditions. Potential UV divergences can arise only from the vacuum part of Π_F (i.e. $n_q, n_{p-q} \to 0$) because the thermal contribution is exponentially suppressed for large loop momenta. Keeping only this part and using the Fourier representation $\rho_p(t) = \int\frac{d\omega}{2\pi}\rho_p(\omega)e^{i\omega t}$ one obtains

$$\begin{aligned}F_p^{inh,G}(t,t') = -\frac{\lambda^2}{2}\mathrm{Re}\int_\Lambda\frac{d^dq}{(2\pi)^d}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\rho_p(\omega)\rho_p(\omega') \\ \times\frac{(e^{i\Omega_q t} - e^{i\omega t})(e^{-i\Omega_q t'} - e^{-i\omega' t'})}{4\omega_q\omega_{p-q}(\Omega_q - \omega)(\Omega_q - \omega')},\end{aligned} \quad (24)$$

where $\Omega_q \equiv \omega_q + \omega_{p-q}$. Note that the integrand has no poles because the numerator vanishes for $\Omega_q \to \omega, \omega'$, respectively. The integration over q is superficially logarithmically divergent. To extract UV sensitive terms we use $\int d\omega\rho_p(\omega) = 0$ to replace

$$\frac{e^{i\Omega_q t} - e^{i\omega t}}{\Omega_q - \omega} \quad \text{by} \quad \frac{e^{i\Omega_q t} - e^{i\omega t}}{\Omega_q - \omega} - \frac{e^{i\Omega_q t}}{\Omega_q} \quad (25)$$

and similarly for the other such fraction appearing in Eq. (24). After some simple algebra, this allows us to rewrite the integral above in an equivalent form, with the second line of (24) replaced by

$$\frac{(\omega e^{i\Omega_q t}-\Omega_q e^{i\omega t})(\omega' e^{-i\Omega_q t'}-\Omega_q e^{-i\omega' t'})}{4\omega_q\omega_{p-q}\Omega_q^4} \times\left[1+\frac{\Omega_q(\omega+\omega')-\omega\omega'}{(\Omega_q-\omega)(\Omega_q-\omega')}\right]. \tag{26}$$

Using $\rho_p(\omega)\propto\omega^{-3}$ for large ω, one shows that the second term in the square bracket of (26) leads to absolutely convergent contributions to $F_p^{inh,G}$, $\partial_t F_p^{inh,G}$ and $\partial_t\partial_{t'}F_p^{inh,G}$. Potential divergences therefore can only arise from the first term in this bracket. Using $\int\frac{d\omega}{2\pi}\omega\rho_p(\omega)=-i$, we obtain

$$F_p^{inh,G}(t,t')=\frac{\gamma}{\pi}[\rho_p(t)\rho_p(t')L_p-\rho_p(t)S_p(t') -\rho_p(t')S_p(t)+C_p(t-t')]+\cdots \tag{27}$$

where the ellipsis stands for absolutely convergent contributions, and we defined the integrals

$$\begin{aligned} L_p &\equiv 32\pi^2\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{\omega_q\omega_{p-q}\Omega_q^2}\sim\ln\frac{\Lambda}{m_\varphi},\\ S_p(t) &\equiv 32\pi^2\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{\sin(\Omega_q t)}{\omega_q\omega_{p-q}\Omega_q^3},\\ C_p(t-t') &\equiv 32\pi^2\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{\cos(\Omega_q(t-t'))}{\omega_q\omega_{p-q}\Omega_q^4}. \end{aligned} \tag{28}$$

For $d=4$, L_p is logarithmically divergent for large Λ, while S_p and C_p are absolutely convergent for all t, t'. Nevertheless, $\dot S_p(0)=L_p$ and $\ddot C_p(0)=-L_p$ are logarithmically divergent, which affects the correlators $\partial_t F_p$ and $\partial_t\partial_{t'}F_p$ (see below). The term $\propto\ddot C_p$ in $\partial_t\partial_{t'}F_p^{inh,G}$ matches the logarithmic divergence of the corresponding vacuum correlator for equal times (21). In the following, we discuss how these divergences affect the nonequilibrium correlators and demonstrate explicitly how they can be removed by the homogeneous and non-Gaussian contributions in (22) for a proper choice of initial conditions. Before that, we briefly discuss the Gaussian case.

A. Gaussian initial condition

On general grounds, one expects that a physical initial state should differ from the vacuum correlations by a finite, cutoff-independent amount. Implementing this idea rigorously would require us to take initial n-point correlations into account for all n. In practice, one has to cut at some finite n. Let us first consider the Gaussian case

$$\text{(G1)}\quad \begin{aligned} F_p(0,0) &= F_p^{\rm vac}(0,0)+\Delta_p^{(0)}\\ \partial_t F_p(0,0) &= \partial_t F_p^{\rm vac}(0,0)+\Delta_p^{(1)},\\ \partial_t\partial_{t'}F_p(0,0) &= \partial_t\partial_{t'}F_p^{\rm vac}(0,0)+\Delta_p^{(2)},\\ \alpha_n &= 0\quad \text{for } n\geq 3, \end{aligned} \tag{29}$$

where only the connected two-point function is nonzero initially, and $\Delta_p^{(i)}$ are cutoff-independent functions that parametrize the nonequilibrium initial state. The logarithmic divergence contained in $\partial_t\partial_{t'}F_p^{\rm vac}(0,0)$, cf. (21), leads to a logarithmic divergence in the homogeneous solution (22),

$$F_p^{\rm hom}(t,t')=\frac{\gamma}{\pi}\rho_p(t)\rho_p(t')\ln\frac{\Lambda}{m_\varphi}+\text{finite}. \tag{30}$$

This divergence has precisely the same time-dependence as the one $\propto L_p$ in $F_p^{inh,G}$, cf. (27), but when summing both contributions there is in fact no cancellation. Instead both divergences add up, and therefore the choice (G1) does not admit a continuum limit for $F_p(t,t')$. This can also be seen in the numerical solution, shown in Fig. 3 (dashed lines).

Is it possible to remedy this shortcoming without going beyond the Gaussian initial state? To answer this question, we consider an alternative initial condition for the mixed derivative (and with the other derivatives initialized as in (G1)) where we add "by hand" a piece that removes the logarithmic divergence in $F_p(t,t')$ at all times,

$$\text{(G2)}\quad \partial_t\partial_{t'}F_p(0,0)=\partial_t\partial_{t'}F_p^{\rm vac}(0,0)-2\frac{\gamma}{\pi}L_p+\Delta_p^{(2)}. \tag{31}$$

Indeed, $F_p(t,t')$ possesses a continuum limit, as can also be observed in Fig. 3 (upper graph, dotted lines). However, closer inspection shows that this choice leads to the appearance of initial-time singularities in the two-point functions involving the canonical momentum, in particular from (27) and (31) it follows that

$$\partial_t F_p^{(\rm G2)}(t,t')|_{t\to 0}=-\frac{\gamma}{\pi}\rho_p(t')L_p+\cdots \tag{32}$$

is logarithmically divergent for $\Lambda\to\infty$, as our numerical simulation also confirms (not shown). Moreover, the momentum-momentum correlator

$$\begin{aligned} \partial_t\partial_{t'}F_p^{(\rm G2)}(t,t') &= -\frac{\gamma}{\pi}[\dot\rho_p(t')\dot S_p(t)+\dot\rho_p(t)\dot S_p(t') +\ddot C_p(t-t')]+\cdots\\ &\to\begin{cases} -\frac{\gamma}{\pi}L_p+\cdots & t,t'\to 0\\ +\frac{\gamma}{\pi}L_p+\cdots & t\to t', \end{cases}\qquad t\gg 1/\Lambda\,, \end{aligned} \tag{33}$$

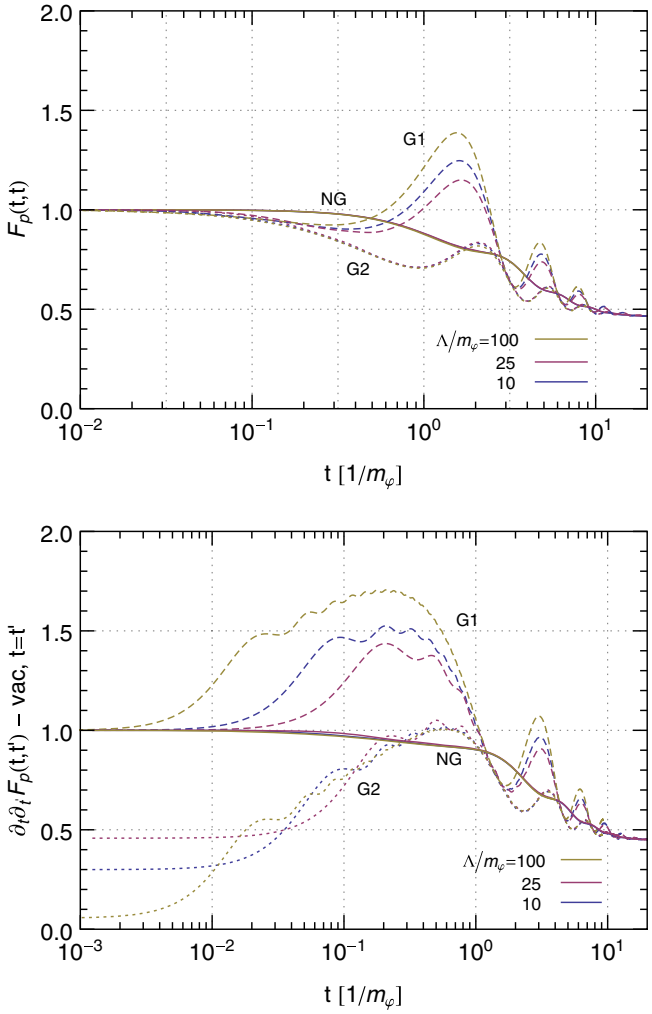

FIG. 3. Time-evolution of the statistical propagator $F_p(t,t)$ and the momentum-momentum correlator $\partial_t\partial_{t'}F_p(t,t') - \partial_t\partial_{t'}F_p^{\rm vac}(t,t')|_{t=t'}$ (for $p=0$) at equal times for three different values of the cutoff $\Lambda/m_\varphi = 10, 25, 100$, and three different initial conditions (G1) (dashed), (G2) (dotted) and (NG) (solid). We used $\Delta^{(0)}_{p=0} = n^{in}/m_\varphi$, $\Delta^{(1)}_{p=0} = 0$, $\Delta^{(2)}_{p=0} = n^{in}m_\varphi$ with $n^{in} = 0.5$, $\gamma/m_\varphi = 0.28$, $T_\chi = m_\chi = 0$.

exhibits a cutoff-dependent "jump" from the initial value imposed at $t = t' = 0$ to the value that matches the vacuum correlator (21) at "late" times (see Fig. 3, lower graph, dotted lines). Note that both (G1) and (G2) lead to a cutoff-dependence which cannot be removed by a cosmological constant counterterm in the contribution of the field- and momentum correlator to the energy density, respectively.

B. Non-Gaussian initial condition

In the following we demonstrate that the non-Gaussian initial condition

$$\begin{aligned}
\text{(NG)}\quad F_p(0,0) &= F_p^{\rm vac}(0,0) + \Delta_p^{(0)} \\
\partial_t F_p(0,0) &= \partial_t F_p^{\rm vac}(0,0) + \Delta_p^{(1)}, \\
\partial_t\partial_{t'}F_p(0,0) &= \partial_t\partial_{t'}F_p^{\rm vac}(0,0) + \Delta_p^{(2)}, \\
\alpha_3 &= \alpha_3^{\rm vac}, \qquad \alpha_n = 0 \quad \text{for } n \geq 4,
\end{aligned} \tag{34}$$

characterized by two-point functions as for (G1) and an initial three-point correlation equal to the one in vacuum avoids the pathologies in the Gaussian case and admits a well-behaved continuum limit. Using the matching procedure developed in [25],

$$i\alpha_3^{ijk,\rm vac} = \frac{-2\lambda}{\omega_q + \omega_{p-q} + \nu_p}, \tag{35}$$

for $ijk = sss, aas, saa$. All higher n-point functions are set to zero initially.

The inhomogeneous part of the solution (22) now contains an additional piece involving α_3. An analogous computation as above shows that

$$\begin{aligned}
F_p^{inh,nG}(t,t') = \frac{\gamma}{\pi}[&-2\rho_p(t)\rho_p(t')L_p \\
&+ \rho_p(t)S_p(t') + \rho_p(t')S_p(t)] + \cdots
\end{aligned} \tag{36}$$

Remarkably, the term $\propto L_p$ has the same structure as in $F_p^{inh,G}(t,t')$, cf., (27), but with a relative factor -2. Together with the divergence in the inhomogeneous Gaussian part (27), this is precisely what is needed to cancel the logarithmic divergence of the homogeneous part (30). In addition, it is important to note that the terms proportional to S_p cancel with those in $F_p^{inh,G}(t,t')$.

This has several consequences which we want to stress: (i) $F_p(t,t')$ and $\partial_t F_p(t,t')$ converge to a finite continuum limit, (ii) $\partial_t\partial_{t'}F_p(t,t')$ has a time-independent logarithmic divergence for $t = t'$ which matches precisely the one in vacuum, i.e. the difference $\partial_t\partial_{t'}F_p(t,t') - \partial_t\partial_{t'}F_p^{\rm vac}(t,t')$ also converges for all $t, t' \geq 0$. (iii) there are no initial-time singularities. These features can be observed also for the numerical solutions (see Fig. 3, solid lines), which are almost indistinguishable when varying the cutoff. Furthermore, (i) and (ii) imply that the energy density is finite at all times and renormalized by the same counterterms as in equilibrium. We emphasize that the initial three-point correlation sizeably affects the solution $F_p(t,t')$ not only at early times, but up to the thermalization timescale $t \sim 1/\gamma$.

V. INITIAL TIME VS CONTINUUM LIMIT

For an initial value problem to be meaningful, it is necessary that the correlation functions possess a well-defined behavior in the limit $t \to 0$. Since the super-renormalizable theory considered here allows us also to

take the continuum limit $\Lambda \to \infty$, it is instructive to discuss the interplay of both limits. We first consider the unequal-time correlation functions for which $t' = 0$. In this case, the solution (22) simplifies to

$$F_p^\Lambda(t,0) = \sigma_p^\Lambda(t) F_p^\Lambda(0,0) + \rho_p^\Lambda(t) \partial_t F_p^\Lambda(0,0), \quad (37)$$

where we indicated the UV regulator in the superscript. By solving the Kadanoff-Baym equations perturbatively in λ for small time t one obtains up to $\mathcal{O}(\lambda^2)$,

$$\sigma_p^\Lambda(t)^{\text{Gauss}} = \cos\nu_p t - \gamma J_\Lambda(t) + \cdots,$$
$$\rho_p^\Lambda(t) = \frac{\sin\nu_p t}{\nu_p} - \gamma I_\Lambda(t) + \cdots, \quad (38)$$

where $\gamma = \lambda^2/(256\pi)$, $J_\Lambda(t) = dI_\Lambda/dt$ and

$$I_\Lambda(t) = \frac{64\pi}{\nu_p}\int_\Lambda \frac{d^dq}{(2\pi)^d} \frac{\nu_p \sin(\Omega_q t) - \Omega_q \sin(\nu_p t)}{\omega_q \omega_{p-q} \Omega_q^2 (\nu_p^2 - \Omega_q^2)}. \quad (39)$$

Here we omitted again thermal contributions to the self-energies which are absolutely convergent even when taking time derivatives. For the mixed correlator of the canonical momentum and the field $\partial_t F_p(t,0)$ one needs to consider also $K_\Lambda(t) \equiv d^2 I_\Lambda/dt^2$. In $d = 4$, I_Λ and J_Λ are absolutely convergent, while the superficial degree of divergence of K_Λ is zero. Nevertheless, due to the structure of the integrand, it turns out that K_Λ has a continuum limit for all times t. For $p = m_\chi = 0$ one finds the explicit results

$$I_\infty(t) = \frac{2}{m_\varphi^2}\sin^2\left(\frac{m_\varphi t}{2}\right)\text{sgn}(t),$$
$$J_\infty(t) = \frac{1}{m_\varphi}\sin(m_\varphi t)\text{sgn}(t),$$
$$K_\infty(t) = \begin{cases} \cos(m_\varphi t)\text{sgn}(t) & t \neq 0 \\ 0 & t = 0 \end{cases}. \quad (40)$$

The absolutely convergent integrals I_∞ and J_∞ are continuous for $t \to 0$, i.e. the continuum limit $\Lambda \to \infty$ and the initial time limit $t \to 0$ commute, as expected. This is not the case for K_Λ, for which $K_\infty(t)|_{t\to 0, t>0} \to 1$ while $K_\Lambda(0)|_{\Lambda\to\infty} = 0$. This implies that

$$F_p^\infty(t,0)|_{t\to 0} = F_p^\Lambda(0,0)|_{\Lambda\to\infty}, \quad (41)$$

i.e. the initial-time and the continuum limit commute for the field correlator. For the mixed correlator of the canonical momentum $\partial_t\varphi$ and the field, on the other hand, one finds

$$\partial_t F_p^\infty(t,0)|_{t\to 0} - \partial_t F_p^\Lambda(0,0)|_{\Lambda\to\infty} = -\gamma F_p^\infty(0,0), \quad (42)$$

for a Gaussian initial state. This implies that the initial condition imposed at a finite value of the cutoff does not correspond to the value of the correlation function that is approached in the continuum limit for arbitrarily small but positive time t. In practice, the time evolution of $\partial_t F_p^\Lambda(t,0)$ corresponds to a rapid "jump" by a finite amount $-\gamma F_p^\infty(0,0)$ within a timescale $t \lesssim 1/\Lambda$, cf. Fig. 4 (dashed lines). The continuum limit (grey dot-dashed line) is approached for any $t > 0$ for $\Lambda \to \infty$, such that the actual value of the correlator at small but nonzero time differs from the initial condition imposed at $t = 0$. Since this difference is finite, it is in principle not in conflict with renormalizability. However, it implies that in the continuum limit the correlator becomes discontinuous. This means that, even when imposing as initial condition for example $\partial_t F_p^\Lambda(0,0) = 0$ the continuum limit effectively corresponds to a nonzero initial value $\partial_t F_p^\infty(0,0)^{\text{eff}} \equiv \lim_{t\to 0, t>0} \partial_t F_p^\infty(t,0) = -\gamma F_p^\infty(0,0)$.

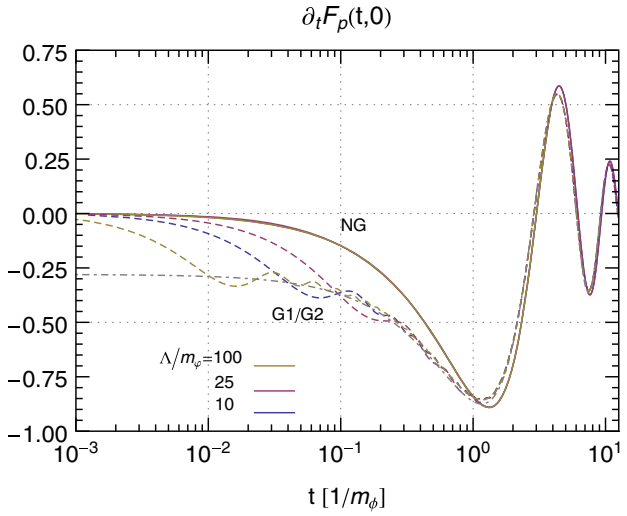

FIG. 4. Time-evolution of the unequal-time correlation function $\partial_t F_p(t,0)$ for the same set of initial conditions and values of Λ as in Fig. 3. Note that both Gaussian initial conditions (G1) and (G2) coincide for this correlator (dashed lines), and approach the continuum limit shown by the grey dot-dashed line for $\Lambda \to \infty$. For the non-Gaussian initial condition (NG) the (solid) lines for the three values $\Lambda/m_\varphi = 100, 25, 10$ lie almost on top of each other.

We checked that this behavior persists also when using instead dimensional regularization and for nonzero p and m_χ. Furthermore, it does not depend on the perturbative expansion performed in (38), which would become secular for large t. To show this we consider again the case $p = m_\chi = T = 0$, for which the continuum limit of the spectral function can be computed analytically, and is given by the Fourier transformation of (19),

$$\rho_{p=0}^\infty(t) = \frac{\sin(m_\varphi t)}{m_\varphi} e^{-\frac{1}{2}\gamma t}. \quad (43)$$

For a Gaussian initial condition one obtains similarly $\sigma_{p=0}^\infty(t)^{\text{Gauss}} = -\gamma\rho_{p=0}^\infty(t)/2 + \cos(m_\varphi t)e^{-\frac{1}{2}\gamma t}$. Using these

expressions one recovers (41) and (42) using in particular that $\partial_t \sigma^{\infty}_{p=0}(t)^{\text{Gauss}} \to -\gamma$ for $t \to 0, t > 0$.

For a non-Gaussian initial condition $\sigma_p(t)$ receives an additional contribution in (23) involving the self-energy $\Pi_F^{\lambda\alpha}$. When including the non-Gaussian initial three-point correlation (35) it is, up to order $\mathcal{O}(\lambda^2)$, given by

$$\sigma_p^{\Lambda}(t)^{nG} = \lambda^2 \int_{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{\cos(\Omega_q t) - \cos(\nu_p t)}{4\omega_q \omega_{p-q}(\Omega_q + \nu_p)(\nu_p^2 - \Omega_q^2)}. \quad (44)$$

Using (38) one finds that in the sum $\sigma_p^{\Lambda}(t) = \sigma_p^{\Lambda}(t)^{\text{Gauss}} + \sigma_p^{\Lambda}(t)^{nG}$ the superficial degree of divergence is lowered by one unit, and therefore the loop integral contributing to $\partial_t \sigma_p$ becomes absolutely convergent. This ensures that $\partial_t \sigma_p^{\infty}(t) \to 0$ for $t \to 0$, and therefore we find that the continuum and initial time limits commute for the non-Gaussian initial condition (NG), i.e.

$$\partial_t F_p^{\infty}(t,0)|_{t\to 0} = \partial_t F_p^{\Lambda}(0,0)|_{\Lambda\to\infty} \quad \text{for (NG)}. \quad (45)$$

This is also confirmed by our numerical analysis shown by the solid lines in Fig. 4, which display almost no sensitivity on the cutoff for any $t \geq 0$.

Let us now briefly discuss the general case with $t, t' \geq 0$. For the Gaussian initial conditions (G1) the continuum limit does not exist at any $t, t' > 0$. For (G2), on the other hand, the continuum limit for $F_p(t,t')$ exists and commutes also with the initial-time limit $t, t' \to 0$. However, as discussed in the previous section, for $\partial_t \partial_{t'} \Delta F_p(t,t')$, where $\Delta F_p = F_p - F_p^{\text{vac}}$, the continuum limit exists only for $t, t' > 0$. When taking the limit $t, t' \to 0$ after the continuum limit, the correlation function becomes singular. Thus, for (G2), the combined continuum- and initial time limit of the momentum correlator $\partial_t \partial_{t'} \Delta F_p(t,t')$ is singular. Similarly, for the first derivative $\partial_t F_p(t,t')$ the limit $t \to 0$, at finite t', leads to a logarithmic divergence when choosing the initial conditions according to (G2), as discussed above. The combined limit $t \to 0$, $\Lambda \to \infty$ at finite t' is therefore again singular, independent of the order in which it is taken.

As was shown in the previous section, the non-Gaussian initial conditions (NG) render all two-point correlation functions involving the field φ and its canonical momentum $\partial_t \varphi$ finite (when subtracting the vacuum contribution for the momentum correlator). For these correlators a well-defined continuum limit can be taken. In particular, all contributions to F_p, $\partial_t F_p$ and $\partial_t \partial_{t'} \Delta F_p(t,t')$ are absolutely convergent for all $t, t' \geq 0$, including the initial time $t = t' = 0$ itself. This implies in particular that the initial-time and the continuum limit commute. These observations imply that the non-Gaussian initial conditions (NG) allow for a continuum limit in a uniform manner for all times, including the initial time, and are therefore suitable to set up a renormalized initial value problem.

VI. CONCLUSION

We have shown for the first time how a proper account of initial, non-Gaussian vacuum correlations yields a UV finite time-evolution of the field- and momentum two-point correlator for all times, starting from an initial state that can be arbitrarily far from equilibrium, within a nonsecular approximation scheme that captures thermalization at late times. The scheme is based on an expansion of initial correlations around the interacting vacuum state, complemented by usual vacuum counterterms, and is well-suited for analytical and numerical evaluation. We have applied it in a relatively simple context in order to illustrate and solve the problem of the interplay between UV singularities and initial conditions in an analytical way. But the heart of our approach, namely the expansion of initial correlations around the interacting vacuum state, is quite general and can be straightforwardly extended to other theories or settings. In particular, we checked that the main features, including the property that only a finite number of initial correlators are required for renormalization can be generalized when taking the backreaction of the various fields into account, that is a fully self-consistent setting. This issue together with the discussion of the renormalizable case will be considered in a future work. This opens the way to the formulation of a renormalized initial value problem in many physically relevant quantum field theories involving scalar and spin $1/2$ fields. The case of gauge theories requires additional features related to how the symmetries manifest in the 2PI framework [26], which are still to be understood and lie beyond the scope of the present work.

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