

Gauge-covariant decomposition and magnetic monopole for $G(2)$ Yang-Mills field

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We provide a gauge-covariant decomposition of the Yang-Mills field with the exceptional gauge group $G(2)$, which extends the field decomposition proposed by Cho, Duan-Ge, and Faddeev-Niemi for the $SU(N)$ Yang-Mills field. As an application of the decomposition, we derive a new expression of the non-Abelian Stokes theorem for the Wilson loop operator in an arbitrary representation of $G(2)$. The resulting new form is used to define gauge-invariant magnetic monopoles in the $G(2)$ Yang-Mills theory. Moreover, we obtain the quantization condition to be satisfied by the resulting magnetic charge. The method given in this paper is general enough to be applicable to any semisimple Lie group other than $SU(N)$ and $G(2)$.

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I. INTRODUCTION

Understanding the mechanism underlying quark confinement from first principles of QCD is still a challenging problem in theoretical particle physics [1]. As a possible step towards this goal, it will be efficient to extract the dominant field mode \mathcal{V} responsible for confinement from the Yang-Mills field \mathcal{A} to clarify the physics behind the phenomena of confinement. The well-known mathematical identity called the Cartan decomposition [2] is used to decompose the field variable \mathcal{A} valued in the Lie algebra $\mathcal{G} = \text{Lie}(G)$ of a gauge group G into the simultaneously diagonalizable part in the Cartan subalgebra $\mathcal{H} = \text{Lie}(H)$ and the remaining off-diagonal part in the orthogonal complement of $\text{Lie}(H)$. However, the Cartan decomposition is not suited for studying the nonperturbative features of the gauge field theory with local gauge invariance, because the Cartan decomposition cannot retain the original form after the gauge transformation, namely, the local rotation of the Cartan-Weyl basis for the Lie algebra.

In view of this, the novel decomposition called the Cho-Duan-Ge-Faddeev-Niemi (CDGFN) decomposition [3–14] is quite attractive, because the CDGFN decomposition given in the form $\mathcal{A} = \mathcal{V} + \mathcal{X}$ is gauge covariant; i.e., it keeps its form under gauge transformation or local color rotation. In the CDGFN decomposition, the unit Lie-algebra-valued field \mathbf{n}_j called the color direction field, or the color field for short, plays a crucial role for retaining the local gauge covariance of the decomposition. For $G = SU(N)$, the color field $\mathbf{n}_j(x)$ is constructed from the maximally commuting generators H_j ($j = 1, \dots, \text{rank}G$) in the Cartan subalgebra $\mathcal{H} = \text{Lie}(H)$ according to the local adjoint rotation by a group element g of the gauge group G at every point x of spacetime,

$$\mathbf{n}_j(x) = g(x)H_jg^\dagger(x), \quad g(x) \in G. \quad (1)$$

The color direction field belongs to the subset of $\text{Lie}(G)$ that is topologically equivalent to G/\tilde{H} , where a subgroup \tilde{H} called the maximal stability subgroup of G is specified from the degeneracy among the eigenvalues of a representation matrix of H_j . In other words, the color field \mathbf{n}_j is regarded as the local embedding of the Cartan direction H_j in the internal space of the non-Abelian group G . From this viewpoint, the Cartan decomposition is identified with a global limit of the CDGFN decomposition; this urges us to consider that the Abelian projection method [15] is nothing but a gauge-fixed version of the gauge-covariant CDGFN decomposition. The application of the novel decomposition to the Yang-Mills non-Abelian gauge field paves the way for understanding quark confinement in a gauge-independent manner. In fact, this method has been extensively used to investigate quark confinement in the $SU(N)$ Yang-Mills theory in the last decade; see, e.g., [14] for a review.

A promising mechanism for understanding quark confinement is known as dual superconductivity [16]. It is a hypothesis based on the electromagnetic dual analog of the type II superconductor, in which the magnetic field applied to the bulk of the superconductor is squeezed to form the magnetic vortex due to the Meissner effect of excluding the magnetic field from the superconductor [17]. The color electric field created by a pair of a quark and an antiquark would be squeezed to form an electric flux tube or a hadron string with its ends on a quark and an antiquark. For the dual superconductivity to work, therefore, one needs magnetic objects (say, magnetic monopoles) to be condensed in the vacuum of the Yang-Mills theory, which is supposed to be dual to the ordinary superconductivity caused by condensation of electric objects consisting of a pair of electrons called the Cooper pair [17].

The magnetic monopole in the pure Yang-Mills theory has been mostly constructed by the Abelian projection

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method [15], which explicitly breaks the original non-Abelian gauge group G to the maximal torus subgroup H . However, this is not the only way to define magnetic monopoles in pure Yang-Mills theory without the Higgs field. In fact, one can define gauge-invariant magnetic monopoles in pure Yang-Mills theory without breaking the original gauge symmetry by using the non-Abelian Stokes theorem [18–25] for the Wilson loop operator, which is in itself gauge invariant [1]. The gauge-invariant magnetic monopole is specified by the maximal stability subgroup \tilde{H} , which is uniquely determined for the highest-weight state of a given representation for a quark source. For quarks in the fundamental representation of $SU(N)$, especially, the maximal stability subgroup \tilde{H} is given by $\tilde{H} = U(N-1)$, which is distinct from the maximal torus group $U(1)^{N-1}$ for $N \geq 3$.

The contribution of magnetic monopoles to the Wilson loop average can be calculated by using the path-integral framework, using the reformulation of the Yang-Mills theory based on change of variables in accord with the field decomposition. We find that there are some options for reformulating the $SU(N)$ Yang-Mills theory ($N \geq 3$) corresponding to different choices of the color direction field compatible with the maximal stability group, although there is only one way to reformulate the $SU(2)$ Yang-Mills theory. Whichever options we use, we need the field decomposition formula, which allows us to decompose an arbitrary element \mathcal{F} of a Lie algebra \mathcal{G} to the part $\mathcal{F}_{\tilde{H}}$ in the Lie algebra of \tilde{H} and the remaining part $\mathcal{F}_{G/\tilde{H}}$; see, e.g., [13,23] and also [14] for a review.

Another popular object of topological nature believed to be responsible for confinement is the center vortex [26,27], which is associated to the center subgroup of G . In fact, the confinement/deconfinement phase transition at finite temperature in the $SU(N)$ Yang-Mills theory is associated with the restoration/spontaneous breaking of the center symmetry $Z(N)$, which is signaled by vanishing/nonvanishing of the Polyakov loop average. See, e.g., [28] for reviews. We suppose, however, that magnetic monopoles and magnetic vortices cannot be independent topological objects. They could be different views of a single physical object, just like two sides of a coin, to be simultaneously defined in a self-consistent way [14,24]. However, this statement remains still a conjecture to be proved.

The purpose of this paper is to extend the gauge-covariant field decomposition of the Yang-Mills field and the non-Abelian Stokes theorem for the Wilson loop operator developed so far for $SU(N)$ to the exceptional group $G(2)$, which is a preliminary step toward reformulating the $G(2)$ Yang-Mills theory [29] to discuss the mechanism for confinement/deconfinement. Our interest in the exceptional group $G(2)$ lies in the fact that the $G(2)$ Yang-Mills theory has linear potential [30–32] and that the center vortex confinement mechanism has been argued to

work for $G(2)$ in [32], although $G(2)$ has a trivial center subgroup consisting only of the identity element [33,34]. We want to define the gauge-invariant magnetic monopole in the $G(2)$ Yang-Mills theory and then examine whether the magnetic monopole defined in our framework can be a universal topological object responsible for confinement, irrespective of the gauge group. This investigation will help us to prove or disprove the above conjecture on the interrelation between magnetic monopoles and magnetic vortices.

The present paper is organized as follows. In Sec. II, we first determine the maximal stability subgroup for a given representation of $G(2)$, after presenting some basic properties of $G(2)$. In Sec. III, we subsequently derive the gauge-covariant decomposition formula for the $G(2)$ Yang-Mills field corresponding to each maximal stability subgroup. We show that the $G(2)$ Yang-Mills field has different gauge-covariant decompositions depending on the maximal stability subgroup, which are more complicated than those obtained for the $SU(N)$ Yang-Mills field. In fact, it turns out that the decomposition formula for $G(2)$ cannot be obtained as a simple extension of that for $SU(N)$. This is because the fact that all roots of $SU(N)$ have the same norm was used in deriving the decomposition formula for $SU(N)$. However, some roots of $G(2)$ have different norms from the other roots. Consequently, the relevant decomposition for $G(2)$ cannot be obtained by using the double commutators, in sharp contrast to $SU(3)$. Nevertheless, multiple commutators with the Cartan generators enable us to obtain the desired decomposition. Remarkably, we have found that the decomposition formula for $G(2)$ can be obtained using sextuple commutators with the Cartan generators or the color fields. Moreover, the method presented in this paper for obtaining the decomposition formula for $G(2)$ can be applied to any semisimple Lie algebra, and therefore the reformulation of the Yang-Mills theory would be possible for an arbitrary semisimple gauge group.

In Sec. IV, we derive a non-Abelian Stokes theorem for the Wilson loop operator of the $G(2)$ Yang-Mills field, which is written using sextuple commutators with the color direction fields, as an application of the decomposition formula. This enables us to define gauge-invariant magnetic monopoles in the $G(2)$ Yang-Mills theory in Sec. V. What kind of magnetic monopoles can be defined is determined by the stability subgroup of G . We show that the magnetic charge derived from the gauge-invariant magnetic monopole is subject to a novel quantization condition, which is similar to, but different from, the quantization condition for the Dirac magnetic monopole and the 't Hooft–Polyakov magnetic monopole. The final section is devoted to conclusions and discussion. Some technical derivations are collected in Appendixes A, B, and C.

II. THE EXCEPTIONAL GROUP $G(2)$

A. Basic properties of $G(2)$

In this section, we give some basic properties of the exceptional group $G(2)$. We begin with the Dynkin diagram of $G(2)$ given by Fig. 1. It indicates that $G(2)$ has two simple roots (i.e., rank 2) with the opening angle $5\pi/3$. In this paper, we use

$$\begin{aligned} \alpha^1 &= \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) =: \alpha^{(1)}, \\ \alpha^2 &= \left(0, \frac{1}{\sqrt{3}}\right) =: \alpha^{(5)} \end{aligned} \quad (2)$$

as simple roots. We see that the other positive roots are obtained as

$$\begin{aligned} \alpha^1 + \alpha^2 &= \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) =: \alpha^{(6)}, \\ \alpha^1 + 2\alpha^2 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) =: \alpha^{(4)}, \\ \alpha^1 + 3\alpha^2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) =: \alpha^{(3)}, \\ 2\alpha^1 + 3\alpha^2 &= (1, 0) =: \alpha^{(2)}. \end{aligned} \quad (3)$$

Figure 2 is the root diagram of $G(2)$. Hence, there are two Cartan generators H_k ($k = 1, 2$) and twelve shift operators E_α ($\alpha \in \mathcal{R}$), where \mathcal{R} is the root system, i.e., the set of positive and negative root vectors. They satisfy the commutation relation called the Cartan standard form,

$$\begin{aligned} [H_j, H_k] &= 0 \quad (j, k = 1, 2), \\ [H_k, E_\alpha] &= \alpha_k E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= \alpha \cdot H \\ [E_\alpha, E_\beta] &\propto \begin{cases} E_{\alpha+\beta} & (\alpha + \beta \in \mathcal{R}) \\ 0 & (\text{otherwise}) \end{cases}, \end{aligned} \quad (4)$$

where α_k denotes the k th component of the root vector α and $\alpha \cdot H$ is the inner product defined by $\alpha \cdot H := \alpha^k H_k$. In this paper we consider a unitary representation. Therefore, representation matrices satisfy the Hermiticity

$$R(H_k)^\dagger = R(H_k), \quad R(E_\alpha)^\dagger = R(E_{-\alpha}), \quad (5)$$

and the normalization

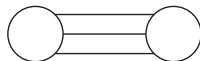


FIG. 1. The Dynkin diagram of $G(2)$.

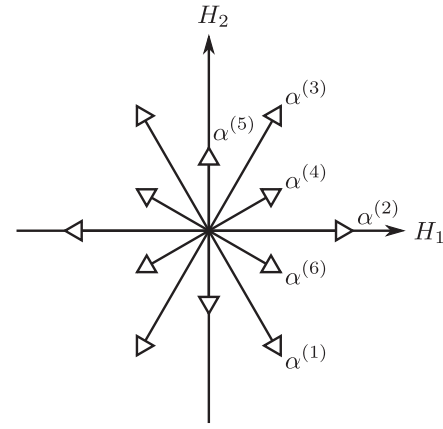


FIG. 2. The root diagram of $G(2)$.

$$\begin{aligned} \kappa \text{tr}(R(H_j)R(H_k)) &= \delta_{jk}, \\ \kappa \text{tr}(R(E_\alpha)R(E_\beta)) &= \begin{cases} 1 & (\beta = -\alpha) \\ 0 & (\text{otherwise}) \end{cases}, \end{aligned} \quad (6)$$

where the value of κ depends on the representation. Using this property, we can define the inner product in the Lie algebra as

$$(\mathcal{F}_1, \mathcal{F}_2) = \kappa \text{tr}(R(\mathcal{F}_1)R(\mathcal{F}_2)), \quad (7)$$

which is independent of the representation R .

The weight vector $\vec{\mu}$ of a representation specified by the Dynkin index $[m_1, \dots, m_r]$ of the Lie group with the rank r is obtained from the relation

$$\frac{2\vec{\alpha}^k \cdot \vec{\mu}}{\vec{\alpha}^k \cdot \vec{\alpha}^k} = m_k. \quad (8)$$

For $G(2)$, the highest-weight vector μ^j ($j = 1, 2$) of a representation with the Dynkin index $[1, 0]$ or $[0, 1]$ satisfies

$$\frac{2\alpha^k \cdot \mu^j}{\alpha^k \cdot \alpha^k} = \delta_{kj}, \quad (9)$$

and hence is determined as

$$\mu^1 = (1, 0), \quad \mu^2 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right). \quad (10)$$

The weight diagrams are determined by μ^j ($j = 1, 2$), as given in Fig. 3. The highest-weight vector μ^1 corresponds to the 14-dimensional adjoint representation **14** with the Dynkin index $[1, 0]$, while the highest-weight vector μ^2 corresponds to the 7-dimensional fundamental representation **7** with the Dynkin index $[0, 1]$. An arbitrary irreducible representation of $G(2)$ is labeled by the two Dynkin indices $[n, m]$ and its highest weight Λ can be written as

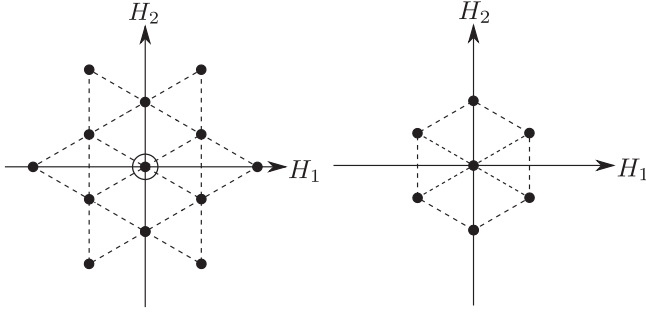


FIG. 3. The weight diagrams of the fundamental representations.

$$\Lambda = n\mu^1 + m\mu^2 = \left(\frac{2n+m}{2}, \frac{m}{2\sqrt{3}} \right). \quad (11)$$

Notice that $G(2)$ contains the Lie group $SU(3)$ as a subgroup. We can see from the root diagram that the Lie algebra $su(3)$ of $SU(3)$, denoted as $su(3) = \text{Lie}(SU(3))$, is generated by a set of elements in $su(3)$,

$$\begin{aligned} \{H_1, H_2, E_{\alpha^{(1)}}, E_{\alpha^{(2)}}, E_{\alpha^{(3)}}, E_{-\alpha^{(1)}}, E_{-\alpha^{(2)}}, E_{-\alpha^{(3)}}\} &\subset su(3) \\ &= \text{Lie}(SU(3)). \end{aligned} \quad (12)$$

Therefore, a representation of $G(2)$ is written as a direct sum of representations of $SU(3)$. For example, the fundamental representations of $G(2)$ are written as

$$7 = 3 + 3^* + 1, \quad (13)$$

$$14 = 8 + 3 + 3^*. \quad (14)$$

B. Maximal stability subgroups

It is known [13,14] that one can construct a number of reformulations of the Yang-Mills theory which are discriminated by the maximal stability subgroup. Therefore, it is important to know which subgroup is identified with the maximal stability subgroup for each representation. In view of this, we first derive a certain property to be satisfied by the generators belonging to the Lie algebra of the maximal stability subgroup of $G(2)$. By using this property, then, we determine the maximal stability subgroup for each representation of $G(2)$.

The maximal stability subgroup \tilde{H} for the representation R of a group G is defined to be a subgroup whose element $h \in \tilde{H}$ leaves the highest-weight state $|\Lambda\rangle$ of the representation R invariant up to a phase factor¹

¹Strictly speaking, we should write $R(h)|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}$, but if we do so the presentation become rather cumbersome. Therefore we omit $R(\cdot)$ throughout this subsection.

$$h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}. \quad (15)$$

Hence, an element of its Lie algebra $\tilde{h} = \text{Lie}(\tilde{H})$ can be written as a linear combination of the Cartan generators and shift-up and -down operators E_α and $E_{-\alpha}$ (where α is a positive root) such that $E_\alpha|\Lambda\rangle = 0$ and $E_{-\alpha}|\Lambda\rangle = 0$. Here notice that if there is E_α in the linear combination, then there is also $E_{-\alpha}$; that is to say, the mutually Hermitian-conjugate generators E_α and $E_{-\alpha}$ must appear in pairs in the linear combination, since all matrices in a unitary representation of the Lie algebra are Hermitian.

We show in the following that $E_\alpha|\Lambda\rangle = 0$ and $E_{-\alpha}|\Lambda\rangle = 0$ if and only if $\Lambda \cdot \alpha = 0$. Here, we should remember that $(\alpha \cdot H)/\alpha^2$ and $E_{\pm\alpha}/|\alpha|$ satisfy the commutation relations of $su(2)$. We see from this fact that if $\alpha \cdot H|\mu\rangle = 0$ and $E_\alpha|\mu\rangle = 0$, then $|\mu\rangle$ belongs to the space of the trivial representation of $SU(2)$ and, hence, $E_{-\alpha}|\mu\rangle = 0$. Because $|\Lambda\rangle$ is highest-weight state, $E_\alpha|\Lambda\rangle = 0$. Hence, if $\Lambda \cdot \alpha = 0$ then $E_{-\alpha}|\Lambda\rangle = 0$. In the same way, the converse can be proven.

Thus we arrive at the conclusion that $X \in \tilde{h}$ can be written as a linear combination of the Cartan generators H_j and shift operators $E_{\pm\alpha}$ with positive root vectors α that are orthogonal to the highest-weight vector Λ ,

$$\alpha \cdot \Lambda = 0. \quad (16)$$

Thus, it is easy to see that all representations of $G(2)$ are classified into the following three categories:

- (1) For the highest-weight $\Lambda = m\mu^2$, the positive root orthogonal to the highest weight is $\alpha^1 = \alpha^{(1)}$ alone. Hence, the maximal stability subgroup is a $U(2)$ with the generators $H_1, H_2, E_{\alpha^{(1)}}$, and $E_{-\alpha^{(1)}}$,

$$\tilde{H} = U(2);$$

$$\text{Lie}(U(2)) = u(2) \supset \{H_1, H_2, E_{\alpha^{(1)}}, E_{-\alpha^{(1)}}\}, \quad (17)$$

which agrees with a subset of $SU(3)$ specified by (12).

- (2) For the highest-weight $\Lambda = n\mu^1$, the positive root orthogonal to the highest weight is $\alpha^2 = \alpha^{(5)}$ alone. Hence, the maximal stability subgroup is another $U(2)$ with the generators $H_1, H_2, E_{\alpha^{(5)}}$, and $E_{-\alpha^{(5)}}$,

$$\tilde{H} = U'(2);$$

$$\text{Lie}(U'(2)) = u'(2) \supset \{H_1, H_2, E_{\alpha^{(5)}}, E_{-\alpha^{(5)}}\}, \quad (18)$$

which differs from a subset of $SU(3)$ specified by (12).

- (3) For the highest-weight $\Lambda = n\mu^1 + m\mu^2$ ($n \neq 0 \neq m$), the maximal stability subgroup is equal to the maximal torus subgroup $U(1) \times U(1)$ generated by the Cartan subalgebra $\{H_1, H_2\}$,

$$\begin{aligned} \tilde{H} &= U(1) \times U(1); \\ \text{Lie}(U(1) \times U(1)) &\supset \{H_1, H_2\}. \end{aligned} \quad (19)$$

This fact is confirmed as follows. We can write any positive root as $k\alpha^1 + l\alpha^2$, where k and l are non-negative integers that are not zero simultaneously. Hence, the relation $\Lambda \cdot \alpha = (n\mu^1 + m\mu^2) \cdot (k\alpha^1 + l\alpha^2) = nk + ml$ implies that all positive roots are not orthogonal to the highest weight when $n \neq 0$ and $m \neq 0$. Thus, in this case, the generators of the maximal stability subgroup are given by H_1 and H_2 .

III. DECOMPOSITION FORMULA

Let \mathcal{F} be an arbitrary element of the Lie algebra. To write the Wilson loop using the color direction fields, and to reformulate the $G(2)$ Yang-Mills theory, we have to decompose \mathcal{F} into the part $\mathcal{F}_{\tilde{H}}$ belonging to $\tilde{h} = \text{Lie}(\tilde{H})$ and the remaining part $\mathcal{F}_{G/\tilde{H}}$ using its commutators with H_k . This is achieved by using double commutators in the case of $SU(N)$, see [14]. However, in the case of $G(2)$, we have to use sextuple commutators. The proof of this is given in Appendix A. In this section, we give the explicit form of such a decomposition for any representation.

A. Decomposing $SU(3)$

Before proceeding to the $G(2)$ case, we reconsider the $SU(3)$ case from the viewpoint of this paper. For $SU(3)$, it is known [14] that the maximal stability subgroup is $U(2)$ or $U(1) \times U(1)$. In the case of the maximal stability subgroup $U(2)$ with generators $H_1, H_2, E_{\alpha^{(2)}}$, and $E_{-\alpha^{(2)}}$, the decomposition formula is written as

$$\mathcal{F}_{G/\tilde{H}} = \frac{4}{3} [H_2, [H_2, \mathcal{F}]], \quad (20)$$

while in the case of the maximal stability subgroup $U(1) \times U(1)$, the decomposition formula is written as

$$\mathcal{F}_{G/\tilde{H}} = \sum_{j=1,2} [H_j, [H_j, \mathcal{F}]]. \quad (21)$$

The derivation of these formulas is written in Appendixes C and D in [14]. We rederive them using another method which can be applied also to $G(2)$. First, we consider the commutator of an arbitrary element of the Cartan subalgebra with \mathcal{F} . An arbitrary element of the Cartan subalgebra can be written as $\nu \cdot H$, where ν is an arbitrary two-dimensional vector. Using the Cartan decomposition

$$\mathcal{F} = \sum_{j=1,2} \mathcal{F}_j H_j + \sum_{\alpha \in \mathcal{R}_+} (\mathcal{F}_\alpha^* E_\alpha + \mathcal{F}_\alpha E_{-\alpha}) \quad (22)$$

and the commutation relation (4), we can write the commutator as

$$[\nu \cdot H, \mathcal{F}] = \sum_{\alpha \in \mathcal{R}_+} (\nu \cdot \alpha) (\mathcal{F}_\alpha^* E_\alpha - \mathcal{F}_\alpha E_{-\alpha}). \quad (23)$$

Here, we choose $\gamma^1 := (\sqrt{3}/2, 1/2)$ as ν , which is orthogonal to $\alpha^{(1)}$, i.e., $\gamma^1 \cdot \alpha^{(1)} = 0$. Then the commutator reads

$$\begin{aligned} [\gamma^1 \cdot H, \mathcal{F}] &= (\gamma^1 \cdot \alpha^{(2)}) (\mathcal{F}_{\alpha^{(2)}}^* E_{\alpha^{(2)}} - \mathcal{F}_{\alpha^{(2)}} E_{-\alpha^{(2)}}) \\ &\quad + (\gamma^1 \cdot \alpha^{(3)}) (\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} - \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}}), \end{aligned} \quad (24)$$

where the terms corresponding to $\alpha^{(1)}$ disappear. By taking the commutator once more, we can eliminate another term. For the vector $\gamma^2 := (0, 1)$, which is orthogonal to $\alpha^{(2)}$, i.e., $\gamma^2 \cdot \alpha^{(2)} = 0$, the double commutator is written as

$$\begin{aligned} [\gamma^2 \cdot H, [\gamma^1 \cdot H, \mathcal{F}]] &= (\gamma^2 \cdot \alpha^{(3)}) (\gamma^1 \cdot \alpha^{(3)}) \\ &\quad \times (\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} + \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}}) \\ &= \frac{3}{4} (\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} + \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}}). \end{aligned} \quad (25)$$

Thus we obtain

$$\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} + \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}} = \frac{4}{3} [\gamma^2 \cdot H, [\gamma^1 \cdot H, \mathcal{F}]]. \quad (26)$$

In this way, we can extract the element of \mathcal{F} corresponding to a particular positive root by taking the double commutator. For the other positive roots, a similar identity holds,

$$\mathcal{F}_{\alpha^{(1)}}^* E_{\alpha^{(1)}} + \mathcal{F}_{\alpha^{(1)}} E_{-\alpha^{(1)}} = -\frac{4}{3} [\gamma^3 \cdot H, [\gamma^2 \cdot H, \mathcal{F}]], \quad (27)$$

$$\mathcal{F}_{\alpha^{(2)}}^* E_{\alpha^{(2)}} + \mathcal{F}_{\alpha^{(2)}} E_{-\alpha^{(2)}} = \frac{4}{3} [\gamma^1 \cdot H, [\gamma^3 \cdot H, \mathcal{F}]], \quad (28)$$

where we have introduced $\gamma^3 := (\sqrt{3}/2, -1/2)$, which is orthogonal to $\alpha^{(3)}$. Using these expressions, we can write the decomposition formula for any case of the maximal stability subgroup. For $\tilde{H} = U(2)$, the decomposition formula is written as

$$\begin{aligned} \mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + (\mathcal{F}_{\alpha^{(2)}}^* E_{\alpha^{(2)}} + \mathcal{F}_{\alpha^{(2)}} E_{-\alpha^{(2)}}) \\ &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + [H_1, [H_1, \mathcal{F}]] - \frac{1}{3} [H_2, [H_2, \mathcal{F}]] \\ \mathcal{F}_{G/\tilde{H}} &= (\mathcal{F}_{\alpha^{(1)}}^* E_{\alpha^{(1)}} + \mathcal{F}_{\alpha^{(1)}} E_{-\alpha^{(1)}}) \\ &\quad + (\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} + \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}}) \\ &= \frac{4}{3} [H_2, [H_2, \mathcal{F}]], \end{aligned} \quad (29)$$

while for $\tilde{H} = U(1) \times U(1)$, the decomposition formula is written as

$$\begin{aligned}
\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j \\
\mathcal{F}_{G/\tilde{H}} &= (\mathcal{F}_{\alpha^{(1)}}^* E_{\alpha^{(1)}} + \mathcal{F}_{\alpha^{(1)}} E_{-\alpha^{(1)}}) \\
&\quad + (\mathcal{F}_{\alpha^{(2)}}^* E_{\alpha^{(2)}} + \mathcal{F}_{\alpha^{(2)}} E_{-\alpha^{(2)}}) \\
&\quad + (\mathcal{F}_{\alpha^{(3)}}^* E_{\alpha^{(3)}} + \mathcal{F}_{\alpha^{(3)}} E_{-\alpha^{(3)}}) \\
&= \sum_{j=1,2} [H_j, [H_j, \mathcal{F}]], \tag{30}
\end{aligned}$$

where we have used the commuting property

$$\begin{aligned}
[H_j, [H_k, \mathcal{F}]] &= [[H_j, H_k], \mathcal{F}] + [H_k, [H_j, \mathcal{F}]] \\
&= [H_k, [H_j, \mathcal{F}]] \tag{31}
\end{aligned}$$

following from $[H_j, H_k] = 0$.

B. Decomposing $G(2)$

Now we consider the $G(2)$ case. We want to write the coset part $\mathcal{F}_{G/\tilde{H}}$ of \mathcal{F} as a linear combination of multiple commutators with the Cartan generators,

$$\mathcal{F}_{G/\tilde{H}} = \sum_{j_1, \dots, j_n \in \{1,2\}} \eta_{j_1 \dots j_n} [H_{j_1}, \dots, [H_{j_n}, \mathcal{F}]] \dots, \tag{32}$$

where the sum is over independent terms by taking account of the commuting property (31). We can obtain (32) for any choice of \tilde{h} if, for every positive root β , the relevant shift part \mathcal{R}_β is written in the form

$$\mathcal{R}_\beta := \mathcal{F}_\beta^* E_\beta + \mathcal{F}_\beta E_{-\beta} = \sum_{j_1, \dots, j_n} \tilde{\eta}_{j_1 \dots j_n} [H_{j_1}, \dots, [H_{j_n}, \mathcal{F}]] \dots. \tag{33}$$

In the following, we give a derivation of (33). In a way similar to that written above for $SU(3)$, we can indeed obtain (33). By using the commutation relation (4), the commutator is calculated as

$$[\nu \cdot H, \mathcal{F}] = \sum_{\alpha \in \mathcal{R}_+} (\nu \cdot \alpha) (\mathcal{F}_\alpha^* E_\alpha - \mathcal{F}_\alpha E_{-\alpha}). \tag{34}$$

If ν is chosen to be orthogonal to a particular α , then the corresponding terms of E_α and $E_{-\alpha}$ disappear from this expression. Thus, by taking the commutator repeatedly, we can eliminate all shift terms except one, \mathcal{R}_β , which corresponds to a particular positive root β . Because there are six positive roots in $G(2)$, we have to eliminate five shift terms. We can do so using the quintuple commutator

$$\begin{aligned}
&[[\nu^1 \cdot H, [\nu^2 \cdot H, [\nu^3 \cdot H, [\nu^4 \cdot H, [\nu^5 \cdot H, \mathcal{F}]]]]]] \\
&= (\nu^1 \cdot \beta)(\nu^2 \cdot \beta)(\nu^3 \cdot \beta)(\nu^4 \cdot \beta)(\nu^5 \cdot \beta) (\mathcal{F}_\beta^* E_\beta - \mathcal{F}_\beta E_{-\beta}), \tag{35}
\end{aligned}$$

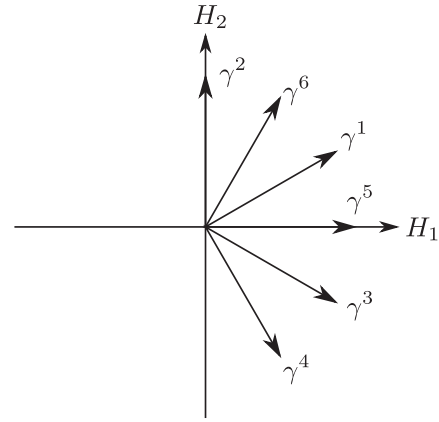


FIG. 4. Unit vectors that are orthogonal to one of the positive roots.

where ν^1, \dots, ν^5 are appropriate two-dimensional vectors.² In this expression, the sign of the term of E_β is opposite to that of $E_{-\beta}$. To make both signs equal, we need to take the commutator once more. We choose a two-dimensional vector ν which is nonorthogonal to β to obtain the non-vanishing commutator of $\nu \cdot H$ and (35),

$$\begin{aligned}
[\nu \cdot H, (35)] &= (\nu \cdot \beta)(\nu^1 \cdot \beta)(\nu^2 \cdot \beta)(\nu^3 \cdot \beta) \\
&\quad \times (\nu^4 \cdot \beta)(\nu^5 \cdot \beta) (\mathcal{F}_\beta^* E_\beta + \mathcal{F}_\beta E_{-\beta}). \tag{36}
\end{aligned}$$

Thus we obtain the key relation

$$\begin{aligned}
\mathcal{R}_\beta &= \frac{1}{N} [\nu \cdot H, [\nu^1 \cdot H, [\nu^2 \cdot H, [\nu^3 \cdot H, [\nu^4 \cdot H, [\nu^5 \cdot H, \mathcal{F}]]]]]], \\
N &:= (\nu \cdot \beta)(\nu^1 \cdot \beta)(\nu^2 \cdot \beta)(\nu^3 \cdot \beta)(\nu^4 \cdot \beta)(\nu^5 \cdot \beta). \tag{37}
\end{aligned}$$

Although this expression is nothing but the desired one [Eq. (33)], it should be remarked that the coefficients $\tilde{\eta}_{j_1 \dots j_6}$ are not uniquely determined. If we multiply ν by a constant, the coefficients $\tilde{\eta}_{j_1 \dots j_6}$ do not change. This point will be observed more concretely shortly.

To obtain the expression (37) for each positive root concretely, we introduce six unit vectors γ^a ($a = 1, \dots, 6$) such that γ^a is positive and orthogonal to one of the positive roots, say $\alpha^{(a)}$ ($a = 1, \dots, 6$),

$$\begin{aligned}
\gamma^1 &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right), & \gamma^2 &= (0, 1), & \gamma^3 &= \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right), \\
\gamma^4 &= \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), & \gamma^5 &= (1, 0), & \gamma^6 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \tag{38}
\end{aligned}$$

²The reason why we need a quintuple commutator is that we can eliminate only one term by taking the commutator once. This is because the roots of $G(2)$ are two-dimensional. If the dimension of the root vectors is larger, we can eliminate more than one term by taking the commutator once.

see Fig. 4. Consequently, these vectors satisfy the following conditions:

$$\gamma^a \perp \alpha^{(a)}, \quad \gamma^4 \parallel \alpha^{(1)}, \quad \gamma^5 \parallel \alpha^{(2)}, \quad \gamma^6 \parallel \alpha^{(3)}, \quad \gamma^1 \parallel \alpha^{(4)}, \quad \gamma^2 \parallel \alpha^{(5)}, \quad \gamma^3 \parallel \alpha^{(6)}. \quad (39)$$

For example, $\mathcal{R}_{\alpha^{(1)}}$ is obtained as

$$\begin{aligned} \mathcal{R}_{\alpha^{(1)}} &= N^{-1} [\gamma \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]], \\ N &= (\gamma \cdot \alpha^{(1)})(\gamma^2 \cdot \alpha^{(1)})(\gamma^3 \cdot \alpha^{(1)})(\gamma^4 \cdot \alpha^{(1)})(\gamma^5 \cdot \alpha^{(1)})(\gamma^6 \cdot \alpha^{(1)}), \end{aligned} \quad (40)$$

where γ is an arbitrary two-dimensional vector that is not orthogonal to $\alpha^{(1)}$.

To obtain more explicit form, we put an arbitrary two-dimensional vector γ in the form $\gamma = a\gamma^1 + b\gamma^4$ (orthogonal decomposition of γ), where γ^1 is orthogonal to $\alpha^{(1)}$ and γ^4 is parallel to $\alpha^{(1)}$: $\gamma^1 \perp \alpha^{(1)}$ and $\gamma^4 \parallel \alpha^{(1)}$. Here $b \neq 0$ to avoid $\gamma \cdot \alpha^{(1)} \equiv b\gamma^4 \cdot \alpha^{(1)} = 0$. Using $\gamma^1 \cdot \alpha^{(1)} = 0$, we have $\gamma^2 \cdot \alpha^{(1)} = -\sqrt{3}|\alpha^{(1)}|/2$, $\gamma^3 \cdot \alpha^{(1)} = \sqrt{3}|\alpha^{(1)}|/2$, $\gamma^4 \cdot \alpha^{(1)} = |\alpha^{(1)}|$, $\gamma^5 \cdot \alpha^{(1)} = |\alpha^{(1)}|/2$, $\gamma^6 \cdot \alpha^{(1)} = -|\alpha^{(1)}|/2$, and $\gamma \cdot \alpha^{(1)} = b\gamma^4 \cdot \alpha^{(1)} = b|\alpha^{(1)}|$, which yields

$$N = b \frac{3}{16} |\alpha^{(1)}|^6 = \frac{3b}{16}. \quad (41)$$

Combining the result $N^{-1} = \frac{16}{3b}$ with $\gamma \cdot H = a\gamma^1 \cdot H + b\gamma^4 \cdot H$, therefore, we obtain

$$\mathcal{R}_{\alpha^{(1)}} = \frac{16}{3} [\gamma^4 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] + c_1 [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]], \quad (42)$$

where $c_1 = 16a/3b$. We can see from this expression that the nonuniqueness of an expression of $\mathcal{R}_{\alpha^{(1)}}$ comes from the fact that the following sextuple commutator is identically vanishing:

$$\begin{aligned} \mathcal{Z} := [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] &= \sum_{j_1, \dots, j_6 \in 1,2} \bar{\zeta}_{j_1 \dots j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] = 0, \\ \bar{\zeta}_{111112} &= \frac{3}{16}, \bar{\zeta}_{111122} = -\frac{5}{8}, \bar{\zeta}_{122222} = \frac{3}{16}. \end{aligned} \quad (43)$$

Thus, the nonuniqueness of the decomposition formula is attributed to degree of freedom due to one parameter c_1 .

In the same way as the above, we obtain

$$\mathcal{R}_{\alpha^{(2)}} = \frac{16}{3} [\gamma^5 \cdot H, [\gamma^1 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] + c_2 \mathcal{Z}, \quad (44)$$

$$\mathcal{R}_{\alpha^{(3)}} = -\frac{16}{3} [\gamma^6 \cdot H, [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] + c_3 \mathcal{Z}, \quad (45)$$

$$\mathcal{R}_{\alpha^{(4)}} = \frac{16}{3} (\sqrt{3})^6 [\gamma^1 \cdot H, [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^5 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] + c_4 \mathcal{Z}, \quad (46)$$

$$\mathcal{R}_{\alpha^{(5)}} = \frac{16}{3} (\sqrt{3})^6 [\gamma^2 \cdot H, [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^6 \cdot H, \mathcal{F}]]]]]] + c_5 \mathcal{Z}, \quad (47)$$

$$\mathcal{R}_{\alpha^{(6)}} = -\frac{16}{3} (\sqrt{3})^6 [\gamma^3 \cdot H, [\gamma^1 \cdot H, [\gamma^2 \cdot H, [\gamma^3 \cdot H, [\gamma^4 \cdot H, [\gamma^5 \cdot H, \mathcal{F}]]]]]] + c_6 \mathcal{Z}, \quad (48)$$

where c_2, \dots, c_6 are arbitrary constants. Here, we have used the commuting property (31).

Collecting an appropriate set of \mathcal{R}_{α} , we obtain the desired decomposition formula corresponding to each category of representations given in (17)–(19):

- (1) For $\tilde{H} = U(2) \subset SU(3)$ with the generators $H_1, H_2, E_{\alpha^{(1)}}$, and $E_{-\alpha^{(1)}}$, the highest weight is $\Lambda = (m/2, m/(2\sqrt{3}))$. Here, we redefine the highest weight as $(0, m/\sqrt{3})$ in order to obtain a simpler form. This is possible because the new one is obtained by acting a Weyl group element on an old one. Thus \tilde{H} has the generators $H_1, H_2, E_{\alpha^{(2)}}$, and $E_{-\alpha^{(2)}}$. The \tilde{H} -commutative part $\mathcal{F}_{\tilde{H}}$ and the coset part $\mathcal{F}_{G/\tilde{H}}$ of \mathcal{F} are given by

$$\begin{aligned}\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + \mathcal{R}_{\alpha^{(2)}}, \\ \mathcal{F}_{G/\tilde{H}} &= \mathcal{R}_{\alpha^{(1)}} + \mathcal{R}_{\alpha^{(3)}} + \mathcal{R}_{\alpha^{(4)}} + \mathcal{R}_{\alpha^{(5)}} + \mathcal{R}_{\alpha^{(6)}}.\end{aligned}\quad (49)$$

The explicit form is given by

$$\begin{aligned}\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + \sum_{j_1, \dots, j_6} \tilde{\zeta}_{j_1, \dots, j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] + c_2 \mathcal{Z}, \\ \tilde{\zeta}_{111111} &= 1, \tilde{\zeta}_{111122} = -\frac{10}{3}, \tilde{\zeta}_{112222} = 1, \\ \mathcal{F}_{G/\tilde{H}} &= \sum_{j_1, \dots, j_6} \zeta_{j_1, \dots, j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] + c \mathcal{Z}, \\ \zeta_{111122} &= \frac{721}{3}, \zeta_{112222} = -154, \zeta_{222222} = 27,\end{aligned}\quad (50)$$

where the other ζ_{j_1, \dots, j_6} s and $\tilde{\zeta}_{j_1, \dots, j_6}$ are zero and $c := c_1 + c_3 + c_4 + c_5 + c_6$. The simplest choice is $c = 0$ and $c_2 = 0$.

- (2) For $\tilde{H} = U'(2) (\not\subset SU(3))$ with the generators $H_1, H_2, E_{\alpha^{(5)}}$, and $E_{-\alpha^{(5)}}$,

$$\begin{aligned}\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + \mathcal{R}_{\alpha^{(5)}}, \\ \mathcal{F}_{G/\tilde{H}} &= \mathcal{R}_{\alpha^{(1)}} + \mathcal{R}_{\alpha^{(2)}} + \mathcal{R}_{\alpha^{(3)}} + \mathcal{R}_{\alpha^{(4)}} + \mathcal{R}_{\alpha^{(6)}}.\end{aligned}\quad (51)$$

We obtain

$$\begin{aligned}\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j + \sum_{j_1, \dots, j_6} \tilde{\zeta}'_{j_1, \dots, j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] + c_5 \mathcal{Z}, \\ \tilde{\zeta}'_{111122} &= 27, \tilde{\zeta}'_{112222} = -90, \tilde{\zeta}'_{222222} = 27, \\ \mathcal{F}_{G/\tilde{H}} &= \sum_{j_1, \dots, j_6} \zeta'_{j_1, \dots, j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] + c' \mathcal{Z}, \\ \zeta'_{111111} &= 1, \zeta'_{111122} = 210, \zeta'_{112222} = -63,\end{aligned}\quad (52)$$

where the other ζ'_{j_1, \dots, j_6} s and $\tilde{\zeta}'_{j_1, \dots, j_6}$ are zero. We can take the simplest choice $c' = 0$ and $c_5 = 0$.

- (3) For $\tilde{H} = U(1) \times U(1)$,

$$\begin{aligned}\mathcal{F}_{\tilde{H}} &= \sum_{j=1,2} (\mathcal{F}, H_j) H_j, \\ \mathcal{F}_{G/\tilde{H}} &= \mathcal{R}_{\alpha^{(1)}} + \mathcal{R}_{\alpha^{(2)}} + \mathcal{R}_{\alpha^{(3)}} + \mathcal{R}_{\alpha^{(4)}} + \mathcal{R}_{\alpha^{(5)}} + \mathcal{R}_{\alpha^{(6)}}.\end{aligned}\quad (53)$$

We obtain

$$\begin{aligned}\mathcal{F}_{G/\tilde{H}} &= \sum_{j_1, \dots, j_6} \zeta''_{j_1 \dots j_6} [H_{j_1}, [H_{j_2}, [H_{j_3}, [H_{j_4}, [H_{j_5}, [H_{j_6}, \mathcal{F}]]]]]] + c'' \mathcal{Z}, \\ \zeta''_{111111} &= 1, \zeta''_{111122} = 237, \zeta''_{112222} = -153, \zeta''_{222222} = 27,\end{aligned}\quad (54)$$

where the other $\zeta''_{j_1 \dots j_6}$ s are zero. We can take the simplest choice $c'' = 0$.

Using the decomposition formula, we can define the field decomposition in a similar way to the case of the gauge group $SU(N)$. For this purpose, we define the color direction field for $G(2)$ as

$$\mathbf{n}_j(x) := \text{Ad}_{g(x)}(H_j), \quad (55)$$

where $\text{Ad}_{g(x)}$ is the adjoint representation of $g(x)$ which is an arbitrary group-valued field.³ For any Lie-algebra-valued field $\mathcal{F}(x)$, by applying the decomposition formula to $\text{Ad}_{g^{-1}(x)}(\mathcal{F}(x))$ and operating $\text{Ad}_{g(x)}$ on the both sides, we can decompose $\mathcal{F}(x)$ into the part $\mathcal{F}_{\tilde{H}}(x)$ belonging to $\text{Ad}_{g(x)}(\tilde{\mathfrak{h}})$ and the remaining part $\mathcal{F}_{G/\tilde{H}}(x)$,

$$\begin{aligned}\mathcal{F}(x) &= \mathcal{F}_{\tilde{H}}(x) + \mathcal{F}_{G/\tilde{H}}(x), \\ \mathcal{F}_{\tilde{H}}(x) &= \sum_{j=1,2} (\mathcal{F}(x), \mathbf{n}_j(x)) \mathbf{n}_j(x) + \sum_{j_1, \dots, j_6} \xi_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathbf{n}_{j_6}(x), \mathcal{F}(x)]] \dots], \\ \mathcal{F}_{G/\tilde{H}}(x) &= \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathbf{n}_{j_6}(x), \mathcal{F}(x)]] \dots],\end{aligned}\quad (56)$$

where ξ_{j_1, \dots, j_6} and η_{j_1, \dots, j_6} are appropriate coefficients specified by the maximal stability subgroup. We decompose the Yang-Mills field $\mathcal{A}_\mu(x)$ into two pieces, $\mathcal{V}_\mu(x)$ and $\mathcal{X}_\mu(x)$,

$$\mathcal{A}_\mu(x) = \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x), \quad (57)$$

where the decomposed fields $\mathcal{V}_\mu(x)$ and $\mathcal{X}_\mu(x)$ are obtained as the solution of the defining equations,

$$0 = \mathcal{D}_\mu[\mathcal{V}] \mathbf{n}_j(x) := \partial_\mu \mathbf{n}_j(x) - ig_{\text{YM}}[\mathcal{V}_\mu(x), \mathbf{n}_j(x)], \quad (58)$$

$$0 = \mathcal{X}_\mu(x)_{\tilde{H}} \Leftrightarrow \mathcal{X}_\mu(x) = \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathbf{n}_{j_6}(x), \mathcal{X}_\mu(x)]] \dots]. \quad (59)$$

Using the first defining equation (58), we find

$$\mathcal{D}_\mu[\mathcal{A}] \mathbf{n}_j(x) = \mathcal{D}_\mu[\mathcal{V}] \mathbf{n}_j(x) - ig_{\text{YM}}[\mathcal{X}_\mu(x), \mathbf{n}_j(x)] = ig_{\text{YM}}[\mathbf{n}_j(x), \mathcal{X}_\mu(x)]. \quad (60)$$

By substituting this relation into the second defining equation (59), $\mathcal{X}_\mu(x)$ is rewritten as

$$\mathcal{X}_\mu(x) = -ig_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), \mathcal{D}_\mu[\mathcal{A}] \mathbf{n}_{j_6}(x)]] \dots]. \quad (61)$$

Then $\mathcal{V}_\mu(x)$ is written as

³This definition of the color direction field is consistent with the definition adopted in the previous works for the gauge group $SU(N)$ because

$$R(\mathbf{n}_j(x)) = R(\text{Ad}_{g(x)}(H_j)) = R(g(x))R(H_j)R(g(x))^\dagger.$$

Here we have used the same notation R to denote the group representation and the corresponding algebra representation, which does not cause the confusion because the domains are different.

$$\begin{aligned}\mathcal{V}_\mu(x) &= \mathcal{A}_\mu(x) - \mathcal{X}_\mu(x) \\ &= \mathcal{A}_\mu(x) + ig_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), \mathcal{D}_\mu[\mathcal{A}]\mathbf{n}_{j_6}(x)]] \dots.\end{aligned}\quad (62)$$

Thus $\mathcal{V}_\mu(x)$ and $\mathcal{X}_\mu(x)$ are written in terms of the original Yang-Mills field $\mathcal{A}_\mu(x)$ and the color fields $\mathbf{n}_j(x)$.

Notice that $\mathcal{V}_\mu(x)$ is further cast into

$$\begin{aligned}\mathcal{V}_\mu(x) &= \mathcal{A}_\mu(x) + \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathcal{A}_\mu(x), \mathbf{n}_{j_6}(x)]]] \dots \\ &\quad + ig_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), \partial_\mu \mathbf{n}_{j_6}(x)]] \dots \\ &= \sum_{j=1,2} (\mathcal{A}_\mu(x), \mathbf{n}_j(x)) \mathbf{n}_j(x) + \sum_{j_1, \dots, j_6} \xi_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathbf{n}_{j_6}(x), \mathcal{A}_\mu(x)]]] \dots \\ &\quad + ig_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), \partial_\mu \mathbf{n}_{j_6}(x)]] \dots,\end{aligned}\quad (63)$$

where we have applied the formula (56) to $\mathcal{A}_\mu(x)$ in the last step. Therefore, $\mathcal{V}_\mu(x)$ is decomposed into $\mathcal{C}_\mu(x)$ and $\mathcal{B}_\mu(x)$,

$$\begin{aligned}\mathcal{V}_\mu(x) &= \mathcal{C}_\mu(x) + \mathcal{B}_\mu(x) \\ \mathcal{C}_\mu(x) &:= \sum_{j=1,2} (\mathcal{A}_\mu(x), \mathbf{n}_j(x)) \mathbf{n}_j(x) + \sum_{j_1, \dots, j_6} \xi_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), [\mathbf{n}_{j_6}(x), \mathcal{A}_\mu(x)]]] \dots, \\ \mathcal{B}_\mu(x) &:= ig_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \eta_{j_1 \dots j_6} [\mathbf{n}_{j_1}(x), \dots, [\mathbf{n}_{j_5}(x), \partial_\mu \mathbf{n}_{j_6}(x)]] \dots.\end{aligned}\quad (64)$$

For the sake of convenience, we define the field $\mathbf{m}(x)$ for the highest-weight $\Lambda = (\Lambda_1, \Lambda_2)$ by

$$\mathbf{m}(x) := \Lambda_j \mathbf{n}_j(x). \quad (65)$$

Here $\mathcal{C}_\mu(x)$ commutes with $\mathbf{m}(x)$,

$$[\mathcal{C}_\mu(x), \mathbf{m}(x)] = 0, \quad (66)$$

while $\mathcal{B}_\mu(x)$ is orthogonal to $\mathbf{n}_j(x)$,

$$(\mathcal{B}_\mu(x), \mathbf{n}_j(x)) = 0. \quad (67)$$

The first term in the right-hand side of $\mathcal{C}_\mu(x)$ corresponds to the element of the Cartan subalgebra $\text{Lie}(H)$ and the second term to the remaining part $\text{Lie}(\tilde{H}) - \text{Lie}(H)$, which vanishes when the maximal stability group coincides with the maximal torus group $\tilde{H} = H$. [This is the case for the maximal option of $SU(N)$]. Notice that $\mathcal{B}_\mu(x)$ is the extension of the $SU(N)$ Cho connection to $G(2)$. An appropriate set of the above fields will be used in the reformulation of the $G(2)$ Yang-Mills theory.

We suppose that the dominant mode for quark confinement is the restricted field $\mathcal{V}_\mu(x)$ extracted from the original $G(2)$ Yang-Mills field $\mathcal{A}_\mu(x)$ through the decomposition given above. In fact, this observation is exemplified for the $G(2)$ Wilson loop operator by using the

non-Abelian Stokes theorem in the same manner as in $SU(N)$, as given in the next section.

IV. NON-ABELIAN STOKES THEOREM

In this section, we derive the non-Abelian Stokes theorem for the Wilson loop operator in an arbitrary representation of the $G(2)$ gauge group using the color direction fields \mathbf{n}_k .

A. General gauge group

Before proceeding to the case of the gauge group $G(2)$, we discuss the general case.

It is known [14,23,25] that the Wilson loop operator defined for any Lie algebra valued Yang-Mills field \mathcal{A} and the irreducible (unitary) representation R is cast into the following (path-integral) representation⁴:

⁴Strictly speaking, we should write $F_{\mu\nu}^g$ in (68) as

$$\begin{aligned}F_{\mu\nu}^g &= \kappa \{ \partial_\mu \text{tr}(R(\mathbf{m}(x))R(\mathcal{A}_\nu(x))) - \partial_\nu \text{tr}(R(\mathbf{m}(x))R(\mathcal{A}_\mu(x))) \\ &\quad + ig_{\text{YM}} \text{tr}(R(\mathbf{m}(x))([\Omega_\mu(x), \Omega_\nu(x)])) \}, \\ R(\mathbf{m}(x)) &= \Lambda_j R(g(x)) R(H_j) R(g(x))^\dagger = \Lambda_j R(\text{Ad}_{g(x)}(H_j)), \\ \Omega_\mu(x) &= ig_{\text{YM}}^{-1} R(g(x)) \partial_\mu R(g(x))^\dagger.\end{aligned}$$

To simplify the notation, we omit the symbol $R(\cdot)$ throughout this section, and in Appendixes B and C.

$$\begin{aligned}
W_C[\mathcal{A}] &= \int [d\mu(g)]_\Sigma \exp \left[-ig_{\text{YM}} \int_{\Sigma: \partial\Sigma=C} F^g \right], \\
F^g &= \frac{1}{2} F_{\mu\nu}^g dx^\mu \wedge dx^\nu \\
F_{\mu\nu}^g(x) &= \kappa \{ \partial_\mu \text{tr}(\mathbf{m}(x) \mathcal{A}_\nu(x)) - \partial_\nu \text{tr}(\mathbf{m}(x) \mathcal{A}_\mu(x)) + ig_{\text{YM}} \text{tr}(\mathbf{m}(x) [\Omega_\mu(x), \Omega_\nu(x)]) \}, \\
\Omega_\mu(x) &:= ig_{\text{YM}}^{-1} g(x) \partial_\mu g^\dagger(x), \quad g(x) \in G,
\end{aligned} \tag{68}$$

where $[d\mu(g)]_\Sigma$ is the product measure of the Haar measure on the gauge group G over Σ and Λ in $\mathbf{m} := \Lambda_j \mathbf{n}_j$ is the highest-weight vector of the representation R . Here the gauge-invariant field strength $F_{\mu\nu}^g$ is equal to the non-Abelian field strength $\mathcal{F}_{\mu\nu}[\mathcal{V}] := \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu - ig_{\text{YM}} [\mathcal{V}_\mu, \mathcal{V}_\nu]$ of the restricted field \mathcal{V}_μ (in the decomposition $\mathcal{A} = \mathcal{V} + \mathcal{X}$) projected to the color field \mathbf{m} ,

$$F_{\mu\nu}^g = \kappa \text{tr} \{ \mathbf{m} \mathcal{F}_{\mu\nu}[\mathcal{V}] \} = \Lambda_j f_{\mu\nu}^{(j)}, \quad f_{\mu\nu}^{(j)} = \kappa \text{tr} \{ \mathbf{n}_j \mathcal{F}_{\mu\nu}[\mathcal{V}] \}. \tag{69}$$

Therefore, the restricted field \mathcal{V}_μ is regarded as the dominant mode for quark confinement, since the remaining field \mathcal{X}_μ does not contribute to the Wilson loop operator. The derivation of this fact is given in Appendix C.

Let \mathcal{F} be an arbitrary element of the Lie algebra $\mathcal{G} = \text{Lie}(G)$. Suppose that \mathcal{F} is decomposed as

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}_{\tilde{H}} + \mathcal{F}_{G/\tilde{H}}, \\
\mathcal{F}_{\tilde{H}} &= \sum_{j=1}^r (\mathcal{F}, H_j) H_j + \sum_{j_1, \dots, j_n} \xi_{j_1 \dots j_n} [H_{j_1}, \dots, [H_{j_n}, \mathcal{F}] \dots], \\
\mathcal{F}_{G/\tilde{H}} &= \sum_{j_1, \dots, j_n} \eta_{j_1 \dots j_n} [H_{j_1}, \dots, [H_{j_n}, \mathcal{F}] \dots],
\end{aligned} \tag{70}$$

where r is the rank of the gauge group. This relation for the decomposition has already been proven for $G(2)$ in the previous section, and the method is applicable to any semisimple compact Lie group.

In order to complete the non-Abelian Stokes theorem, we can follow the same procedures as those for $SU(N)$ given in [25], if $g^\dagger \partial_\mu \mathbf{m} g$ does not have the part belonging to the $\text{Lie}(\tilde{H})$,

$$(g^\dagger \partial_\mu \mathbf{m} g)_{\tilde{H}} = 0. \tag{71}$$

This enables us to rewrite $[\Omega_\mu(x), \Omega_\nu(x)]$ in terms of the color fields $\mathbf{n}_i(x) := \text{Ad}_{g(x)}(H_i)$, which is indeed shown in Appendix B.

The relevant relation (71) is verified as follows. By applying (70) to $\text{Ad}_{g^{-1}}(\partial_\mu \mathbf{m})$, we obtain the decomposition

$$\begin{aligned}
(g^\dagger \partial_\mu \mathbf{m} g)_{\tilde{H}} &= \sum_{j=1}^r \kappa \text{tr} (g^\dagger \partial_\mu \mathbf{m} g H_j) H_j \\
&+ \sum_{i_1, \dots, i_n} \xi_{i_1 \dots i_n} [H_{i_1}, \dots, [H_{i_n}, g^\dagger \partial_\mu \mathbf{m} g] \dots]. \tag{72}
\end{aligned}$$

The first term on the right-hand side vanishes, because

$$\begin{aligned}
\text{tr}(g^\dagger \partial_\mu \mathbf{m} g H_j) &= \Lambda_i \text{tr}(g^\dagger \partial_\mu (g H_i g^\dagger) g H_j) \\
&= \Lambda_i \text{tr}(g^\dagger \partial_\mu g H_i H_j + H_i \partial_\mu g^\dagger g H_j) \\
&= \Lambda_i \text{tr}(g^\dagger \partial_\mu g H_i H_j - H_i g^\dagger \partial_\mu g H_j) \\
&= \Lambda_i \text{tr}(g^\dagger \partial_\mu g H_i H_j - g^\dagger \partial_\mu g H_j H_i) \\
&= \Lambda_i \text{tr}(g^\dagger \partial_\mu g [H_i, H_j]) \\
&= 0,
\end{aligned} \tag{73}$$

where we have used $g^\dagger g = 1$ in the second equality, the relation $\partial_\mu g^\dagger g = -g^\dagger \partial_\mu g$ following from $\partial_\mu (g g^\dagger) = 0$ in the third equality, and the cyclicity of the trace in the fourth equality. In addition, by taking account of

$$\begin{aligned}
g^\dagger \partial_\mu \mathbf{m} g &= g^\dagger \partial_\mu (g \Lambda \cdot H g^\dagger) g \\
&= g^\dagger \partial_\mu g \Lambda \cdot H + \Lambda \cdot H \partial_\mu g^\dagger g \\
&= -\partial_\mu g^\dagger g \Lambda \cdot H + \Lambda \cdot H \partial_\mu g^\dagger g \\
&= [\Lambda \cdot H, \partial_\mu g^\dagger g],
\end{aligned} \tag{74}$$

the second term is rewritten as

$$\begin{aligned}
& \sum_{i_1, \dots, i_n} \xi_{i_1 \dots i_n} [H_{i_1}, \dots, [H_{i_n}, g^\dagger \partial_\mu \mathbf{m} g] \dots] \\
&= \sum_{i_1, \dots, i_n} \xi_{i_1 \dots i_n} [H_{i_1}, \dots, [H_{i_n}, [\Lambda \cdot H, \partial_\mu g^\dagger g]] \dots] \\
&= \sum_{i_1, \dots, i_n} \xi_{i_1 \dots i_n} [\Lambda \cdot H, [H_{i_1}, \dots, [H_{i_n}, \partial_\mu g^\dagger g] \dots]] \\
&= [\Lambda \cdot H, (\partial_\mu g^\dagger g)_{\tilde{H}}], \tag{75}
\end{aligned}$$

where we have used the commuting property (31) in the second equality and (70) for $\mathcal{F} = \partial_\mu g^\dagger g$ in the third equality. By substituting the Cartan decomposition of $(\partial_\mu g^\dagger g)_{\tilde{H}}$ in \tilde{h} given by

$$\begin{aligned}
(\partial_\mu g^\dagger g)_{\tilde{H}} &= \sum_{j=1}^r \kappa \text{tr}(\partial_\mu g^\dagger g H_j) H_j + \sum_{\alpha \in \mathcal{R}_+ : E_{\pm\alpha} \in \tilde{h}} ((\partial_\mu g^\dagger g)_\alpha^* E_\alpha \\
&\quad + (\partial_\mu g^\dagger g)_\alpha E_{-\alpha}) \tag{76}
\end{aligned}$$

into (75), we find that the second term also vanishes,

$$\begin{aligned}
(75) &= \left[\Lambda \cdot H, \sum_{\alpha \in \mathcal{R}_+ : E_{\pm\alpha} \in \tilde{h}} ((\partial_\mu g^\dagger g)_\alpha^* E_\alpha + (\partial_\mu g^\dagger g)_\alpha E_{-\alpha}) \right] \\
&= \sum_{\alpha \in \mathcal{R}_+ : E_{\pm\alpha} \in \tilde{h}} \Lambda \cdot \alpha ((\partial_\mu g^\dagger g)_\alpha^* E_\alpha - (\partial_\mu g^\dagger g)_\alpha E_{-\alpha}) \\
&= 0, \tag{77}
\end{aligned}$$

where we have used $\Lambda \cdot \alpha = 0$ for α satisfying $E_\alpha \in \tilde{h}$. Thus we have confirmed (71).

We obtain the final form of Wilson loop operator as⁵

$$\begin{aligned}
W_C[\mathcal{A}] &= \int [d\mu(g)]_\Sigma \exp \left[-i g_{\text{YM}} \int_{\Sigma : \partial\Sigma = C} F^g \right], \\
F^g &= \frac{1}{2} F_{\mu\nu}^g dx^\mu \wedge dx^\nu, \\
F_{\mu\nu}^g(x) &= \kappa \left\{ \partial_\mu \text{tr}(\mathbf{m}(x) \mathcal{A}_\nu(x)) - \partial_\nu \text{tr}(\mathbf{m}(x) \mathcal{A}_\mu(x)) \right. \\
&\quad \left. + i g_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}(\mathbf{m}(x) [\partial_\mu \mathbf{n}_{i_1}(x), \right. \\
&\quad \left. [\mathbf{n}_{i_2}, \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\nu \mathbf{n}_{i_n}(x)] \dots]] \right\}. \tag{78}
\end{aligned}$$

⁵We can rewrite $F_{\mu\nu}^g$ of (78) using the inner product instead of using the trace as

$$\begin{aligned}
F_{\mu\nu}^g &= \partial_\mu (\mathbf{m}, \mathcal{A}) - \partial_\nu (\mathbf{m}, \mathcal{A}) + i g_{\text{YM}} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} (\mathbf{m}(x), \\
&\quad [\partial_\mu \mathbf{n}_{i_1}(x), [\mathbf{n}_{i_2}(x), \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\nu \mathbf{n}_{i_n}(x)] \dots]]],
\end{aligned}$$

so that the Wilson loop depends on the representation only through the highest-weight vector Λ .

The detail of the derivation of (78) is given in Appendix B, which is almost the same as that given in [25] for $SU(N)$, once (71) is established.

B. $G(2)$ case

In each case of representations, we can write the new form for the Wilson loop operator using the decomposition formula based on the above general consideration.

- (1) For $\tilde{H} = U(2) \subset SU(3)$, the Wilson loop operator is written as (78) where $n = 6$ and

$$\eta_{j_1 \dots j_6} = \zeta_{j_1 \dots j_6}, \tag{79}$$

where $\zeta_{j_1 \dots j_6}$ is defined in (50).

- (2) For $\tilde{H} = U(2) \not\subset SU(3)$, the Wilson loop operator is written as (78) where $n = 6$ and

$$\eta_{j_1 \dots j_6} = \zeta'_{j_1 \dots j_6}, \tag{80}$$

where $\zeta'_{j_1 \dots j_6}$ is defined in (52).

- (3) For $\tilde{H} = U(1) \times U(1)$, the Wilson loop operator is written as (78) where $n = 6$ and

$$\eta_{j_1 \dots j_6} = \zeta''_{j_1 \dots j_6}, \tag{81}$$

where $\zeta''_{j_1 \dots j_6}$ is defined in (54).

By using $\mathbf{m} = \Lambda_i \mathbf{n}_i = (2n + m) \mathbf{n}_1 / 2 + m \mathbf{n}_2 / (2\sqrt{3})$, another form is obtained as

$$\begin{aligned}
F_{\mu\nu}^g(x) &= \frac{2n + m}{2} F_{\mu\nu}^{(1)}(x) + \frac{m}{2\sqrt{3}} F_{\mu\nu}^{(2)}(x) \\
F_{\mu\nu}^{(1)}(x) &= \kappa \left\{ \partial_\mu \text{tr}(\mathbf{n}_1(x) \mathcal{A}_\nu(x)) - \partial_\nu \text{tr}(\mathbf{n}_1(x) \mathcal{A}_\mu(x)) \right. \\
&\quad \left. + i g_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \zeta'_{j_1 \dots j_6} \text{tr}(\mathbf{n}_1(x) [\partial_\mu \mathbf{n}_{j_1}(x), \right. \\
&\quad \left. [\mathbf{n}_{j_2}, \dots, [\mathbf{n}_{j_5}(x), \partial_\nu \mathbf{n}_{j_6}(x)] \dots]] \right\} \\
F_{\mu\nu}^{(2)}(x) &= \kappa \left\{ \partial_\mu \text{tr}(\mathbf{n}_2(x) \mathcal{A}_\nu(x)) - \partial_\nu \text{tr}(\mathbf{n}_2(x) \mathcal{A}_\mu(x)) \right. \\
&\quad \left. + i g_{\text{YM}}^{-1} \sum_{j_1, \dots, j_6} \zeta_{j_1 \dots j_6} \text{tr}(\mathbf{n}_2(x) [\partial_\mu \mathbf{n}_{j_1}(x), \right. \\
&\quad \left. [\mathbf{n}_{j_2}, \dots, [\mathbf{n}_{j_5}(x), \partial_\nu \mathbf{n}_{j_6}(x)] \dots]] \right\}. \tag{82}
\end{aligned}$$

V. MAGNETIC MONOPOLES

We can define magnetic-monopole current k as the codifferential of the Hodge dual of F^g ,

$$k = \delta^* F^g. \tag{83}$$

In the D -dimensional spacetime, k is expressed by a differential form, $(D-3)$ -form. For $D=4$, especially, the magnetic monopole current reads

$$k^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}^g. \quad (84)$$

Then, the magnetic charge q_m is defined by

$$q_m := \int d^3x k^0 = \int d^3x \frac{1}{2} \epsilon^{ijkl} \partial_l F_{jk}^g = \int d^2S_l \epsilon^{ijkl} F_{jk}^g. \quad (85)$$

We examine the quantization condition for the magnetic charge. The magnetic charge can have nonzero value because the map defined by

$$m: S^2 \rightarrow G(2)/\tilde{H} = \begin{cases} G(2)/U(2) \\ G(2)/(U(1) \times U(1)) \end{cases} \quad (86)$$

has the nontrivial homotopy group

$$\begin{aligned} \pi_2(G(2)/\tilde{H}) &= \pi_1(\tilde{H}) \\ &= \begin{cases} \pi_1(S(2) \times U(1)) = \pi_1(U(1)) = \mathbb{Z} \\ \pi_1(U(1) \times U(1)) = \mathbb{Z} + \mathbb{Z} \end{cases}. \end{aligned} \quad (87)$$

Because the value of the magnetic charge depends only on the topological character of \mathbf{n}_i , we can use specific group elements g to obtain the quantization condition for the magnetic charge. Now, we consider a case in which $g(x)$ belongs to $SU(3)$. In this case, $F_{\mu\nu}^g$ reduces to

$$\begin{aligned} F_{\mu\nu}^g &= \frac{2n+m}{2} F_{\mu\nu}^{(1)} + \frac{m}{2\sqrt{3}} F_{\mu\nu}^{(2)}, \\ F_{\mu\nu}^{(1)} &= \kappa \{ \partial_\mu \text{tr}(\mathbf{n}_1 \mathcal{A}_\nu) - \partial_\nu \text{tr}(\mathbf{n}_1 \mathcal{A}_\mu) + i g_{\text{YM}}^{-1} \text{tr}(\mathbf{n}_1 [\partial_\mu \mathbf{n}_1, \partial_\nu \mathbf{n}_1] \\ &\quad + \mathbf{n}_1 [\partial_\mu \mathbf{n}_2, \partial_\nu \mathbf{n}_2]) \}, \\ F_{\mu\nu}^{(2)} &= \kappa \{ \partial_\mu \text{tr}(\mathbf{n}_2 \mathcal{A}_\nu) - \partial_\nu \text{tr}(\mathbf{n}_2 \mathcal{A}_\mu) + i g_{\text{YM}}^{-1} \text{tr}(\mathbf{n}_2 [\partial_\mu \mathbf{n}_1, \partial_\nu \mathbf{n}_1] \\ &\quad + \mathbf{n}_2 [\partial_\mu \mathbf{n}_2, \partial_\nu \mathbf{n}_2]) \} \\ &= \kappa \left\{ \partial_\mu \text{tr}(\mathbf{n}_2 \mathcal{A}_\nu) - \partial_\nu \text{tr}(\mathbf{n}_2 \mathcal{A}_\mu) \right. \\ &\quad \left. + \frac{4}{3} i g_{\text{YM}}^{-1} \text{tr}(\mathbf{n}_2 [\partial_\mu \mathbf{n}_2, \partial_\nu \mathbf{n}_2]) \right\}. \end{aligned} \quad (88)$$

Here, notice that two field strengths $F_{\mu\nu}^{(1)}$ and $F_{\mu\nu}^{(2)}$ appear in the non-Abelian Stokes theorem for $SU(3)$. It is shown [14,23] that the two kinds of gauge-invariant charges $q_m^{(1)}$ and $q_m^{(2)}$ obey different quantization conditions,

$$\begin{aligned} q_m &= \frac{2n+m}{2} q_m^{(1)} + \frac{m}{2\sqrt{3}} q_m^{(2)}, \\ q_m^{(1)} &:= \int d^3x \frac{1}{2} \epsilon^{ijk\ell} \partial_\ell F_{jk}^g = \frac{4\pi}{g_{\text{YM}}} \left(\ell - \frac{1}{2} \ell' \right), \\ q_m^{(2)} &:= \int d^3x \frac{1}{2} \epsilon^{ijk\ell} \partial_\ell F_{jk}^g = \frac{4\pi}{g_{\text{YM}}} \frac{1}{2} \sqrt{3} \ell', \quad \ell, \ell' \in \mathbb{Z}. \end{aligned} \quad (89)$$

Thus, we obtain the quantization condition for the magnetic charge in $G(2)$,

$$q_m = \frac{4\pi}{g_{\text{YM}}} \left(\frac{n}{2} (2\ell - \ell') + \frac{m}{2} \ell' \right) = \frac{2\pi}{g_{\text{YM}}} (nk + m\ell), \quad (90)$$

where we have defined $k := 2\ell - \ell'$, which can take an arbitrary integer. The observation based on the homotopy group (87) implies that there need to be two integers in q_m . There exist already two integers in q_m . Therefore, it is enough to consider a case $g(x) \in SU(3)$ for deriving the quantization condition for the magnetic charge in $G(2)$.

VI. CONCLUSIONS AND DISCUSSIONS

For the exceptional group $G(2)$, we first showed that there exist three cases of the maximal stability subgroup. Then, we derived the gauge-covariant decomposition formula, which is written using multiple commutators with the color direction fields, in accord with each stability group. In addition, we have obtained the non-Abelian Stokes theorem for the Wilson loop operator that is written in terms of the relevant color direction fields. These results indicate that there exist three options for the reformulation of the $G(2)$ Yang-Mills theory. In any option, we need two kinds of color fields, since the two Cartan generators are inevitably required in the decomposition formula; this is in marked contrast to the minimal option of the $SU(N)$ group. Nevertheless, each option would be utilized for describing confinement of quarks in the relevant representation of $G(2)$. This is because the non-Abelian Stokes theorem for the Wilson loop operator is attributed to the decomposition formula available for a given representation. This can be confirmed more explicitly when the reformulation is ready to be checked.

The method we have used in this paper for obtaining the decomposition formula is so general that the decomposition formula is written for any semisimple Lie group using multiple commutators with the Cartan generators. In addition, once the decomposition formula given in the above is obtained, we can immediately obtain the expression of the Wilson loop operator written in terms of the color direction fields, because we derived the non-Abelian Stokes theorem in a general way. This observation suggests that the reformulation of the Yang-Mills theory with an arbitrary semisimple gauge group would be possible.

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APPENDIX A: NECESSITY OF SEXTUPLE COMMUTATORS IN THE DECOMPOSITION FORMULA FOR $G(2)$

In Sec. III, we have seen that sextuple commutators are used to obtain the decomposition formula for $G(2)$. In this appendix we show that such a formula for $G(2)$ cannot be obtained by taking the commutator fewer than six times. Taking the commutator an odd number of times, we obtain

$$\begin{aligned} & [H_{j_1}, \dots, [H_{j_{2n+1}}, \mathcal{F}] \dots] \\ &= \sum_{\alpha \in \mathcal{R}_+} (\alpha_{j_1} \dots \alpha_{j_{2n+1}} \mathcal{F}_\alpha^* E_\alpha - \alpha_{j_1} \dots \alpha_{j_{2n+1}} \mathcal{F}_\alpha E_{-\alpha}). \end{aligned} \quad (\text{A1})$$

In this expression, the sign of the term E_α is different from the sign of the term $E_{-\alpha}$; therefore this is not appropriate. Thus we see that we need to consider only the cases of double and quadruple commutators.

First, we consider the case of double commutators. We can decompose an arbitrary element \mathcal{F} of the Lie algebra if and only if there are real numbers k_1, k_2 , and k_3 that satisfy

$$\begin{aligned} & k_1[H_1, [H_1, \mathcal{F}]] + k_2[H_1, [H_2, \mathcal{F}]] + k_3[H_2, [H_2, \mathcal{F}]] \\ &= \sum_{\alpha \in \mathcal{R}_+ : E_{\pm\alpha} \notin \tilde{h}} (\mathcal{F}_\alpha^* E_\alpha + \mathcal{F}_\alpha E_{-\alpha}). \end{aligned} \quad (\text{A2})$$

Using the Cartan decomposition of \mathcal{F} , we find that the left-hand side of (A2) is equal to

$$\sum_{\alpha \in \mathcal{R}_+} ((\alpha_1)^2 k_1 + \alpha_1 \alpha_2 k_2 + (\alpha_2)^2 k_3) (\mathcal{F}_\alpha^* E_\alpha + \mathcal{F}_\alpha E_{-\alpha}), \quad (\text{A3})$$

where we set $\alpha = (\alpha_1, \alpha_2)$. Hence, (A2) is equivalent to

$$\begin{aligned} & (\alpha_1)^2 k_1 + \alpha_1 \alpha_2 k_2 + (\alpha_2)^2 k_3 = 1 \quad \text{for } E_\alpha \notin \tilde{h}, \\ & (\alpha_1)^2 k_1 + \alpha_1 \alpha_2 k_2 + (\alpha_2)^2 k_3 = 0 \quad \text{for } E_\alpha \in \tilde{h}. \end{aligned} \quad (\text{A4})$$

In the case of $\tilde{H} = U(2) \subset SU(3)$, the three equations (A4) for $\alpha^{(1)}$, $\alpha^{(2)}$, and $\alpha^{(3)}$ can be written in a matrix form as

$$\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad (\text{A5})$$

The solution of this equation is $k_1 = k_2 = 0$, $k_3 = 4/3$. This solution is consistent with the decomposition formula for $SU(3)$. However, these values of k_1, k_2 , and k_3 do not satisfy the equation (A4) for $\alpha^{(4)}$, $\alpha^{(5)}$, and $\alpha^{(6)}$. For example, Eq. (A4) for $\alpha^{(5)}$ is given by

$$\frac{1}{3} k_3 = 1, \quad (\text{A6})$$

which is not satisfied by $k_3 = 4/3$. Therefore, there is no solution for all of (A4).

In the case of $\tilde{H} = U(2) \not\subset SU(3)$ and of $\tilde{H} = U(1) \times U(1)$, Eq. (A4) for $\alpha^{(1)}$, $\alpha^{(2)}$, and $\alpha^{(3)}$ reads

$$\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (\text{A7})$$

The solution of this equation is $k_1 = k_3 = 1$, $k_2 = 0$. This solution is also consistent with the decomposition formula for $SU(3)$. However, this does not satisfy Eq. (A4) for $\alpha^{(4)}$, $\alpha^{(5)}$, and $\alpha^{(6)}$. Therefore, there are no solutions for all of (A4) in this case as well. Thus, we confirm that there are no decomposition formulas using double commutators for all representations.

Next, we consider the case of quadruple commutators. There exist decomposition formulas if and only if there are real numbers k_1, k_2, k_3, k_4 , and k_5 that satisfy

$$\begin{aligned} & k_1[H_1, [H_1, [H_1, [H_1, \mathcal{F}]]]] + k_2[H_1, [H_1, [H_1, [H_2, \mathcal{F}]]]] \\ &+ k_3[H_1, [H_1, [H_2, [H_2, \mathcal{F}]]]] + k_4[H_1, [H_2, [H_2, [H_2, \mathcal{F}]]]] \\ &+ k_5[H_2, [H_2, [H_2, [H_2, \mathcal{F}]]]] \\ &= \sum_{\alpha \in \mathcal{R}_+ : E_{\pm\alpha} \notin \tilde{h}} (\mathcal{F}_\alpha^* E_\alpha + \mathcal{F}_\alpha E_{-\alpha}). \end{aligned} \quad (\text{A8})$$

This is equivalent to

$$\begin{aligned} & (\alpha_1)^4 k_1 + (\alpha_1)^3 \alpha_2 k_2 + (\alpha_1)^2 (\alpha_2)^2 k_3 + \alpha_1 (\alpha_2)^2 k_4 + (\alpha_2)^4 k_5 \\ &= 1 \quad \text{for } E_\alpha \notin \tilde{h}, \\ & (\alpha_1)^4 k_1 + (\alpha_1)^3 \alpha_2 k_2 + (\alpha_1)^2 (\alpha_2)^2 k_3 + \alpha_1 (\alpha_2)^2 k_4 + (\alpha_2)^4 k_5 \\ &= 0 \quad \text{for } E_\alpha \in \tilde{h}. \end{aligned} \quad (\text{A9})$$

The equivalent matrix form is given as

$$\frac{1}{16} \begin{pmatrix} 1 & -\sqrt{3} & 3 & -3^{\frac{3}{2}} & 9 \\ 16 & 0 & 0 & 0 & 0 \\ 1 & \sqrt{3} & 3 & 3^{\frac{3}{2}} & 9 \\ 1 & \frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3^{\frac{3}{2}}} & \frac{1}{9} \\ 0 & 0 & 0 & 0 & \frac{16}{9} \\ 1 & -\frac{1}{\sqrt{3}} & \frac{1}{3} & -\frac{1}{3^{\frac{3}{2}}} & \frac{1}{9} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (\text{A10})$$

Calculating the rank of the matrix, we see that these equations do not have solutions. Thus we have confirmed that quadruple commutators are not enough to obtain the desired decomposition formula.

APPENDIX B: DERIVATION OF THE NON-ABELIAN STOKES THEOREM FOR THE GENERAL GAUGE GROUP USING (71)

From the fact [Eq. (71)] that the \tilde{H} part of $g^\dagger \partial_\mu \mathbf{m} g$ is vanishing, we have

$$g^\dagger \partial_\mu \mathbf{m} g = (g^\dagger \partial_\mu \mathbf{m} g)_{G/H} = \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} [H_{i_1}, \dots, [H_{i_n}, g^\dagger \partial_\mu \mathbf{m} g] \dots]. \quad (\text{B1})$$

Multiplying both sides of this equation, by g from the left and by g^\dagger from the right, we obtain

$$\begin{aligned} \partial_\mu \mathbf{m}(x) &= \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} [\mathbf{n}_{i_1}(x), \dots, [\mathbf{n}_{i_n}(x), \partial_\mu \mathbf{m}(x)] \dots] \\ &= \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} [\mathbf{n}_{i_1}(x), \dots, [\mathbf{m}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots] \\ &= \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} [\mathbf{m}, [\mathbf{n}_{i_1}(x), \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]] \\ &= ig_{\text{YM}} [\mathcal{B}_\mu(x), \mathbf{m}(x)], \end{aligned} \quad (\text{B2})$$

where we have used $[\mathbf{n}_i, \partial_\mu \mathbf{m}] = [\mathbf{m}, \partial_\mu \mathbf{n}_i]$ following from $\partial_\mu [\mathbf{n}_i, \mathbf{m}] = 0$ in the second equality, the commuting property (31) in the third equality, and we have introduced

$$\mathcal{B}_\mu(x) := ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} [\mathbf{n}_{i_1}(x), \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]. \quad (\text{B3})$$

in last equality. On the other hand, we find

$$\partial_\mu \mathbf{m}(x) = ig_{\text{YM}} [\Omega_\mu, \mathbf{m}(x)]. \quad (\text{B4})$$

Combining (B2) and (B4), we obtain

$$[\Omega_\mu, \mathbf{m}(x)] = [\mathcal{B}_\mu(x), \mathbf{m}(x)]. \quad (\text{B5})$$

Using this relation, we can rewrite the third term in $F_{\mu\nu}^g(x)$ as

$$\begin{aligned} ig_{\text{YM}} \text{tr}(\mathbf{m}(x) [\Omega_\mu(x), \Omega_\nu(x)]) &= ig_{\text{YM}} \text{tr}([\mathbf{m}(x), \Omega_\mu(x)] \Omega_\nu(x)) \\ &= ig_{\text{YM}} \text{tr}([\mathbf{m}(x), \mathcal{B}_\mu(x)] \Omega_\nu(x)) \\ &= ig_{\text{YM}} \text{tr}([\Omega_\nu(x), \mathbf{m}(x)] \mathcal{B}_\mu(x)) \\ &= \text{tr}(\partial_\nu \mathbf{m}(x) \mathcal{B}_\mu(x)) \\ &= ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}(\partial_\nu \mathbf{m}(x) [\mathbf{n}_{i_1}(x), \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]) \\ &= ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}([\partial_\nu \mathbf{m}(x), \mathbf{n}_{i_1}(x)] [\mathbf{n}_{i_2}, \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]) \\ &= -ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}([\mathbf{m}(x), \partial_\nu \mathbf{n}_{i_1}(x)] [\mathbf{n}_{i_2}, \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]) \\ &= -ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}(\mathbf{m}(x) [\partial_\nu \mathbf{n}_{i_1}(x), [\mathbf{n}_{i_2}, \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\mu \mathbf{n}_{i_n}(x)] \dots]]) \\ &= ig_{\text{YM}}^{-1} \sum_{i_1, \dots, i_n} \eta_{i_1 \dots i_n} \text{tr}(\mathbf{m}(x) [\partial_\mu \mathbf{n}_{i_1}(x), [\mathbf{n}_{i_2}, \dots, [\mathbf{n}_{i_{n-1}}(x), \partial_\nu \mathbf{n}_{i_n}(x)] \dots]]), \end{aligned} \quad (\text{B6})$$

where we have used the cyclicity of the trace in the first, third, sixth, and eighth equalities, (B4) in the fourth equality, $[\partial_\nu \mathbf{m}, \mathbf{n}_{i_1}] = -[\mathbf{m}, \partial_\nu \mathbf{n}_{i_1}]$ following from $\partial_\nu [\mathbf{m}, \mathbf{n}_{i_1}] = 0$ in the seventh equality, and the fact that the first expression is antisymmetric in μ and ν in the last equality. This completes the proof of the non-Abelian Stokes theorem (78).

APPENDIX C: DERIVATION

OF $F_{\mu\nu}^g = \kappa \text{tr}(\mathbf{m} \mathcal{F}_{\mu\nu}[\mathcal{V}])$

In this appendix, we show that the remaining field \mathcal{X} does not contribute to the Wilson loop operator by deriving the equality (69). Using the decomposition (64), we obtain

$$\begin{aligned} \text{tr}(\mathbf{m} \mathcal{F}_{\mu\nu}[\mathcal{V}]) &= \text{tr}(\mathbf{m}(\partial_\mu \mathcal{C}_\nu - \partial_\nu \mathcal{C}_\mu - ig_{\text{YM}}[\mathcal{C}_\mu, \mathcal{C}_\nu] \\ &\quad - ig_{\text{YM}}[\mathcal{C}_\mu, \mathcal{B}_\nu] - ig_{\text{YM}}[\mathcal{B}_\mu, \mathcal{C}_\nu] \\ &\quad + \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu - ig_{\text{YM}}[\mathcal{B}_\mu, \mathcal{B}_\nu])). \end{aligned} \quad (\text{C1})$$

From the fact (66) we see that the third, fourth, and fifth terms vanish. Thus we obtain

$$\begin{aligned} \text{tr}(\mathbf{m} \mathcal{F}_{\mu\nu}[\mathcal{V}]) &= \text{tr}(\mathbf{m}(\partial_\mu \mathcal{C}_\nu - \partial_\nu \mathcal{C}_\mu + \partial_\mu \mathcal{B}_\nu \\ &\quad - \partial_\nu \mathcal{B}_\mu - ig_{\text{YM}}[\mathcal{B}_\mu, \mathcal{B}_\nu])). \end{aligned} \quad (\text{C2})$$

The first term of (C2) reads

$$\begin{aligned} \partial_\mu \text{tr}(\mathbf{m} \mathcal{C}_\nu) &= \partial_\mu (\text{tr}(\mathbf{m} \mathcal{C}_\nu)) \\ &= \text{tr}(\partial_\mu \mathbf{m} \mathcal{C}_\nu + \mathbf{m} \partial_\mu \mathcal{C}_\nu) \\ &= \text{tr}(\mathbf{m} \partial_\mu \mathcal{C}_\nu), \end{aligned} \quad (\text{C3})$$

where we have used $(g^\dagger \partial_\mu \mathbf{m} g)_{\tilde{H}} = 0$ and $\mathcal{C}_\nu \in g\text{Lie}(\tilde{H})g^\dagger$. The third term of (C2) reads

$$\begin{aligned} \text{tr}(\mathbf{m} \partial_\mu \mathcal{B}_\nu) &= -\text{tr}(\partial_\mu \mathbf{m} \mathcal{B}_\nu) \\ &= -ig_{\text{YM}} \text{tr}([\mathcal{B}_\mu, \mathbf{m}] \mathcal{B}_\nu) \\ &= ig_{\text{YM}} \text{tr}(\mathbf{m} [\mathcal{B}_\mu, \mathcal{B}_\nu]), \end{aligned} \quad (\text{C4})$$

where we have used $\text{tr}(\mathbf{m} \mathcal{B}_\nu) = 0$ in the first equality, (B2) in the second equality, and the cyclicity of the trace in the last equality. Thus, we obtain

$$\begin{aligned} (\text{C2}) &= \partial_\mu \text{tr}(\mathbf{m} \mathcal{C}_\nu) - \partial_\nu \text{tr}(\mathbf{m} \mathcal{C}_\mu) + ig_{\text{YM}} \text{tr}(\mathbf{m} [\mathcal{B}_\mu, \mathcal{B}_\nu]) \\ &= \partial_\mu \text{tr}(\mathbf{m} \mathcal{C}_\nu) - \partial_\nu \text{tr}(\mathbf{m} \mathcal{C}_\mu) + ig_{\text{YM}} \text{tr}(\mathbf{m} [\Omega_\mu, \Omega_\nu]) \\ &= \frac{1}{\kappa} F_{\mu\nu}^g, \end{aligned} \quad (\text{C5})$$

where we have used (C3) and (C4) in the first equality and (B5) in the second equality.

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