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Note on scaling arguments in the effective average action formalism

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The effective average action (EAA) is a scale-dependent effective action where a scale k is introduced via an infrared regulator. The k dependence of the EAA is governed by an exact flow equation to which one associates a boundary condition at a scale μ. We show that the μ dependence of the EAA is controlled by an equation fully analogous to the Callan-Symanzik equation which allows one to define scaling quantities straightforwardly. Particular attention is paid to composite operators which are introduced along with new sources. We discuss some simple solutions to the flow equation for composite operators and comment on their implications in the case of a local potential approximation.

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I. INTRODUCTION

The renormalization group (RG) provides an ideal framework to discuss the scaling of operators in quantum field theory. In this work we consider the scaling properties of a quantum field theory within the effective average action (EAA) formalism, which is a functional realization of the Wilsonian renormalization program [1–3]. The EAA is a scale-dependent effective action whose associated action has been modified by the introduction of an infrared cutoff depending on a scale k in such a way that low-momentum modes are suppressed in the integration. In order to discuss the scaling properties within this formalism we show that the EAA satisfies, besides an exact flow equation involving the scale k, an equation which involves the scale μ at which the boundary conditions are imposed. This equation entails an invariance under changes of "floating normalization point" μ very similar to one expressed by the Callan-Symanzik equation. In this sense this equation recalls the connection between the methods of perturbative renormalization and the Wilsonian RG studied in [4].

We carefully investigate composite operators by modifying the original action with a source-dependent term which allows us to consider composite operator insertions via functional derivatives with respect to the source. We will see that the above-mentioned μ invariance allows defining scaling operators straightforwardly. Then we comment on the relation between the so-defined scaling dimension and critical exponents and show that, if total derivative terms are neglected, the results are easily related. Moreover we will show that introducing the sources for composite operators allows us to identify some general types of composite operators among which are the descendant operators of the scaling operators at the fixed point. To make our discussion concrete we revisit the result of the local potential approximation from our perspective.

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The paper is organized as follows. In Sec. II we introduce the flow equation for the EAA and for composite operators. In Sec. III we discuss the μ dependence of the EAA and how scaling dimensions can be identified while in Sec. IV we consider the local potential approximation in view of the previous discussion. In Sec. V we summarize our results and discuss possible outlooks.

II. RG FLOW OF COMPOSITE OPERATORS

In a quantum field theory a composite operator is defined as a function of the field and its derivatives, i.e. $O = O(\varphi, \partial_\mu\varphi)$. Generically, once such an operator is inserted in a correlation function new divergences appear. Owing to these divergences one has to renormalize, besides the couplings of the theory, also the composite operator itself. As a result one finds that a renormalized composite operator, which we denote $[O]$ using square brackets, is a sum of various operators. For instance, at one loop in a six-dimensional φ^3 theory one finds $[\varphi^2] = c_1\varphi + c_2\Delta\varphi + c_3\varphi^2$, where c_i are suitable coefficients [5]. A convenient way to keep track of composite operators and their insertion into Green's functions is to couple them to a source ε adding a term $\varepsilon \cdot O$ to the action as follows[1]:

$$\begin{aligned}\langle O\rangle &= \mathcal{N}\int \mathcal{D}\chi O e^{-S}\\ &= -\frac{\delta}{\delta\varepsilon_x}\mathcal{N}\int \mathcal{D}\chi e^{-S-\varepsilon\cdot O},\end{aligned}$$

where $\mathcal{N}$ is a suitable normalization. Let us consider the generating functional $W[J,\varepsilon]$ for the connected Green's functions associated to the modified action $S + \varepsilon \cdot O$:

[1]Whenever a dot appears in a mathematical expression, e.g. $f \cdot g$, the DeWitt notation is used meaning that integration and index summation is intended.

$$e^{W[J,\varepsilon]} \equiv \int \mathcal{D}\chi e^{-S-\varepsilon\cdot O+J\cdot\chi}.$$

The connected part of the correlation function $\langle O \rangle$ will be given just by the derivative $-\frac{\delta}{\delta\varepsilon}W[J,\varepsilon]$. Via a Legendre transform we obtain the associated effective action, $\Gamma[\varphi,\varepsilon]$:

$$\Gamma[\varphi,\varepsilon] = J\cdot\varphi - W[J,\varepsilon], \qquad \varphi = \delta_J W.$$

Note that we do not perform a Legendre transform with respect to the source ε.[2] Since the insertion of one composite operator is related to a functional differentiation with respect to the source ε let us consider

$$\begin{aligned}\delta_\varepsilon\Gamma[\varphi,\varepsilon] &= \delta_\varepsilon(J\cdot\varphi - W[J,\varepsilon])\\ &= \frac{\delta J}{\delta\varepsilon}\cdot\varphi - \left(\frac{\delta W}{\delta\varepsilon}[J,\varepsilon] + \frac{\delta J}{\delta\varepsilon}\cdot\delta_J W\right)\\ \frac{\delta\Gamma}{\delta\varepsilon}[\varphi,\varepsilon] &= -\frac{\delta W}{\delta\varepsilon}[J,\varepsilon]. \end{aligned} \tag{1}$$

This tells us that we can obtain information regarding the renormalization of composite operators by considering the renormalization of $\delta_\varepsilon\Gamma[\varphi,\varepsilon]$.

As already said we shall work within the functional renormalization group (FRG) framework. In particular we consider the RG flow of the EAA which is a scale-dependent generalization of the standard effective action. One first defines a modified generating functional of connected Green's functions, $W_k[J]$:

$$e^{W_k[J]} \equiv \int \mathcal{D}\chi e^{-S-\Delta S_k+J\cdot\chi},$$

where ΔS_k suppresses the integration of momentum modes $p^2 < k^2$ and is quadratic in the fields with a kernel R_k, i.e. $\Delta S_k = \frac{1}{2}\int \chi R_k \chi$.[3] Note that this cutoff action acts like an infrared cutoff. Let us denote $\tilde{\Gamma}_k$ the Legendre transform of W_k and define the EAA subtracting the cutoff action which we added at the beginning:

$$\Gamma_k \equiv \tilde{\Gamma}_k - \Delta S_k.$$

The k dependence of the functional Γ_k satisfies the following exact equation [1–3]:

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + R_k)^{-1}\partial_t R_k], \tag{2}$$

where $\partial_t = k\partial_k$ is the logarithmic derivative with respect to the cutoff and $\Gamma_k^{(2)}$ is the Hessian of the EAA. To concretely employ Eq. (2) one needs to resort to some approximations and makes an *Ansatz* for Γ_k. On top of this approximation there remains the dependence of the EAA on the choice of the cutoff profile R_k; this freedom can be used as a quality test for a truncation or as an optimization criteria, and we refer to [7–11] for further details. Such a procedure has been proved robust in many fields, especially to determine scaling properties of statistical systems at criticality; see [12,13] for an overview.

The inclusion of composite operators in this framework is straightforward: we simply upgrade the above definitions employing the modified action $S + \varepsilon\cdot O$ instead of S (we refer to [14] for a detailed discussion regarding the flow of composite operators; see also [10,15]). In this manner we obtain a modified generating functional $W_k[J,\varepsilon]$ for connected Green's functions which depends also on the source ε. In full analogy with the derivation of relation (1) we obtain

$$-\frac{\delta}{\delta\varepsilon}W_k[J,\varepsilon] = \frac{\delta}{\delta\varepsilon}\Gamma_k[\varphi,\varepsilon].$$

We also introduce the notation

$$[O_k] \equiv \frac{\delta}{\delta\varepsilon}\Gamma_k[\varphi,\varepsilon],$$

where the subscript k indicates that $[O_k]$ depends on the scale k. In order to obtain the scale dependence of composite operators we observe that the flow equation (2) holds in full generality also for the modified EAA $\Gamma_k[\varphi,\varepsilon]$:

$$\partial_t\Gamma_k[\varphi,\varepsilon] = \frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)}[\varphi,\varepsilon] + R_k)^{-1}\partial_t R_k], \tag{3}$$

where $\Gamma_k^{(2)}[\varphi,\varepsilon]$ denotes the Hessian of the EAA with respect to the field φ. The scale dependence of $[O_k]$ is obtained taking a single functional derivative with respect to the source ε and setting $\varepsilon = 0$ afterwards. In this way we have

$$\begin{aligned}\partial_t\left(\frac{\delta}{\delta\varepsilon}\Gamma_k[\varphi,\varepsilon]\right)\Bigg|_{\varepsilon=0} &= -\frac{1}{2}\mathrm{Tr}\Bigg[(\Gamma_k^{(2)} + R_k)^{-1}\frac{\delta\Gamma_k^{(2)}}{\delta\varepsilon}\\ &\quad\times(\Gamma_k^{(2)} + R_k)^{-1}\partial_t R_k\Bigg]\Bigg|_{\varepsilon=0}.\end{aligned} \tag{4}$$

Let us observe that we can avoid performing a functional derivative with respect to the source ε and just compare order by order in ε. Clearly, since we are interested just in a single insertion of the composite operator, we can limit ourselves to consider an ε such that $\varepsilon^2 = 0$ (the flow equation of order $O(\varepsilon^2)$ is considered in Appendix A). We can rewrite the flow equation as

$$\begin{aligned}\partial_t(\varepsilon\cdot[O_k]) &= -\frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + R_k)^{-1}(\varepsilon\cdot[O_k]^{(2)})\\ &\quad\times(\Gamma_k^{(2)} + R_k)^{-1}\partial_t R_k],\end{aligned} \tag{5}$$

[2] Such a Legendre transform is performed on a bilocal source when considering the 2PI effective action [6] after adding a term $\varphi(x)\cdot\varepsilon(x,y)\cdot\varphi(y)$ to the action.

[3] One usually requires $R_k \approx k^2$ for $p^2 \ll k^2$ and $R_k \approx 0$ for $p^2 \gg k^2$.

where $[O_k]^{(2)}$ is the Hessian of the operator $[O_k]$ with respect to the field φ. Equation (5) can be seen as an RG-improved version of a one-loop equation. To see this let us consider the one-loop EAA $\Gamma_{k,1}$ associated to the modified action $S + \varepsilon \cdot O$:

$$\Gamma_{k,1} = S + \varepsilon \cdot O + \frac{1}{2}\mathrm{Tr}\log(S^{(2)} + R_k + \varepsilon \cdot O^{(2)}),$$

where $S^{(2)}$ and $O^{(2)}$ are the Hessians for the action and the composite operator, respectively. If we now differentiate the above expression with respect to the scale k we obtain

$$\partial_t \Gamma_{k,1} = \frac{1}{2}\mathrm{Tr}[(S^{(2)} + R_k + \varepsilon \cdot O^{(2)})^{-1}\partial_t R_k].$$

Finally, in order to derive the running of the composite operator, we just need to take a functional derivative with respect to the source ε and set this to zero:

$$\partial_t [O_k] = -\frac{1}{2}\mathrm{Tr}[(S^{(2)} + R_k)^{-1} \cdot O^{(2)} \cdot (S^{(2)} + R_k)^{-1}\partial_t R_k]. \tag{6}$$

Equation (6) is fully analogous to Eq. (4) with the microscopic action S in place of the EAA and the bare operator in place of the renormalized one.

As in the case of the flow equation (2), also the flow equation (5) has to be equipped with suitable boundary conditions; see also [14]. When the scale k has been lowered to zero the integration has been fully performed and we have that $\langle O_B \rangle = [O_{k=0}]$, where O_B is the bare operator (see [16] for an analogous observation using the Wilsonian action).

Finally we would like to stress the following point: in principle the renormalization of composite operators must be carried out in addition to the usual renormalization and some "extra" work is needed. Since Eq. (5) is essentially the linearization of the flow equation one may get the impression that the operator dimensions at the fixed point are given by the linearization of the RG, i.e. by the critical exponents.[4] Although critical exponents and scaling dimensions of composite operators are closely related (as we will discuss in Secs. III C and IV A) there are some notable differences. To understand why this is the case let us consider a simple example: in a six-dimensional φ^3 theory the operator $[\varphi^2]$ contains the operator $\Delta\varphi$ [5], with $\Delta = -\partial^2$. If we consider $\Delta\varphi$ as a term appearing in the action, this would result simply in a surface term and as such this term would usually be neglected. When considering composite operators it is no longer so, given that $\varepsilon \cdot \Delta\varphi$ is not a surface term. For the same reason one should distinguish between $\varphi\Delta\varphi$ and $\partial\varphi\partial\varphi$ when dealing with composite operators, in contrast to what one does with couplings appearing in the action. In the standard formulation of quantum field theory one indeed considers [18]

$$\varepsilon \cdot [\varphi^4] = \varepsilon \cdot (Z_{21} Z_\varphi \varphi^2 + Z_{22} Z_\varphi^2 \varphi^4 + Z_{23} Z_\varphi (\partial\varphi)^2 + Z_{24} Z_\varphi \varphi \Delta\varphi), \tag{7}$$

where Z_φ is the wave function renormalization.

Furthermore, Eq. (5) can be used not only for scalar operators, but also for higher spin ones. In this case the source will carry an index, for instance ε^μ in the case of a spin-one operator. It appears clear that the renormalization of such a term cannot be extracted directly from the renormalization of the theory alone and its linearization around the fixed point. Finally it may happen that one is interested in the flow of some particular operator while neglecting some others. In such a case our framework is particularly convenient; see [19] for an example of such an application.

III. FLOATING NORMALIZATION POINT AND SCALING DIMENSIONS

In this section we discuss how RG quantities and scaling dimensions are related in the functional renormalization group framework. In particular we shall show that it is possible to infer an equation which resembles the Callan-Symanzik equation. By reformulating scaling arguments via the Callan-Symanzik equation we will obtain a generic framework which can be applied to any system of interest. Besides this, our equation shows an obvious connection with the standard framework of quantum field theory and may be a useful tool for comparison with results obtained via other approaches. In Sec. III A we describe the invariance of the EAA with respect to suitable changes of boundary conditions of the flow equation and derive a Callan-Symanzik type of equation. In Sec. III B we extend those arguments including composite operators, while in Sec. III C we relate these results to the usual quantities computed via the FRG.

We will often work within the so-called LPA′ truncation where one takes into account up to two derivatives of the field and a generic potential, including the wave function renormalization of the field[5]:

$$\Gamma_k[\varphi] = \int \left[\frac{Z_k}{2}\partial_\mu\varphi\partial^\mu\varphi + U_k(Z_k^{1/2}\varphi)\right]. \tag{8}$$

The extension of our arguments to more general truncations is obvious. In the above *Ansatz* we made explicit the fact

[4]Here critical exponents are associated to deformations of the type $S + \delta S$ and not of the type $S + g \cdot \delta S$, where g is an arbitrary source. The latter case is considered by Wilson and Kogut in [17]. However, in FRG computations one usually takes source-independent deformations.

[5]The acronym LPA, without the prime, is generally used for the local potential approximation without the wave function renormalization.

that the wave function renormalization can be looked at as an inessential coupling and can be removed via a rescaling of the field (see [15] for a similar discussion in the context of the Wilsonian action). One can eventually define the field $\phi = Z_k^{1/2}\varphi$ and insist on having a canonically normalized kinetic term. When the flow equation is expressed in terms of ϕ the effect of the wave function renormalization is fully contained in the appearance of the anomalous dimension. In this work we shall express the EAA as a function of φ or ϕ according to convenience.

A. Invariance under changes of the floating normalization point

Let us take U_k to be a polynomial of a given order whose coefficients are parametrized by a set of dimensionful couplings $\{g_i\}$, whose dimensionless version is denoted $\{\tilde{g}_i\}$.[6] The RG flow can be described by a set of first-order differential equations. In particular the couplings $\{\tilde{g}_i\}$ satisfy an equations of the following form:

$$\partial_t \tilde{g}_i = f_i(\{\tilde{g}_j\}).$$

Let us consider a given trajectory which is associated to a boundary condition

$$\tilde{g}_i(\mu) = \tilde{g}^{(R)}$$

at the scale μ, which we could call "floating normalization point." We observe that this trajectory can be labeled by many other equivalent boundary conditions along the RG trajectory at some other scale μ'. In order to make this clear let us consider a simple example of a dynamical system which mimics our system of beta functions:

$$\dot{x}(t) = f(x), \qquad x(t_0) = x_0. \tag{9}$$

Let $F(t)$ be the generic solution of the first equation in (9) and suppose that we can implement the boundary condition $x(t_0) = x_0$ explicitly via

$$x(t) = F(t) - F(t_0) + x_0. \tag{10}$$

The fact that we can associate to this trajectory many other boundary conditions is expressed by the vanishing of the total derivative with respect to t_0:

$$\begin{aligned} \frac{d}{dt_0}x(t) &= \frac{d}{dt_0}[F(t) - F(t_0) + x_0] \\ &= -\frac{d}{dt_0}F(t_0) + \dot{x}_0 \\ &= -\dot{x}_0 + \dot{x}_0 = 0. \end{aligned}$$

In the second line we took care of both the explicit presence of t_0 in the first term and of the implicit dependence of x_0 on the "normalization point" t_0. Thus we can rewrite the above equation as

$$\frac{d}{dt_0}x(t) = \left(\frac{\partial}{\partial t_0} + \dot{x}_0 \frac{\partial}{\partial x_0}\right)x(t) = 0. \tag{11}$$

Note that the boundary condition x_0 may enter in the solution in many possible ways and not just as shown in (10). The invariance under the operator in the right-hand side of Eq. (11) is guaranteed by the following reasoning: given a trajectory (i.e. a solution of the equation) associated to the boundary condition x_0 at t_0, we observe that the same trajectory can be associated to many other boundary conditions along the same trajectory at some other points $x_0', x_0'', \ldots$; see Fig. 1. This means that the selected trajectory $x(t)$ is invariant under translations of the boundary condition along the trajectory itself. Thus the solution is not invariant under any change of the couple (t_0, x_0) but is invariant under those changes which bring (t_0, x_0) into another point along the solution. Indeed this is exactly what is implemented by the operator of Eq. (11). To see this let us make explicit the presence of (t_0, x_0) in the solution denoting $x(t) = x(t; t_0, x_0)$. Let us translate t_0 infinitesimally and move x_0 accordingly:

$$\begin{aligned} x(t; t_0, x_0) &= x(t; t_0 + \varepsilon, x_0 + \delta x_0) \\ &= x(t; t_0 + \varepsilon, x_0 + \dot{x}_0\varepsilon) \\ &= x(t; t_0, x_0) + \varepsilon(\partial_{t_0} + \dot{x}_0\partial_{x_0})x(t; t_0, x_0), \end{aligned}$$

where we used $x_0 = x(t_0)$ and so $\delta x_0 = \dot{x}(t_0)\varepsilon$. We can rewrite the above equation as

$$(\partial_{t_0} + f(x_0)\partial_{x_0})x(t) = 0, \tag{12}$$

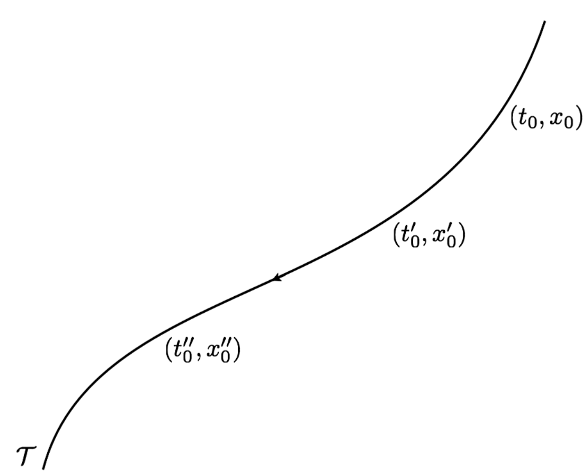

FIG. 1. Trajectory of the dynamical system (9). The couples (t_0, x_0), (t_0', x_0') and (t_0'', x_0'') are possible boundary conditions associated to the trajectory $\mathcal{T}$.

[6]The following arguments apply equally well if one considers an *Ansatz* $\Gamma_k = \sum_i g_i O_i$ parametrized by generic operators O_i.

where we exploited $\dot{x}_0 = f(x_0)$. Thus, Eq. (12) entails in a compact form the invariance under changes of boundary conditions that we just discussed and that is represented in Fig. 1 by the equivalent couples of boundary conditions $(t_0, x_0), (t'_0, x'_0), \ldots$. Now, having in mind RG trajectories, one may wish to associate the boundary condition to a condition given at a microscopic scale Λ. However, this is not necessary and an equivalent condition can be given at any other point along the flow. For this reason we referred to μ as a floating normalization point; in principle it can be chosen anywhere along the trajectory.

In fact, there is nothing preventing us from applying this reasoning directly to our system of equation describing the RG flow of the EAA. Let us denote with the superscript (R) all the boundary conditions; for example $\tilde{g}^{(R)} = \tilde{g}(\mu)$ is the boundary condition associated to $\tilde{g}(k)$. The analogue of Eq. (12) applied to the solution Γ_k of the flow equation is

$$\mu \frac{d}{d\mu}\Gamma_k[\varphi] = \left(\mu\partial_\mu + \tilde{\beta}_i(\tilde{g}^{(R)})\frac{\partial}{\partial \tilde{g}_i^{(R)}} + \partial_{\log\mu} Z^{(R)}\frac{\partial}{\partial Z^{(R)}}\right)\Gamma_k[\varphi] = 0, \quad (13)$$

where $\partial_{\log\mu} Z^{(R)} \equiv \partial_t Z_k(\mu)$. Recalling that in the *Ansatz* (8) the field is always accompanied by a factor $Z_k^{1/2}$ we rewrite Eq. (13) as follows:

$$0 = \left(\mu\partial_\mu + \tilde{\beta}_i(\tilde{g}^{(R)})\frac{\partial}{\partial \tilde{g}_i^{(R)}} + \frac{1}{2}\partial_{\log\mu} Z^{(R)}(Z^{(R)})^{-1}\varphi\cdot\frac{\delta}{\delta\varphi}\right)\Gamma_k[\varphi]. \quad (14)$$

In going from (13) to (14) we have been able to trade $\partial/\partial Z^{(R)}$ with $\delta/\delta\varphi$ by exploiting the fact that the flow equation for the wave function renormalization has the form

$$\partial_t Z_k = f(\tilde{g}) Z_k$$

and the associated solution is

$$Z_k^{(\mathrm{sol})} = Z^{(R)} \exp\left[\int_\mu^k f(\tilde{g}(k'))\frac{dk'}{k'}\right]. \quad (15)$$

This tells us that the solution $Z_k^{(\mathrm{sol})}$ of the wave function renormalization is proportional to the boundary condition $Z^{(R)}$ implying that the previous rewriting is correct.

If we think of the EAA as the sum of all the proper vertices,[7]

$$\Gamma_k = \sum_n \frac{\Gamma_k^{(n)}}{n!}\varphi^n,$$

and we functionally differentiate Eq. (14) n times with respect to φ before setting $\varphi = 0$, we obtain

$$\left(\mu\partial_\mu + \beta_i^{(R)}\frac{\partial}{\partial \tilde{g}_i^{(R)}} - \frac{n}{2}(-(Z^{(R)})^{-1}\partial_{\log\mu} Z^{(R)})\right)\Gamma_k^{(n)} = 0. \quad (16)$$

Equation (16) looks exactly like the Callan-Symanzik equation of usual quantum field theory once $k \to 0$.

Let us now discuss in some detail the meaning of Eq. (16) in the limit $k \to 0$ in presence of an IR fixed point. In the fixed point regime the dimensionless couplings $\tilde{g}$ tend to a constant $\tilde{g}_*$ while the equation for the wave function renormalization takes the simple form

$$\partial_t Z_k = f(\tilde{g}_*) Z_k. \quad (17)$$

Let us impose the boundary condition in the fixed point regime, i.e. μ is enough small that the running of Z_k is given approximatively by Eq. (17) (we will come back to an arbitrary μ in a moment). In this limit the solution of the Eq. (17) is particularly simple and reads

$$Z_k^{(\mathrm{sol})} = Z^{(R)}\left(\frac{k}{\mu}\right)^{-\eta}, \quad (18)$$

where $\eta = -f(\tilde{g}_*)$ and $Z^{(R)}$ is the value of $Z_k^{(\mathrm{sol})}$ at $k = \mu$. As we will see in a moment $\eta/2$ can be identified with the anomalous dimension of the field at the fixed point. In the limit $k \to 0$ all the couplings $\tilde{g}(k)$ appearing in the EAA tend to a constant and they no longer depend on μ. However, there is still some μ dependence in the EAA coming from the wave function renormalization. In this case the invariance under changes in the normalization point μ tells us that

$$\left(\mu\partial_\mu - \frac{n}{2}\eta\right)\Gamma_{k\to 0}^{(n)}[\varphi] = 0. \quad (19)$$

This suggests to identify the anomalous dimension at the fixed point with $\eta = -Z_k^{-1}\partial_t Z_k$ for $k \to 0$. Indeed using (19) and dimensional analysis for a given proper vertex, i.e.

$$\left(\mu\partial_\mu + p\partial_p + n\frac{d-2}{2}\right)\Gamma_{k\to 0}^{(n)}[\varphi] = 0, \quad (20)$$

one can deduce the (p, μ) dependence of $\Gamma_{k\to 0}^{(n)}$ (in (20) we consider functional derivative with respect to the Fourier transform of the field). For instance if we consider $\Gamma_{k\to 0}^{(2)}$ and factor out the overall delta function entailing momentum conservation, we obtain

$$\Gamma^{(2)}[\varphi] \sim p^{2-\eta}.$$

[7]Actually one should consider $\tilde{\Gamma}_k$; this modifies only the two-point function for $k > 0$.

This shows that η can indeed be identified with the anomalous dimension.

At this point the reader may wonder if the fact of having chosen boundary conditions for an arbitrarily small μ played any role in our arguments. The answer is no—it is by no means necessary to impose boundary conditions for μ arbitrarily small—this just leads to a quicker argument. To see this, let us denote $Z^{(R)}$ the boundary condition imposed at some other scale μ. Again in the fixed point regime the couplings in the EAA do not depend any longer on the scale μ or on the boundary condition $\tilde{g}^{(R)}$. Now, however, the wave function renormalization has no longer the simple structure of Eq. (18). In particular the solution $Z_k^{(\mathrm{sol})}$ depends also on the boundary condition $\tilde{g}^{(R)}$ since the integral in the exponent of (15) depends on it. Therefore, for $k \to 0$ we have a solution of the following form:

$$Z_k^{(\mathrm{sol})} = Z^{(R)} y(g^{(R)}) \left(\frac{k}{\mu}\right)^{-\eta} \tag{21}$$

for some function $y(g^{(R)})$.[8] The invariance under floating normalization point transformations tells us that

$$\left(\mu\partial_\mu + \beta_i^{(R)} \frac{\partial}{\partial \tilde{g}_i^{(R)}} - \frac{n}{2}(-(Z^{(R)})^{-1}\partial_{\log\mu} Z^{(R)})\right)\Gamma_{k\to 0}^{(n)} = 0,$$

where the derivatives with respect to the couplings $\tilde{g}_i^{(R)}$ act only on $Z_k^{(\mathrm{sol})}$ since all the other couplings are approaching their fixed point values. Exploiting this we can rewrite the above equation as

$$\left(\mu\partial_\mu + \frac{n}{2}\beta_i^{(R)} \frac{\partial y(\tilde{g}^{(R)})}{\partial \tilde{g}_i^{(R)}} \frac{1}{y(\tilde{g}^{(R)})} - \frac{n}{2}(-(Z^{(R)})^{-1}\partial_{\log\mu} Z^{(R)})\right)\Gamma_{k\to 0}^{(n)} = 0$$

or equivalently as

$$\left(\mu\partial_\mu + \frac{n}{2}\left[\beta_i^{(R)} \frac{\partial y(\tilde{g}^{(R)})}{\partial \tilde{g}_i^{(R)}} \frac{1}{y(\tilde{g}^{(R)})} + f(\tilde{g}^{(R)})\right]\right)\Gamma_{k\to 0}^{(n)} = 0.$$

The above equation tells us that we can express the anomalous dimension also as follows:

$$-\eta = \beta_i^{(R)} \frac{\partial y(\tilde{g}^{(R)})}{\partial \tilde{g}_i^{(R)}} \frac{1}{y(\tilde{g}^{(R)})} + f(\tilde{g}_i^{(R)}). \tag{22}$$

Although not obvious, the right-hand side of Eq. (22) must be independent of $\tilde{g}^{(R)}$. We checked relation (22) in a few examples.[9] It is clear now why choosing the boundary condition with μ arbitrarily small is convenient: if we let $\mu \to 0$ the first term on the right-hand side of (22) vanishes and we are left with $-\eta = f(\tilde{g}_*) = Z_k^{-1}\partial_t Z_k|_{k=0}$. Thus our arguments suggest to identify the anomalous dimension with $-Z_k^{-1}\partial_t Z_k$ in the limit $k \to 0$.

With respect to the standard Callan-Symanzik equation we derived Eq. (16) using dimensionless couplings. However, nothing prevents us from repeating the same reasoning for the dimensionful couplings. It is just more convenient to work with dimensionless quantities since those are the ones of interest in the fixed point regime of the EAA. Moreover, if one repeats our reasoning in the case of dimensionful couplings, one notes that Eq. (20) involves new terms of the type $d_g g^{(R)}\partial/\partial g^{(R)}$, where d_g is the mass dimension of the coupling. However, after eliminating μ one notices that it is just the dimensionless beta function which enters in the scaling equation. Equations describing μ invariance in a somewhat different manner are also known in the Wilson-Polchinski framework; see in particular [20] for a discussion including a choice of parametrization scheme related to the MS scheme in dimensional regularization.

A further motivation for the identification of $\eta = -Z_k^{-1}\partial_t Z_k|_{k=0}$ as the anomalous dimension has been provided in [21]. Thus let us briefly consider how the arguments in Appendix A of [21] apply to our framework since we will employ similar arguments in the next section to provide a further reason in favor of our definitions. Dimensional analysis tells us that

$$\frac{\Gamma_k^{(2)}(p,k,\mu)}{\Gamma_k^{(2)}(p',k,\mu)} = \hat{f}\left(\frac{p'}{p},\frac{p}{k},\frac{p'}{\mu}\right).$$

The arguments of $\hat{f}$ are three possible independent ratios; all the other ratios can be obtained from them. Now we take k to be sufficiently small (fixed point regime) and observe that in the EAA the μ dependence is contained only in the wave function renormalization factors. Thus, in the above ratio, these wave function renormalizations cancel against each other and we have

$$\frac{\Gamma_k^{(2)}(p,k,\mu)}{\Gamma_k^{(2)}(p',k,\mu)} = \hat{f}\left(\frac{p'}{p},\frac{p}{k}\right).$$

Setting $p' = 0$ we obtain

$$\Gamma_k^{(2)}(p,k,\mu) = \Gamma_k^{(2)}(0,k,\mu)\hat{f}\left(0,\frac{p}{k}\right).$$

[8] More formally this function can be defined via $y(g^{(R)}) \equiv \lim_{k\to 0}(Z^{(R)})^{-1}(\frac{k}{\mu})^{-f(\tilde{g}_*)} Z_k^{(\mathrm{sol})}$.

[9] The fact that the right-hand side of Eq. (22) is just a constant can be understood applying the μ invariance operator directly on $Z_{k\to 0}^{(\mathrm{sol})}$: the explicit μ dependence comes solely from the factor μ^η in (21) and, writing down all the terms, one finds Eq. (22).

Let us denote $f(\frac{p}{k}) \equiv \hat{f}(0, \frac{p}{k})$ and observe that in the fixed point regime

$$\Gamma_k^{(2)}(0, k, \mu) \sim k^2 \left(\frac{k}{\mu}\right)^{-\eta}.$$

Therefore we finally have

$$\Gamma_k^{(2)}(p, k, \mu) \sim k^2 \left(\frac{k}{\mu}\right)^{-\eta} f\left(\frac{p}{k}\right).$$

In order for this expression to be well defined in the critical regime, i.e. $k = 0$, we require that $f \sim (p/k)^{2-\eta}$ and thus

$$\Gamma_0^{(2)}(p, 0, \mu) \sim p^2 \left(\frac{p}{\mu}\right)^{-\eta}.$$

This completes the argument to interpret η as the anomalous dimension. These considerations also provide a justification for the argument telling us that $\Gamma_k^{(2)}(p = k) \sim k^{2-\eta}$, which is sometime used in the FRG literature.

B. Fixed point and anomalous dimension of composite operators

In this section we extend our reasoning to the modified EAA $\Gamma_k[\varphi, \varepsilon]$ introduced in Sec. II. We are interested in the scaling of composite operators at a fixed point. Let us consider a composite operator parametrized via a sum of various operators. To gain some insights we shall consider perturbation theory as a first hint. We parametrize a renormalized composite operator of mass dimension d_O with the sum of all possible composite operators of mass dimension smaller or equal to d_O. For instance in a six-dimensional φ^3 theory one can consider [5]

$$[\varphi^2] = Z_a \frac{\varphi^2}{2} + (Z_b m^2)\varphi + Z_c \Delta\varphi. \qquad (23)$$

To obtain complete information one should define simultaneously all the composite operators which form the basis $\{O_i\}$:

$$[O_i] = Z_{ij} O_j.$$

For instance in the example of the operator $[\varphi^2]$ one should consider [5][10]

$$\begin{pmatrix} [\frac{1}{2}\varphi^2] \\ [\varphi] \\ [\Delta\varphi] \end{pmatrix} = \begin{pmatrix} Z_a & Z_b m^2 & Z_c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\varphi^2 \\ \varphi \\ \Delta\varphi \end{pmatrix}.$$

[10]The known scale dependence of the one loop matrix elements can be easily computed via Eq. (5).

Let us note that the entries of Z_{ij} are in general dimensionful. In dimensional regularization dimensionful factors of Z_{ij} are due to the presence of dimensionful couplings like the mass. In our scheme, however, it must be generically expected that some dimensionful factors may depend on the scale k itself.

Parametrizations of composite operators similar to the ones in Eqs. (7) and (23) can be straightforwardly adopted in our flow equation (5). The difference with the standard framework is due to the fact that, in our scheme, closed families of operators under renormalization are generically infinite dimensional and not just a finite set like it happens when using dimensional regularization. To make progress let us consider the flow equation for composite operators where we insert (in)finitely many operators adding $\varepsilon_i Z_{ij} O_j$ to the EAA. After denoting G_k the regularized inverse propagator,

$$G_k \equiv (\Gamma_k^{(2)} + R_k)^{-1},$$

we rewrite Eq. (5) as

$$\partial_t(\varepsilon_i Z_{ij} O_j) = -\frac{1}{2} G_k \cdot (\varepsilon_i Z_{ij} O_j^{(2)}) \cdot G_k \cdot \partial_t R_k. \qquad (24)$$

Now we want to extend to $\Gamma_k[\varphi, \varepsilon]$ Eqs. (14) and (16). This will allow us to identify which are the scaling dimensions of the composite operators. The reasoning of Sec. III A applies straightforwardly to the ε-dependent EAA. Let $\{O_i\}$ be the set of operators which parametrize the composite operators. In analogy with the example (7) a renormalized composite operator is defined as

$$[O_i](\varphi) = Z_{ij} O_j(Z_k^{1/2} \varphi).$$

Since we shall be interested in just one insertion of a composite operator we will limit ourselves to consider sources ε such that $\varepsilon^2 = 0$. In this case we can write

$$\Gamma_k[\varphi, \varepsilon] = \Gamma_k[\varphi] + \varepsilon \cdot Z \cdot O.$$

Let $Z_{ij}^{(R)}$ be the boundary condition associated to the flow of the mixing matrix Z_{ij}. Applying the reasoning of Sec. III A and treating $Z_{ij}^{(R)}$ as all the other boundary conditions we obtain

$$\mu \frac{d}{d\mu} \Gamma_k[\varphi] = \left(\mu \partial_\mu + \beta_i^{(R)} \frac{\partial}{\partial \tilde{g}_i^{(R)}} + \partial_{\log\mu} Z^{(R)} \frac{\partial}{\partial Z^{(R)}} + \partial_{\log\mu} Z_{ij}^{(R)} \cdot \frac{\partial}{\partial Z_{ij}^{(R)}} \right) \Gamma_k[\varphi, \varepsilon] = 0,$$

where $\partial_{\log\mu} Z_{ij}^{(R)} = \partial_t Z_{ij}(k = \mu)$. In Sec. III A we have been able to trade the partial derivative with respect to $Z^{(R)}$ for a functional derivative with respect to the field. Here we would like to do something similar and trade the derivative with respect to $Z_{ij}^{(R)}$ for a functional derivative with respect to the source ε_i. In Sec. III A this step was straightforward since the solution of the wave function renormalization $Z_k^{(\mathrm{sol})}$ is proportional to $Z^{(R)}$. The situation for the solution of the mixing matrix $Z_{ij}^{(\mathrm{sol})}$ is slightly more complicated. Taking a functional derivative with respect to ε_i in Eq. (24) one obtains that the flow of the mixing matrix has the following form:

$$Z_{ij}^{-1}\partial_t Z_{jk} = f_{ik}(\tilde{g}, k). \tag{25}$$

The solution of this type of equation is related to a k-ordered exponential matrix (Dyson's series). The crucial point, however, is the following: a generic boundary condition $Z_{ij}^{(R)}$ appears in the solution via $Z_{ij}^{(\mathrm{sol})} = Z_{im}^{(R)} M_{mj}$ for some matrix M_{mj}. This fact allows us to trade the derivative $\partial/\partial Z_{ij}^{(R)}$ with a suitable functional derivative with respect to the source ε. In particular we can rewrite

$$\mu \frac{d}{d\mu}\Gamma_k[\varphi,\varepsilon] = \left[\mu\partial_\mu + \beta_i^{(R)}\frac{\partial}{\partial \tilde{g}_i^{(R)}} + \frac{1}{2}\frac{\partial_{\log\mu} Z^{(R)}}{Z^{(R)}}\varphi\cdot\frac{\delta}{\delta\varphi} + \varepsilon_i\cdot\gamma_{Z,ij}\cdot\frac{\delta}{\delta\varepsilon_j}\right]\Gamma_k[\varphi,\varepsilon] = 0, \tag{26}$$

where

$$\gamma_{Z,ik} \equiv \partial_{\log\mu} Z_{ij}^{(R)} (Z^{(R)})_{jk}^{-1}.$$

Let us impose the boundary condition in the fixed point regime, i.e. at μ small enough that the running of Z_{ij} is given by Eq. (25) with $\tilde{g} = \tilde{g}_*$. Then, using the result of Sec. III A for η, in the fixed point regime we can rewrite (26) as

$$\mu \frac{d}{d\mu}\Gamma_k[\varphi,\varepsilon] = \left(\mu\partial_\mu - \frac{1}{2}\eta\varphi\cdot\frac{\delta}{\delta\varphi} + \varepsilon_i\cdot\gamma_{Z,ij}\cdot\frac{\delta}{\delta\varepsilon_j}\right)\Gamma_{k\to 0}[\varphi,\varepsilon] = 0. \tag{27}$$

In order to discuss the scaling associated to composite operators we need to take into account both Eq. (27) and dimensional analysis. This is fully analogous to what we did in Sec. III A for the wave function renormalization. However, there is now a crucial difference, namely the fact that the boundary conditions $Z_{ij}^{(R)}$ are now dimensionful parameters which must be taken into account in the dimensional analysis. Let d_i be the mass dimension of the operator O_i, then the matrix element $Z_{ij}^{(R)}$ has mass dimension $(d_i - d_j)$. Focusing our attention only on one insertion of a composite operator, i.e. taking a single functional derivative with respect to the source ε_i and setting ε_i to zero, we obtain

$$\left(\mu\partial_\mu - x\partial_x + (d_i - d_j)Z_{ij}^{(R)}\frac{\partial}{\partial Z_{ij}^{(R)}} - d_i\delta_{ij}\right)\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0,$$

where the last term takes into account the mass dimension of the operator O_j. Introducing the matrix

$$D_{ij} = d_i\delta_{ij},$$

we can rewrite the third term in the brackets via $(d_i - d_j)Z_{ij}^{(R)} = D_{ia}Z_{aj}^{(R)} - Z_{ia}^{(R)}D_{aj}$ and trade the derivative of $Z_{ij}^{(R)}$ with $(Z_{ij}^{(R)})^{-1}$. In this manner we obtain (in matrix notation)

$$(\mu\partial_\mu - x\partial_x + (DZ^{(R)} - Z^{(R)}D)(Z^{(R)})^{-1} - D)\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0,$$

$$(\mu\partial_\mu - x\partial_x - Z^{(R)}D(Z^{(R)})^{-1})\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0.$$

Now we eliminate μ from the above equation using (27) and we obtain

$$(x\partial_x + Z^{(R)}D(Z^{(R)})^{-1} + \gamma_Z)\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0. \tag{28}$$

The eigenvalues of the matrix $Z^{(R)}D(Z^{(R)})^{-1} + \gamma_Z$ yield the full, i.e. classical plus anomalous, dimensions of the scaling operators. To see this let us assume that $Z^{(R)}D(Z^{(R)})^{-1} + \gamma_Z$ is diagonalizable, i.e. $Z^{(R)}D(Z^{(R)})^{-1} + \gamma_Z = AEA^{-1}$, where $E_{ab} = e_a\delta_{ab}$ is the eigenvalue matrix. Then we can manipulate Eq. (28) as follows:

$$(x\partial_x\delta_{ij} + A_{ia}E_{ab}A_{bj}^{-1})\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0,$$

$$(x\partial_x + e_b)A_{bj}^{-1}\frac{\delta\Gamma_{k\to 0}}{\delta\varepsilon_j} = 0.$$

The last equation tells us that $A_{bj}^{-1}[O_j]$ is a scaling operator and we can identify e_b with its full dimension, which we denote Δ_b. In analogy with the case of the wave function renormalization, this shows that the full dimensions of the scaling operators are given by the eigenvalues of the matrix $ZDZ^{-1} + \gamma_Z$ in the limit $k \to 0$.

It turns out convenient to consider also the dimensionless analogue of Z_{ij}, which we denote N_{ij}. Introducing the matrix

$$K_{ij} = k^{d_i}\delta_{ij}, \qquad d_i \equiv \text{mass dimension of } O_i(\varphi), \tag{29}$$

the mixing matrix Z_{ij} can be rewritten via a dimensionless matrix N_{ij} defined as follows:

$$N_{il} \equiv K_{ij}^{-1} Z_{jk} K_{kl}.$$

In particular we want to show that the spectrum of $ZDZ^{-1} + \gamma_Z$ is the same as $D + \gamma_N$, where $\gamma_N = \partial_t N N^{-1}$. First we observe that

$$ZDZ^{-1} + \gamma_Z = ZDZ^{-1} + \partial_t Z Z^{-1} = Z(D + Z^{-1}\partial_t Z)Z^{-1}.$$

Thus the matrix $ZDZ^{-1} + \gamma_Z$ has the same spectrum as $D + Z^{-1}\partial_t Z$. Now we shall check that indeed these matrices yield the same scaling dimensions, i.e. same spectrum, as $D + \partial_t N N^{-1}$. In matrix notation we have

$$\begin{aligned} Z^{-1}\partial_t Z &= (KN^{-1}K^{-1})\partial_t(KNK^{-1}) \\ &= Z^{-1}DZ + KN^{-1}\partial_t N K^{-1} - D \\ &= Z^{-1}K(D + \gamma_N)K^{-1}Z - D, \end{aligned}$$

from which our claim follows. Similarly, the matrix $N^{-1}DN + N^{-1}\partial_t N$ has the same spectrum as $D + \gamma_N$.

All these manipulations are quite formal and we would like to provide a more intuitive argument for our definition. Let us consider the composite operator $[\varphi^2]$ and parametrize it via the following simple *Ansatz*: $[\varphi^2] = Z_{\varphi^2} Z_k \varphi^2$. Now we shall follow the arguments used at the end of Sec. III A. Let us consider the dimensionless quantity $\Gamma_k^{(2,1)}$, where in the superscript we indicated the number of functional derivatives with respect to φ and ε, respectively. $\Gamma_k^{(2,1)}$ satisfies

$$\Gamma_k^{(2,1)} = \frac{G_k^{(2,1)}(p_1, p_2)}{G_k(p_1)G_k(p_2)},$$

where

$$G_k^{(2,1)}(p_1, p_2) = \int dx_1 dx_2 dy e^{ip_1x_1 + ip_2x_2}\langle[\varphi^2(y)]\varphi(x_1)\varphi(x_2)\rangle.$$

We consider $p = p_1 = -p_2$ and observe that in the fixed point regime

$$\frac{\Gamma_k^{(2,1)}(p, k, \mu)}{\Gamma_k^{(2,1)}(p', k, \mu)} = \hat{f}\left(\frac{p'}{p}, \frac{p}{k}\right),$$

where the right-hand side does not depend on μ since the various wave functions renormalizations cancel against each other in the ratio. Once again we have

$$\Gamma_k^{(2,1)}(p, k, \mu) = \Gamma_k^{(2,1)}(p', k, \mu)\hat{f}\left(\frac{p'}{p}, \frac{p}{k}\right),$$

and setting $p' = 0$ we obtain

$$\Gamma_k^{(2,1)}(p, k, \mu) = \Gamma_k^{(2,1)}(0, k, \mu)\hat{f}\left(0, \frac{p}{k}\right).$$

Let us denote $f(\frac{p}{k}) \equiv \hat{f}(0, \frac{p}{k})$ and $\gamma_{\varphi^2} = Z_{\varphi^2}^{-1}\partial_t Z_{\varphi^2}$. We observe that in the fixed point regime

$$\Gamma_k^{(2,1)}(0, k, \mu) \sim \left(\frac{k}{\mu}\right)^{\gamma_{\varphi^2} - \eta}.$$

Repeating the reasoning of Sec. III A we obtain

$$\Gamma_0^{(2,1)}(p, 0, \mu) \sim \left(\frac{p}{\mu}\right)^{\gamma_{\varphi^2} - \eta}.$$

Thus we identify $\gamma_{\varphi^2} = Z_{\varphi^2}^{-1}\partial_t Z_{\varphi^2}$ with the anomalous dimension of $[\varphi^2]$ as expected.

C. Scaling dimensions from the flow equation

In the previous section we deduced the quantities which define the full dimension of scaling operators. Here we shall show how these quantities are most easily computed in the FRG framework. Let us recall that at the fixed point it is convenient to work via dimensionless objects which are defined using the cutoff and suitable rescalings. In particular we define

$$\begin{aligned} \varphi(x) &= \tilde{\varphi}(\tilde{x})k^{\frac{d-2}{2}}, \\ \varepsilon_i(x) &= \tilde{\varepsilon}_i(\tilde{x})k^{d-d_i}, \\ x &= \tilde{x}k^{-1}. \end{aligned}$$

Being $\varphi(x)$, $\varepsilon(x)$ and x independent of the scale k by definition, we obtain

$$\begin{aligned} \partial_t\tilde{\varphi}(\tilde{x}) &= -\left(\frac{d-2}{2}\right)\tilde{\varphi}(\tilde{x}), \\ \partial_t\tilde{\varepsilon}_i(\tilde{x}) &= -(d - d_i)\tilde{\varepsilon}_i(\tilde{x}), \\ \partial_t\tilde{x} &= \tilde{x}. \end{aligned}$$

An operator $O = \partial_x^m \varphi^n$ will satisfy

$$\partial_t(\partial_x^m \varphi^n) = 0,$$

implying

$$\partial_t\left(k^m k^{\frac{d-2}{2}n}\partial_{\tilde{x}}^m\tilde{\varphi}^n\right) = 0,$$

$$\partial_t(\partial_{\tilde{x}}^m\tilde{\varphi}^n) = -d_O(\partial_{\tilde{x}}^m\tilde{\varphi}^n),$$

where $d_O = m + \frac{d-2}{2}n$. For a LPA truncation of $\tilde{O}_l$ we can rewrite this last term also as[11]

[11]In the LPA′ this term gets a further contribution coming from the anomalous dimension. If derivatives are present, it might be convenient to express this term via $\frac{\delta\tilde{O}_l}{\delta\tilde{\phi}}(-\frac{d-2+\eta}{2}\tilde{\phi}) - \tilde{p}^\mu\frac{\partial\tilde{O}_l}{\partial\tilde{p}^\mu}$.

$$\partial_t \tilde{O}_l = \frac{\delta \tilde{O}_l}{\delta \tilde{\varphi}} \left(-\frac{d-2}{2} \tilde{\varphi} \right).$$

In order to express the flow equation (24) in terms of dimensionless variables it is convenient to perform the following manipulation:

$$\int d^d x \varepsilon_i Z_{ij} O_i = \int d^d \tilde{x} \tilde{\varepsilon}_i K_{ij}^{-1} Z_{jk} K_{kl} \tilde{O}_l = \int d^d \tilde{x} \tilde{\varepsilon}_i N_{il} \tilde{O}_l.$$

The left-hand side of the flow equation reads

$$\partial_t \int d^d x \varepsilon_i Z_{ij} O_i = \int d^d \tilde{x} \left[d_i \tilde{\varepsilon}_i N_{il} \tilde{O}_l + \tilde{\varepsilon}_i (\partial_t N_{il} N_{lm}^{-1}) N_{mn} \tilde{O}_n - \frac{d-2}{2} \tilde{\varepsilon}_i N_{il} \frac{\delta \tilde{O}_l}{\delta \tilde{\varphi}} \tilde{\varphi} \right]. \quad (30)$$

In the above expression the first term comes from the logarithmic k derivative of $d^d\tilde{x}$ and $\tilde{\varepsilon}_i$ and the third term from the derivative acting on $\tilde{O}_l$. In the second term of (30) we inserted an identity $N^{-1} \cdot N$ in order to make explicit the presence of $\gamma_N = \partial_t N \cdot N^{-1}$, which enters in the definition of scaling dimension as shown in the previous section. At this point it proves convenient to introduce the new basis of operators $\tilde{B}_i = N_{il} \tilde{O}_l$ and bring the last term in (30) to the right-hand side of (24). After these manipulations we are left with an equation of the form

$$\int d^d \tilde{x} [\tilde{\varepsilon}_i (d_i \delta_{ij} + \partial_t N_{il} N_{lj}^{-1}) \tilde{B}_j] = \int d^d \tilde{x} \tilde{\varepsilon}_i \frac{\delta \tilde{B}_i}{\delta \tilde{\varphi}} \left(\frac{d-2}{2} \tilde{\varphi} \right) + \mathrm{Tr}[\cdots], \quad (31)$$

where the last term indicates the right-hand side of Eq. (24). Clearly the right-hand side of Eq. (24) is proportional to

$$N_{il} \frac{\delta^2 \tilde{O}_l}{\delta \tilde{\varphi}^2} = \frac{\delta^2 \tilde{B}_i}{\delta \tilde{\varphi}^2}.$$

We observe that the quantity in round brackets on the left-hand side of (31) is precisely the matrix $D + \partial_t N \cdot N^{-1}$, whose eigenvalues are the full dimensions of the scaling operators. Let us assume that the matrix $d_i \delta_{ij} + \partial_t N_{il} N_{lj}^{-1}$ is diagonalizable and let us denote λ_i, Λ_{ij} and A_{ij} the eigenvalues, the eigenvalue matrix and the eigenvector matrix, respectively. After taking a functional derivative with respect to the source $\tilde{\varepsilon}_i$ we can rewrite Eq. (31) as follows:

$$A_{ia} \Lambda_{ab} A_{bj}^{-1} \tilde{B}_j = \frac{\delta \tilde{B}_i}{\delta \tilde{\varphi}} \left(\frac{d-2}{2} \tilde{\varphi} \right) - \frac{1}{2} G_k \cdot \frac{\delta^2 \tilde{B}_i}{\delta \tilde{\varphi}^2} \cdot G_k \cdot \partial_t R_k, \quad (32)$$

where the last term in the right-hand side is meant solely to represent schematically the structure of the "trace term" in (31). At this point it is convenient to multiply Eq. (32) by A_{mi}^{-1} and introduce a new set of operators $D_m \equiv A_{mi}^{-1} \tilde{B}_i$. Writing explicitly the right-hand side of (32) in the LPA we have

$$\lambda_i D_i(\tilde{\varphi}) = D_i'(\tilde{\varphi}) \left(\frac{d-2}{2} \tilde{\varphi} \right) - c_d \frac{D_i''(\tilde{\varphi})}{(1 + \tilde{U}_k''(\tilde{\varphi}))^2}, \quad (33)$$

where $c_d^{-1} = (4\pi)^{d/2} \Gamma(d/2 + 1)$ (more details are given in Sec. IV). Remarkably Eq. (33) is expressed directly in terms of the full dimension λ_i of the scaling operators and thus its solutions yield directly the scaling dimensions of the operator content of the fixed point theory. In Sec. IV we will see that, considering composite operators of the form $O(\varphi)$ within the LPA′, the eigenvalues λ_i are directly connected to the critical exponents θ_i via $\lambda_i = d - \theta_i$. Note also that one can also arrive at Eq. (33) by taking a functional derivative with respect to $\tilde{\varepsilon}_i$ of (30) and applying the matrix N^{-1} to it; in this case one diagonalizes the matrix $N^{-1} D N + N^{-1} \partial_t N$.

Let us conclude this section by commenting on some possible contacts with other works in the literature. In particular it would be nice to set up our discussion in a geometric language along the lines considered in [22–24] (see also [25] for a slightly different approach). In these works, roughly speaking, composite operators are thought of as living in the tangent space associated to the theory space. The quantity $\gamma_a^b \equiv \partial_{g_a} \beta^b$ naturally appears and one can derive a Callan-Symanzik type of equation by considering the RG as a one-parameter group of diffeomorphisms [22]. The matrix γ_a^b can be interpreted as the anomalous dimension mixing matrix. In our approach this can be understood via the following argument. Let us limit ourselves to parametrize composite operators via operators which are not total derivatives and take the sources ε_i to be constants. Moreover let us parametrize the EAA via $\Gamma_k = \sum_i g_i O_i$. From Eq. (24) we obtain

$$Z_{ia}^{(-1)} (\partial_t Z_{aj}) O_j = -\frac{1}{2} \frac{1}{\Gamma_k^{(2)} + R_k} \cdot (O_i^{(2)}) \cdot \frac{1}{\Gamma_k^{(2)} + R_k} \cdot \partial_t R_k. \quad (34)$$

Now let us write the flow equation (2) for $\Gamma_k = \sum_i g_i O_i$:

$$\sum_j \beta^j O_j = \frac{1}{2} \frac{1}{\sum_j g_j O_j^{(2)} + R_k} \cdot \partial_t R_k.$$

Taking a derivative with respect to g_i we obtain

$$\sum_j \partial_{g_i}\beta^j O_j = -\frac{1}{2}\frac{1}{\Gamma_k^{(2)}+R_k}\cdot O_i^{(2)}\cdot\frac{1}{\Gamma_k^{(2)}+R_k}\cdot\partial_t R_k. \tag{35}$$

Comparing (34) to (35) one concludes that $\gamma_a^b = \partial_{g_a}\beta^b$ equals $Z_{ac}^{(-1)}\partial_t Z_{cb}$. However, let us note once again that, for the above argument to go through, we had to neglect some total derivative operators whose contribution might be important. Finally more work is needed to spell out the possible geometrical interpretations of our arguments; we hope to come back to these issues in the future.

IV. SCALING SOLUTIONS AND COMPOSITE OPERATORS

In this section we consider approximate solutions of the fixed point equation and how one can use them to estimate the anomalous dimensions of various operators at the fixed point. We parametrize composite operators as functions of the field but not of its derivatives. As we shall see, this choice can be put in one-to-one correspondence with results known within the LPA′ truncation and we shall comment them in view of the discussion of the previous section.

Scaling solutions are (approximate) solutions of the flow equation which include infinitely many couplings; in the LPA′ case they are generic functions of the field. In order to find such solutions one has typically to solve a differential equation coming from the flow equation and integrate it numerically. More details are given in Sec. IV A. In Sec. IV B we discuss some simple solutions of the composite operator flow equation. In Appendix B we consider some numerical results obtained in the literature within the LPA′ truncation for some statistical systems at criticality and discuss them in connection to our framework.

A. Scaling solutions

We consider a scalar field theory and limit our discussion to the so-called LPA′ truncation, where we take into account up to two derivatives of the field and a generic potential including the wave function renormalization:

$$\Gamma_k[\varphi] = \int\left[\frac{Z_k}{2}\partial_\mu\varphi\partial^\mu\varphi + U_k(Z_k^{1/2}\varphi)\right].$$

Let us denote $\tilde\phi = Z_k^{1/2}\tilde\varphi$. The flow equation for the potential in dimensionless variables is given by (throughout this work we consider the optimized cutoff [8]) [26,27]

$$\partial_t\tilde U_k = -d\tilde U_k + \frac{d-2+\eta}{2}\tilde\phi\tilde U_k' + c_d\frac{1-\frac{\eta}{d+2}}{1+\tilde U_k''},$$
$$\eta = -\frac{\partial_t Z_k}{Z_k} = c_d\frac{(\tilde U_k''')^2}{(1+\tilde U_k'')^4}, \tag{36}$$

where $c_d^{-1} = (4\pi)^{d/2}\Gamma(d/2+1)$ and the field has been set to its minimum in the equation for η. The prime denotes a derivative with respect to the argument. We employed the optimized cutoff in order to derive Eq. (36) without resorting to a numerical integration of the flow equation. We shall not study the regulator dependence of Eq. (36) nor shall we consider any optimization procedure. Regarding these issues, we refer the interested reader to the literature [7–11].

As far as the results obtained with this truncation are concerned the situation is the following: some of the critical exponents are already in (relative) quantitative agreement with exact or best values available while others are less precise. The anomalous dimension η has usually a rather large error. This is a known shortcoming of the LPA′ truncation which can be overcome with more general *Ansätze* and/or employing more refined truncation schemes as those developed in [21,28].

It is worth to observe that the derivation of the flow equations (5) and (33) is very similar to the linearization of the flow equation itself and thus to the linearized RG and the associated critical exponents. Let us spell out the relation between the equation of eigenperturbation of the RG and composite operators for the LPA′ truncation. Let $\delta\tilde U_k = (k/k_0)^{-\theta}\delta v$ be the eigenperturbation, then the linearized form of the equation reads

$$-\theta\delta\tilde U_k = -d\delta\tilde U_k + \frac{d-2+\eta}{2}\tilde\phi\delta\tilde U_k' - c_d\frac{1}{1+\tilde U_k''}\delta\tilde U_k''\frac{1-\frac{\eta}{d+2}}{1+\tilde U_k''}. \tag{37}$$

It is thus clear that under our approximations—i.e. composite operators are parametrized as functions of the field—the above flow equation for $\delta\tilde U_k$ and Eq. (33) for D_i are the same provided that we make the identification $d-\theta_i = \lambda_i$. This relation can be extended to any truncation with the following caveat. The linearized flow equation for an eigenperturbation $\delta\tilde U_k$ coincides with the one obtained for composite operators once we restrict the composite operator source ε to be constant, hence neglecting total derivatives terms. Note that this implies by no means that such operators are not important. As we shall see in the following, among these total derivative operators we will find the descendant operators of the field φ as well as other interesting scaling operators. Moreover in a nonperturbative setting it is difficult to argue whether or not an operator gives a sizable contribution.

B. Some simple composite operators

In this section we discuss some exact solutions to Eq. (5) and the associated equivalent relations. Let us recall that we express a renormalized composite operator $[O_i]$ via the following generic parametrization: $Z_{ij}O_i(Z_k^{1/2}\varphi) = Z_{ij}O_j(\phi)$; a simple example was given in Eq. (7).

To begin with we note that there are two very simple solutions to Eq. (5). The first solution is the identity operator which does not require any renormalization and as such its anomalous dimension is zero. This solution corresponds to a constant solution of Eq. (37) with eigenvalue $\theta = d$ so that its anomalous dimension, given by $d - \theta$, is simply zero. The second solution is proportional to the field ϕ itself. In this case $Z_{ij} = 1/Z_k^{-1/2}$ implying that the anomalous dimension of the field operator is $-Z_k^{-1}\partial_t Z_k/2 = \eta/2$ as expected. In terms of Eq. (37) the solution is proportional to $\tilde{\phi}$; in this case the last term of (37) vanishes and the eigenvalue problem is solved by $\theta = (d+2-\eta)/2$ implying $\lambda = d - \theta = (d-2+\eta)/2$, this being the full scaling dimension of the field operator.

Along the same lines there are some further operators worth commenting which are solutions of Eq. (5). The operator $O_{\Delta^n} \equiv Z_{O_{\Delta^n}} \Delta^n \phi$, where Δ^n is the nth power of the Laplacian, is a solution of Eq. (5) with $Z_{O_{\Delta^n}} = 1/Z_k^{1/2}$. Indeed once again we observe that the right-hand side of (5) vanishes due to the fact that O_{Δ^n} is made of a single field φ. The anomalous dimension of this operator is thus $\gamma_{O_{\Delta^n}} = \eta/2$ and the full scaling dimension is simply $2n + (d-2+\eta)/2$. This suggests to identify the operators O_{Δ^n} with the descendants of the field φ of a hypothetical conformal field theory (CFT) describing the fixed point (as we discuss in Appendix B 1, in $d = 2$ these operators are secondary operators but are not all of them).

Further consistent solutions for the flow equation for composite operators can be constructed using derivative operators. In particular we want to show that if $[O]$ is a renormalized composite operator, i.e. an exact solution of Eq. (5), then $\Delta[O]$ is also a renormalized composite operator. To see this we have to check that $\Delta[O]$ is also an exact solution of Eq. (5). This can be noted as follows: it is convenient to integrate by parts the source-dependent term of the EAA, namely, $\varepsilon \cdot \Delta[O] = \Delta\varepsilon \cdot [O]$. If we call $\hat{\varepsilon} \equiv \Delta\varepsilon$, we notice that the flow equation for $\Delta[O]$ is nothing but the flow equation for $[O]$ written via the new source $\hat{\varepsilon}$. Since the solution $[O]$ is valid for arbitrary sources and therefore also for $\hat{\varepsilon}$, we have that $\Delta[O]$ is a solution of the flow equation as well.[12] In Appendix B we discuss these operators in connection with the results known for some critical models in two and three dimensions.

Finally in the LPA′ truncation there is always an eigendirection associated to the derivative of the dimensionless potential, $\tilde{U}'$, with critical exponent $\theta = (d-2+\eta)/2$; see [29,30] for a general discussion including both the Wilsonian action and the EAA. Then the scaling dimension of $[U']$ is $\Delta_{[U']} = (d+2-\eta)/2$ whose anomalous part reads $\gamma_{[U']} = -\eta/2$. In our framework we can consider the "equation of motion" operator

$$O_{\delta\Gamma_k} = \frac{\delta\Gamma_k}{\delta\varphi}[\varphi] = Z_k^{1/2}\frac{\delta\Gamma_k}{\delta\phi}[\phi].$$

In order to check that $O_{\delta\Gamma_k}$ is an exact solution of Eq. (5) we note that the right-hand side of (5) can be found directly from the left-hand side using the known running of the EAA. Indeed we observe the following:

$$\partial_t\left(\varepsilon \cdot \frac{\delta\Gamma_k}{\delta\varphi}[\varphi]\right) = \left(\varepsilon \cdot \partial_t \frac{\delta\Gamma_k}{\delta\varphi}[\varphi]\right) = \varepsilon \cdot \left(-\frac{1}{2} G_k \cdot \frac{\delta^3\Gamma_k}{\delta\varphi^3}[\varphi] \cdot G_k \cdot \partial_t R_k\right).$$

The last term of this expression is exactly the right-hand side of Eq. (5) for the operator $O_{\delta\Gamma_k}$. This means that no other operator mixes with $O_{\delta\Gamma_k}$, which is thus an exact solution of the equation. In particular we note that $Z_k^{1/2}$ can be identified with the mixing matrix Z_{ij} and that the associated anomalous dimension is $\gamma_{O_{\delta\Gamma_k}} = -\eta/2$. The scaling dimension of $O_{\delta\Gamma_k}$ is $\Delta_{O_{\delta\Gamma_k}} = (d+2-\eta)/2$. Let us note that this operator is clearly redundant since the redefinition $\varphi \to \varphi + \varepsilon$ of the field can eliminate this term from the effective action:

$$\Gamma_k[\varphi + \varepsilon] = \Gamma_k[\varphi] + \varepsilon \cdot \frac{\delta\Gamma_k[\varphi]}{\delta\varphi}.$$

Being redundant this operator should not be considered in the spectrum of observable scaling operators; see also [30]. In Appendix B we report some examples where such an operator is identified within the LPA′ truncation. However, the identification of this redundant operator may not be so straightforward in other truncations.

V. CONCLUSIONS AND OUTLOOK

In this work we have considered the dependence of the EAA on the floating normalization point μ at which the boundary condition is imposed. Particular attention has been paid to the renormalization of composite operators. In Sec. II we have described the general features of the flow equation for the EAA generalized to include sources for composite operators. In Sec. III we have shown that the EAA satisfies a sort of Callan-Symanzik equation which entails the invariance under changes of the floating normalization point μ. This mechanism unveils how anomalous scaling shows up in the EAA formalism. We have also shown how the scaling of composite operators is related to critical exponents. Finally, in Sec. IV we have considered the local potential approximation in view of our discussion and we have found some simple solutions to the flow equation for composite operators. It turns out that one can systematically identify a redundant operator present in the spectrum of eigenperturbations (as already observed in [30]) and it is possible to straightforwardly extend the

[12]Note that it is nontrivial to build composite operators out of other composite operators. A simple example is given by φ which is possibly the simplest "composite" operator. In particular, given φ, simple products like φ^n are not solutions of Eq. (5), whose right-hand side induces new operators via the mixing.

solutions of the equation to include a class of total derivatives operators which we identify with the descendants of primary operators of the fixed point theory.

The invariance under changes of the floating normalization point μ makes explicit an intriguing link between standard quantum field theory and the FRG approach. Indeed, when the scale k is lowered to zero one is left with the standard effective action which satisfies the Callan-Symanzik equation. In the FRG scheme all the couplings compatible with the symmetries of the system are generated and so the Callan-Symanzik equation involves, in principle, infinitely many couplings whereas more common schemes involve just a finite set of couplings (working with a renormalizable theory). The latter possibility, perturbation theory being a particular solution of the flow equation, can be recovered provided one solves the flow iteratively as outlined in [31–34]. We observe that in order to solve the flow equation for the EAA a truncation must be chosen and thus, in practice, one cannot account for all the couplings generated by the flow. We feel that this link between a Wilsonian type of RG and the Gell-Mann and Low formulation goes in the direction outlined in [4].

We also remark that, in principle, our analysis tells that one is not allowed to discard total derivative operators in the spectrum of eigenperturbations. Depending on the cases these operators may or may not give important corrections to the scaling dimensions of fixed point scaling operators. However, in a nonperturbative setting, as the FRG is, we think that one should be aware of these possibly important contributions.

Finally it may be interesting to consider some particular operators like the stress energy tensor (see [35,36] for a FRG perspective) or nonlocal operators like Wilson loops or Polyakov loops (see [10]). This goes however beyond the scope of the present work. Possibly the flow equation (5) could be applied in gauge theory to test the approximate restoration of Becchi-Rouet-Stora-Tyutin (BRST) symmetry in the limit $k \to 0$. In this case one needs to evaluate the BRST composite operators by coupling them to a (Grassmannian odd) source and following the flow down to $k \to 0$. Finally, given the flexible nature of the EAA formalism, we feel that the reasonings outlined in this work may be applied to many areas of interest.

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APPENDIX A: ε^2 TERMS

So far we have been studying the flow of the modified action $S + \varepsilon \cdot O$ up to first order. Multiple insertions of the composite operators can be obtained by several functional differentiations with respect to the source ε. Thus it is worth studying also the RG flow of higher-order terms in ε. Let us recall

$$\Gamma[\varphi, \varepsilon] = J \cdot \varphi - W[J, \varepsilon], \qquad \varphi = \delta_J W,$$
$$\delta_\varepsilon \Gamma[\varphi, \varepsilon] = -\delta_\varepsilon W[J, \varepsilon].$$

If we take a further functional derivative, we obtain

$$\begin{aligned}
\delta_\varepsilon^2 \Gamma[\varphi, \varepsilon] &= -\frac{\delta^2 W[J, \varepsilon]}{\delta \varepsilon^2} - \frac{\delta^2 W[J, \varepsilon]}{\delta J \delta \varepsilon} \cdot \frac{\delta J}{\delta \varepsilon} \\
&= -\frac{\delta^2 W[J, \varepsilon]}{\delta \varepsilon^2} - \left(\frac{\delta}{\delta \varepsilon} \frac{\delta W[J, \varepsilon]}{\delta J} \right) \cdot \frac{\delta J}{\delta \varepsilon} \\
&= -\frac{\delta^2 W[J, \varepsilon]}{\delta \varepsilon^2} - \left(\frac{\delta}{\delta \varepsilon} \varphi \right) \cdot \frac{\delta J}{\delta \varepsilon} = -\frac{\delta^2 W[J, \varepsilon]}{\delta \varepsilon^2}.
\end{aligned}$$

In the last line we used the fact that $\varphi = \delta_J W[J, \varepsilon]$ is a given function and thus it has no dependence on the source ε. We see that crucial information regarding the insertion of two composite operators can be obtained straightforwardly deriving twice with respect to ε. Let us consider the flow equation:

$$\partial_t \left(\frac{\delta^2}{\delta \varepsilon^2} \Gamma_k[\varphi, \varepsilon] \right) = \left[-\frac{1}{2} G_k \cdot \frac{\delta^2 \Gamma_k^{(2)}}{\delta \varepsilon^2} \cdot G_k \cdot \partial_t R_k + G_k \cdot \frac{\delta \Gamma_k^{(2)}}{\delta \varepsilon} \cdot G_k \cdot \frac{\delta \Gamma_k^{(2)}}{\delta \varepsilon} \cdot G_k \cdot \partial_t R_k \right]. \tag{A1}$$

If we expand the EAA in terms of ε, we obtain an expression of the following form:

$$\Gamma_k[\varphi, \varepsilon] = \Gamma_k[\varphi] + \int_x \varepsilon(x) O_k(x) + \int_{x,y} \varepsilon(x) \varepsilon(y) B_k(x, y) + O(\varepsilon^3). \tag{A2}$$

The flow equation (5) gives us the running of O_k appearing at the first order in ε. The term B_k in (A2) can be determined from Eq. (A1). Let us note that if we set

$$\Gamma_k[\varphi, \varepsilon] = \Gamma_k[\varphi] + \int_x \varepsilon(x) O_k(x) + \int_x \varepsilon(x)^2 B_k(x) + O(\varepsilon^3),$$

the vertex in the first term of (A1) would amount just to a contact term, i.e. a term proportional to the Dirac delta. As such we could discard this term at separate points. However, it is important to stress that most likely it is crucial to keep $B_k(x, y)$ as a semilocal (as opposed to local) term. A simple example of this is given in [15] where the author considers the Wilsonian action keeping track of the source J conjugate to the field. In order to obtain the correct two-point correlation function from two J differentiations, it is crucial to keep a semilocal term at order J^2.

Equation (A1) is potentially the starting point to study in a nonperturbative setting the correlation functions with two insertions of composite operators. In turn this implies the possibility of studying the operator product expansion coefficients along the lines of [37–40]. In this sense the study of the $O(\varepsilon)$ terms is a first step in this direction.

APPENDIX B: LPA AND COMPOSITE OPERATORS

In this Appendix we consider the results of the LPA′ truncation and compare them with the exact results coming from CFT. In particular we shall consider the critical and tricritical Ising model in two dimensions and the Ising model in three dimensions. According to the discussion of Secs. III B, III C and IV A in the LPA′ truncation for composite operators we are led to identify the full dimension of the scaling operators λ_i as $d - \theta_i$, where θ_i is a critical exponent. It is thus straightforward to use the known results regarding critical exponents to deduce anomalous dimensions of composite operators under these approximations. We shall consider the results of Ref. [29] whose methods allow one to find several eigendirections in a systematic manner; see also [41].

1. Critical and tricritical Ising models

In two dimensions, by means of CFT techniques [42], it has been possible to exactly compute the scaling dimension of the operators in the theory. In particular the critical and tricritical Ising models correspond to the minimal models having central charge $c = 1 - 6/(m(m+1))$ with $m = 3$ and $m = 4$. In the following we compare the exact results with our approximations and comment on the results. The correspondence between composite operators of the Landau-Ginzburg Hamiltonian and the scaling fields of the CFT is due to Zamolodchikov [43].

Let us consider the Ising model. Table I shows the first four scaling operators (the identity operator is not shown) found using Eqs. (33) and (37). As already anticipated the estimate for the anomalous dimension of the field has a large error. This is a known feature of the LPA′ truncation and better results can be obtained by employing a more general kinetic term of the form $K(\varphi)\partial\varphi\partial\varphi$ [44].

Given that the anomalous dimension has such an error one may expect that also the critical exponents, and thus the anomalous dimension of composite operators, are not precise. Actually this depends on which quantity we consider: certain quantities converge to relatively precise values already in simple truncations while others need more refined approximations. In the present case we observe that the anomalous dimension of $[\varphi^2]$ is close to its correct value. The operator that we denoted $[\varphi^3]$ is simply the redundant operator $O_{\delta\Gamma_k}$ and as such it does not appear among the physical scaling operators.

Note that the arguments outlined in Sec. IV B allow us to easily identify some of the descendant operators associated to $[\varphi]$ and $[\varphi^2]$. More precisely our arguments identify the secondary operators which are not quasiprimaries (these are derivative operators of the type $L_{-1}\Phi$, where L_n are the generators of the Virasoro algebra and where we omitted the antiholomorphic generator). Other operators, like $L_{-2}\phi_{2,2}$, should be present in the spectrum of eigenoperators and in principle should be seen. Unfortunately, the other operators present in the spectrum of Eq. (37) have scaling dimensions which are not easily put in correspondence with CFT results. As noted in [44] this may be due to the fact that higher-dimension operators correspond to operators having also many derivatives, which are not present in our truncation. Ideally, solving the flow equation for composite operators one should find the spectrum of scaling dimensions known from CFT together with the associated degeneracy at each level. Of course this is an incredibly hard task, but one can aim to obtain approximate results.

We shall now consider a similar analysis for the tricritical Ising model. The exact values are compared with the results of the LPA′ truncation in Table II. We observe that the anomalous dimension of $[\varphi]$ is by about a factor of 2 bigger than the exact result; a more refined computation yields much better predictions [44]. We observe that besides η also the anomalous dimension of $[\varphi^2]$ is rather poor, while those for $[\varphi^3]$ and $[\varphi^4]$ are closer to the exact values. Once again

TABLE I. Scaling dimension in the critical Ising model. The first column indicates the composite operator whose exact scaling dimension is reported in the second column. The third column lists the scaling dimensions obtained within the LPA′ using the critical exponents computed in Ref. [29].

Operator	Exact	LPA′
$[\varphi] \sim \phi_{2,2}$	$\frac{1}{8} = 0.125$	$\frac{\eta}{2} = 0.22$
$[\varphi^2] \sim \phi_{1,3}$	1	1.05
$[\varphi^3]$		1.78
$[\varphi^4]$		2.68

TABLE II. Scaling dimension in the tricritical Ising model. The first column indicates the composite operator whose exact scaling dimension is reported in the second column. The third column lists the scaling dimension obtained within the LPA′ using the critical exponents computed in Ref. [29].

Operator	Exact	LPA′
$[\varphi] \sim \phi_{2,2}$	$\frac{3}{40} = 0.075$	$\frac{\eta}{2} = 0.156$
$[\varphi^2] \sim \phi_{3,3}$	$\frac{1}{5} = 0.2$	0.33
$[\varphi^3] \sim \phi_{2,1}$	$\frac{7}{8} = 0.875$	0.84
$[\varphi^4] \sim \phi_{3,2}$	$\frac{6}{5} = 1.2$	1.32
$[\varphi^5]$		1.84
$[\varphi^6] \sim \phi_{3,1}$	3	2.45

better values can be found by considering a more refined truncation which includes mixing with derivatives [44]. The operator $[\varphi^5]$ can be identified with the redundant operator $O_{\delta\Gamma_k}$. The discussion regarding descendant operators that we did for the critical Ising model applies also in this case.

2. Three-dimensional Ising model

The three-dimensional Ising model is a paradigmatic application of the renormalization group and has been studied via various truncations in the FRG literature. These studies allowed a rather precise determination of the anomalous dimension and the critical exponents [45–49]. In this section we use the results of Ref. [29] to obtain the anomalous dimension of composite operators following the discussion of Sec. III B. The results are shown in Table III where we compare the LPA′ truncation with the rigorous results found by conformal bootstrap techniques [50–53].[13]

As in the other models that we have discussed, the anomalous dimension of the field is poorly determined under our approximations. We note that $[\varphi^2]$ is relatively close to the exact value while for the other operators the results are not so precise. The operator $[\varphi^3]$ can be identified with the redundant operator $O_{\delta\Gamma_k}$. Being redundant this operator must be discarded and indeed it finds no counterpart in the part of Table III dedicated to the bootstrap approach. Furthermore the result for $[\varphi^6]$ has a large error compared to the bootstrap results [51,53].[14] Moreover the arguments of Sec. IV B allow us to easily upgrade our solution to include the descendants of the fields in Table III. However, given the difficulties encountered with the two-dimensional Ising model, we feel that one should take the identifications of Table III with a grain of salt, especially regarding $[\varphi^6]$.

TABLE III. Scaling dimensions in the three-dimensional Ising model. The first column indicates the (primary) operators whose scaling dimension is reported in the second column. The results come from the bootstrap approach [50–53]. (Only the first two digits are shown but more can be found in [51,53]). In the third column we indicate the composite operators whose scaling dimension is obtained from the LPA′ using the critical exponents computed in Ref. [29].

Operator	Bootstrap	Operator	LPA′
σ	0.52	$[\varphi]$	0.56
ε	1.41	$[\varphi^2]$	1.45
		$[\varphi^3]$	2.44
ε'	3.83	$[\varphi^4]$	3.53
σ'	4.05	$[\varphi^5]$	4.69
ε''	4.61	$[\varphi^6]$	5.9
	≈ 7		

[13] Interestingly a proof of conformal invariance of the Ising model has been given in [54] using functional renormalization techniques.

[14] In the last row of Table III two values coming from the bootstrap approach are shown. The difference in the result depends on whether an operator disappears or not from the spectrum [53]. This discrepancy should eventually disappear. Our result is however rather "wrong" in both cases.

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