

Finite energy quantization on a topology changing spacetime

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The “trousers” spacetime is a pair of flat two-dimensional cylinders (“legs”) merging into a single one (“trunk”). In spite of its simplicity this spacetime has a few features (including, in particular, a naked singularity in the “crotch”) each of which is presumably unphysical, but for none of which a mechanism is known able to prevent its occurrence. Therefore, it is interesting and important to study the behavior of the quantum fields in such a space. Anderson and DeWitt were the first to consider the free scalar field in the trousers spacetime. They argued that the crotch singularity produces an infinitely bright flash, which was interpreted as evidence that the topology of space is dynamically preserved. Similar divergencies were later discovered by Manogue, Copeland, and Dray who used a more exotic quantization scheme. Later yet the same result obtained within a somewhat different approach led Sorkin to the conclusion that the topological transition in question is suppressed in quantum gravity. In this paper I show that the Anderson-DeWitt divergence is an artifact of their choice of the Fock space. By choosing a different one-particle Hilbert space one gets a quantum state in which the components of the stress-energy tensor (SET) are bounded in the frame of a free-falling observer.

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I. INTRODUCTION AND CONCLUSIONS

The two-dimensional spacetime \mathcal{M} called trousers is obtained from the strip

$$ds^2 = dt^2 - dx^2 \quad t \in \mathbb{R}, \quad x \in [-P, P]$$

by, first, deleting the points $t = 0, x = \pm P$ and the ray $t \leq 0, x = \mu P$, then attaching a copy of the deleted ray to either bank of the cut and, finally, smoothly gluing each of the six rays to its counterpart so that the resulting space consists of three cylinders; see Fig. 1.

The trousers spacetime merits the most detailed consideration because in spite of its simplicity it possesses two features, interesting and important, but poorly understood.

- (1) The topology of its spacelike sections changes with time. It is $\mathbb{S}^1 \sqcup \mathbb{S}^1$ at negative t and \mathbb{S}^1 at positive. This type of topology change is particularly significant, because it may have to do with the appearance of a wormhole or (if the t axis is directed to the past as in [1]) with the final stage of the wormholes (including the Schwarzschild black hole) evaporation [2];
- (2) the spacetime is singular, as one might expect, and the singularity—loosely speaking it is located at the crotch of the trousers—is naked and quasiregular. Presumably either of these properties makes it unphysical, but no mechanism is found that would protect the Universe from the appearance of such singularities.

The evolution of a quantum field in trousers was first considered by Anderson and DeWitt (AD). In their well-known paper [1] they conjectured that the above-mentioned

singularity emits an infinitely bright flash. Their reasoning was as follows [3]: “an ‘in’ mode function propagating to the right splits into components propagating to the right in each leg. Although continuous in the trunk region, such mode functions generally have discontinuities [...] in the legs. [...] Every ‘out’ mode function is continuous in each leg (vanishing in one of them) but has discontinuities in the trunk region. [...] When these functions are differentiated the discontinuities give rise to delta functions. Since the

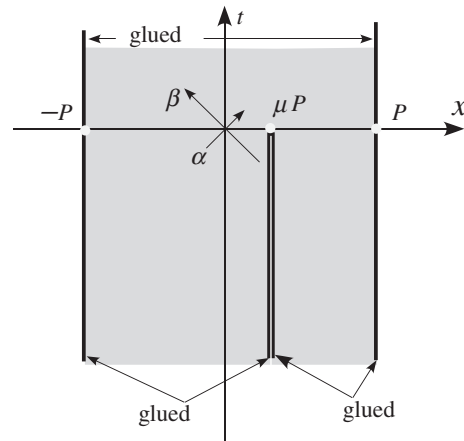


FIG. 1. Constructing the trousers spacetime from a flat strip. The left leg $\mathcal{M}_L \equiv (t \leq 0, x \in [-P, \mu P])$, the right leg $\mathcal{M}_R \equiv (t \leq 0, x \in [\mu P, P])$, and the trunk $\mathcal{M}_T \equiv (t \geq 0, x \in [-P, P])$ are obtained by gluing together the rays bounding the corresponding strips. The white circles depict the removed points. They cannot be returned back after the relevant identifications are performed and thus a naked quasiregular singularity appears.

terms of the mode sum for $\langle \text{in, vac} | \mathbf{T}_{\text{ren}}^{00} | \text{in, vac} \rangle$ are bilinear in differentiated mode functions, the square of the delta function automatically appears.”

The divergence of $\langle \text{in, vac} | \mathbf{T}_{00}^{\text{ren}} | \text{in, vac} \rangle$ does not automatically exclude the topology changes: it may happen that *some* of them are free from that divergence [4] or that the relevant quantity is the matrix element $\langle \text{in, vac} | \mathbf{T}_{00}^{\text{ren}} | \text{out, vac} \rangle$ as opposed to the expectation value of the stress-energy tensor [5]. It is also possible that the fields in such an unusual space must be quantized in some special way [6] (one such unusual quantization was proposed in [7]; the resulting $\langle \mathbf{T}_{00}^{\text{ren}} \rangle$, though, diverges all the same). There is a good consensus, however, that the AD flashes are an indicator of some “flaw” in the trousers spacetime [1,8,9]. The goal of this paper is to show that this is not the case.

A. Conclusions

In regard to the divergence of the energy density the trousers turn out to be as “nice” as, say, the Schwarzschild space. The latter is not compromised by the fact that in *some* states (such as the Boulware vacuum) the energy density diverges at the horizon. What matters is the existence of states *free* of such divergences. Accordingly, we rehabilitate the trousers spacetime by explicitly constructing a state in which $\langle \mathbf{T}_{ik}^{\text{ren}} \rangle$ are bounded. Note that the existence of such a state does not contradict the argument quoted above, owing to the word generally used in the latter.

II. THE QUANTIZATION

A. The plan

The field ϕ considered in this paper obeys (classically) the wave equation

$$\square\phi = 0. \quad (1)$$

Though the spacetime under discussion is nonglobally hyperbolic, the piecewise smooth (see below) complex-valued solutions of (1) are fixed uniquely—this is proven in Sec. II B—by the data at any surface $t = \text{const} \leq 0$. In this sense the singularity is “harmless” [8,10] and we can proceed exactly as in the globally hyperbolic case. In doing so we are guided by the textbook [11], in particular, the units and sign conventions are those used there.

To canonically quantize the field one must first expand it as a series in vacuum modes, that is, find a set of functions $\{\phi_k\}$ on \mathcal{M} that are an orthonormal basis in a Hilbert space \mathcal{H} . It is the choice of \mathcal{H} that encodes the physics of the problem and determines the resulting theory.

The field operator $\hat{\phi}$ in quantum field theory (QFT) is taken to be an (operator-valued) *distribution*. But the Hilbert space in discussion is usually built on the basis of *smooth* solutions of the classical equation of motion (that is, \mathcal{H} may contain nonsmooth functions, but only those to which a sequence of smooth ones converges). Such a

choice seems inadequate in studying “thunderbolts” with their discontinuities. Therefore, Manogue *et al.* in [7] expanded the space by allowing the vacuum modes to have jumps. However, the derivatives of such modes will have δ -like singularities exactly where the modes are discontinuous, which makes the Klein-Gordon scalar product ill defined; see (2). Physically such singularities seem unwarranted too. Indeed, the crotch singularity can play the role of a source, so the solutions to the (now inhomogeneous) wave equation are expected to have irregularities on the null geodesics emanating from the “missing point.” However, the energy density proportional to the square of the delta function is a *too* strong irregularity. So, in this paper we propose an intermediate approach and require classical solutions to be *continuous*, though not continuously differentiable. Specifically, let \mathcal{C} be the space of bounded continuous complex-valued functions on \mathcal{M} that are smooth solutions to Eq. (1) on the whole X except perhaps at the points of past incomplete inextendible null geodesics (i.e., loosely speaking, null geodesics emanating from the singularity), where the derivatives of those functions may have jump discontinuities. Correspondingly, we are looking for a space \mathcal{H} such that

- (1) \mathcal{H} is a Hilbert space with respect to the Klein-Gordon scalar product

$$(f, g)_{\text{KG}} \equiv i \int_{-P}^P (g^* \dot{f} - f \dot{g}^*)|_{t=0} dx \quad (2)$$

(we indicate the particular value of t because at this stage we cannot guarantee that the integral does not depend on the choice of that value, the spacetime being nonglobally hyperbolic). The positive definiteness of this form is a nontrivial restriction on \mathcal{H} .

- (2) Up to a constant any function f in \mathcal{C} is the sum of a function from \mathcal{H} and a function from \mathcal{H}^* ,

$$\text{for all } f \in \mathcal{C} \quad f = f^+ + (f^-)^* + c,$$

$$f^+, f^- \in \mathcal{H}, \quad c = \text{const}. \quad (3)$$

f^+ and $(f^-)^*$ are often called, respectively, “positive and negative frequency” parts of f . Conversely, \mathcal{H} must not include “superfluous functions;” in other words, \mathcal{H} must not have a proper subspace satisfying (3).

1. Remark

The word *past* in the definition of \mathcal{C} signifies some time asymmetry in our approach that is not related to the asymmetry of the underlying spacetime.

B. Reduction to initial conditions

In this subsection we represent in a convenient form the space \mathcal{C} : we use the periodicity of the functions constituting

\mathcal{C} to express them in terms of their restrictions to the surface $t = 0$ (thus trading functions of two variables each for a pair of functions of one variable).

To begin with we note that any $\phi \in \mathcal{C}$ being a solution to the wave equation

$$\partial_\alpha \partial_\beta \phi = 0, \quad \alpha \equiv t + x, \quad \beta \equiv t - x$$

is the sum of a right-moving and a left-moving (i.e., depending—within each of the cylinders—on the α - or, respectively, β -coordinate of its argument) function. Put more formally, it has the form

$$\begin{aligned} \phi|_{\mathcal{M}_\lambda}(p) &= a_\lambda(\alpha(p)) + b_\lambda(\beta(p)) + c, \\ \lambda &\equiv L, R, T. \end{aligned} \quad (4)$$

Here c is an arbitrary constant and $a_\lambda(\alpha)$, $b_\lambda(\beta)$ for each value of λ are a pair of functions such that, first,

$$\begin{aligned} a_\lambda(\alpha(p)) &= a|_{\mathcal{M}_\lambda}(p), \quad b_\lambda(\beta(p)) = b|_{\mathcal{M}_\lambda}(p), \\ \text{where } a, b &\in \mathcal{C} \end{aligned} \quad (5a)$$

[note that the *entire* functions $a(p)$, $b(p)$ do not have to be of the form $a(\alpha(p))$ and $b(\beta(p))$]; moreover, they may have a discontinuity on the ray $t \leq 0$, $x = \mu P$] and, second, they satisfy the following normalization conditions:

$$\begin{aligned} a_L(\mu P) &= \frac{\int_{-P}^{\mu P} a_L(\alpha) d\alpha}{(1 + \mu)P} + \frac{\int_{\mu P}^P a_R(\alpha) d\alpha}{(1 - \mu)P}, \\ b_L(\mu P) &= \frac{\int_{-P}^{\mu P} b_L(\beta) d\beta}{(1 + \mu)P} + \frac{\int_{\mu P}^P b_R(\beta) d\beta}{(1 - \mu)P}. \end{aligned} \quad (5b)$$

The reason for choosing these particular conditions will become clear later—see (A5); for now notice only that for any ϕ Eqs. (4)–(5b) define a unique c . The subset of \mathcal{C} consisting of all functions ϕ for which $c = 0$ is denoted $\mathring{\mathcal{C}}$.

Finally, the topology of our spacetime requires ϕ to have some periodicity properties. In order to satisfy them we take a_λ and b_λ to be periodic functions of α and β , respectively [12]. The period

$$\begin{aligned} \text{of } a_L \text{ and } b_L &\text{ is } (1 + \mu)P, \\ \text{of } a_R \text{ and } b_R &\text{ is } (1 - \mu)P, \\ \text{and of } a_T \text{ and } b_T &\text{ is } 2P. \end{aligned} \quad (5c)$$

Now let us introduce the aforementioned functions of one variable. To this end denote by \mathcal{F} the space of all continuous functions $A(x): [-P, P] \rightarrow \mathbb{C}$ (it is convenient to imagine A as defined on the surface $t = 0$; in doing so one, strictly speaking, must keep in mind that this surface lacks the points $x = \pm P, \mu P$; we omit this trivial reservation from now on) which

- (1) are smooth, except, perhaps, at the points $x = \mu P$ where the derivatives are allowed to have jump discontinuities;
- (2) satisfy the condition

$$\begin{aligned} A(\mu P) &= \frac{1}{(1 + \mu)P} \int_{-P}^{\mu P} A(x) dx \\ &+ \frac{1}{(1 - \mu)P} \int_{\mu P}^P A(x) dx; \end{aligned} \quad (6)$$

- (3) obey the “periodicity condition”

$$\begin{aligned} A^{(n)}(-P) &= A^{(n)}(\mu P - 0), \\ A^{(n)}(\mu P + 0) &= A^{(n)}(P), \quad n = 0, 1, \dots \end{aligned} \quad (7)$$

Each pair $A, B \in \mathcal{F}$ defines uniquely a function $\Psi(A, B) \in \mathring{\mathcal{C}}$ in the following way: a_λ are defined to be the extensions by periodicity, see (5c), of the functions, respectively,

$$\begin{aligned} a_T &\equiv A \quad \text{at } x \in (-P, P], \quad a_L \equiv A \quad \text{at } x \in (-P, \mu P], \\ a_R &\equiv A \quad \text{at } x \in (\mu P, P]. \end{aligned}$$

The functions b_λ are dealt with in exactly the same manner [the only difference is in the sign: $b_\lambda(\beta)|_{t=0} = B(-x)$]. Now $\phi|_{\mathcal{M}_\lambda}$ are built by (4) with $c = 0$ and, finally, $\Psi(A, B)$ is defined to be the result of gluing together all three restrictions $\phi|_{\mathcal{M}_\lambda}$.

Conversely, any $\phi \in \mathring{\mathcal{C}}$ defines uniquely a pair $A, B \in \mathcal{F}$ such that $\Psi(A, B) = \phi$. This is done by decomposing ϕ into the right-moving and the left-moving parts $a(p)$ and $b(p)$, see (5a), and defining A, B to be their restrictions,

$$A(x(p)) \equiv a(p)|_{t=0}, \quad B(x(p)) \equiv b(p)|_{t=0}.$$

We have thus established that $\Psi: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathring{\mathcal{C}}$ is an isomorphism. It can be transformed into an isometry by an appropriate choice of the inner product in \mathcal{F} . Indeed, substituting the obvious expressions

$$\begin{aligned} \phi|_{t=0}(x) &= A(x) + C(x), \\ \dot{\phi}|_{t=0}(x) &= A'(x) - C'(x), \quad \text{where } C(x) \equiv B(-x) \end{aligned} \quad (8)$$

into (2) one finds

$$(\phi_1, \phi_2)_{\text{KG}} = \langle A_1, A_2 \rangle + \langle B_1, B_2 \rangle, \quad (9)$$

$$\text{where } \langle F_1, F_2 \rangle \equiv i \int_{-P}^P [F_2^*(x) F_1'(x) - F_2'^*(x) F_1(x)] dx. \quad (10)$$

1. Remark

The formulas (8) show that the singularity in \mathcal{M} is harmless in the sense of [8]: there exists a unique solution to the wave equation for any Cauchy data fixed at a surface $t = t_0 \leq 0$. It is, of course, this property that enables one to

$$u_{Lk}(x) \equiv \begin{cases} (4\pi k)^{-\frac{1}{2}} e^{-ik\xi} & \text{at } x \in [-P, \mu P] \quad \text{i.e., } \xi \in [-\pi, \pi]; \\ (4\pi k)^{-\frac{1}{2}} e^{-ik\pi}, & \text{at } x \in [\mu P, P], \quad \text{where } \xi \equiv \frac{\pi}{1+\mu} \left(\frac{2}{P}x + 1 - \mu \right); \end{cases}$$

and likewise

$$u_{Rk}(x) \equiv \begin{cases} (4\pi k)^{-\frac{1}{2}} e^{-ik\zeta} & \text{at } x \in [\mu P, P] \quad \text{i.e., } \zeta \in [-\pi, \pi], \\ (4\pi k)^{-\frac{1}{2}} e^{-ik\pi}, & \text{at } x \in [-P, \mu P], \quad \text{where } \zeta \equiv \frac{\pi}{1-\mu} \left(\frac{2}{P}x - 1 - \mu \right). \end{cases}$$

The set $\{u_{Rk}, u_{Lk}\}, k = 1, 2, \dots$ is orthonormal with respect to the scalar product \langle, \rangle defined in (10) and we denote by \mathcal{H}_F the (auxiliary) Hilbert space obtained by declaring that set to be a basis.

The modes ϕ_k we are after are now defined as

$$\phi_{Qk} \equiv \begin{cases} \Psi(u_{Qk}, 0), & k > 0 \\ \Psi(0, u_{Q|k}^*), & k < 0, \end{cases} \quad \text{where } Q \equiv R, L \quad (11)$$

(i.e., ϕ_{Qk} are obtained from u_{Qk} by replacing $x \rightarrow -\beta = x - t$ for $k < 0$ and $x \rightarrow \alpha = x + t$ for $k > 0$ and extending the resulting functions by periodicity). So, every mode is at first a harmonic wave that moves in the corresponding leg (left or right depending on whether the subscript is L or R) to the left or to the right depending on the sign of k . In the trunk, however, the behavior of the mode becomes more exotic. It is just a constant here except in a spiral strip bounded by two null geodesics emanating from the crotch. Within that strip the mode is still a piece of a harmonic wave whose fronts are just those geodesics. The mode, though continuous—which enables the SET to remain bounded, contrary to the AD conjecture—is not smooth. So, one does not expect the energy to be conserved, but this is natural for a nonstatic spacetime.

1. Example

Assume $Q = L, k = -3$. Then we find first that $u_{Q|k}^*$ is the function (we again perceive its domain as the segment $t = 0$ with the values at the missing points $x = \pm P, \mu P$ being defined by continuity) equal to $\sqrt{12\pi} e^{3i\xi(x)}$ at $x \in [-P, \mu P]$ and $-\sqrt{12\pi}$ otherwise. Correspondingly, for every $p \in \mathcal{M}$ we define $\phi_{L,-3}(p)$ to be equal to

study QFT on \mathcal{M} without adopting a number of *ad hoc* assumptions.

C. The choice of the Hilbert space and its basis

Let $\{u_{Lk}\}$ and $\{u_{Rk}\}, k = 1, 2, \dots$ be the sets of functions defined as follows:

$$\Psi(0, u_{L3}^*) = \begin{cases} \sqrt{12\pi} e^{3i\xi(x_p)}, & \text{at } x_p \in [-P, \mu P]; \\ -\sqrt{12\pi}, & \text{otherwise,} \end{cases}$$

where x_p is the x -coordinate of the point at which the null α -directed geodesic through p meets the segment $t = 0$. Whence, in particular, $\phi_{L,-3} = -\sqrt{12\pi}$ in the entire non-hatched region in Fig. 2. To write down the explicit expression for $\phi_{L,-3}$ in the remaining part of \mathcal{M} one replaces $x \rightarrow -\beta = x - t$ in the relevant u and specifies the periodicity condition

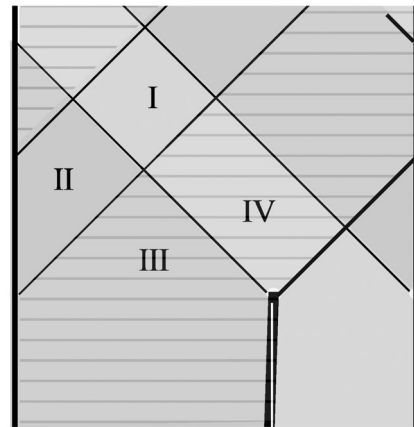


FIG. 2. The slanted rays are the null geodesics at which the derivatives of functions of \mathcal{C} are allowed to have jumps. In the dark and in the hatched regions ϕ_{Rk} with, respectively, positive and negative k are constant. In the complements to those regions ϕ_{Lk} with k of the same sign are constant.

$$\phi_{L,-3} = \sqrt{12\pi} e^{\frac{3i\pi(\mu-1-2\beta/P)}{1+\mu}}, \quad \beta \in \begin{cases} [-\mu P + P(1+\mu)n, P + P(1+\mu)n], & \text{in } \mathcal{M}_L; \\ [-P + Pn, P + Pn], & \text{in } \mathcal{M}_T, n \in \mathbb{Z}. \end{cases}$$

2. Remark

The problem obtained by restricting the consideration to the right (for definiteness) leg and choosing the vacuum to be that defined by the set of modes $\{\phi_{Rk}\}$ is well studied; see [11]. The index Q absent in that case may seem to double “the number” of modes (for $\mu = 0$, say). Note, however, that the separation between the frequencies of the modes in the trousers is twice that between the ones in the cylinder.

Now let the sought-for Hilbert space \mathcal{H} be defined as the completion of the linear span of the set

$$\{\phi_{Rk}\} \cup \{\phi_{Lk}\}, \quad k = 1, 2, \dots$$

by the norm generated by the scalar product (2). That it is indeed a scalar product (which requires positive definiteness) follows from the easily verified equality

$$(\phi_{Qk}, \phi_{\check{Q}n})_{\text{KG}} = \delta_{kn} \delta_{\check{Q}Q}, \quad \text{for all } \check{Q}, Q \quad k, n = 1, 2, \dots \quad (12)$$

According to the plan outlined in the end of Sec. II A it only remains to prove (3), which is done in the appendix.

D. The vacuum SET

The (nonrenormalized) vacuum expectation value of the SET is $\langle T_{\mu\nu} \rangle = \sum_n \hat{T}_{\mu\nu}[\phi_n]$, where

$$\hat{T}_{\mu\nu}[f] \equiv f_{,\mu} f^*_{,\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\kappa\lambda} f_{,\kappa} f^*_{,\lambda}, \quad (13)$$

and the summation is over all modes [11]. The series diverges of course and to renormalize the result one introduces a cutoff factor into the divergent sum by replacing $t \rightarrow t - i\delta$ and lets $\delta \rightarrow 0$ at the end of the calculation, that is, after subtracting the limit at $P \rightarrow \infty$. Thus,

$$\langle \mathbf{T}_{\mu\nu}^{\text{ren}} \rangle = T_{>\mu\nu}^L + T_{<\mu\nu}^L + T_{>\mu\nu}^R + T_{<\mu\nu}^R, \quad (14)$$

where

$$T_{\lesseqgtr\mu\nu}^Q \equiv \lim_{\delta \rightarrow 0} \left[\left(1 - \lim_{P \rightarrow \infty} \right) \sum_{n \leq 0} \hat{T}_{\mu\nu}[\phi_{Qn}, \delta] \right],$$

where $\hat{T}_{\mu\nu}[\phi_{Qn}, \delta]$ is the result of the substitution $t \rightarrow t - i\delta$ into $\hat{T}_{\mu\nu}[\phi_{Qn}]$. Let us find the four terms in turn. In the coordinate basis

$$\hat{T}_{\mu\nu}[e^{-iC(t-x)}] = C^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where C is a real constant [note that the second term of (13) vanishes]. Correspondingly, the term $T_{<\mu\nu}^L$, when nonzero (i.e., in the hatched region in Fig. 1) is

$$\lim_{\delta \rightarrow 0} \left\{ \left(1 - \lim_{P \rightarrow \infty} \right) \sum_{n < 0} \hat{T}_{\mu\nu} \left[\frac{\exp\{i|n| \frac{\pi}{1+\mu} (\mu - 1 - \frac{2}{P} (\beta - i\delta))\}}{\sqrt{4\pi|n|}} \right] \right\} = -\frac{\pi}{12P^2(1+\mu)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The term $T_{>\mu\nu}^L$ differs from $T_{<\mu\nu}^L$ in two respects: (1) it vanishes in the light gray region and (2) the exponents in the expression for modes depend on $t + x$ instead of $t - x$, which changes the sign of the tx components. Correspondingly,

$$T_{>\mu\nu}^L = -\frac{\pi}{12P^2(1+\mu)^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{in the dark region and } = 0 \text{ otherwise.}$$

Changing the sign of μ we find

$$T_{<\mu\nu}^R = -\frac{\pi}{12P^2(1-\mu)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{in the nonhatched region and } = 0 \text{ otherwise.}$$

$$T_{>\mu\nu}^R = -\frac{\pi}{12P^2(1-\mu)^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{in the light region and } = 0 \text{ otherwise.}$$

Summing these terms up one finally gets (the description and numbering refer to Fig. 2)

$$\text{light nonhatched regions I} \quad \langle \mathbf{T}_{tt}^{\text{ren}} \rangle = \langle \mathbf{T}_{xx}^{\text{ren}} \rangle = -\frac{\pi}{6P^2(1-\mu)^2}, \quad \langle \mathbf{T}_{tx}^{\text{ren}} \rangle = \langle \mathbf{T}_{xt}^{\text{ren}} \rangle = 0;$$

$$\text{dark hatched regions III} \quad \langle \mathbf{T}_{tt}^{\text{ren}} \rangle = \langle \mathbf{T}_{xx}^{\text{ren}} \rangle = -\frac{\pi}{6P^2(1+\mu)^2}, \quad \langle \mathbf{T}_{tx}^{\text{ren}} \rangle = \langle \mathbf{T}_{xt}^{\text{ren}} \rangle = 0;$$

$$\text{light hatched regions IV} \quad \langle \mathbf{T}_{tt}^{\text{ren}} \rangle = \langle \mathbf{T}_{xx}^{\text{ren}} \rangle = -\frac{\pi}{6P^2} \frac{1+\mu^2}{(1-\mu^2)^2}, \quad \langle \mathbf{T}_{tx}^{\text{ren}} \rangle = \langle \mathbf{T}_{xt}^{\text{ren}} \rangle = -\frac{\pi}{3P^2} \frac{\mu}{(1-\mu^2)^2};$$

$$\text{and dark nonhatched regions II} \quad \langle \mathbf{T}_{tt}^{\text{ren}} \rangle = \langle \mathbf{T}_{xx}^{\text{ren}} \rangle = -\frac{\pi}{6P^2} \frac{1+\mu^2}{(1-\mu^2)^2}, \quad \langle \mathbf{T}_{tx}^{\text{ren}} \rangle = \langle \mathbf{T}_{xt}^{\text{ren}} \rangle = \frac{\pi}{3P^2} \frac{\mu}{(1-\mu^2)^2}.$$

Thus, in the reference frame of a free-falling observer the stress-energy tensor is componentwise bounded even though in the general case it suffers discontinuities of the first kind. If the legs are equal ($\mu = 0$) there is no thunderbolt—the SET is perfectly regular.

ACKNOWLEDGMENTS

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APPENDIX: THE RELATION BETWEEN THE CLASSICAL AND THE QUANTUM SPACES

By construction $\mathring{\mathcal{C}}$ is dense in \mathcal{H} (since all $\phi_{Qk} \in \mathring{\mathcal{C}}$). In this appendix a converse, in a sense, property (3) is established.

For an arbitrary function $A \in \mathcal{F}$ let us prove that

$$A = A^+ + A^-, \quad \text{where } A^+, (A^-)^* \in \mathcal{H}_F. \quad (\text{A1})$$

Obviously, this implies (3) for all f of the type $f = \Psi(A, 0)$ [it suffices to set $f^\pm = \tilde{\Psi}(A^\pm, 0)$ where $\tilde{\Psi}$ is the extension by continuity of Ψ to the entire $\mathcal{H}_F \otimes \mathcal{H}_F$]. The case of $f = \Psi(0, B)$ is perfectly analogous and f of the general type is just the sum of those two plus a constant. So, the validity of (A1) proves (3).

Proof of (A1).—The mode u_1 and the function A are smooth except at $x = \mu P$ where either of them has a—nonzero in the case of u_1 —jump in its first derivative [see (7)],

$$w'(\mu P + 0) - w'(\mu P - 0) = w'(P) - w'(-P),$$

$$\text{where } w \equiv A, u_1.$$

So, we can find a continuously differentiable linear combination

$$A_1 \in C^1, \quad A_1(P) = A_1(-P), \quad A_1'(P) = A_1'(-P),$$

$$\text{where } A_1 \equiv A - C_u u_1, \quad C_u = \text{const.}$$

Evidently, (A1) is true if and only if it is true with A replaced by A_1 . Thus, it involves no loss of generality to assume that $A \in C^1$.

Now, consider the Fourier coefficients

$$F_{L,k} \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} A|_{x \in [-P, \mu P]}(x(\xi)) e^{ik\xi} d\xi,$$

$$F_{R,k} \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} A|_{x \in [\mu P, P]}(x(\zeta)) e^{ik\zeta} d\zeta$$

of (the restrictions of) A . By [[13], n°708] it follows from the continuous differentiability [14] of A that

$$F_{Q,k} = O(k^{-3}). \quad (\text{A2})$$

This rate of convergence implies that for some functions X_Q^\pm

$$S_Q^+(k_0) \equiv \sum_{k=1}^{k_0} F_{Q,k} u_{Qk} \sqrt{4\pi|k|} \quad \text{and}$$

$$S_Q^-(k_0) \equiv \sum_{k=-k_0}^{-1} F_{Q,k}^* u_{Q,-k} \sqrt{4\pi|k|} \quad \text{converge}$$

$$\text{uniformly to, respectively, } X_Q^+ \quad \text{and} \quad X_Q^- \quad (\text{A3})$$

and

$$[S_Q^\pm(k_0)]' \quad \text{converges uniformly to } Y_Q^\pm \equiv [X_Q^\pm]'. \quad (\text{A4})$$

The expression $S_L^+ + S_L^* + \frac{1}{(1+\mu)^P} \int_{-P}^{\mu P} A(x) dx$ on the interval $x \in [-P, \mu P]$ is a partial sum of the Fourier series of $A|_{x \in [-P, \mu P]}$. So, it converges to the said function there while the first two terms tend, respectively, to X_L^+ and X_L^* .

On the other interval (i.e., at $x \in [\mu P, P]$) it converges to $A(\mu P)$. Similar considerations apply to $S_R^+ + S_R^{*-}$ and hence

$$\begin{aligned} A|_{x \in [-P, P]} &= X_L^+ + X_L^{*-} + \frac{1}{(1+\mu)P} \int_{-P}^{\mu P} A(x) dx \\ &\quad + X_R^+ + X_R^{*-} + \frac{1}{(1-\mu)P} \int_{\mu P}^P A(x) dx - A(\mu P) \\ &= (X_L^+ + X_R^+) + (X_L^- + X_R^-)^* \end{aligned} \quad (\text{A5})$$

[the last equality follows from (6)]. Comparing this with (A1) we see that the latter is proven once we show that

$$X_Q^+, X_Q^{*-} \in \mathcal{H}_F. \quad (\text{A6})$$

So, recall that by construction $S_Q^+, S_Q^{*-} \in \mathcal{H}_F$; see (A3). At the same time by (A3) and (A4)

$$\max |X_Q^\pm - S_Q^\pm(k_0)|, \max |X_Q^{\prime \pm} - S_Q^{\prime \pm}(k_0)| \xrightarrow{k_0 \rightarrow \infty} 0$$

and hence S_Q^+ and S_Q^{*-} converge to, respectively, X_Q^+ and X_Q^{*-} in the metric of \mathcal{H}_F ; see (10). Thus, the containment (A6) follows from the completeness of the Hilbert space \mathcal{H}_F .

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