Spin-3/2 fields in *D*-dimensional Schwarzschild black hole spacetimes

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In previous works we have studied spin-3/2 fields near four-dimensional Schwarzschild black holes. The techniques we developed in that case have now been extended here to show that it is possible to determine the potential of spin-3/2 fields near D-dimensional black holes by exploiting the radial symmetry of the system. This removes the need to use the Newman-Penrose formalism, which is difficult to extend to D-dimensional spacetimes. In this paper we will derive a general D-dimensional gauge-invariant effective potential for spin-3/2 fields near black hole systems. We then use this potential to determine the quasinormal modes and absorption probabilities of spin-3/2 fields near a D-dimensional Schwarzschild black hole.

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I. INTRODUCTION

With the introduction of supergravity theories there has been a lot of interest in understanding the gravitino, the supersymmetric partner of the graviton. This particle is predicted to be a spin-3/2 particle and would behave like a Rarita-Schwinger field. In many supergravity theories, these fields act as a source of torsion and curvature of the spacetime [1,2]. It is also predicted that the gravitino is the lightest or second lightest supersymmetric particle. This makes the gravitino an ideal candidate for studying supersymmetric gravitational theories. Much of the research into gravitinos has been focused on the particle and not its interaction with curved spacetime, specifically near black holes. The motivations for this paper are twofold. First, it is to fill the gap in understanding the role that dimensions and spin play during gravitational interactions. Work on this has already been done for the case of spin-0, 1/2, 1 and 2 fields [3], so work on the spin-3/2 field would help fill this gap. Furthermore, by studying the higher-dimensional cases we may highlight the special features of the four-dimensional spacetime. Second, as $N = 2$ supergravity can be viewed as a partial realization of gravitational and electromagnetic unification, Crispino *et al.* [4] have shown that the unpolarized gravitational and electromagnetic scattering cross sections are equal in the extremal limit. Since in their consideration the spin-1 electromagnetic and the spin-2 gravitational fields are joined by the spin-3/2 gravitino field, a direct study of these spin-3/2 fields may shed more light on the symmetry behind this equality.

In previous works we have investigated this spacetime interaction for four-dimensional Schwarzschild black holes with a spin-3/2 field [5]. We did this by exploiting the radial symmetry of our system, which allowed us to separate the metric into a radial-time part and an angular part. We could then calculate the eigenvalues of our eigenspinor vectors in the angular part by using the works of Camporesi *et al.* [6]. Relating the eigenvalues of our radial-time part to those for the angular part we could easily determine the effective potential for our fields near the black hole. Using the same approach we hope to be able to investigate spacetimes with dimensions greater than 4.

This paper aims to derive a gauge-invariant D-dimensional effective potential for spin-3/2 fields near Schwarzschild black holes. Using this potential we can study the evolution of our spin-3/2 particles as they propagate through the curved spacetime. This evolution through spacetime is characterized by an oscillation of the spacetime. Near black holes these oscillations have a single frequency with a damping term, and are called quasinormal modes (QNMs). The QNMs characterize the parameters of the black hole [7]. A variety of numerical and semianalytic techniques can been used to determine the numerical values for the emitted QNMs [8,9]. We will use the WKB method and a method developed by some of us called the improved asymptotic iterative method (improved AIM) to calculate these values [10].

As we have done previously, we will also look at the absorption probabilities of our spin-3/2 particles, as this will give us an insight into the grey body factors and emission cross section of our black holes. These are required in order to understand the stability and evolution of our black holes.

The paper will be structured as follows. In the next section we determine the eigenvalues for the spinor vectors on an N sphere. In Sec. III we calculate the potential functions for spin-3/2 fields in this spacetime. We then use the potential to determine the QNMs and absorption

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potential associated to our spin-3/2 particles; the results are given in Secs. IV and V. Final discussions and conclusions are given in Sec. VI.

II. SPINOR-VECTOR EIGENMODES ON S^N

A. The N sphere

The metric that describes the angular part of our spacetimes is simply the metric for a sphere. The metric for the N sphere, S^N, is given as

$$d\Omega_N^2 = \sin^2\theta_N d\tilde{\Omega}_{N-1}^2 + d\theta_N^2, \tag{2.1}$$

where $d\tilde{\Omega}_{N-1}$ is the metric of S^{N-1}. In the rest of this section we will denote terms for S^{N-1} with tildes. Nonzero Christoffel symbols for the S^{N-1} are

$$\Gamma^{\theta_N}_{\theta_i\theta_j} = -\sin\theta_N\cos\theta_N\tilde{g}_{\theta_i\theta_j};$$
$$\Gamma^{\theta_j}_{\theta_i\theta_N} = \cot\theta_N\tilde{g}^{\theta_j}_{\theta_i}; \qquad \Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}. \tag{2.2}$$

In order to determine our covariant derivatives on S^N we need to determine the appropriate spin connections. We will use the n-bein formalism in order to relate components on our curved space to those of an orthonormal basis [11]. Our metric is related as follows:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}, \qquad e^a_\mu e^\mu_b = \delta^a_b, \qquad e^a_\mu e^\nu_a = \delta^\nu_\mu, \tag{2.3}$$

where greek letters represent our world indices and latin letters represent our Lorentz indices. For S^N the n bein is given as

$$e^{\theta_N}_N = 1, \qquad e^{\theta_i}_i = \frac{1}{\sin(\theta)}\tilde{e}^{\theta_i}_i. \tag{2.4}$$

We can relate gamma matrices on the orthogonal basis to gamma matrices of those on S^N as follows:

$$\gamma^{\theta_i} = e^{\theta_i}_i\gamma^i = \frac{1}{\sin\theta_N}\tilde{e}^{\theta_i}_i\gamma^i, \qquad \gamma^{\theta_N} = e^{\theta_N}_N\gamma^N = \gamma^N, \tag{2.5}$$

where the gamma matrices on the orthogonal basis obey the Clifford algebra. Spin connections are calculated as [11]

$$\omega_{\mu ab} = e^\alpha_a(\partial_\mu e_{\alpha b} - \Gamma^\rho_{\mu\alpha}e_{\rho b}). \tag{2.6}$$

Nonzero spin connections on S^N are then determined to be

$$\omega_{\theta_i jN} = \cos\theta_N\tilde{e}_{\theta_i j}, \qquad \omega_{\theta_i jk} = \tilde{\omega}_{\theta_i jk}. \tag{2.7}$$

The covariant derivative for the spinor-vector field is

$$\nabla_\mu\psi_\nu = \partial_\mu\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho + \omega_\mu\psi_\nu, \tag{2.8}$$

where

$$\omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}, \qquad \Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \tag{2.9}$$

We can now determine the eigenvalues for our spinors and spinor vectors on S^N. The eigenvalues for our spinors have already been determined by Camporesis and Higuchi [6], a brief overview is given in the Appendix. We use the eigenvalues of our spinors to determine the eigenvalues for our spinor vectors.

We denote the spinor vectors as ψ_μ, where each of the components are spinors. To begin our investigation into spinor vectors we find two orthogonal eigenspinor vectors on S^2, which can be written as linear combinations of the basis $\gamma_\mu\psi_{(\lambda)}$ and $\nabla_\mu\psi_{(\lambda)}$, where $\psi_{(\lambda)}$ is the eigenspinor on S^2. Note that we use (λ) to highlight that λ is not an index. These are "non-transverse and traceless eigenmodes" (non-TT modes), as they do not satisfy the transverse and traceless conditions. These two eigenspinor vectors can be generalized to the S^N case, and are analogous to the longitudinal eigenmode for vector fields on spheres. These "non-TT eigenmodes" form a complete set of eigenmodes on S^2. For higher-dimensional surfaces these modes do not represent a complete set and we must introduce TT eigenmodes. We are also required to consider the behavior of the S^N spinor-vector components on S^{N-1}. Consider the surface S^3; we expect the following spinor vector $\psi_{\theta_i} = (\psi_{\theta_3}, \psi_{\theta_2}, \psi_{\theta_1})$, with $\psi_{\theta_1}, \psi_{\theta_2}$, and ψ_{θ_3} representing spinors. Furthermore on S^3 we expect our "TT components," ψ_{θ_1} and ψ_{θ_2}, to behave like spinor vectors on S^2. Since S^2 only has non-TT eigenmodes we should represent ψ_{θ_1} and ψ_{θ_2} as linear combinations of non-TT eigenmodes on S^3. As such ψ_{θ_3} acts like a spinor on S^2, and we represent it using a linear combination of spinor eigenmodes on S^2. This gives us our first type of "TT eigenmode" which we can call the "TT mode I." The complete set of spinor vectors on S^3 is therefore given by two non-TT eigenmodes and one TT eigenmode I. On S^4 spinors ψ_{θ_1}, ψ_{θ_2}, and ψ_{θ_3} behave like spinor vectors and ψ_{θ_4} behaves like a spinor on S^3. We therefore have two types of spinor vectors on S^3, which we can represent in two ways. First, we can represent ψ_{θ_1}, ψ_{θ_2}, and ψ_{θ_3} as linear combinations of non-TT eigenmodes on S^3, and ψ_{θ_4} represented with a linear combination of spinor eigenmodes on S^3, this is the TT eigenmode I on S^4. We could also represent ψ_{θ_1}, ψ_{θ_2}, and ψ_{θ_3} as TT eigenmodes on S^3. Since they are already the TT eigenmodes, ψ_{θ_4} must go to zero. In this case we call it the TT eigenmode II. Hence, the complete set of eigenmodes on S^4 is given by two non-TT eigenmodes, one TT eigenmode I and one TT eigenmode II.

Generally, eigenmodes on S^N are represented by two non-TT eigenmodes, one TT-eigenmode I, and $N-3$ "TT eigenmode II" when $N > 2$. We can now determine the eigenvalues for our spinor vectors. In the following section we will denote values relating to the surface S^{N-1} with tildes.

B. Spinor-vector non-TT eigenmodes on S^N

We denote eigenvalues for the non-TT eigenmode spinor vectors as $i\xi$. The eigenvalue equation for our eigenspinor vectors with the Dirac operator is

$$\gamma^\mu \nabla_\mu \psi_\nu = i\xi\psi_\nu. \tag{2.10}$$

We can construct eigenspinor vectors on S^N using the following linear combination:

$$\psi_\nu = \nabla_\nu \psi_{(\lambda)} + a\gamma_\nu \psi_{(\lambda)}, \tag{2.11}$$

where $\psi_{(\lambda)}$ is an eigenspinor on S^N. Plugging Eq. (2.11) into Eq. (2.10) we have

$$\begin{aligned} i\xi\psi_\nu = \gamma^\mu[\nabla_\mu, \nabla_\nu]\psi_{(\lambda)} + \nabla_\nu(\gamma^\mu \nabla_\mu \psi_{(\lambda)}) \\ + a\{\gamma^\mu, \gamma_\nu\}\nabla_\mu \psi_{(\lambda)} - a\gamma_\nu(\gamma^\mu \nabla_\mu \psi_{(\lambda)}). \end{aligned} \tag{2.12}$$

The commutator can be rewritten in terms of the Riemann curvature tensor $R_{\mu\nu}{}^{\sigma\rho}$ as,

$$\gamma^\mu[\nabla_\mu, \nabla_\nu]\psi_{(\lambda)} = \frac{1}{8} R_{\mu\nu}{}^{\sigma\rho}[\gamma_\sigma, \gamma_\rho]\psi_{(\lambda)} = \frac{1}{2}(N-1)\gamma_\nu \psi_{(\lambda)}. \tag{2.13}$$

Then Eq. (2.12) becomes

$$\gamma^\mu \nabla_\mu \psi_\nu = (i\lambda + 2a)\left(\nabla_\nu \psi_{(\lambda)} + \frac{-i\lambda a + \frac{1}{2}(N-1)}{i\lambda + 2a}\gamma_\nu \psi_{(\lambda)}\right), \tag{2.14}$$

where $i\lambda$ is the spinor eigenvalue on S^N Eq. (A9). Comparing with Eq. (2.11), we can show that the non-TT eigenvalues are

$$i\xi = i\lambda + 2a = \pm i\sqrt{j^2 + \frac{(N-1)^2}{4}}, \quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots. \tag{2.15}$$

We can write the non-TT eigenspinor vectors as

$$\begin{aligned} \psi_\nu^{(1)} &= \nabla_\nu \psi_{(\lambda)} + \frac{1}{2}(-i\lambda + \sqrt{(N-1) - \lambda^2})\gamma_\nu \psi_{(\lambda)}, \\ \psi_\nu^{(2)} &= \nabla_\nu \psi_{(\lambda)} + \frac{1}{2}(-i\lambda - \sqrt{(N-1) - \lambda^2})\gamma_\nu \psi_{(\lambda)}. \end{aligned} \tag{2.16}$$

C. Spinor-vector TT eigenmode I on S^N

Here we denote the eigenvalues as $i\zeta$, to distinguish them from the non-TT eigenmodes. So our spinor-vector equation with Dirac operator is

$$\gamma^\mu \nabla_\mu \psi_\nu = i\zeta\psi_\nu. \tag{2.17}$$

The transverse traceless condition is

$$\nabla^\mu \psi_\mu = \gamma^\mu \psi_\mu = 0. \tag{2.18}$$

With the spinor eigenmodes noted in the Appendix, we will separate out our lower dimensional part and separately consider the N even and the N odd cases.

1. N odd

Using Eq. (A10)and Eq. (2.4) we find that Eq. (2.18) becomes

$$\gamma^\mu \psi_\mu = 0 \Rightarrow \psi_{\theta_N} = -\frac{1}{\sin\theta_N}\gamma^N \tilde{\gamma}^{\theta_i}\psi_{\theta_i}. \tag{2.19}$$

Using the Christoffel symbols in Eq. (2.2), Eq. (2.17) and Eq. (2.18) become

$$\nabla^\mu \psi_\mu = 0 \Rightarrow \left(\partial_{\theta_N} + \left(N - \frac{1}{2}\right)\cot\theta_N\right)\psi_{\theta_N} = -\frac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\psi_{\theta_i}, \tag{2.20}$$

$$\gamma^\mu \nabla_\mu \psi_{\theta_N} = i\zeta\psi_{\theta_N} \Rightarrow \gamma^N\left(\partial_{\theta_N} + \left(\frac{N+1}{2}\right)\cot\theta_N\right)\psi_{\theta_N} + \frac{1}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{\theta_i}\psi_{\theta_N} = i\zeta\psi_{\theta_N}, \tag{2.21}$$

$$\gamma^\mu \nabla_\mu \psi_{\theta_i} = i\zeta\psi_{\theta_i} \Rightarrow \gamma^N\left(\partial_{\theta_N} + \left(\frac{N-1}{2}\right)\cot\theta_N\right)\psi_{\theta_i} + 2\cot\theta_N \gamma^N \tilde{\Sigma}^{\theta_j}_{\theta_i}\psi_{\theta_j} + \frac{1}{\sin\theta_N}\tilde{\gamma}^{\theta_j}\tilde{\nabla}_{\theta_j}\psi_{\theta_i} = i\zeta\psi_{\theta_i}. \tag{2.22}$$

ψ_{θ_N} behaves like a spinor and we write it as a linear combination of the eigenspinors on S^{N-1}. The ψ_{θ_i} terms behave like spinor vectors and we write them in terms of non-TT mod eigenspinor vectors on S^{N-1},

$$\psi_{\theta_N} = \frac{1}{\sqrt{2}}(1 + i\gamma^N)A^{(1)}\tilde{\psi}_{(\lambda)} + \frac{1}{\sqrt{2}}(1 - i\gamma^N)A^{(2)}\tilde{\psi}_{(\lambda)}, \tag{2.23}$$

$$\psi_{\theta_i} = \frac{1}{\sqrt{2}}(1+i\gamma^N)(C^{(1)}\tilde{\nabla}_{\theta_i}\tilde{\psi}_{(\lambda)} + D^{(1)}\tilde{\gamma}_{\theta_i}\tilde{\psi}_{(\lambda)}) + \frac{1}{\sqrt{2}}(1-i\gamma^N)(C^{(2)}\tilde{\nabla}_{\theta_i}\tilde{\psi}_{(\lambda)} + D^{(2)}\tilde{\gamma}_{\theta_i}\tilde{\psi}_{(\lambda)}). \quad (2.24)$$

The coefficients $A^{(1,2)}, C^{(1,2)}$, and $D^{(1,2)}$ are functions of θ_N only, and $\tilde{\psi}_{(\lambda)}$ is the spinor eigenmode on S^{N-1}. Using these two definitions we can derive the eigenspinor-vector equations, (2.19)–(2.22), by setting $A^{(1,2)} = (\sin\theta_N)^{-\frac{N+1}{2}}\mathbb{A}^{(1,2)}$. The coefficients of the eigenspinor vector are

$$\mathbb{A}^{(1,2)} = \left(\sin\frac{\theta_N}{2}\right)^{|\frac{1}{2}\pm\tilde{\lambda}|+\frac{1}{2}}\left(\cos\frac{\theta_N}{2}\right)^{|\frac{1}{2}\mp\tilde{\lambda}|+\frac{1}{2}} P_n^{|\frac{1}{2}\pm\tilde{\lambda}|,|\frac{1}{2}\mp\tilde{\lambda}|}(\cos\theta_N), \quad (2.25)$$

$$C^{(1,2)} = \frac{\sin\theta_N}{\tilde{\lambda}^2 - \frac{1}{4}(N-1)^2}\left(\frac{N-1}{2}\cos\theta_N \mp \tilde{\lambda}\right)A^{(1,2)} \mp \frac{N-1}{N-2}\frac{\zeta\sin^2\theta_N}{\tilde{\lambda} - \frac{1}{4}(N-1)^2}A^{(2,1)}, \quad (2.26)$$

$$D^{(1,2)} = -\frac{i\sin\theta_N}{\tilde{\lambda}^2 - \frac{1}{4}(N-1)^2}\left(\tilde{\lambda}\cos\theta_N \mp \frac{N-1}{2}\right)A^{(1,2)} \pm \frac{i\zeta\tilde{\lambda}\sin^2\theta_N}{(N-2)(\tilde{\lambda}^2 - \frac{1}{4}(N-1)^2)}A^{(2,1)}, \quad (2.27)$$

where $P_n^{|\frac{1}{2}\pm\tilde{\lambda}|,|\frac{1}{2}\mp\tilde{\lambda}|}(\cos\theta_N)$ is the Jacobi polynomial. The eigenvalues are

$$\zeta = \pm\left(n + |\tilde{\lambda}| + \frac{1}{2}\right), \quad (2.28)$$

where $|\tilde{\lambda}| = \tilde{n} + (N-1)/2$, $\tilde{n} = 0, 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$. Equation (2.28) can be rewritten as

$$\zeta = \pm\left(j + \frac{N-1}{2}\right), \qquad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots. \quad (2.29)$$

As such we have determined the eigenvalue for our eigenspinor vectors for $N \geq 3$ and N odd.

2. N even

We use the gamma matrices and spin connections as given in Eq. (A3) and Eq. (A4) and set $\psi_\mu = (\psi_{\theta_i}, \psi_{\theta_N})$, as such

$$\psi_{\theta_N} = \begin{pmatrix} \psi_{\theta_N}^{(1)} \\ \psi_{\theta_N}^{(2)} \end{pmatrix}, \qquad \psi_{\theta_i} = \begin{pmatrix} \psi_{\theta_i}^{(1)} \\ \psi_{\theta_i}^{(2)} \end{pmatrix}. \quad (2.30)$$

This allows us to rewrite Eq. (2.17) and Eq. (2.18) as

$$\gamma^\mu\psi_\mu = 0 \Rightarrow \psi_{\theta_N}^{(1,2)} = \pm\frac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\psi_{\theta_i}^{(1,2)}, \quad (2.31)$$

$$\nabla^\mu\psi_\mu = 0 \Rightarrow \left(\partial_{\theta_N} + \left(N - \frac{1}{2}\right)\cot\theta_N\right)\psi_{\theta_N}^{(1,2)} = -\frac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\psi_{\theta_i}^{(1,2)}, \quad (2.32)$$

$$\gamma^\mu\nabla_\mu\psi_{\theta_N} = i\zeta\psi_{\theta_N} \Rightarrow \left(\partial_{\theta_N} + \left(\frac{N+1}{2}\right)\cot\theta_N\right)\psi_{\theta_N}^{(1,2)} \mp \frac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{\theta_i}\psi_{\theta_N}^{(1,2)} = i\zeta\psi_{\theta_N}^{(2,1)}, \quad (2.33)$$

$$\gamma^\mu\nabla_\mu\psi_{\theta_i} = i\zeta\psi_{\theta_i} \Rightarrow \left(\partial_{\theta_N} + \left(\frac{N-1}{2}\right)\cot\theta_N\right)\psi_{\theta_i}^{(1,2)} + 2\cot\theta_N\tilde{\Sigma}_{\theta_i}{}^{\theta_j}\psi_{\theta_j}^{(1,2)} \mp \frac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_j}\tilde{\nabla}_{\theta_j}\psi_{\theta_i}^{(1,2)} = i\zeta\psi_{\theta_i}^{(2,1)}. \quad (2.34)$$

We set

$$\psi_{\theta_N}^{(1)} = A^{(1)}\tilde{\psi}_{(\lambda)}; \qquad \psi_{\theta_N}^{(2)} = iA^{(2)}\tilde{\psi}_{(\lambda)}$$
$$\psi_{\theta_i}^{(1)} = C^{(1)}\tilde{\nabla}_{\theta_i}\tilde{\psi}_{(\lambda)} + D^{(1)}\tilde{\gamma}_{\theta_i}\tilde{\psi}_{(\lambda)};$$
$$\psi_{\theta_i}^{(2)} = iC^{(2)}\tilde{\nabla}_{\theta_i}\tilde{\psi}_{(\lambda)} + iD^{(2)}\tilde{\gamma}_{\theta_i}\tilde{\psi}_{(\lambda)}. \quad (2.35)$$

Substituting these into Eqs. (2.31) and (2.34), we have the same results as we did for the N odd case. That is, we find that the eigenvalues are the same as those for the N odd case. We note that $N > 2$, since as discussed earlier, there are no TT eigenmodes for the surface S^2.

D. Spinor-vector TT-modes II on S^N

As discussed earlier in this section TT mode II are only possible for $N \geq 4$. We start by letting the eigenspinor vector $\psi_{\theta_N} = 0$, and the TT mode eigenspinor vector on S^{N-1} will still be an eigenspinor vector on S^N with suitable coefficients.

1. N odd

Setting spinor vector ψ_{θ_i} as

$$\psi_{\theta_i} = \left(\frac{1}{\sqrt{2}}B^{(1)}(1+i\gamma^N) + \frac{1}{\sqrt{2}}B^{(2)}(1-i\gamma^N)\right)\tilde{\psi}_{\theta_i}, \quad (2.36)$$

where $B^{(1)}$ and $B^{(2)}$ are functions of θ_N only. Substituting into Eq. (2.22) by setting $B^{(1,2)} = (\sin\theta_N)^{-(\frac{N-3}{2})}\mathbb{B}^{(1,2)}$, we have

$$\mathbb{B}^{(1,2)} = \left(\sin\frac{\theta_N}{2}\right)^{|\frac{1}{2}\pm\tilde{\zeta}|+\frac{1}{2}}\left(\cos\frac{\theta_N}{2}\right)^{|\frac{1}{2}\mp\tilde{\zeta}|+\frac{1}{2}} P_n^{|\frac{1}{2}\pm\tilde{\zeta}|,|\frac{1}{2}\mp\tilde{\zeta}|}(\cos\theta_N), \tag{2.37}$$

where $P_n^{|\frac{1}{2}\pm\tilde{\zeta}|,|\frac{1}{2}\mp\tilde{\zeta}|}(\cos\theta_N)$ is again the Jacobi polynomial. $\tilde{\zeta}$ is the spinor-vector eigenvalue of the TT mode I on S^{N-1}, which is given as $\tilde{\zeta} = \tilde{j} + \frac{N-2}{2}$. Such that the eigenvalues on TT mode II are

$$i\zeta = \pm i\left(n + |\tilde{\zeta}| + \frac{1}{2}\right) = \pm i\left(j + \frac{N-1}{2}\right),$$
$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \tag{2.38}$$

which are the same as those of TT mode I on S^N.

2. *N even*

Setting

$$\psi_{\theta_i}^{(1)} = B^{(1)}\tilde{\psi}_{\theta_i}; \qquad \psi_{\theta_i}^{(2)} = iB^{(2)}\tilde{\psi}_{\theta_i}, \tag{2.39}$$

and using Eqs. (2.31) to (2.34), we find that the eigenvalue is still given as Eq. (2.38). Using the eigenvalues for our spinors and spinor vectors we can determine our potential for spin-3/2 particles near Schwarzschild black holes.

III. THE RADIAL EQUATION AND THE POTENTIAL FUNCTION

In this section we are going to obtain the radial equation and the effective potential for the spin-3/2 field in the D-dimensional Schwarzschild black hole spacetime. Since the mode function of the spin-3/2 field will be represented by the spinor-vector wave functions, we have to do the construction analogous to the details with the spinor and the vector fields. In the study of Maxwell fields, it has been shown that there are two physical modes with different mode functions [12,13]. One is related to the scalar spherical harmonics, and another one is related to the vector spherical harmonics, these are also known as the "longitudinal" and "transverse" parts of a vector field [14]. In our case there are non-TT eigenmodes and TT eigenmodes on S^N, where we may obtain two physical modes related to these eigenmodes for our spin-3/2 field case.

A. Massless Rarita-Schwinger field for *D* dimensions

To begin we need to define our metric as

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2 d\bar{\Omega}_N^2, \tag{3.1}$$

where $f = 1 - (\frac{2M}{r})^{D-3}$ and $D = N+2$ is the dimensions of the spacetime. The $d\bar{\Omega}_N$ is the metric for the N sphere, and we denote terms from the N sphere with over bars. We will use the massless form of the Rarita-Schwinger equation to represent the spin-3/2 field,

$$\gamma^{\mu\nu\alpha}\nabla_\nu\psi_\alpha = 0, \tag{3.2}$$

where the antisymmetric Dirac gamma product is given as

$$\gamma^{\mu\nu\alpha} = \gamma^{[\mu}\gamma^\nu\gamma^{\alpha]} = \gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\mu g^{\nu\alpha} + \gamma^\nu g^{\mu\alpha} - \gamma^\alpha g^{\mu\nu}. \tag{3.3}$$

We choose the following gamma matrices:

$$\gamma^0 = i\sigma^3 \otimes \mathbb{1} \Rightarrow \gamma^t = \frac{1}{\sqrt{f}}(i\sigma^3 \otimes \mathbb{1}),$$
$$\gamma^i = \sigma^1 \otimes \bar{\gamma}^i \Rightarrow \gamma^{\theta_i} = \frac{1}{r}(\sigma^1 \otimes \bar{\gamma}^{\theta_i}),$$
$$\gamma^{D-1} = \sigma^2 \otimes \mathbb{1} \Rightarrow \gamma^r = \sqrt{f}(\sigma^2 \otimes \mathbb{1}), \tag{3.4}$$

with $\mathbb{1}$ being the $2^{(\frac{D-2}{2})} \times 2^{(\frac{D-2}{2})}$ unit matrix for the case of D even and the $2^{(\frac{D-3}{2})} \times 2^{(\frac{D-3}{2})}$ unit matrix for the case of D odd. $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices and $\bar{\gamma}^{\theta_i}$ are the Dirac matrices for the N sphere. The nonzero spin connections are

$$\omega_t = -\frac{f'}{4}(\sigma^1 \otimes \mathbb{1}),$$
$$\omega_{\theta_i} = \mathbb{1} \otimes \bar{\omega}_{\theta_i} + \frac{\sqrt{f}}{2}(i\sigma^3 \otimes \bar{\gamma}_{\theta_i}). \tag{3.5}$$

And the nonzero triple gamma products are given as

$$\gamma^{t\theta_i r} = -\frac{1}{r}(\mathbb{1} \otimes \bar{\gamma}^{\theta_i}),$$
$$\gamma^{t\theta_i\theta_j} = \frac{1}{r^2\sqrt{f}}(i\sigma^3 \otimes \bar{\gamma}^{\theta_i\theta_j}),$$
$$\gamma^{r\theta_i\theta_j} = \frac{\sqrt{f}}{r^2}(\sigma^2 \otimes \bar{\gamma}^{\theta_i\theta_j}),$$
$$\gamma^{\theta_i\theta_j\theta_k} = \frac{1}{r^3}(\sigma^1 \otimes \bar{\gamma}^{\theta_i\theta_j\theta_k}), \tag{3.6}$$

where $\bar{\gamma}^{\theta_i\theta_j} = \bar{\gamma}^{\theta_i}\bar{\gamma}^{\theta_j} - \bar{g}^{\theta_i\theta_j}$ is the antisymmetric product of two Dirac matrices.

B. With non-TT eigenfunctions

We represent our radial, temporal, and angular parts as ψ_r, ψ_t, and ψ_{θ_i}. The radial and temporal parts will behave as spinors on S^N and we write them as

$$\psi_r = \phi_r \otimes \bar{\psi}_{(\lambda)} \quad \text{and} \quad \psi_t = \phi_t \otimes \bar{\psi}_{(\lambda)}, \tag{3.7}$$

where $\bar{\psi}_{(\lambda)}$ is an eigenspinor on the S^N, with eigenvalues $i\bar{\lambda}$. The angular part, however, will behave as a spinor vector on S^N and can be written as Eq. (2.16). However, it is more convenient to write it as

$$\psi_{\theta_i} = \phi_\theta^{(1)} \otimes \bar{\nabla}_{\theta_i}\bar{\psi}_{(\lambda)} + \phi_\theta^{(2)} \otimes \bar{\gamma}_{\theta_i}\bar{\psi}_{(\lambda)}, \tag{3.8}$$

where $\phi_\theta^{(1)}, \phi_\theta^{(2)}$ are functions of r and t which behave like 2-spinors. This is the same form as we have used for spinors when studying the four-dimensional spacetime [5]. Using Eq. (3.2) we will derive our equations of motion and then try to rewrite them as Schrödinger-like equations. We will initially work in the Weyl gauge, where $\phi_t = 0$, to determine our equations of motion, and will then find a gauge-invariant form.

1. Equations of motion

First, consider the case where $\mu = t$ in Eq. (3.2),

$$\gamma^{t\nu\alpha}\nabla_\nu\psi_\alpha = 0. \tag{3.9}$$

By using Eq. (3.7) and Eq. (3.8), with the angular part separated, we have our first equation of motion in terms of $\phi_r, \phi_\theta^{(1)}$, and $\phi_\theta^{(2)}$;

$$\begin{aligned} 0 = &-\left[i\bar{\lambda} + \frac{\sqrt{f}}{2}(D-2)(i\sigma^3)\right]\phi_r \\ &+ \left[i\bar{\lambda}\partial_r - \frac{1}{4}\frac{(D-2)(D-3)}{r\sqrt{f}}(i\sigma^3) + (D-3)\frac{i\bar{\lambda}}{2r}\right]\phi_\theta^{(1)} \\ &+ \left[(D-2)\partial_r + \frac{i\bar{\lambda}(D-3)}{r\sqrt{f}}(i\sigma^3) \right. \\ &\left. + \frac{(D-2)(D-3)}{2r}\right]\phi_\theta^{(2)}. \end{aligned} \tag{3.10}$$

Next consider the case where $\mu = r$ in Eq. (3.2),

$$\gamma^{r\nu\alpha}\nabla_\nu\psi_\alpha = 0. \tag{3.11}$$

The second equation of motion is

$$\begin{aligned} 0 = &\left[-\frac{i\bar{\lambda}}{\sqrt{f}}\partial_t + \frac{i\bar{\lambda}f'}{4\sqrt{f}}\sigma^1 - \frac{(D-3)(D-2)}{4r}\sigma^2 \right. \\ &\left. + (D-3)\frac{i\bar{\lambda}\sqrt{f}}{2r}\sigma^1\right]\phi_\theta^{(1)} \\ &+ \left[-\frac{(D-2)}{\sqrt{f}}\partial_t + (D-2)\frac{f'}{4\sqrt{f}}\sigma^1 + (D-3)\frac{i\bar{\lambda}}{r}\sigma^2 \right. \\ &\left. + (D-2)(D-3)\frac{\sqrt{f}}{2r}\sigma^1\right]\phi_\theta^{(2)}. \end{aligned} \tag{3.12}$$

Finally for $\mu = \theta_i$,

$$\gamma^{\theta_i\nu\alpha}\nabla_\nu\psi_\alpha = 0. \tag{3.13}$$

Giving us our final two equations of motion,

$$\begin{aligned} 0 = &\left(\frac{1}{r\sqrt{f}}(i\sigma^3)\partial_t + \frac{\sqrt{f}}{r}\sigma^2\partial_r + \frac{f'}{4r\sqrt{f}}\sigma^2 \right. \\ &\left. + (D-4)\frac{\sqrt{f}}{2r^2}\sigma^2\right)\phi^{(1)} - \left(\frac{D-4}{r^2}\sigma^1\right)\phi_\theta^{(2)} \\ &- \left(\frac{\sqrt{f}}{r}\sigma^2\right)\phi_r \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} 0 = &-\left(\frac{i\bar{\lambda}}{r\sqrt{f}}(i\sigma^3)\partial_t + \frac{i\bar{\lambda}\sqrt{f}}{r}\sigma^2\partial_r + \frac{i\bar{\lambda}f'}{4r\sqrt{f}}\sigma^2 \right. \\ &\left. + (D-3)(D-4)\frac{1}{4r^2}\sigma^1 + (D-4)\frac{i\bar{\lambda}\sqrt{f}}{2r^2}\sigma^2\right)\phi_\theta^{(1)} \\ &- \left(\frac{D-3}{r\sqrt{f}}(i\sigma^3)\partial_t + (D-3)\frac{\sqrt{f}}{r}\sigma^2\partial_r + (D-3)\frac{f'}{4r\sqrt{f}}\sigma^2 \right. \\ &\left. - (D-4)\frac{i\bar{\lambda}}{r^2}\sigma^1 + (D-3)(D-4)\frac{\sqrt{f}}{2r^2}\sigma^2\right)\phi_\theta^{(2)} \\ &+ \left(\partial_t - \frac{(D-3)f}{2r}\sigma^1 - \frac{f'}{4}\sigma^1 + \frac{i\bar{\lambda}\sqrt{f}}{r}\sigma^2\right)\phi_r. \end{aligned} \tag{3.15}$$

We now have our four equations of motion, Eqs. (3.10), (3.12), (3.14), and (3.15), in terms of ϕ_r, $\phi_\theta^{(1)}$, and $\phi_{\theta_2}^{(2)}$. The functions ϕ_r, $\phi_\theta^{(1)}$, and $\phi_\theta^{(2)}$ are not gauge invariant. In the next section we investigate the required gauge invariance and determine the appropriate transformations in order to create our gauge-invariant radial equation.

2. Gauge-invariant variable

If we consider a system where only gravitational forces are present then

$$\gamma^{\mu\nu\alpha}\nabla_\nu\nabla_\alpha\varphi = \frac{1}{8}\gamma^{\mu\nu\alpha}R_{\nu\alpha\rho\sigma}\gamma^\rho\gamma^\sigma\varphi, \tag{3.16}$$

where φ is a Dirac spinor. This allows our spinors vectors to transform as

$$\psi'_\mu = \psi_\mu + \nabla_\mu\varphi, \tag{3.17}$$

and still have Eq. (3.2) remain true, given that Eq. (3.16) is equal to zero. This is not the case if our metric is charged, and we would need to introduce terms containing the electromagnetic field strength.

We can simplify the expression given in Eq. (3.16) by exploiting the symmetry of the Riemann tensor,

$$\begin{aligned} &\gamma^\mu\gamma^\nu\gamma^\alpha(R_{\mu\nu\alpha\beta} + R_{\nu\alpha\mu\beta} + R_{\alpha\mu\nu\beta}) = 0 \\ &\qquad\Rightarrow \gamma^\mu\gamma^\nu\gamma^\alpha R_{\mu\nu\alpha\beta} = -2\gamma^\alpha R_{\alpha\beta}. \end{aligned} \tag{3.18}$$

We also have that

$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta R_{\mu\nu\alpha\beta} = -2R. \tag{3.19}$$

Using these two identities Eq. (3.16) becomes

$$\frac{1}{8}\gamma^{\mu\nu\alpha}R_{\nu\alpha\rho\sigma}\gamma^\rho\gamma^\sigma = \frac{1}{4}(2\gamma^\alpha R^\mu_\alpha - \gamma^\mu R)\varphi. \tag{3.20}$$

This is zero for Ricci flat spacetimes like the D-dimensional Schwarzschild spacetime. However, for de Sitter and anti–de Sitter spacetimes it does not vanish, so we would need to modify the covariant derivative in those cases in order to respect the gauge symmetry. This means we can perform the above transformation on our spinor vector.

First, consider the transformation of ϕ_r and ϕ_t. Take $\varphi = \phi \otimes \overline{\psi}_{(\lambda)}$, then Eq. (3.17) becomes

$$\begin{aligned}\psi_t' &= \psi_t + \nabla_t\psi \Rightarrow \phi_t{}' = \phi_t + \partial_r\phi - \frac{f'}{4}\sigma^1\phi,\\ \psi_r' &= \psi_r + \nabla_r\psi \Rightarrow \phi_r{}' = \phi_r + \partial_r\phi.\end{aligned} \tag{3.21}$$

Next we consider the transformation of our angular components of ψ_μ. They are given as

$$\begin{aligned}\psi_{\theta_i}{}' = \psi_{\theta_i} + \nabla_{\theta_i}\varphi, &\Rightarrow \phi_\theta'^{(1)} \otimes \overline{\nabla}_{\theta_i}\overline{\psi}_{(\lambda)} + \phi_\theta'^{(2)} \otimes \overline{\gamma}_{\theta_i}\overline{\psi}_{(\lambda)} = (\phi_\theta^{(1)} + \phi) \otimes \overline{\nabla}_{\theta_i}\overline{\psi}_{(\lambda)} + \left(\phi_\theta^{(2)} + \frac{\sqrt{f}}{2}(i\sigma^3)\phi\right) \otimes \overline{\gamma}_{\theta_i}\overline{\psi}_{(\lambda)},\\ &\Rightarrow \phi_\theta'^{(1)} = \phi_\theta^{(1)} + \phi; \qquad \phi_\theta'^{(2)} = \phi_\theta^{(2)} + \frac{\sqrt{f}}{2}(i\sigma^3)\phi.\end{aligned} \tag{3.22}$$

So clearly ϕ_t, ϕ_r, $\phi_\theta^{(1)}$, and $\phi_\theta^{(2)}$ are not gauge invariant. We need to perform a transformation of these spinors in order to obtain gauge-invariant functions. We use the combination we have used in the four-dimensional spacetime [5]

$$\Phi = -\frac{\sqrt{f}}{2}(i\sigma^3)\phi_\theta^{(1)} + \phi_\theta^{(2)}. \tag{3.23}$$

Note that there is no dimensional dependence for our gauge-invariant variable.

3. Effective potential

Using the gauge-invariant variable Φ, Eq. (3.10), Eq. (3.12), and Eq. (3.14) become

$$\left((D-2)\partial_r + (D-3)\frac{i\bar{\lambda}}{r\sqrt{f}}(i\sigma^3) + \frac{(D-2)(D-3)}{2r}\right)\Phi + \left(i\bar{\lambda} + \frac{D-2}{2}\sqrt{f}i\sigma^3\right)\partial_r\phi_\theta^{(1)} = \left(i\bar{\lambda} + \frac{D-2}{2}\sqrt{f}(i\sigma^3)\right)\phi_r, \tag{3.24}$$

$$\begin{aligned}&\left(-\frac{D-2}{\sqrt{f}}\partial_t + \frac{(D-2)(D-3)\sqrt{f}}{2r}\sigma^1 + \frac{(D-2)f'}{4\sqrt{f}}\sigma^1 + (D-3)\frac{i\bar{\lambda}}{r}\sigma^2\right)\Phi\\ &+ \left(-\left(\frac{i\bar{\lambda}}{\sqrt{f}} + \frac{D-2}{2}i\sigma^3\right)\partial_t + \frac{i\bar{\lambda}f'}{4\sqrt{f}}\sigma^1 - \frac{D-2}{8}f'\sigma^2\right)\phi_\theta^{(1)} = 0,\end{aligned} \tag{3.25}$$

and

$$\left(\frac{1}{\sqrt{f}}(i\sigma^3)\partial_t + \sqrt{f}\sigma^2\partial_r + \frac{f'}{4\sqrt{f}}\sigma^2\right)\phi_\theta^{(1)} - \left(\frac{D-4}{r}\sigma^1\right)\Phi - \sqrt{f}\sigma^2\phi_r = 0. \tag{3.26}$$

We have used that $f' = (D-3)(1-f)/r$ to simplify our equations. We can now use Eq. (3.24), Eq. (3.25), and Eq. (3.26) to derive a gauge-invariant equation of motion in terms of only Φ,

$$\begin{aligned}0 = &\left(\frac{D-2}{2}\sqrt{f} + \bar{\lambda}\sigma^3\right)\left[-(D-2)\sigma^1\partial_t + \frac{D-2}{4}f' + \frac{D-3}{r}\sqrt{f}\left(\frac{D-2}{2}\sqrt{f} - \bar{\lambda}\sigma^3\right)\right]\Phi\\ &- \left(\frac{D-2}{2}\sqrt{f} - \bar{\lambda}\sigma^3\right)\left[(D-2)f\partial_r + \frac{D-3}{r}\sqrt{f}\left(\frac{D-2}{2}\sqrt{f} - \bar{\lambda}\sigma^3\right)\right]\Phi\\ &- \left(\frac{D-2}{2}\sqrt{f} - \bar{\lambda}\sigma^3\right)\left[\frac{D-4}{r}\sqrt{f}\left(\frac{D-2}{2}\sqrt{f} + \bar{\lambda}\sigma^3\right)\right]\Phi.\end{aligned} \tag{3.27}$$

Component wise Φ is given as

$$\Phi = \begin{pmatrix} \Phi_1(r)e^{-i\omega t} \\ \Phi_2(r)e^{-i\omega t} \end{pmatrix}. \tag{3.28}$$

Equation (3.27) then becomes

$$\left(\frac{B}{A}\right) f\partial_r\Phi_1 - \frac{f'}{4}\Phi_1 - 2\left(\frac{B}{A}\right)\frac{D-3}{D-2}\frac{\sqrt{f}\lambda}{r}\Phi_1 + \frac{D-4}{D-2}\frac{\sqrt{f}}{r}B\Phi_1 = i\omega\Phi_2, \tag{3.29}$$

$$\left(\frac{A}{B}\right) f\partial_r\Phi_2 - \frac{f'}{4}\Phi_2 + 2\left(\frac{A}{B}\right)\frac{D-3}{D-2}\frac{\sqrt{f}\lambda}{r}\Phi_2 + \frac{D-4}{D-2}\frac{\sqrt{f}}{r}A\Phi_2 = i\omega\Phi_1, \tag{3.30}$$

where we have set

$$A = \frac{D-2}{2}\sqrt{f} + \bar{\lambda} \quad \text{and} \quad B = \frac{D-2}{2}\sqrt{f} - \bar{\lambda}. \tag{3.31}$$

We can further simplify the above equation by defining

$$\tilde{\Phi}_1 = r^{\frac{D-4}{2}}\left(\frac{f^{1/4}}{\frac{D-2}{2}\sqrt{f} + \bar{\lambda}}\right)\Phi_1;$$
$$\tilde{\Phi}_2 = r^{\frac{D-4}{2}}\left(\frac{f^{1/4}}{\frac{D-2}{2}\sqrt{f} - \bar{\lambda}}\right)\Phi_2. \tag{3.32}$$

Equation (3.29) and Eq. (3.30) then become

$$\left(\frac{d}{dr_*} - W\right)\tilde{\Phi}_1 = i\omega\tilde{\Phi}_2;$$
$$\left(\frac{d}{dr_*} + W\right)\tilde{\Phi}_2 = i\omega\tilde{\Phi}_1, \tag{3.33}$$

where r_* is the tortoise coordinate and is defined as $d/dr_* = fd/dr$. W is known as the superpotential in supersymmetric quantum mechanics and is determined to be

$$W = \frac{|\bar{\lambda}|\sqrt{f}}{r}\left(\frac{(\frac{2}{D-2})^2|\bar{\lambda}|^2 - 1 - \frac{D-4}{D-2}(\frac{2M}{r})^{D-3}}{(\frac{2}{D-2})^2|\bar{\lambda}|^2 - f}\right). \tag{3.34}$$

This allows us to write our Schrödinger-like equation to describe our particles, where we have called this equation our radial equation, given as

$$-\frac{d^2}{dr_*^2}\tilde{\Phi}_1 + V_1\tilde{\Phi}_1 = \omega^2\tilde{\Phi}_1;$$
$$-\frac{d^2}{dr_*^2}\tilde{\Phi}_2 + V_2\tilde{\Phi}_2 = \omega^2\tilde{\Phi}_2, \tag{3.35}$$

with isospectral supersymmetric partner potentials [15]

$$V_{1,2} = \pm f\frac{dW}{dr} + W^2, \tag{3.36}$$

where $f = 1 - (2M/r)^{D-3}$ in D-dimensional space. As $\bar{\lambda} = n + (D-2)/2$ and $n = 0, 1, 2, \ldots$, we rewrite it as $\bar{\lambda} = j + (D-3)/2$, where $j = 1/2, 3/2, 5/2, \ldots$ such that $V(r)$ is explicitly given as

$$\begin{aligned} V_{(1,2)} = {} & \frac{X\sqrt{f}(j+\frac{D-3}{2})}{r^2(X+Y)^2} \\ & \times\left[X\left(\left(j+\frac{D-3}{2}\right)\sqrt{f} \pm \left(\frac{D-1}{2}\right)Y \mp 1\right)\right. \\ & \left.\mp\left(\frac{2D^2-13D+19}{D-2}\right)Y^2\right] \\ & + \frac{(j+\frac{D-3}{2})\sqrt{f}Y^2}{r^2(X+Y)^2}\left(\frac{D-4}{D-2}\right) \\ & \times\left[\left(j+\frac{D-3}{2}\right)\left(\frac{D-4}{D-2}\right)\sqrt{f} \pm 1 \mp \left(\frac{D-1}{2}\right)Y\right] \\ & + \frac{X\sqrt{f}(j+\frac{D-3}{2})}{r^2(X+Y)^2}[\pm 2(D-4)Y], \end{aligned} \tag{3.37}$$

$$X = \left(\frac{2}{D-2}\right)^2\left(j-\frac{1}{2}\right)\left(j+\frac{2D-5}{2}\right),$$
$$Y = \left(\frac{2M}{r}\right)^{D-3}. \tag{3.38}$$

Setting $D = 4$ we find that our potential is the same as in Refs. [5,16];

$$\begin{aligned} V_{1,2} = {} & \frac{(j-\frac{1}{2})(j+\frac{1}{2})(j+\frac{3}{2})\sqrt{f}}{r^2((j-\frac{1}{2})(j+\frac{3}{2})+(\frac{2M}{r}))^2} \\ & \times\left[\pm\frac{2M^2}{r^2} + \left(j-\frac{1}{2}\right)\left(j+\frac{3}{2}\right)\right. \\ & \left.\times\left(\left(j+\frac{1}{2}\right)\sqrt{f} \pm \left(\frac{3M}{r}\right) \mp 1\right)\right]. \end{aligned} \tag{3.39}$$

C. With the TT eigenfunctions

1. Equations of motion

We set the radial and temporal parts, ψ_r and ψ_t, to be the same as the non-TT eigenfunctions case given in Eq. (3.7). The angular part, ψ_{θ_i}, can be written in terms of the TT mode eigenspinor vector on S^N as

$$\psi_{\theta_i} = \phi_\theta \otimes \bar{\psi}_{\theta_i}, \tag{3.40}$$

where $\bar{\psi}_{\theta_i}$ is the TT mode eigenspinor vector which includes the TT mode I and TT mode II, and ϕ_θ behaves like a 2-spinor. As we have done for the previous case, we will use the Weyl gauge and consider the cases $\mu = t$, $\mu = r$, and $\mu = \theta_i$ for Eq. (3.2). Applying the TT conditions on a sphere, namely, $\bar{\gamma}^{\theta_i}\bar{\psi}_{\theta_i} = \bar{\nabla}^{\theta_i}\bar{\psi}_{\theta_i} = 0$, we have $\psi_r = 0$. We find that the equation of motion in this case is

$$\left(i\sigma^3\partial_t + f\sigma^2\partial_r + \frac{f'}{4}\sigma^2 + \frac{D-4}{2r}f\sigma^2 + \frac{\sqrt{f}}{r}i\bar{\zeta}\sigma^1\right)\phi_\theta = 0. \tag{3.41}$$

In this case ϕ_θ is gauge invariant, so we directly derive the radial equation in the next section.

2. Effective potential

Assuming ϕ_θ is given as

$$\phi_\theta = \sigma^2 \begin{pmatrix} \Psi_{\theta_1} e^{-i\omega t} \\ \Psi_{\theta_2} e^{-i\omega t} \end{pmatrix}. \tag{3.42}$$

Equation (3.41) can then be rewritten as

$$\begin{aligned} &\left(f\partial_r + \frac{f'}{4} + \frac{D-4}{2r}f - \frac{\sqrt{f}}{r}\bar{\zeta}\right)\Psi_{\theta_1} = i\omega\Psi_{\theta_2}, \\ &\left(f\partial_r + \frac{f'}{4} + \frac{D-4}{2r}f + \frac{\sqrt{f}}{r}\bar{\zeta}\right)\Psi_{\theta_2} = i\omega\Psi_{\theta_1}. \end{aligned} \tag{3.43}$$

These expressions can be simplified using the following transformations:

$$\bar{\Psi}_{\theta_1} = r^{\frac{D-4}{2}} f^{\frac{1}{4}}\Psi_{\theta_1}, \qquad \bar{\Psi}_{\theta_2} = r^{\frac{D-4}{2}} f^{\frac{1}{4}}\Psi_{\theta_2}. \tag{3.44}$$

We have the radial equations,

$$\begin{aligned} -\frac{d^2}{dr_*^2}\bar{\Psi}_{\theta_1} + \mathbb{V}_1\bar{\Psi}_{\theta_1} &= \omega^2\bar{\Psi}_{\theta_1}; \\ -\frac{d^2}{dr_*^2}\bar{\Psi}_{\theta_2} + \mathbb{V}_2\bar{\Psi}_{\theta_2} &= \omega^2\bar{\Psi}_{\theta_2}, \end{aligned} \tag{3.45}$$

where

$$\mathbb{V}_{1,2} = \pm f\frac{d\mathbb{W}}{dr} + \mathbb{W}^2, \tag{3.46}$$

and

$$\mathbb{W} = \frac{\sqrt{f}}{r}\bar{\zeta}. \tag{3.47}$$

As $\bar{\zeta} = j + (D-3)/2$ where $j = 1/2, 3/2, 5/2, \ldots$. Our spinor-vector potentials are then explicitly given as

$$\begin{aligned} \mathbb{V}_{1,2} = \pm &\frac{\sqrt{1-(\frac{2M}{r})^{D-3}}}{r^2}\left(j + \frac{D-3}{2}\right) \\ &\times\left[\frac{D-1}{2}\left(\frac{2M}{r}\right) - 1 \right. \\ &\left.\pm\sqrt{1-\left(\frac{2M}{r}\right)^{D-3}}\left(j+\frac{D-3}{2}\right)\right]. \end{aligned} \tag{3.48}$$

This is the same potential as obtained in Ref. [17], where the radial equation of a spin-1/2 field on the general dimensional Schwarzschild black hole spacetime is considered. We can say that the radial equation for the spin-3/2 field is equivalent to that of the spinor field case when the eigenmode on S^N is the TT mode, with $\psi_t = \psi_r = 0$, and only ψ_{θ_i} remains.

IV. QUASINORMAL MODES

In this section we focus on the QNMs for our non-TT eigenfunctions spinor vectors, where we will use the new potential that we have derived for the massless spin-3/2 fields. The potential that we have derived for the TT eigenfunctions is the same as that seen for the spin-1/2 Dirac field; we therefore refer the reader to Ref. [17] for the results of the TT eigenfunctions.

A. Methods

We have used two methods in order to determine the numerical values of our QNMs. We have used the WKB method to third and sixth order, and the improved AIM to calculate the numerical values of our QNMs. The third order WKB method was developed by Iyer and Will [18] and the sixth order was developed by Konoplya [19].

1. Implementation of the improved AIM

The improved AIM has been developed in the following papers: [10,20–22]. In order to use this technique we must first perform a coordinate change so that we are operating on a compact space; we choose $\xi^2 = 1 - 2M/r$. Our boundary conditions require that our particles are purely in-going at the horizon and purely out-going at infinity. Since our particles would exhibit plane wave behavior at these boundaries, we can write their wave functions as

$$\begin{aligned} \tilde{\Phi}_1 &\sim e^{i\omega r_*} \quad \text{for } r_* \to \infty; \\ \tilde{\Phi}_1 &\sim e^{-i\omega r_*} \quad \text{for } r_* \to -\infty. \end{aligned} \tag{4.1}$$

With r_* as the tortoise coordinate, where the general formula for an D-dimensional tortoise coordinate is given in Ref. [23]

$$r_* = r + \sum_{n=1}^{D-3} \frac{e^{2\pi i \frac{n}{D-3}}}{D-3} 2M \ln\left(r - 2M e^{2\pi i \frac{n}{D-3}}\right). \qquad (4.2)$$

Plugging Eq. (4.2) into our wave functions for our particles gives us our general behavior of the particles for our D-dimensional space

$$\tilde{\Phi}_1 \sim e^{\pm \frac{2iM\omega}{1-\xi^2}} \prod_{n=1}^{D-3} (1 - (1-\xi^2)\Theta(n))^{\pm \frac{2iM\omega\Theta(n)}{D-3}} \times ((1-\xi^2)\Theta(n))^{\pm \frac{2iM\omega\Theta(n)}{D-3}}, \qquad (4.3)$$

where $\Theta(n) = e^{\frac{2\pi i n}{D-3}}$. Clearly at the boundaries of our system, namely, $\xi = 1, 0$, we would encounter asymptotic behavior. We extract this asymptotic behavior from ϕ and write

$$\tilde{\Phi}_1 = \beta(\xi)\chi(\xi), \qquad (4.4)$$

where $\beta(\xi)$ contains our asymptotic behavior, and $\chi(\xi)$ satisfies the equation

$$\chi''(\xi) = \lambda_0 \chi(\xi) + s_0. \qquad (4.5)$$

The functions λ_0 and s_0 are determined to be

$$\lambda_0 = -\left(2\frac{\beta'(\xi)}{\beta(\xi)} + A\right), \qquad (4.6)$$

$$s_0 = -\left(\frac{\beta''(\xi)}{\beta(\xi)} + \frac{\beta'(\xi)}{\beta(\xi)} A + B\right), \qquad (4.7)$$

with

$$A = \frac{\xi''}{\xi'} + \frac{f'}{f}; \qquad B = \frac{1}{(f\xi')^2}(\omega^2 - V). \qquad (4.8)$$

We then apply the AIM to Eq. (4.5) and after 200 iteration we obtain the results given in Tables I–III.

B. Results

We present the results of our WKB and AIM calculation in Tables I–III. For fixed angular quantum number l with specific dimension D, when the mode number n gets larger, the real parts of the frequencies decrease and the imaginary parts, or the damping rates, increase. This indicates that the $n = 0$ mode has the largest probability of being observed. We also show the behavior of first few QNMs with various dimensions in Fig. 1, which shows that both the frequency and the damping rate increase with the number of dimensions.

Comparing the WKB and AIM methods, we see that the WKB method returns results which are not as accurate as the AIM, however, the WKB is much easier to implement compared to the AIM. In order to obtain accurate results for QNMs using the AIM it is necessary for us to perform a large number of iterations of the AIM, which has the drawback of requiring large amounts of computation time. When $D = 4, 5, 6$ we see that the AIM and WKB method

TABLE I. Low-lying ($n \le l$, with $l = j - 3/2$) spin-3/2 field quasinormal mode frequencies using the WKB and the AIM methods with $D = 4, 5$.

		4 Dimensions					5 Dimensions		
l	n	Third order WKB	Sixth order WKB	AIM	l	n	Third order WKB	Sixth order WKB	AIM
0	0	0.3087 − 0.0902i	0.3113 − 0.0902i	0.3112 − 0.0902i	0	0	0.4409 − 0.1529i	0.4641 − 0.1436i	0.4641 − 0.1435i
1	0	0.5295 − 0.0938i	0.5300 − 0.0938i	0.5300 − 0.0937i	1	0	0.7530 − 0.1653i	0.7558 − 0.1652i	0.7558 − 0.1651i
1	1	0.5103 − 0.2858i	0.5114 − 0.2854i	0.5113 − 0.2854i	1	1	0.6902 − 0.5112i	0.6989 − 0.5075i	0.6988 − 0.5074i
2	0	0.7346 − 0.0949i	0.7348 − 0.0949i	0.7347 − 0.0948i	2	0	1.0322 − 0.1700i	1.0332 − 0.1700i	1.0332 − 0.1700i
2	1	0.7206 − 0.2870i	0.7210 − 0.2869i	0.7210 − 0.2869i	2	1	0.9869 − 0.5182i	0.9900 − 0.5172i	0.9899 − 0.5172i
2	2	0.6960 − 0.4844i	0.6953 − 0.4855i	0.6952 − 0.4855i	2	2	0.9076 − 0.8835i	0.9070 − 0.8868i	0.9070 − 0.8868i
3	0	0.9343 − 0.0954i	0.9344 − 0.0954i	0.9343 − 0.0953i	3	0	1.2998 − 0.1723i	1.3003 − 0.1723i	1.3003 − 0.1723i
3	1	0.9233 − 0.2876i	0.9235 − 0.2876i	0.9235 − 0.2875i	3	1	1.2639 − 0.5221i	1.2654 − 0.5217i	1.2653 − 0.5216i
3	2	0.9031 − 0.4834i	0.9026 − 0.4840i	0.9025 − 0.4839i	3	2	1.1985 − 0.8839i	1.1974 − 0.8858i	1.1974 − 0.8857i
3	3	0.8759 − 0.6835i	0.8733 − 0.6870i	0.8732 − 0.6870i	3	3	1.1111 − 1.2586i	1.1009 − 1.2748i	1.1008 − 1.2748i
4	0	1.1315 − 0.0956i	1.1315 − 0.0956i	1.1315 − 0.0956i	4	0	1.5617 − 0.1736i	1.5620 − 0.1736i	1.5619 − 0.1736i
4	1	1.1224 − 0.2879i	1.1225 − 0.2879i	1.1225 − 0.2879i	4	1	1.5319 − 0.5244i	1.5327 − 0.5242i	1.5326 − 0.5242i
4	2	1.1053 − 0.4828i	1.1050 − 0.4831i	1.1049 − 0.4830i	4	2	1.4761 − 0.8841i	1.4751 − 0.8852i	1.4750 − 0.8851i
4	3	1.0817 − 0.6812i	1.0798 − 0.6830i	1.0798 − 0.6829i	4	3	1.3998 − 1.2546i	1.3919 − 1.2639i	1.3919 − 1.2638i
4	4	1.0530 − 0.8828i	1.0485 − 0.8891i	1.0484 − 0.8890i	4	4	1.3070 − 1.6346i	1.2874 − 1.6672i	1.2873 − 1.6672i
5	0	1.3273 − 0.0958i	1.3273 − 0.0958i	1.3273 − 0.0957i	5	0	1.8204 − 0.1744i	1.8205 − 0.1744i	1.8205 − 0.1744i
5	1	1.3196 − 0.2881i	1.3196 − 0.2881i	1.3196 − 0.2881i	5	1	1.7947 − 0.5259i	1.7952 − 0.5258i	1.7951 − 0.5257i
5	2	1.3048 − 0.4824i	1.3045 − 0.4826i	1.3045 − 0.4825i	5	2	1.7461 − 0.8842i	1.7453 − 0.8848i	1.7452 − 0.8848i
5	3	1.2839 − 0.6795i	1.2826 − 0.6805i	1.2826 − 0.6805i	5	3	1.6783 − 1.2515i	1.6724 − 1.2571i	1.6723 − 1.2570i
5	4	1.2582 − 0.8794i	1.2547 − 0.8832i	1.2547 − 0.8831i	5	4	1.5950 − 1.6274i	1.5793 − 1.6478i	1.5793 − 1.6478i
5	5	1.2284 − 1.0821i	1.2221 − 1.0915i	1.2220 − 1.0914i	5	5	1.4983 − 2.0108i	1.4696 − 2.0621i	1.4695 − 2.0621i

TABLE II. Low-lying ($n \leq l$, with $l = j - 3/2$) spin-3/2 field quasinormal mode frequencies using the WKB and the AIM methods with $D = 6,7$.

6 Dimensions					7 Dimensions				
l	n	Third order WKB	Sixth order WKB	AIM	l	n	Third order WKB	Sixth order WKB	AIM
0	0	0.5916 − 0.2260i	0.5714 − 0.2197i	0.5713 − 0.2197i	0	0	0.7725 − 0.2978i	0.7530 − 0.3037i	0.7008 − 0.3036i
1	0	0.9479 − 0.2274i	0.9548 − 0.2229i	0.9547 − 0.2229i	1	0	1.1441 − 0.2893i	1.1415 − 0.2831i	1.1231 − 0.2976i
1	1	0.8210 − 0.7101i	0.8416 − 0.6761i	0.8415 − 0.6761i	1	1	0.9465 − 0.9065i	0.9267 − 0.8783i	0.9266 − 0.8782i
2	0	1.2745 − 0.2336i	1.2771 − 0.2329i	1.2771 − 0.2328i	2	0	1.4998 − 0.2921i	1.5026 − 0.2891i	1.5026 − 0.2891i
2	1	1.1832 − 0.7155i	1.1934 − 0.7085i	1.1933 − 0.7084i	2	1	1.3503 − 0.8967i	1.3624 − 0.8752i	1.3623 − 0.8752i
2	2	1.0199 − 1.2310i	1.0220 − 1.2203i	1.0220 − 1.2202i	2	2	1.0742 − 1.5569i	1.0498 − 1.5066i	1.0498 − 1.5065i
3	0	1.5862 − 0.2376i	1.5874 − 0.2373i	1.5874 − 0.2373i	3	0	1.8419 − 0.2960i	1.8438 − 0.2949i	1.8438 − 0.2949i
3	1	1.5140 − 0.7220i	1.5187 − 0.7198i	1.5187 − 0.7197i	3	1	1.7229 − 0.9007i	1.7323 − 0.8923i	1.7322 − 0.8922i
3	2	1.3804 − 1.2296i	1.3803 − 1.2284i	1.3802 − 1.2283i	3	2	1.4961 − 1.5419i	1.4953 − 1.5198i	1.4953 − 1.5198i
3	3	1.1990 − 1.7635i	1.1734 − 1.7885i	1.1734 − 1.7885i	3	3	1.1835 − 2.2325i	1.1146 − 2.2262i	1.1146 − 2.2261i
4	0	1.8900 − 0.2400i	1.8906 − 0.2399i	1.8906 − 0.2399i	4	0	2.1755 − 0.2991i	2.1766 − 0.2986i	2.1765 − 0.2986i
4	1	1.8298 − 0.7266i	1.8323 − 0.7257i	1.8323 − 0.7256i	4	1	2.0760 − 0.9061i	2.0818 − 0.9025i	2.0817 − 0.9024i
4	2	1.7160 − 1.2302i	1.7152 − 1.2305i	1.7151 − 1.2304i	4	2	1.8838 − 1.5389i	1.8845 − 1.5300i	1.8845 − 1.5300i
4	3	1.5584 − 1.7552i	1.5396 − 1.7714i	1.5396 − 1.7714i	4	3	1.6124 − 2.2095i	1.5732 − 2.2118i	1.5731 − 2.2118i
4	4	1.3647 − 2.3005i	1.3099 − 2.3698i	1.3099 − 2.3698i	4	4	1.2773 − 2.9207i	1.1449 − 2.9964i	1.1449 − 2.9964i
5	0	2.1890 − 0.2417i	2.1894 − 0.2416i	2.1894 − 0.2416i	5	0	2.5035 − 0.3013i	2.5042 − 0.3011i	2.5041 − 0.3010i
5	1	2.1371 − 0.7299i	2.1387 − 0.7294i	2.1386 − 0.7293i	5	1	2.4177 − 0.9105i	2.4213 − 0.9087i	2.4212 − 0.9087i
5	2	2.0378 − 1.2311i	2.0369 − 1.2315i	2.0368 − 1.2315i	5	2	2.2504 − 1.5392i	2.2507 − 1.5351i	2.2506 − 1.5351i
5	3	1.8982 − 1.7499i	1.8841 − 1.7603i	1.8841 − 1.7603i	5	3	2.0107 − 2.1974i	1.9842 − 2.2013i	1.9841 − 2.2013i
5	4	1.7246 − 2.2867i	1.6825 − 2.3309i	1.6825 − 2.3308i	5	4	1.7101 − 2.8899i	1.6167 − 2.9406i	1.6166 − 2.9405i
5	5	1.5215 − 2.8396i	1.4372 − 2.9604i	1.4372 − 2.9604i	5	5	1.3583 − 3.6159i	1.1548 − 3.7942i	1.1547 − 3.7941i

TABLE III. Low-lying ($n \leq l$, with $l = j - 3/2$) spin-3/2 field quasinormal mode frequencies using the WKB and the AIM methods with $D = 8$, 9.

8 Dimensions					9 Dimensions				
l	n	Third order WKB	Sixth order WKB	AIM	l	n	Third order WKB	Sixth order WKB	AIM
0	0	0.9675 − 0.3597i	0.9577 − 0.3647i	0.9675 − 0.3597i	0	0	1.1706 − 0.4149i	1.1654 − 0.4181i	1.1706 − 0.4148i
1	0	1.3483 − 0.3498i	1.3372 − 0.3477i	1.3483 − 0.3498i	1	0	1.5593 − 0.4062i	1.5473 − 0.4069i	1.5593 − 0.4062i
1	1	1.0776 − 1.0933i	1.0370 − 1.0848i	1.0775 − 1.0933i	1	1	1.2078 − 1.2636i	1.1700 − 1.2482i	1.2078 − 1.2635i
2	0	1.7213 − 0.3485i	1.7199 − 0.3446i	1.7213 − 0.3484i	2	0	1.9438 − 0.4026i	1.9376 − 0.3998i	1.9437 − 0.4026i
2	1	1.5065 − 1.0692i	1.5036 − 1.0437i	1.5064 − 1.0692i	2	1	1.6581 − 1.2319i	1.6427 − 1.2082i	1.6580 − 1.2318i
2	2	1.0973 − 1.8731i	1.0016 − 1.8170i	1.0972 − 1.8730i	2	2	1.0963 − 2.1769i	0.9333 − 2.1004i	1.0962 − 2.1768i
3	0	2.0845 − 0.3507i	2.0856 − 0.3484i	2.0844 − 0.3507i	3	0	2.3216 − 0.4029i	2.3202 − 0.4000i	2.3215 − 0.4028i
3	1	1.9099 − 1.0665i	1.9192 − 1.0496i	1.9099 − 1.0664i	3	1	2.0850 − 1.2224i	2.0882 − 1.2003i	2.0849 − 1.2223i
3	2	1.5665 − 1.8349i	1.5456 − 1.7789i	1.5665 − 1.8348i	3	2	1.6038 − 2.1138i	1.5438 − 2.0212i	1.6037 − 2.1137i
3	3	1.0897 − 2.6898i	0.9046 − 2.6246i	1.0896 − 2.6897i	3	3	0.9361 − 3.1458i	0.5638 − 3.0044i	0.9361 − 3.1457i
4	0	2.4398 − 0.3534i	2.4410 − 0.3522i	2.4398 − 0.3534i	4	0	2.6931 − 0.4046i	2.6934 − 0.4027i	2.6930 − 0.4046i
4	1	2.2931 − 1.0700i	2.3017 − 1.0611i	2.2931 − 1.0699i	4	1	2.4923 − 1.2226i	2.5009 − 1.2076i	2.4922 − 1.2225i
4	2	2.0011 − 1.8219i	2.0000 − 1.7896i	2.0010 − 1.8218i	4	2	2.0795 − 2.0861i	2.0671 − 2.0180i	2.0795 − 2.0861i
4	3	1.5816 − 2.6380i	1.4930 − 2.5837i	1.5815 − 2.6380i					
5	0	2.7896 − 0.3558i	2.7904 − 0.3552i	2.7895 − 0.3557i	5	0	3.0595 − 0.4066i	3.0601 − 0.4055i	3.0594 − 0.4066i
5	1	2.6627 − 1.0745i	2.6690 − 1.0698i	2.6627 − 1.0745i	5	1	2.8851 − 1.2260i	2.8934 − 1.2168i	2.8850 − 1.2259i
5	2	2.4091 − 1.8186i	2.4114 − 1.8008i	2.4090 − 1.8185i	5	2	2.5261 − 2.0759i	2.5286 − 2.0323i	2.5260 − 2.0759i
5	3	2.0378 − 2.6105i	1.9886 − 2.5773i	2.0377 − 2.6105i	5	3	1.9901 − 3.0021i	1.8953 − 2.8832i	1.9901 − 3.0020i

are in strong agreement with each other, however, for higher values of D we begin to see discrepancies between the AIM and WKB results. Because of these inconsistencies, we have omitted some of the results for the eight- and nine-dimensional cases. These inconsistencies may caused by the limitation of the WKB method when $n \sim l$ and/or when l is large, in the higher-dimensional cases.

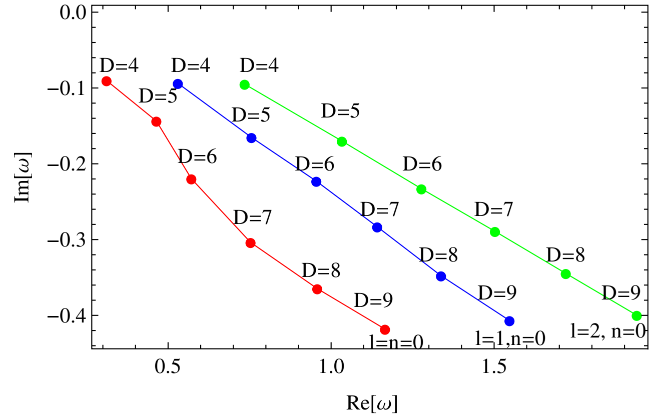

FIG. 1. Low-lying QNMs of black hole spacetimes for $D = 4$ to 9.

V. ABSORPTION PROBABILITIES

In this section we consider the absorption probabilities associated with the non-TT eigenfunctions. In Ref. [24] a similar analysis is done for the spin-1/2 field which is equivalent to our TT eigenfunction case for the spin-3/2 field. The analytic study of field absorption probabilities near black holes was pioneered by Unruh in 1976 [25]. However, his method was only able to determine the absorption probabilities for low energy particles. So in order to determine the entire spectrum of absorption probabilities we need to use the WKB method. We will give a brief overview of the Unruh method to determine the form of our absorption probabilities, and then provide the absorption probabilities we calculate when using the WKB method.

A. Unruh method

To implement the Unruh method we must consider three regions around the black holes: the near region, where $f(r) \to 0$, the central region, where $V(r) \gg \omega$, and the far region where $f(r) \to 1$. Approximations are obtained for each of the regions and then coefficients are determined by comparing and evaluating the solutions at the boundaries. We will for convenience denote V_1 as V in the following section and write either potential explicitly where ambiguity may occur. For the case with non-TT eigenfunctions, we use V_1, given in Eq. (3.36).

1. Near region

In the near region $f(r) \to 0$, so Eq. (3.35) becomes

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\tilde{\Phi}_I = 0, \tag{5.1}$$

with the in-going boundary condition near the event horizon. The solution in this case becomes

$$\tilde{\Phi}_I = A_I e^{-i\omega r_*}. \tag{5.2}$$

2. Central region

In this region we have that $V(r) \gg \omega$ and hence Eq. (3.35) becomes

$$\left(\frac{d}{dr_*} + W\right)\left(\frac{d}{dr_*} - W\right)\tilde{\Phi}_{II} = 0. \tag{5.3}$$

Defining H as

$$H = \left(\frac{d}{dr_*} - W\right)\tilde{\Phi}_{II}, \tag{5.4}$$

the solution of Eqs. (5.3) and (5.4) is

$$H = B_{II}\left(\frac{1+\sqrt{f}}{1-\sqrt{f}}\right)^{\frac{j}{D-3}+\frac{1}{2}}\left(\frac{(\frac{2}{D-2})(j+\frac{D-3}{2}) - \sqrt{f}}{(\frac{2}{D-2})(j+\frac{D-3}{2}) + \sqrt{f}}\right), \tag{5.5}$$

where substituting Eq. (5.5) into Eq. (5.4), we have a first order differential equation with solution

$$\tilde{\Phi}_{II} = A_{II}\left(\frac{1+\sqrt{f}}{1-\sqrt{f}}\right)^{\frac{j}{D-3}+\frac{1}{2}}\left(\frac{(\frac{2}{D-2})(j+\frac{D-3}{2}) - \sqrt{f}}{(\frac{2}{D-2})(j+\frac{D-3}{2}) + \sqrt{f}}\right) + B_{II}\Psi, \tag{5.6}$$

$$\Psi = \left(\frac{1+\sqrt{f}}{1-\sqrt{f}}\right)^{\frac{j}{D-3}+\frac{1}{2}}\left(\frac{(\frac{2}{D-2})(j+\frac{D-3}{2}) - \sqrt{f}}{(\frac{2}{D-2})(j+\frac{D-3}{2}) + \sqrt{f}}\right) \times\left[\int^r \frac{1}{f}\left(\frac{1-\sqrt{f}}{1+\sqrt{f}}\right)^{\frac{2j}{D-3}+1}\left(\frac{(\frac{2}{D-2})(j+\frac{D-3}{2}) + \sqrt{f}}{(\frac{2}{D-2})(j+\frac{D-3}{2}) - \sqrt{f}}\right)^2 dr'\right]. \tag{5.7}$$

3. Far region

In the far region $f(r) \to 1$, Eq. (3.35) becomes

$$\frac{d^2}{dr_*^2}\tilde{\Phi}_{III} - \left[\frac{((j+\frac{D-4}{2})^2 - \frac{1}{4})}{r^2} - \omega^2\right]\tilde{\Phi}_{III} = 0. \tag{5.8}$$

In this region $r_* \sim r$, the solution can be expressed as a Bessel function

$$\tilde{\Phi}_{III} = A_{III}\sqrt{r}J_{j+\frac{D-4}{2}}(\omega r) + B_{III}\sqrt{r}N_{j+\frac{D-4}{2}}(\omega r). \tag{5.9}$$

We can set the incoming amplitude of our field $\tilde{\Phi}_{III}$ at $r \to \infty$ to one. This gives us that

$$A_{III} + iB_{III} = \sqrt{2\pi\omega}. \tag{5.10}$$

Taking $r \to 1$ in the near region gives us that

$$A_I = A_{II}, \qquad B_{II} = -i\omega A_I. \tag{5.11}$$

Matching the solutions for regions II and III by taking $f = 1 - (1/r)^{D-3}$ and $r \to \infty$, we find that the absorption probability is given as

$$|A_j(\omega)|^2 = 4\pi C^2 \omega^{2j+D-3}(1 + \pi C^2 \omega^{2j+D-3})^{-2} \approx 4\pi C^2 \omega^{2j+D-3}, \tag{5.12}$$

where

$$C = \frac{1}{2^{\frac{D-1}{D-3}j+\frac{D-1}{2}}\Gamma(j+\frac{D-2}{2})}\left(\frac{j+\frac{2D-5}{2}}{j-\frac{1}{2}}\right), \tag{5.13}$$

with Γ denoting the gamma function and ω less than 1. This can be checked by taking $f = 1 - 2/r$ and $D = 4$, then we obtain the solution for the four-dimensional case we studied in Ref. [5].

B. WKB method

When using the WKB method it is more convenient to take $Q(x) = \omega^2 - V$ such that Eqs. (3.35) and (3.45) become

$$\left(\frac{d^2}{dr_*^2} + Q\right)\tilde{\Phi}_1 = 0. \tag{5.14}$$

For low energy particles $\omega \ll V$, we can use the first order WKB approximation. The result for this absorption probability is given in Ref. [26]

$$|A_j| = e^{[-2\int_{x_1}^{x_2}\frac{dx'}{f(x')}\sqrt{-Q(x')}]}, \tag{5.15}$$

where $x = \omega r$ with x_1 with x_2 being the turning points. That is, $Q(x_1, x_2) = 0$ or $V_{x_1,x_2} = \omega^2$ for a given energy ω and potential V. For particles of energy $\omega^2 \sim V$ the formula of Eq. (5.15) no longer converges and therefore has no solution. For this energy region we will need to use a higher order WKB approximation. We use the method developed by Iyer and Will [18]; the absorption probability is given as

$$|A_j(\omega)|^2 = \frac{1}{1 + e^{2S(\omega)}}, \tag{5.16}$$

where

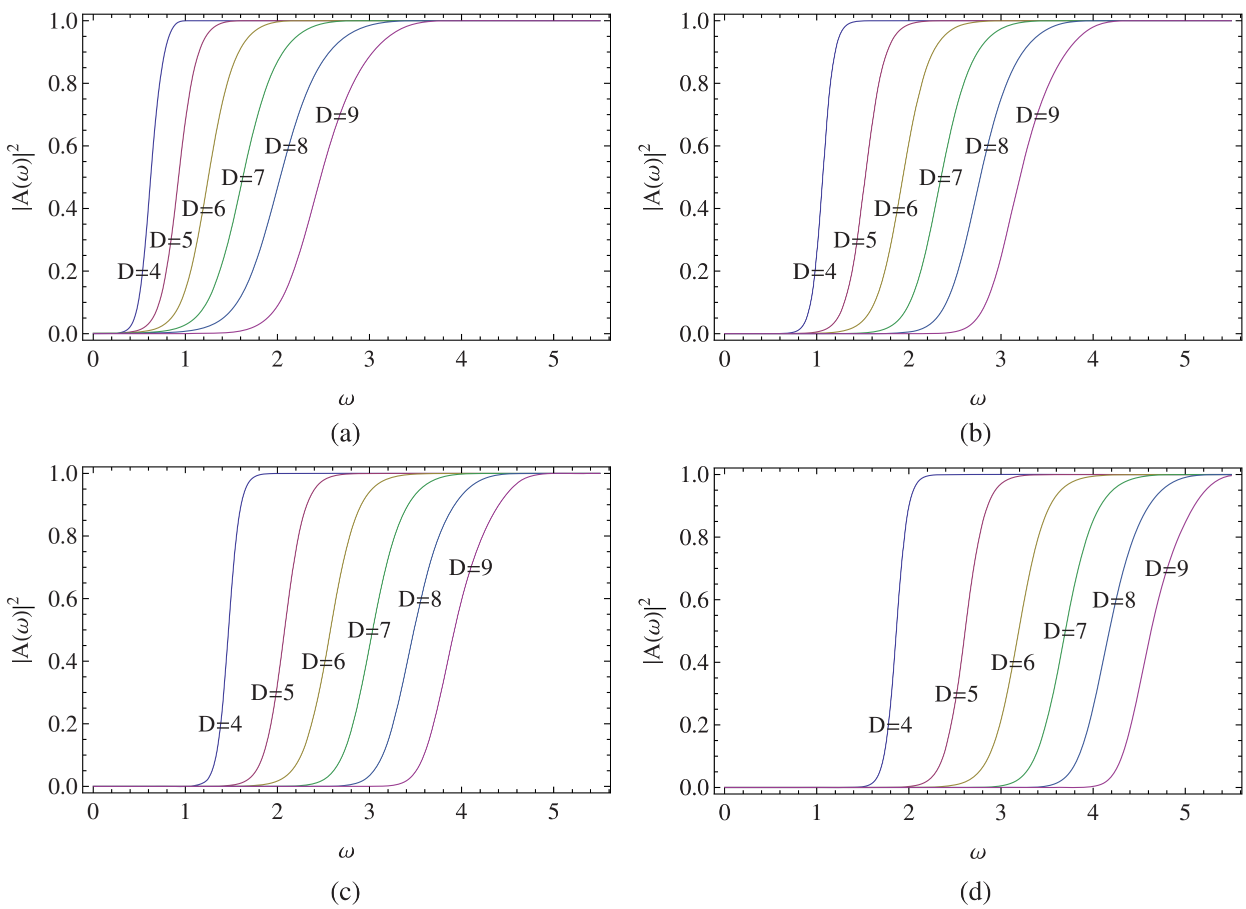

FIG. 2. Spin-3/2 field absorption probabilities with various dimensions. (a) $j = \frac{3}{2}$, (b) $j = \frac{5}{2}$, (c) $j = \frac{7}{2}$, (d) $j = \frac{9}{2}$.

$$S(\omega) = \pi k^{1/2}\left[\frac{1}{2}z_0^2 + \left(\frac{15}{64}b_3^2 - \frac{3}{16}b_4\right)z_0^4\right] + \pi k^{1/2}\left[\frac{1155}{2048}b_3^4 - \frac{315}{256}b_3^2 b_4 + \frac{35}{128}b_4^2 + \frac{35}{64}b_3 b_5 - \frac{5}{32}b_6\right]z_0^6 + \pi k^{-1/2}\left[\frac{3}{16}b_4 - \frac{7}{64}b_3^2\right] - \pi k^{-1/2}\left[\frac{1365}{2048}b_3^4 - \frac{525}{256}b_3^2 b_4 + \frac{85}{128}b_4^2 + \frac{95}{64}b_3 b_5 - \frac{25}{32}b_6\right]z_0^2, \quad (5.17)$$

where z_0^2, b_n and k are defined by the components of the Taylor series expansion of $Q(r)$ near r_0,

$$Q = Q_0 + \frac{1}{2}Q_0'' z^2 + \sum_{n=3}\frac{1}{n!}\left(\frac{d^n Q}{dx^n}\right)_0 z^n = k\left[z^2 - z_0^2 + \sum_{n=3} b_n z^n\right]. \quad (5.18)$$

That is,

$$z = r - r_0; \qquad z_0^2 \equiv -2\frac{Q_0}{Q_0''}; \qquad k \equiv \frac{1}{2}Q_0'';$$

$$b_n \equiv \left(\frac{2}{n! Q_0''}\right)\left(\frac{d^n Q}{dr_*^n}\right)_0; \qquad \frac{d}{dr_*} = \left(1 - \left(\frac{2M}{r}\right)^{D-3}\right)\frac{d}{dr}, \quad (5.19)$$

where 0 denotes the maximum Q and the primes denote derivatives.

In Fig. 2 we can see that an increase in the value of j results in an increase in the minimum required energy for total adsorption; we have observed and discussed this result in Ref. [5]. From Fig. 2 we can clearly see that an increase in the number of dimensions results in an increase in the minimum required energy for total adsorption, similar to that seen for an increase in j. This occurs since, in both cases, our effective potential is getting larger and therefore the particles require more energy to tunnel through the effective potential.

VI. CONCLUSION AND DISCUSSION

In this paper we have shown that by using the eigenvalues and eigenmodes of spinor vectors on an N sphere we

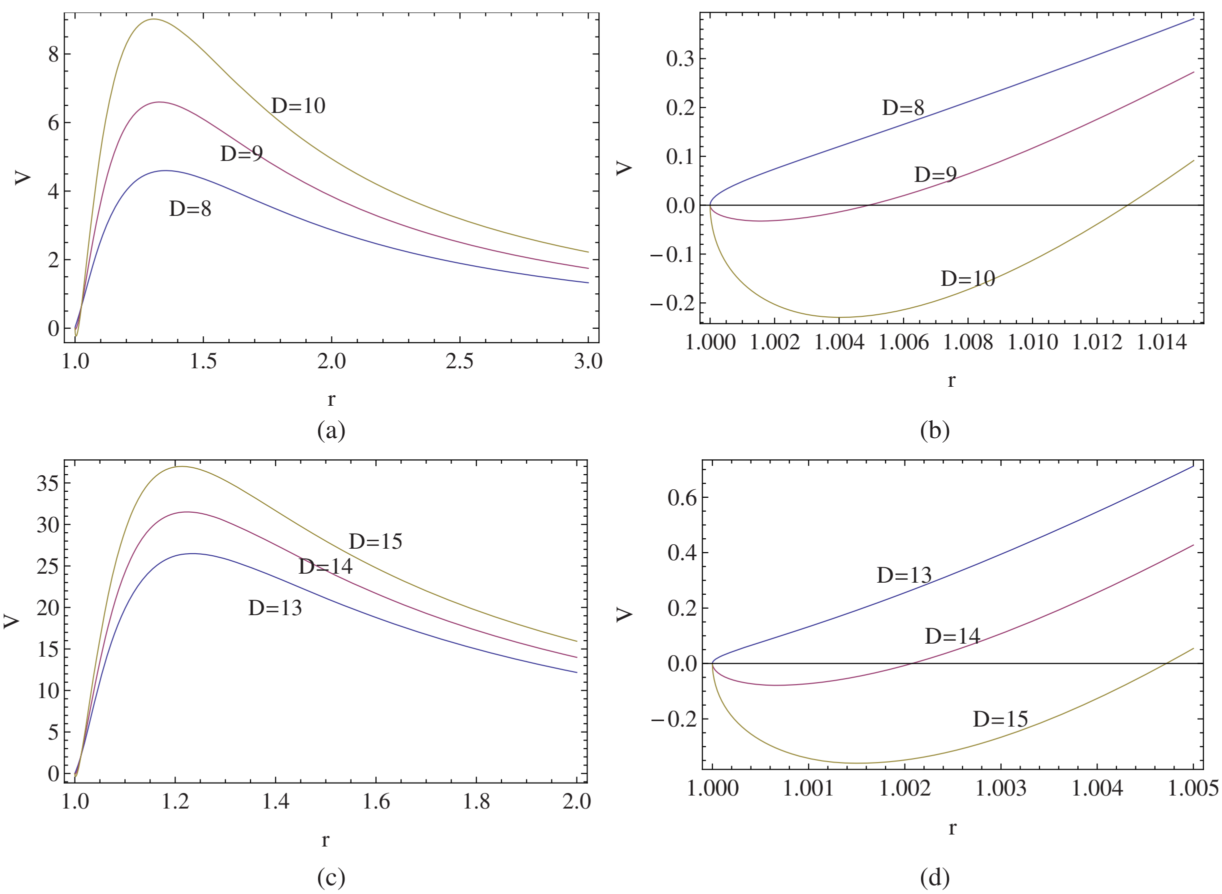

FIG. 3. Spin-3/2 effective potential V_1 in higher dimensions. (a) $j = \frac{3}{2}$, (b) $j = \frac{3}{2}$ when near the horizon, (c) $j = \frac{5}{2}$, (d) $j = \frac{5}{2}$ when near the horizon.

can determine the effective potential for spin-3/2 fields in spherically symmetric space times, with dimensions larger than 4. We have shown that there is a strong agreement between the potential that we have calculated and those calculated in other papers studying four-dimensional spacetimes [5,10,27]. We have also investigated the QNMs for emitted fields from our D-dimensional black holes. Since the real part of our QNMs is a frequency we can see that the energy of emitted fields and the dimension of the spacetime are directly related. This result is again seen when studying the absorption probabilities of particles near a Schwarzschild black hole. This suggests interesting results for the grey body factors of our D-dimensional black holes, where in order to make conclusions about the grey body factors we would need to study the cross sections of our black holes.

Our method requires that the spacetime be spherically symmetrical which means we could use this method to study Reissner-Nordström, anti–de Sitter, and de Sitter black holes. In the case of the Reissner-Nordtröm black holes it has been shown that for the extremal four-dimensional black holes the QNM frequencies are the same for spin-0, 1/2, 3/2, and 2 [27]. We would like to see if this is true for higher-dimensional extremal Reissner-Nordström black holes. In order to do this we must determine the covariant derivative related to the Reissner-Nordström black hole spacetime, where as stated earlier we must introduce terms with the Maxwell stress tensor. This derivative is given in Refs. [28].

We can use our calculated potentials to determine the stability of the higher-dimensional Schwarzschild black holes, a similar analysis is done in Refs. [29,30]. The effective potential V_1 in Eq. (3.37) has a local minimum near the horizon when $j = 3/2$ and $D = 9$. For higher dimensions we see that this minimum becomes more negative when the number of dimensions are increased. We see the same thing occurs when $j = 5/2$ and $D \geq 14$, as can be seen in Fig. 3. These behaviors are similar to the integer spin fields in some maximally symmetric spacetimes [29]. While the effective potentials studied in this paper are all barrier-like, the effective potentials of these higher-dimensional spacetimes do warrant further studies.

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APPENDIX: EIGEN SPINORS ON N-DIMENSIONAL SPHERES

The covariant derivative for our massless spinors is

$$\nabla_\mu \psi = \partial_\mu \psi + \omega_\mu \psi, \tag{A1}$$

where

$$\omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}, \qquad \Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \tag{A2}$$

In order to determine the eigenvalues for our spinors on S^N we must consider the case of N even and the case of N odd. We begin with the case of N even.

1. N even

In this case our gamma matrices are given as

$$\gamma^N = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & i\tilde{\gamma}^i \\ -i\tilde{\gamma}^i & 0 \end{pmatrix}, \tag{A3}$$

where $\mathbb{1}$ is the $2^{\frac{N-2}{2}} \times 2^{\frac{N-2}{2}}$ identity matrix. Our spin connections are then given as

$$\omega_{\theta_i} = \begin{pmatrix} \tilde{\omega}_{\theta_i} + \frac{i}{2}\cos\theta_N \tilde{\gamma}_{\theta_i} & 0 \\ 0 & \tilde{\omega}_{\theta_i} - \frac{i}{2}\cos\theta_N \tilde{\gamma}_{\theta_i} \end{pmatrix}. \tag{A4}$$

Using the definition of the Dirac derivative given in Eq. (A1), we have our spinor equation as

$$\gamma^\mu \nabla_\mu \psi_{(\lambda)} = \left[\left(\partial_{\theta_N} + \frac{N-1}{2}\cot\theta_N\right)\gamma^N + \frac{i}{\sin\theta_N}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{\theta_i}\right]\psi_{(\lambda)} = i\lambda\psi_{(\lambda)}. \tag{A5}$$

We can choose to express $\psi_{(\lambda)}$ as

$$\psi_{(\lambda)} = \begin{pmatrix} \psi^{(1)}_{(\lambda)} \\ \psi^{(2)}_{(\lambda)} \end{pmatrix} = \begin{pmatrix} A_{(\lambda)}(\theta_N)\tilde{\psi}_{(\lambda)} \\ -iB_{(\lambda)}(\theta_N)\tilde{\psi}_{(\lambda)} \end{pmatrix}, \tag{A6}$$

where $\tilde{\psi}_{(\lambda)}$ is the eigenspinor for the surface of S^{N-1}. Substituting Eq. (A6) into Eq. (A5) we find that

$$\begin{aligned} \left(\partial_{\theta_N} + \frac{N-1}{2}\cot\theta_N + \frac{\tilde{\lambda}}{\sin\theta_N}\right)A_{(\lambda)} &= \lambda B_{(\lambda)}, \\ \left(\partial_{\theta_N} + \frac{N-1}{2}\cot\theta_N - \frac{\tilde{\lambda}}{\sin\theta_N}\right)B_{(\lambda)} &= -\lambda A_{(\lambda)}. \end{aligned} \tag{A7}$$

We can solve this by expressing B_λ as a Jacobi polynomial

$$B_{(\lambda)}(\theta_N) = \left(\cos\frac{1}{2}\theta_N\right)^l\left(\sin\frac{1}{2}\theta_N\right)^{l+1} \times P_{n-l}^{((N/2)+l,(N/2)+l-1)}(\cos\theta_N) = (-1)^{n-l}A_{(\lambda)}(\pi-\theta_N). \quad (A8)$$

We require that $(n-l)\geq 0$ by restriction from our Jacobi polynomial and the eigenvalue can then be written as

$$i\lambda = \pm i\left(n+\frac{N}{2}\right), \qquad n=0,1,2,\ldots. \quad (A9)$$

2. N odd

In the case where N is odd our gamma matrices are given as

$$\gamma^N = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}; \qquad \gamma^i = \tilde{\gamma}^i, \quad (A10)$$

where $\mathbb{1}$ is the identity matrix of size $2^{\frac{N-3}{2}}\times 2^{\frac{N-3}{2}}$. The nonzero spin connection is determined to be

$$\omega_i = \tilde{\omega}_{\theta_i} - \frac{1}{2}\cos\theta_N\gamma^N\tilde{\gamma}_{\theta_i}. \quad (A11)$$

Substituting the result for our spin connection into Eq. (A1) we find that the spinor equation is

$$\gamma^\mu\nabla_\mu\psi_{(\lambda)} = \left[\left(\partial_{\theta_N}+\frac{N-1}{2}\cot\theta_N\right)\gamma^N + \frac{1}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{\theta_i}\right]\psi_{(\lambda)} = i\lambda\psi_{(\lambda)}. \quad (A12)$$

Choosing $\psi_{(\lambda)}$ as

$$\psi_{(\lambda)} = \frac{1}{\sqrt{2}}(1+i\gamma^N)A_{(\lambda)}(\theta_N)\tilde{\psi}_{(\lambda)} + \frac{1}{\sqrt{2}}(1-i\gamma^N)B_{(\lambda)}(\theta_N)\tilde{\psi}_{(\lambda)}, \quad (A13)$$

which is the same relation between A_λ and B_λ as we had in Eq. (A7). Hence, the eigenvalues will be the same as those for the case of N even, given in Eq. (A9).

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