Cosmological wormholes in $f(R)$ theories of gravity

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Motivated by recent proposals of possible wormhole existence in galactic halos, we analyze the cosmological evolution of wormhole solutions in modified $f(R)$ gravity. We construct a dynamical wormhole that asymptotically approaches a Friedmann-Lemaître-Robertson-Walker (FLRW) universe, with supporting material going to the perfect isotropic fluid described by the equation of state for a radiation- and matter-dominated universe respectively. Our analysis is based on an approximation of a small wormhole—a wormhole that can be treated as matched with the FLRW metric at some radial coordinate much smaller than the Hubble radius, so that cosmological boundary conditions are satisfied. With a special interest in viable wormhole solutions, we refer to the results of the reconstruction procedure and use $f(R)$ functions which lead to the experimentally confirmed ΛCDM expansion history of the Universe. Solutions we find imply no need for exotic matter near the throat of considered wormholes, while in the limit of $f(R) = R$ this need is always present during radiation- and matter-dominated epochs.

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I. INTRODUCTION

The notion of Lorentzian wormholes (or Morris-Thorne wormholes or simply WH) arose when Morris and Thorne explored the possibility of time travel for humans using the principles of general relativity (GR) [1]. Einstein's theory of GR predicts that the structure and geometry of spacetime in the presence of matter are not rigid but are elastic and deformable. The more compact the object is, the more strong the curvature of space is, which essentially leads to the idea of black holes. However in the latter case, the fabric of spacetime loses its meaning at the curvature singularity. If somehow the formation of a singularity is avoided then it would be possible to travel in and out of the throat, so that there is no restriction to observer's motion on the manifold. The possibility of such a solution to the Einstein field equations was for the first time explored by Flamm [2] recently after the discovery of GR, but it was later shown that his solution was unstable. The first more detailed analysis of the wormhole solution was done later by Einstein and Rosen [3].

A typical wormhole is a tubelike structure which is asymptotically flat from both sides. The radius of the wormhole's throat could be a constant or variable depending on its construction and is termed static or nonstatic respectively. GR predicts that to form a WH, an exotic form of matter (violating the energy conditions) must be present near the throat of the WH. The problem is the dearth of a reasonable source that sustains the wormhole geometry. One possible candidate is the phantom energy (which is a cosmic dynamical scalar field with negative kinetic energy in its Lagrangian) and is the primed candidate of explaining cosmic accelerated expansion as well [4]. Since the existence of phantom energy is questionable and no other suitable exotic matter candidate is available, an alternative approach is commonly followed: investigation if the modifications of laws of gravity (i.e. GR), proposed primarily for explanation of accelerated expansion and avoiding singularities, can support the WH geometries. Since the WH is a nonvacuum solution of Einstein field equations, the presence of some form of energy-matter is necessary to construct a WH. In the framework of modified gravity, the matter content is assumed to satisfy the energy conditions near the WH's throat, while the higher curvature correction terms in the Lagrangian are required to sustain the WH geometry.

In recent years, theories of modified gravity have gotten enormous attention to model cosmic accelerated expansion, explaining flat rotation curves of galaxies, wormhole formation and other esoteric phenomenon near black holes [5,6]. A well-known theory of modified gravity is the $f(R)$ gravity, where R is the Ricci scalar. The simple idea on which this theory is based is generalizing the integral of

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action for GR, so that the Ricci curvature scalar is replaced by some arbitrary function, $f(R)$. Field equations obtained in this fashion have a higher degree of complexity, and admit a richer set of solutions than the standard GR. Harko *et al.* [7] constructed solutions of static wormholes threaded by ordinary matter (satisfying the energy conditions) in $f(R)$ gravity, whereby the curvature/gravitational fluid supports the nonstandard wormhole geometries. Note that the gravitational fluid cannot be considered as exotic. De Benedictis *et al.* also obtained new static wormhole solutions in the power-law R^m gravity [8]. Rahaman *et al.* [9] obtained new static wormholes in $f(R)$ theory using the noncommutative geometry. It was also demonstrated that viable $f(R)$ theories of gravity, obtained by demanding the consistency with observational data for the Solar System and cosmological evolution, admit the existence of wormhole solutions that do not require exotic matter [10]. Additionally, in [11] a traversable wormhole solution in the framework of mimetic $f(R)$ gravity was found. Besides, the static wormhole geometries have also been theorized within curvature-matter coupling theories such as $f(R,T)$, where T is the trace of the stress-energy-momentum tensor [12]. Additionally, wormhole geometries have also been explored in generalized teleparallel gravity [13], Gauss-Bonnet gravity [14] and Lovelock-Brans-Dicke gravity [15]. In addition, wormholes minimally coupled to pions supported by a negative cosmological constant were obtained in Ref. [16]. Spherically symmetric evolving wormholes were studied in the context of standard GR in [17], and similar analysis was also extended to the case of rotating axially symmetric wormholes in GR [18]. It was later shown that the violation of the null energy condition is also necessary in nonsymmetric and time-dependent wormholes [19]. Some specific solutions for dynamical wormholes were also investigated in $f(R)$ gravity [20], as well as $f(T)$ theories of gravity [21].

It was recently proposed by Rahaman *et al.* that galactic halos possess some of the characteristics needed to support traversable wormholes [22], which was followed by discussion on the possibility of their detection [23]. If these galactic wormholes exist it is natural to assume that they would be produced in the early Universe, and therefore the question of their properties and evolution during the expansion of the Universe naturally arises. In any case, even if these proposed galactic wormholes do not exist, investigation of wormholes in the cosmological context is an interesting topic in its own right. Therefore, we are curious about the construction of evolving WH in $f(R)$ gravity, in the framework of standard cosmological assumptions. By construction, the evolving WH metric is similar to the FLRW metric; hence the WH expands as the Universe expands. The WH spacetime is threaded by anisotropic matter which asymptotically becomes isotropic, in order to match the usual description of a cosmological ideal fluid. Since the isotropicity about every point in spacetime implies homogeneity (but not vice versa), the matter distribution is also homogeneous asymptotically. The asymptotic limit may correspond to the cosmological horizon, but it could also be much smaller than the horizon in the limit of a small wormhole—with geometry approaching Friedmann-Lemaître-Robertson-Walker (FLRW) on some small distance from the wormhole throat.

The plan of the paper is as follows. In Sec. II, we give an overview of the $f(R)$ theory and its field equations for the static wormhole geometry. In Sec. III, we introduce the cosmological wormholes in $f(R)$ gravity. In Sec. IV, we numerically solve the wormhole equations for radiation and matter eras and then we analyze the energy conditions. Finally we give the conclusion in Sec. V. We work with units $c = 1 = G$ and use the signature $(-,+,+,+)$.

II. STATIC WORMHOLES IN $f(R)$ GRAVITY

The action in $f(R)$ gravity reads

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g} f(R) d^4x + \int \mathcal{L}_m d^4x, \tag{1}$$

where $\kappa = 8\pi G$, $\mathcal{L}_m$ is the Lagrangian density for the matter and $f(R)$ is a smooth function which depends on the scalar curvature. We can notice that if we set $f(R) = R$ we recover the Einstein-Hilbert action.

By varying this action with respect to the metric we find

$$f_R(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f_R(R) = \kappa T_{\mu\nu}, \tag{2}$$

where $f_R(R) = df(R)/dR$ and $T_{\mu\nu}$ is the energy-momentum tensor of the matter.

The Morris-Thorne metric which can describe the spacetime of a static wormhole is given by [1]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{1}{1-\frac{b(r)}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

where $\Phi(r)$ is the redshift function and $b(r)$ is a shape function, which are both functions of the radial coordinate, r. In the wormhole geometry, the radial coordinate needs to nonmonotonically decrease from infinity to a minimal value r_0 at the throat, where $b(r_0) = r_0$, and then increase to infinity. Although the metric is singular at $r = r_0$, proper distance as an invariant quantity must be well behaved, and therefore the following integral must be real and regular outside the throat,

$$l(r) = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{1-\frac{b(r)}{r}}}, \tag{4}$$

and from this one obtains the condition

$$1 - \frac{b(r)}{r} \geq 0. \tag{5}$$

Far from the throat, space must be asymptotically flat which implies the condition $b(r)/r \to 0$ as $l \to \pm\infty$. By definition, the throat represents the minimum radius in this wormhole geometry and this leads to the flaring-out condition [1],

$$\frac{b(r) - b'(r)r}{b(r)^2} > 0. \tag{6}$$

Hereafter, a prime denotes a derivative with respect to the argument of a function, so that $b'(r) = db/dr$. It is also assumed that a wormhole has no horizons, so that it can be traversable and for this reason $\Phi(r)$ must be finite everywhere. As already mentioned in the Introduction, static wormholes in Einstein's general relativity require exotic fluid, i.e. fluid which is violating the standard condition on the stress-energy tensor, the weak energy condition (WEC). This condition is given by $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any spacelike vector k^μ [24]. Apart from construction of cosmological wormholes in $f(R)$ gravity, we will also be interested in whether these wormholes, which could be produced in the early Universe and determined by its dynamics, could satisfy WEC or not. Moreover, we will also consider the limit $f(R) = R$ as a comparison.

III. $f(R)$ COSMOLOGICAL WORMHOLES

Static wormhole geometry (3) can easily be generalized to the evolving case,

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + a^2(t)\left[\frac{1}{1-\frac{b(r)}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{7}$$

where $a(t)$ is the scale factor which controls the dynamic of the wormhole. Since parameters $b(r)$ and $\Phi(r,t)$ determine the geometrical properties of the wormhole, we will naturally demand that they obey the same conditions as in the static case with $b \to a(t)b(r)$ and $r \to a(t)r$. We can notice that if we set $a(t) \to 1$ and $\Phi(r,t) = \Phi(r)$, we recover the static Morris-Thorne metric (3). In addition, for an observer far away from the throat of the wormhole we demand that $\Phi(t,r) \to 0$ and $b(r) \to 0$ fast enough, so that for the large enough distances the metric converts to the FLRW one. In other words, this metric allows wormhole solutions to be approximately embedded in FLRW universes. This geometry is clearly not asymptotically flat, and therefore we here use somehow an extended definition of a wormhole which does not imply asymptotic flatness. In any case, this implies only generalization of the wormhole's asymptotic behavior, and makes no changes in the throat properties, which is the most interesting and fundamental attribute of a wormhole. Related question of formal definitions regarding dynamical wormholes, including wormholes which are asymptotically FLRW, was previously discussed in [17,25]. We can choose a radius $r_c \ll 1/H$ for all times [where $H = \dot{a}(t)/a(t)$ is the cosmological Hubble parameter] such that if we take some $r > r_c$, the redshift and shape function can be neglected and we can approximately treat the wormhole as confined within the region $r < r_c$ with no influence on the global geometry of FLRW spacetime. Therefore, we can demand that the dynamical properties of this wormhole, apart from the evolution of the redshift function, are determined only by the expansion of the Universe, so that the scale factor $a(t)$ is equal in both regions. We can call this approximate geometry—where r_c can be effectively treated as a wormhole's asymptotic infinity—a small cosmological wormhole. Objects like this could be created on microscopic scales in the early Universe and then subsequently enlarged during expansion of the Universe. For instance, if such wormholes were created at or before the electroweak scale their physical size $a(t)r_c$ should be much less than a centimeter, a typical scale of the Hubble radius at that time. By the end of matter-dominated era such wormholes would then be characterized by the typical size of astrophysical objects.

Thus, on the cosmological scales, the energy-momentum tensor is represented with an isotropic ideal fluid with equation of state

$$p_c(t) = w\rho_c(t), \tag{8}$$

where w is the state parameter and $\rho_c(t)$ and $p_c(t)$ are the energy density and pressure of the cosmological fluid respectively. Here, index c denotes that we are dealing with functions on the cosmological scales of the FLRW metric to distinguish them from the components of the anisotropic energy momentum tensor defined in the region $r < r_c$, which we will denote with the subscript "wh."

If we use the conservation law for the energy-momentum tensor for the cosmological fluid, $\nabla_\mu T^{\mu\nu}_{(c)} = 0$, we find that the energy density of this fluid is given by

$$\rho_c(t) = \rho_0 a(t)^{-3(1+w)}, \tag{9}$$

where ρ_0 is the current energy density. It is important to mention that in both regions $r > r_c$ and $r < r_c$, the form of $f(R)$ should naturally be same.

The $f(R)$ field equations (2) for the cosmological flat FLRW spacetime ($r > r_c$ region) yield

$$H(t)^2 = \frac{1}{3f_R(R)}\left(\frac{1}{2}[R_c f_R(R) - f(R)] - 3H(t)\dot{R}_c f_{RR} + \rho_c(t)\right), \tag{10}$$

where $R_c = 6(\dot{H} + 2H^2)$ is the scalar curvature of the flat FLRW spacetime, $f_{RR} = d^2f(R)/dR^2$ and dots represent differentiation with respect to the cosmic time.

In the region $r < r_c$, matter supporting the wormhole will be an anisotropic fluid such as the energy-momentum tensor given by $T^{\mu}_{\nu} = \mathrm{diag}(-\rho_{wh}(r,t), p_{r_{wh}}(r,t), p_{l_{wh}}(r,t), p_{l_{wh}}(r,t))$, where $\rho_{wh}(r,t)$, $p_{r_{wh}}(r,t)$ and $p_{l_{wh}}(r,t)$ are the energy density, radial pressure and lateral pressure respectively. It follows that for an anisotropic fluid, WEC is given by the conditions

$$\rho_{wh}(r,t) \geq 0, \tag{11}$$

$$\rho_{wh}(r,t) + p_{r_{wh}}(r,t) \geq 0, \tag{12}$$

$$\rho_{wh}(r,t) + p_{l_{wh}}(r,t) \geq 0. \tag{13}$$

The $f(R)$ field equations (2) in this region ($r < r_c$) for the nonstatic metric (7) are given by

$$\begin{aligned}-\rho = {}& -\frac{1}{2}f + a^{-2}\left(1 - \frac{b(r)}{r}\right)R'^2 f_{RRR} \\ &+ \frac{f_R}{2r^2a^2}[(r(b'(r)-4) + 3b(r))\Phi' - 2r(r-b(r))\Phi'^2 \\ &+ 2r(b(r)-r)\Phi''] + \frac{f_{RR}}{2r^2a^2}[2r(r-b(r))R'' \\ &- (r(b'(r)-4)+3b(r))R' - 6r^2 a\dot{a}\dot{R}\,e^{-2\Phi}] \\ &+ \frac{3f_R e^{-2\Phi}}{a}(\ddot{a} - \dot{a}\,\dot{\Phi}),\end{aligned} \tag{14}$$

$$\begin{aligned}p_r = {}& -\frac{1}{2}f + \frac{b(r)f_R}{2r^3a^2}[2r^2\Phi'^2 + 2r^2\Phi'' - r\Phi' - 2] \\ &+ \frac{e^{-2\Phi}f_R}{2r^2a^2}[e^{2\Phi}(b'(r)(r\Phi'+2) \\ &- 2r^2(\Phi'^2 + \Phi'')) + 4r^2\dot{a}^2] + \frac{f_R e^{-2\Phi}}{a}(\ddot{a} - \dot{a}\,\dot{\Phi}) \\ &- e^{-2\Phi}f_{RRR}\dot{R}^2 + \frac{f_{RR}R'}{a^2}\left(1 - \frac{b(r)}{r}\right)\left(\Phi' + \frac{2}{r}\right) \\ &+ \frac{e^{-2\Phi}f_{RR}}{a}[a(\dot{R}\,\dot{\Phi} - \ddot{R}) - 2\dot{a}\,\dot{R}],\end{aligned} \tag{15}$$

$$\begin{aligned}p_l = {}& -\frac{1}{2}f + \frac{f_R}{2r^3a^2}[b(r)(2r\Phi'+1) + (b'(r) - 2r\Phi')r] \\ &+ \frac{e^{-2\Phi}f_R}{2a^2}[4\dot{a}^2 + 2a(\ddot{a} - \dot{a}\,\dot{\Phi})] \\ &+ f_{RRR}\left[\frac{R'^2}{a^2}\left(1 - \frac{b(r)}{r}\right) - \dot{R}^2 e^{-2\Phi}\right] \\ &+ \frac{f_{RR}}{2r^2a^2}[R'(r(2r\Phi' - b'(r) + 2) - b(r)(2r\Phi'+1)) \\ &+ 2r(r-b(r))R''] - \frac{f_{RR}}{r^3a^2}[4r^3 a\dot{a}\,\dot{R} \\ &- 2r^3a^2\dot{R}\,\dot{\Phi} + 2r^3a^2\ddot{R}],\end{aligned} \tag{16}$$

$$2\left(\frac{\dot{a}}{a}\right)f_R\Phi' = f_{RRR}\dot{R}R' - f_{RR}\left[\frac{\dot{a}R'}{a} + \dot{R}\Phi' - \dot{R}'\right], \tag{17}$$

where $f_{RRR} = d^3f(R)/dR^3$, and R is the Ricci scalar for the evolving wormhole geometry (7). We will also consider the following boundary conditions (at the boundary $r = r_c$):

$$\rho_{wh}(r_c,t) = \rho_c(t), \tag{18}$$

$$p_{r_{wh}}(r_c,t) = p_c(t), \tag{19}$$

$$p_{l_{wh}}(r_c,t) = p_c(t). \tag{20}$$

These conditions imply that the anisotropic fluid supporting the wormhole goes continuously to the isotropic fluid of the Universe. Moreover, they also tell us that the density-pressure relationship goes to the cosmological equation of state at $r = r_c$,

$$p_{r_{wh}}(r_c,t) = p_{l_{wh}}(r_c,t) = w\rho_{wh}(r_c,t). \tag{21}$$

There are four field equations (14)–(17) containing seven unknown functions; therefore we need to close the system by choosing some ansatz. Hereafter, we will consider that $\Phi(r,t) = h(r)\phi(t)$ where $h(r)$ will be some chosen function which decreases fast to negligible values for $r \geq r_c$. It is also necessary to prescribe the form of the modified gravity, $f(R)$, which from (10) determines $a(t)$ and therefore the expansion of the small cosmological wormhole. With special interest in viable wormhole solutions we will consider choices for $f(R)$ which lead to experimentally confirmed ΛCDM expansion history of the Universe. Thus, this choice of $f(R)$ will lead to known forms of $a(t) = a_0(t/t_0)^n$ in the radiation- ($n = 1/2$) and matter-dominated epochs ($n = 2/3$). Following the results of this reconstruction procedure we take [26–30]

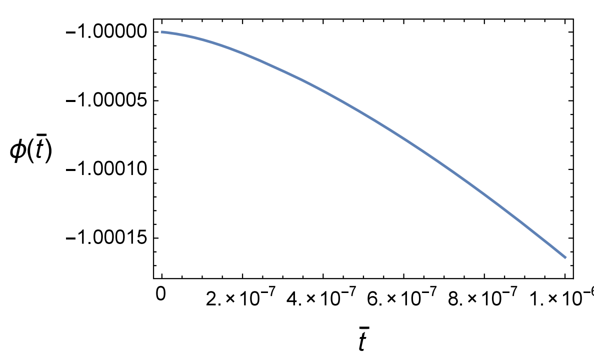


FIG. 1. Time-dependent part of the redshift function $\phi(t)$ as a function of the dimensionless time $\bar{t} = t/t_0$ for a radiation-dominated era with a shape function $b(r) = r_0^{m+1}/r^m$ and $h(r) = \varphi_0 e^{(-r/r_0+1)^n}$. This pictures represents a slice $r = r_0$ with $m = 2$, $n = 1/2$, $\phi(\bar{t} = 10^{-9}) = -1$, $\dot{\phi}(\bar{t} = 10^{-9}) = -10$ and $\varphi_0 = 1$.

$$f(R)_{\rm rad} = R - 3D(\kappa\rho_m^0)^{4/3}R^{-1/3}, \tag{22}$$

during the radiation-dominated phase, where ρ_m^0 is the current energy density of matter and D is an arbitrary constant. For the matter-dominated phase we will consider [26–30]

$$f(R)_{\rm mat} = R - \zeta\left(\frac{R}{\Lambda}\right)^{1-p}, \tag{23}$$

where we have defined the constant ζ as

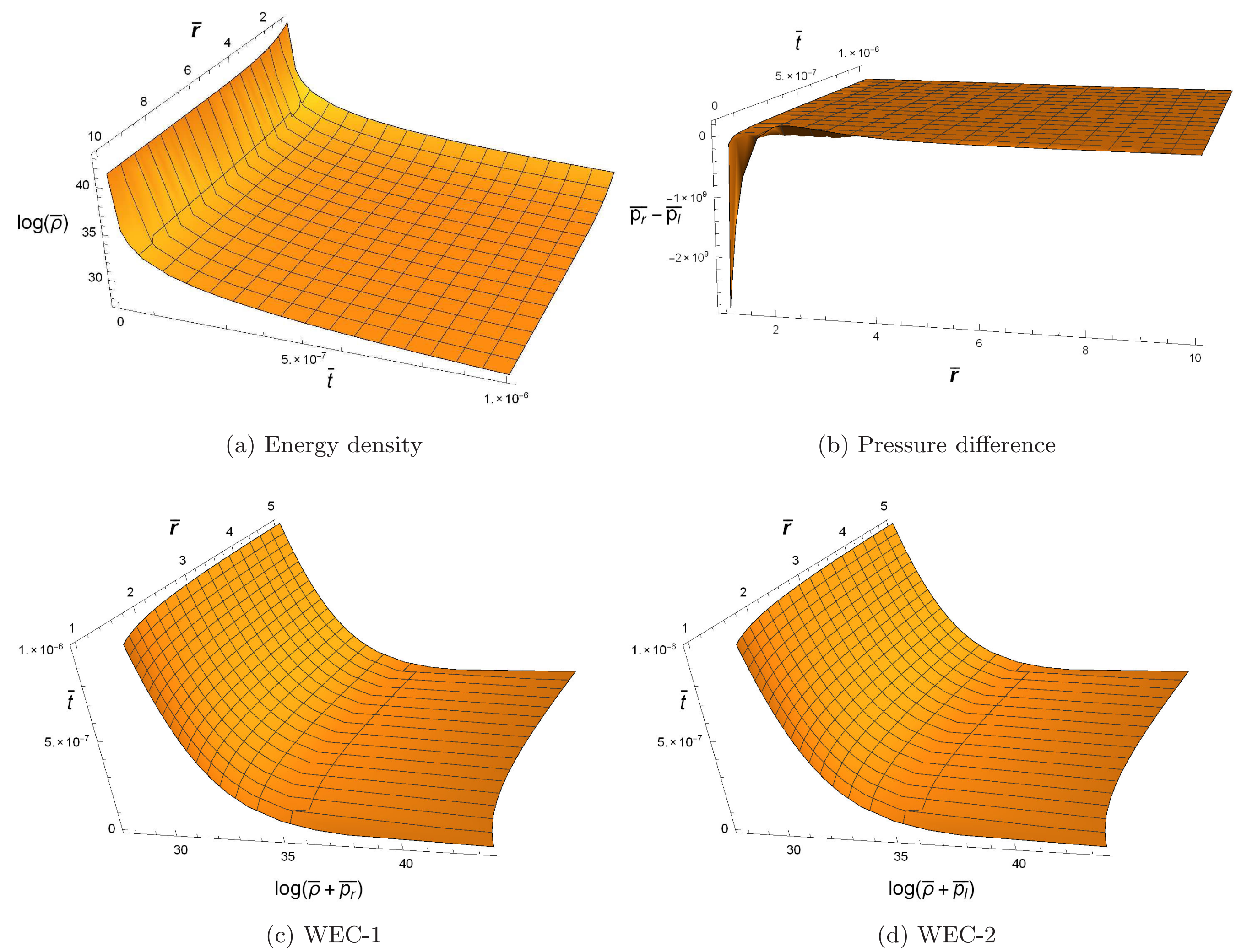
$$\zeta = \frac{240D\Gamma(\frac{\sqrt{73}}{2})}{\Gamma(\frac{7+\sqrt{73}}{4})\Gamma(\frac{13+\sqrt{73}}{4})}\kappa^2\rho_r^0\frac{(R_0-4\Lambda)}{\Lambda^{p-1}}\left(\frac{\rho_m^0}{\rho_r^0}\right)^{3(p-1)}, \tag{24}$$

where ρ_r^0 is the current energy density of radiation and $p = (5+\sqrt{73})/12 \approx 1.13$. It can be shown that $f(R)_{\rm rad}$ and $f(R)_{\rm mat}$ are the special limits of one more general $f(R)$ form in terms of hypergeometric functions, capable of describing both radiation- and matter-dominated eras [26].

IV. NUMERICAL SOLUTIONS

A. Radiation-dominated era

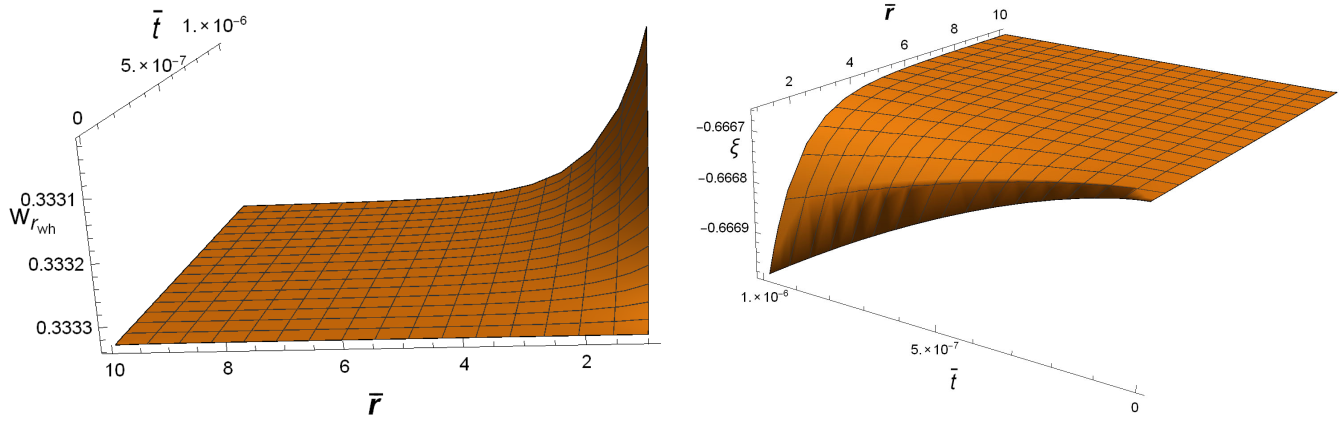
In this section, we will consider the evolution of a cosmological wormhole during the radiation-dominated era, where a cosmological perfect fluid is described by the equation of state (EOS) parameter $w = 1/3$, leading to the expansion of the scale factor as $a(t) = a_0(t/t_0)^n$. We will consider a small wormhole for which $r_c = 10 \times r_0$ with



(a) Energy density (b) Pressure difference

(c) WEC-1 (d) WEC-2

FIG. 2. Logarithmic of the dimensionless energy density $\bar{\rho} = \rho/\rho_c$, difference in dimensionless pressures $\bar{p}_l(t,r) - \bar{p}_r(t,r)$, logarithmic WEC-1 ($\bar{p}_r + \bar{\rho}$) and logarithmic WEC-2 ($\bar{p}_l + \bar{\rho}$) as a function of the dimensionless time $\bar{t} = t/t_0$ and the dimensionless radius $\bar{r} = r/r_0$ respectively. The parameters used were $m = 2$, $n = 1/2$, $\rho_m^0 = 0.27$, $\varphi_0 = 1$ and $t/t_0 = 10^{-9}$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, and a redshift function given by $\Phi(t,r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 1 and the function $f(R)$ given by (22)].



(a) WH state parameter (b) Exoticity

FIG. 3. Wormhole state parameter $w_{r_{wh}} = p_{r_{wh}}/\rho_{r_{wh}}$ and exoticity parameter $\xi = (p - \rho)/|\rho|$ for a radiation-dominated era as a function of the dimensionless time $\bar{t} = t/t_0$ and the dimensionless radius $\bar{r} = r/r_0$. The parameters used were $m = 2$, $n = 1/2$, $\varphi_0 = 1$, $\rho_m^0 = 0.27$ and $t/t_0 = 10^{-9}$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, and a redshift function given by $\Phi(t, r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 1 and the function $f(R)$ given by (22)].

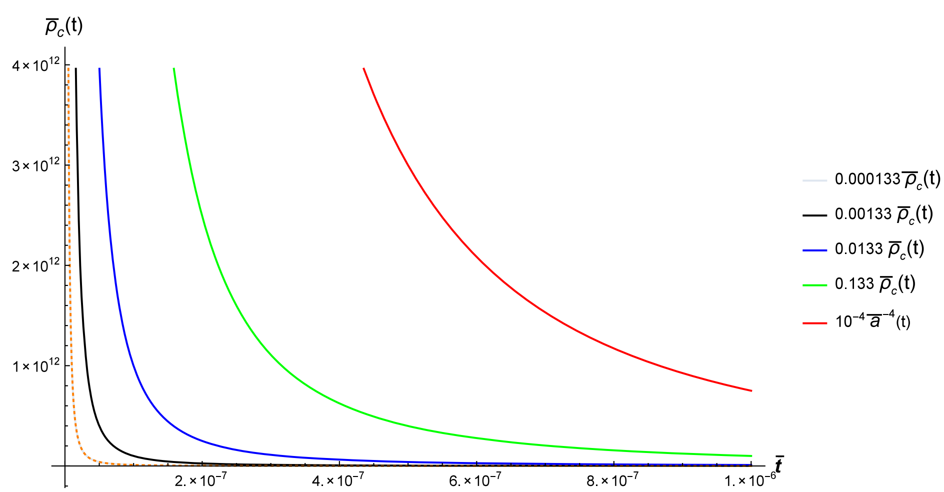


FIG. 4. Family of curves with the same evolution as the cosmological dimensionless density $\bar{\rho}_c(t)$ evaluated at $r_c = 10$ versus dimensionless time $\bar{t} = t/t_0$. The curve $10^{-4}\bar{a}(t)^{-4} = 10^{-4}\bar{t}^{-2}$ is overlapped with the curve $0.000133\bar{\rho}_c(t)$, so that $C = 0.000133$ is a suitable value. The parameters used were $m = 2$, $n = 1/2$, $\varphi_0 = 1$, $\rho_m^0 = 0.27$ and $t/t_0 = 10^{-9}$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, and a redshift function given by $\Phi(t, r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 1 and the function $f(R)$ given by (22)].

$r_0 \ll 1/H$ for all times. We choose the shape function to be $b(r) = r_0^{m+1}/r^m$ and take $h(r) = \varphi_0 e^{(-r/r_0+1)^n}$, where m and n are integers. Also, we take $D = (1/3)(\kappa\rho_m^0)^{-4/3}$ and parameters for the energy densities today are set according to the standard cosmological values [31]. Our results will be presented until the end of the radiation-dominated epoch, $t/t_0 = 10^{-6}$, where t_0 is the time passed from the big bang until now.

We first consider the nondiagonal modified Einstein's equation (13), which is a partial differential equation with respect to $\Phi(r, t)$ [note that Ricci curvature is itself dependent on $\Phi(r, t)$, so this equation contains derivatives with respect to both the radial and time coordinates, as well as the mixed terms]. In order to solve this equation we will solve it at the throat, $r = r_0$, and assume that the same time dependence will remain valid for all other positions. This is consistent with the approximation that $\Phi(r, t)$ goes fast to zero for $r > r_0$, so that the only important region is that in the vicinity of the throat. After $\phi(t)$ has been numerically obtained in this manner, it can be used to solve the

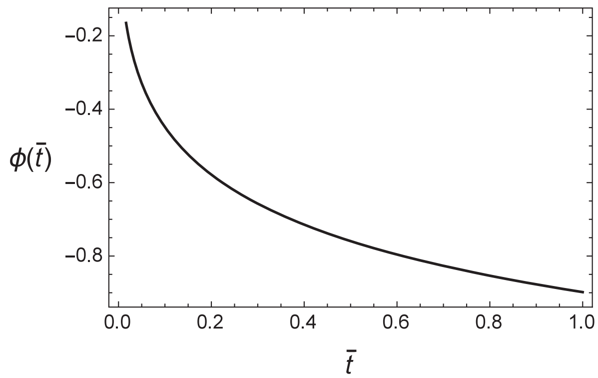


FIG. 5. Time-dependent part of the redshift function $\phi(t)$ versus the dimensionless time $\bar{t} = t/t_0$ for a matter-dominated era with a shape function $b(r) = r_0^{m+1}/r^m$ and $h(r) = \varphi_0 e^{(-r/r_0+1)^n}$. This pictures represents a slice $r = r_0$ and the parameters used were $m = 2$, $n = 2/3$, $\phi(\bar{t} = 10^{-6}) = 1$, $\dot{\phi}(\bar{t} = 10^{-6}) = -1$ and $\varphi_0 = 1$.

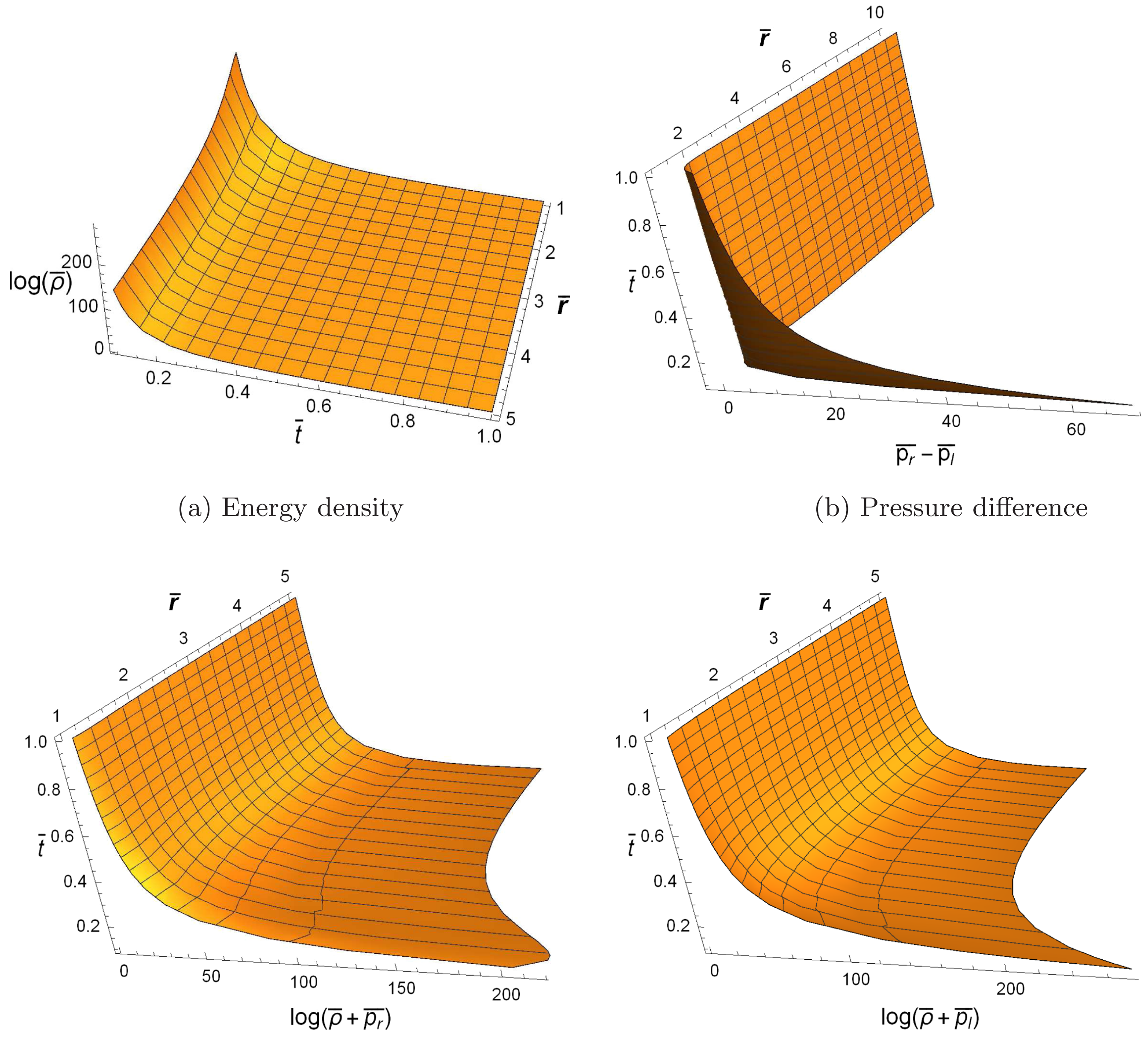
remaining modified Einstein's equations, which represent only algebraic equations for the components of the energy-momentum tensor of anisotropic fluid. In Figs. 2(a), 2(c) and 2(d) we show the obtained time evolution and the space dependence of $\rho_{wh}(r,t)$, and check for the possible WEC violation related to both pressures: $\rho_{wh}(r,t) + p_{r_{wh}}(r,t)$ and $\rho_{wh}(r,t) + p_{l_wh}(r,t)$. We also show $w_{r_{wh}} = p_{r_{wh}}/\rho_{wh}$ [see Fig. 3(a)] as well as the difference between radial and lateral pressure, $p_{r_{wh}} - p_{l_{wh}}$, as a natural measure of anisotropicity of the fluid [see Fig. 2(b)]. One can clearly see that the matter supporting the wormhole goes to the ideal isotropic cosmological fluid with $w_{r_{wh}} = w_{l_{wh}} \approx 1/3$ when approaching r_c. Moreover, WEC is always satisfied, so there is no need for introducing any form of exotic matter to support the wormhole. The exoticity parameter ξ describes the physical nature of matter near and far away from the wormhole's throat. If $\xi > 0, (\xi < 0)$ then the matter surrounding the wormhole is exotic (nonexotic). From Fig. 3(b), it is obvious that $\xi < 0$ for all values of r; thus matter (in this case, radiation) remains nonexotic and the wormhole is stable even with radiation and does not require exotic matter.

In the region $r > r_c$ the flat FLRW metric is by construction replacing the dynamical wormhole metric to which it is connected, leading to the known solutions that we do not plot here. In the shown plots we choose $t/t_0 = 10^{-9}$ as a suitable origin, but with the presented conclusions remaining valid for the earlier times after the big bang. The last issue that then remains to be discussed is the exact numerical establishment of the boundary condition $\rho_{wh}(r_c,t) = \rho_c(t)$. This can be achieved by choosing the appropriate values for the free parameters m, n, ϕ_0 and $\dot{\phi}(t_0)$. Since this adjusting of the free parameters is not very enlightening, for the sake of the simplicity we choose to demonstrate this in a different manner. We absorb the dependence on the free parameters in a constant term, C, which is multiplying our solution for energy density, $\rho_{wh}(r_c,t)$, obtained with an arbitrary set of parameters. It can be seen in Fig. 4 that with choosing the different constant parameters one obtains the family of curves with the same evolution as the cosmological density, $\rho_c(t)$. For a suitable value of C curves $\rho_{wh}(r_c,t)$ and $\rho_c(t) = \rho_0 a^{-4}$ exactly coincide.

B. Matter-dominated era

While considering the matter-dominated era, characterized with $w = 0$ and $a(t) = a_0(t/t_0)^n$, we again analyze the case of a small dynamical wormhole, $r_c = 10 \times r_0$, with shape and redshift functions given by $b(r) = r_0^{m+1}/r^m$ and $h(r) = \varphi_0 e^{(-r/r_0+1)^n}$ respectively. We also set $\zeta = 1$. We are interested in the evolution of dynamic wormhole solutions from the beginning of the matter-domination era, $t/t_0 = 10^{-6}$, until now, $t/t_0 = 1$. Using the same approach as above to solve the nondiagonal equation for $\phi(t)$ (see Fig. 5), we compute the components of the energy-momentum tensor for matter supporting the considered cosmological wormhole. We depict the time evolution of energy density, pressure difference and the logarithmic form of weak energy conditions in Fig. 6. Similar to the solutions in the radiation-domination phase, there is no WEC violation present for the supporting matter during the matter-dominated era. In Fig. 7(b) is sketched the exoticity parameter for the matter era. We can notice a similar behavior as the previous section since ξ is always negative. Thus, the matter remains nonexotic and hence the wormhole is stable.

From the plot for time evolution of the radial EOS parameter, $w_{r_{wh}} = p_{r_{wh}}/\rho_{wh}$, shown in Fig. 7(a), one can see that it at first approaches $w_{r_{wh}} = 0$, but then goes to the negative values. This comes from the fact that the constant term in the $f(R)$ expansion, which effectively describes the cosmological constant, starts to dominate over the matter when approaching t_0. This is, of course, completely consistent with our current understanding of the cosmological evolution. We demonstrate this with showing $w_{r_{wh}}$ for the case when $\Lambda = 0$ (see Fig. 9), where it is clear that the wormhole EOS goes to the cosmological one when there is only matter present. The result depicted in Fig. 6(b) furthermore shows that the difference between radial and lateral pressure goes to zero, and therefore wormhole supporting matter approaches the ideal cosmological fluid as the radial coordinate goes to r_c, so that it can be smoothly matched with the FLRW universe for $r > r_c$. Finally, in the same manner as for the radiation-domination era, we demonstrate that the boundary condition $\rho_{wh}(r_c,t) = \rho_c(t)$ can be exactly numerically satisfied in Fig. 8.



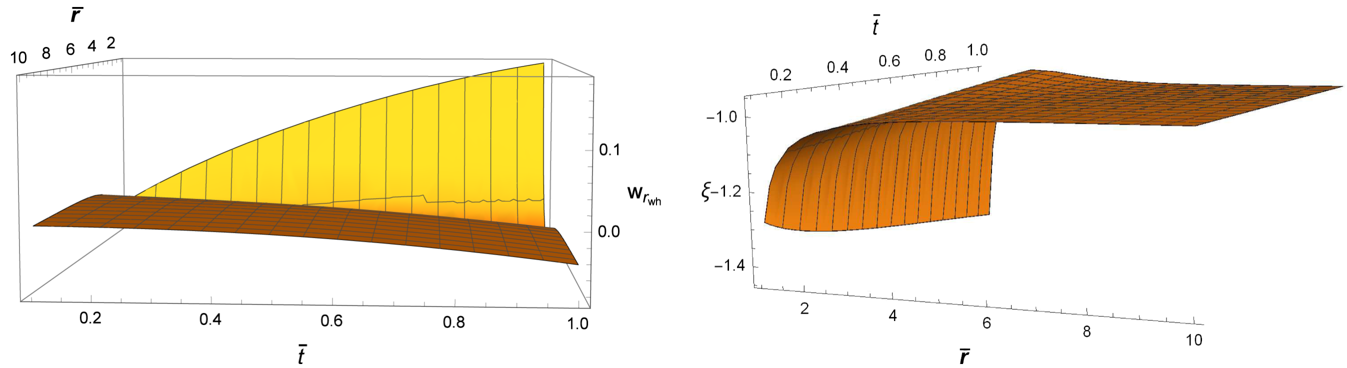
(a) Energy density (b) Pressure difference

(c) WEC-1 (d) WEC-2

FIG. 6. Dimensionless energy density $\bar{\rho} = \rho/\rho_c$, difference in dimensionless pressures $\bar{p}_l - \bar{p}_r$, WEC-1 ($\bar{p}_r + \bar{\rho}$) and WEC-2 ($\bar{p}_l + \bar{\rho}$) as a function of the dimensionless time $\bar{t} = t/t_0$ and dimensionless radius $\bar{r} = r/r_0$ respectively. In this case, we have used the parameters $m = 2$, $n = 2/3$, $\Lambda = 0.001$, $r_0 = \varphi_0 = \zeta = 1$, $p = (5 + \sqrt{73})/12$, and $t/t_0 = 0.1$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, and a redshift function given by $\Phi(t, r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 5 and the function $f(R)$ given by (23)].

C. $f(R) = R$ limit

A case of special interest in the theoretical framework that we use is given by the limit of Einstein's GR, $f(R) = R$. As shown in [19], in general WEC is necessarily violated near the throat in time-dependent wormholes. This is also confirmed in the cosmological solutions we obtain. Taking the special case of standard GR, wormhole solutions cannot in general be supported by the considered anisotropic ideal fluid represented by only diagonal components of the energy-momentum tensor. In order to solve the off-diagonal field equation (17) in the case of a general dynamic wormhole, one needs to introduce an additional off-diagonal component of the anisotropic fluid, $T^t_r = J(r, t)$, which corresponds to the energy flux of the fluid. The off-diagonal equation now reads



(a) WH state parameter (b) Exoticity

FIG. 7. Figures show the wormhole state parameter $w_{r_{wh}} = p_{r_{wh}}/\rho_{r_{wh}}$ and the exoticity parameter $\xi = (p - \rho)/|\rho|$ for a matter-dominated era as a function of the dimensionless time $\bar{t} = t/t_0$ and the dimensionless radius $\bar{r} = r/r_0$. The parameters used were $m = 2$, $n = 2/3$, $\Lambda = 0.001$, $\varphi_0 = \zeta = 1$ and $t/t_0 = 0.1$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, the shape function $b(r) = r_0^{m+1}/r^m$, and the redshift function given by $\Phi(t, r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 5 and the function $f(R)$ given by (23)].

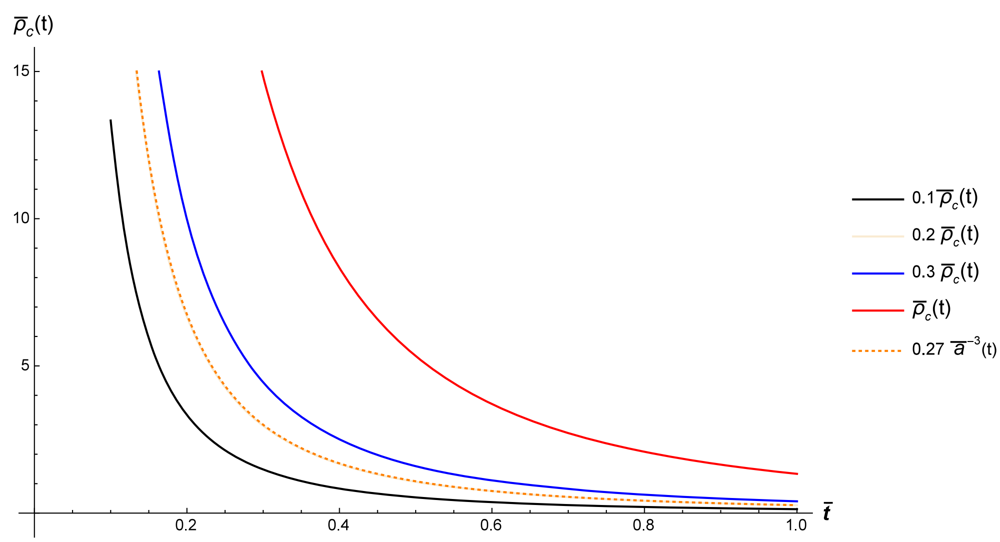


FIG. 8. Family of curves with the same evolution as the cosmological dimensionless density $\bar{\rho}_c(t)$ evaluated at $r_c = 10$ versus dimensionless time $\bar{t} = t/t_0$. The curve $0.27\bar{a}(t)^{-3} = 0.27\bar{t}^{-2}$ is overlapped with the curve $0.2\bar{\rho}_c(t)$, so that $C = 0.2$ is a suitable value. The parameters used were $m = 2$, $n = 2/3$, $\Lambda = 0.001$, $\varphi_0 = \zeta = 1$, and $t/t_0 = 0.1$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, and a redshift function given by $\Phi(t, r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 5 and the function $f(R)$ given by (23)].

$$2\left(\frac{\dot{a}}{a}\right)\Phi' = J(r, t). \qquad (25)$$

This means that the WEC, $T^{\mu\nu}k_\mu k_\nu \geq 0$, will now also include this energy flux:

$$\rho_{wh}(r, t) \geq 0, \qquad (26)$$

$$\rho_{wh}(r, t) + p_{r_{wh}}(r, t) + 2J(r, t) \geq 0, \qquad (27)$$

$$\rho_{wh}(r, t) + p_{l_{wh}}(r, t) + 2J(r, t) \geq 0. \qquad (28)$$

We again take $h(r) = \varphi_0 e^{(-r/r_0+1)^n}$, but now we also have the freedom to prescribe $\phi(t)$. Since $a(t)$ defines the natural time scale of the system under consideration it seems natural to take $\phi(t) = a_0/a(t)$. We show solutions

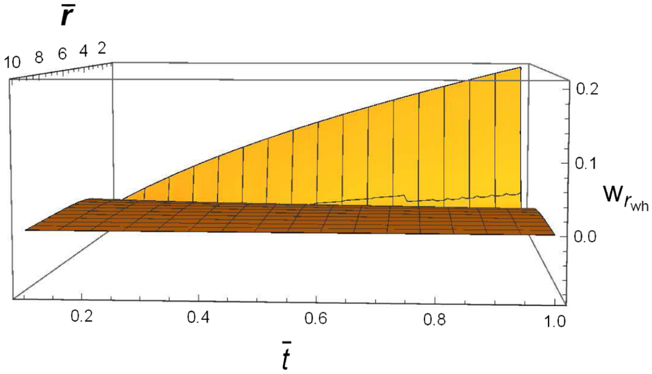
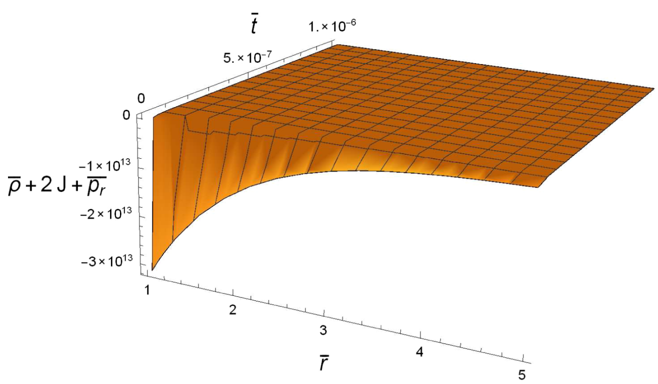


FIG. 9. Wormhole state parameter $w_{r_{wh}} = p_{r_{wh}}/\rho_{r_{wh}}$ for a matter-dominated era with $\Lambda = 0$ as a function of the dimensionless time $\bar{t} = t/t_0$ and the dimensionless radius $\bar{r} = r/r_0$. The parameters used were $m = 2$, $n = 2/3$, $\varphi_0 = \zeta = 1$ and $t/t_0 = 0.1$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, the shape function $b(r) = r_0^{m+1}/r^m$, and the redshift function given by $\Phi(t,r) = \varphi_0 e^{(-r/r_0+1)^n}\phi(t)$ [with $\phi(t)$ being displayed in Fig. 5 and the function $f(R)$ given by (23)].

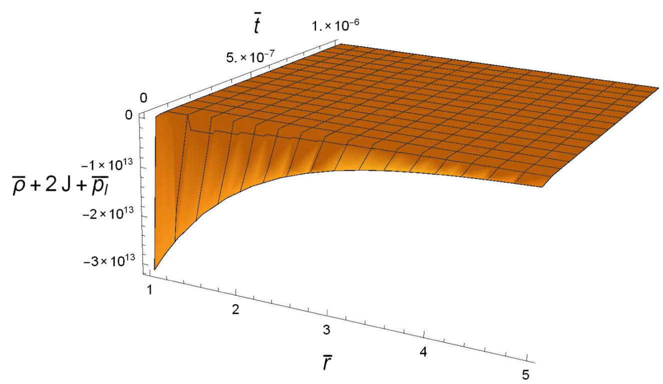
of the field equations for the $f(R) = R$ case, for a radiation-dominated universe in Figs. 10(a) and 10(b), and for the matter-dominated universe in Figs. 11(a) and 11(b). One can see that in the case of standard general relativity our solutions lead to the WEC violation. Therefore, an important difference between Einstein's GR and cosmologically viable reconstructed $f(R)$ theory lies in the fact that only the latter admits a wormhole solution supported by regular matter during the whole period of matter and radiation domination.

V. DISCUSSION

Discussion of the possible existence of wormholes as astrophysical objects naturally leads to the question of their properties and evolution during the expansion of the Universe. We speculate that cosmological wormholes could be created in the conditions characterizing the early Universe plasma, and then subsequently evolve during the radiation- and matter-dominance epochs. In order to describe wormholes in a cosmological context it is necessary to match evolving wormhole geometry with the FLRW spacetime. In the present work this was achieved by using the approximation of a small wormhole—where geometry is determined by the requirement that for some radial distance away from the throat of the wormhole, r_c, shape and redshift functions become so small that they can be taken to vanish, therefore enabling wormhole matching to the FLRW geometry. When approaching r_c, anisotropic fluid supporting the wormhole needs to go to the ideal isotropic fluid of the Universe, with the equation of state parameter going to the one characterizing radiation- and matter-dominance epochs respectively. Our analysis was done in the framework of $f(R)$ modified gravity, which is of special interest for several reasons: it represents a simple and natural mathematical generalization of standard GR, it does not introduce any new physical assumptions or entities, and it is capable of describing the observed accelerated expansion of the Universe and avoiding singularities which appear in the standard GR. Interested in viable cosmological solutions we choose the form of (R) functions given by the reconstruction procedure, which leads to the known evolution of the cosmological scale factor. We considered the wormhole solutions that were described by the same scale factor as the Universe, i.e. that

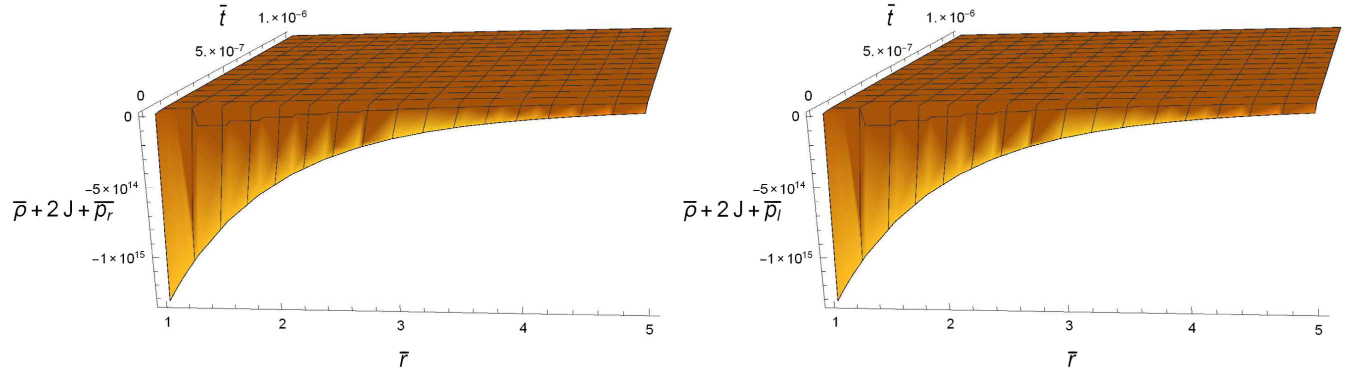


(a) WEC-1



(b) WEC-2

FIG. 10. WEC-1 ($\bar{p}_r + \bar{\rho} + 2J$) and WEC-2 ($\bar{p}_l + \bar{\rho} + 2J$) for a radiation era as a function of the dimensionless time $\bar{t} = t/t_0$ and the dimensionless radius $\bar{r} = r/r_0$ respectively. The parameters used were $m = 2$, $n = 1/2$, and $t/t_0 = 10^{-9}$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, a redshift function given by $\Phi(r,t) = a(t)^{-1}\varphi_0 e^{(-r/r_0+1)^n}$ and the function $f(R) = R$.



(a) WEC-1 (b) WEC-2

FIG. 11. WEC-1 ($\bar{p}_r + \bar{\rho} + 2J$) and WEC-2 ($\bar{p}_l + \bar{\rho} + 2J$) for a matter era as a function of the dimensionless time $\bar{t} = t/t_0$ and dimensionless radius $\bar{r} = r/r_0$ respectively. In this case, we have used the parameters $m = 2$, $n = 2/3$ and $t/t_0 = 0.1$ as the origin with a critical density $\rho_c = 3/(\kappa t_0^2)$, a shape function $b(r) = r_0^{m+1}/r^m$, a redshift function given by $\Phi(t, r) = a^{-1}(t)\varphi_0 e^{(-r/r_0+1)^n}$ and the function $f(R) = R$.

had their dynamics determined by the expansion of the Universe. Choosing simple functions for the shape parameter and space-dependent part of the redshift function we constructed examples of cosmological wormhole solutions in the radiation- and matter-dominance epochs. It was demonstrated that there is no WEC violation for material supporting these wormholes, while in the $f(R) = R$ limit WEC will always be violated in both epochs. Such cosmological wormholes could then be created as microscopic objects in the early Universe, and then they could increase their size during the evolution of the Universe, until they reach the size of average astrophysical objects today. In this paper, we have focused on the evolution of wormholes in the radiation- and matter-dominated phases of the Universe. Working with the $f(R)$ framework we assumed that dark energy is effectively described by modification of the action for the GR. If the Universe is dominated by some form of exotic matter such as phantom energy, then phantom wormholes within $f(R)$ theory would be of theoretical interest as well. The presented investigation of cosmological wormholes opens up many potential avenues for further work. It would be important to extend the analysis to the era of inflation and late time expansion, described by the scale factor exponentially dependent on time. Another interesting direction would be to look for evolving wormhole construction within the curvature-matter coupling models with actions such as $f(R)\mathcal{L}_m$ or $f(R,T)$, where T is the trace of the energy-momentum tensor. Moreover thermodynamic aspects of such wormholes at their apparent or event horizons along with further extensions to higher dimensions would be of some interest too. One feature of special physical interest would be to discuss astrophysical properties and the possibility of detection for discussed solutions, as well as their interaction with surrounding matter—for instance via accretion of different material on the cosmological wormholes.

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