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Analytic treatment of the system of a Kerr-Newman black hole and a charged massive scalar field

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Charged rotating Kerr-Newman black holes are known to be superradiantly unstable to perturbations of charged massive bosonic fields whose proper frequencies lie in the bounded regime $0 < \omega < \min\{\omega_c \equiv m\Omega_H + q\Phi_H, \mu\}$, where $\{\Omega_H, \Phi_H\}$ are, respectively, the angular velocity and electric potential of the Kerr-Newman black hole, and $\{m, q, \mu\}$ are, respectively, the azimuthal harmonic index, the charge-coupling constant, and the proper mass of the field. In this paper we study analytically the complex resonance spectrum which characterizes the dynamics of linearized charged massive scalar fields in a near-extremal Kerr-Newman black hole spacetime. Interestingly, it is shown that near the critical frequency ω_c for superradiant amplification and in the eikonal large-mass regime, the superradiant instability growth rates of the explosive scalar fields are characterized by a nontrivial (nonmonotonic) dependence on the dimensionless charge-to-mass ratio q/μ. In particular, for given parameters $\{M, Q, J\}$ of the central Kerr-Newman black hole, we determine analytically the optimal charge-to-mass ratio q/μ of the explosive scalar field which maximizes the growth rate of the superradiant instabilities in the composed Kerr-Newman-black-hole-charged-massive-scalar-field system.

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I. INTRODUCTION

Recent analytical [1] and numerical [2] studies of the coupled Einstein–Maxwell–Klein-Gordon field equations have revealed that, thanks to the intriguing mechanism of superradiance in curved black-hole spacetimes [3–5], charged rotating black holes can support stationary bound-state configurations of charged massive bosonic (integer-spin) fields which are everywhere regular outside the black-hole horizon [6–8].

These stationary bosonic field configurations [1,2] are characterized by proper frequencies which coincide with the critical (threshold) frequency ω_c for the superradiant scattering phenomenon in the black hole spacetime [3–5]. In particular, stationary charged field configurations linearly coupled to a charged rotating Kerr-Newman black hole of mass M, electric charge Q, and angular momentum $J = Ma$, are characterized by the simple relation [1,2,9]

$$\omega_{\text{field}} = \omega_c \equiv m\Omega_H + q\Phi_H, \tag{1}$$

where $\{\omega_{\text{field}}, m, q\}$ are, respectively, the proper frequency, the azimuthal harmonic index, and the charge-coupling constant of the stationary charged scalar field mode [10], and [11]

$$\Omega_H = \frac{a}{r_+^2 + a^2}, \qquad \Phi_H = \frac{Qr_+}{r_+^2 + a^2} \tag{2}$$

are, respectively, the angular velocity and electric potential of the Kerr-Newman black hole.

The proper frequencies of these stationary bosonic field configurations are also characterized by the inequality [1,2,6–8]

$$\omega_{\text{field}}^2 < \mu^2 \tag{3}$$

(here μ is the proper mass of the bosonic field [12]), a property which guarantees that these external bound-state configurations cannot radiate their energies to spatial infinity [1,2,6–8].

Interestingly, the stationary bosonic field configurations (1) studied in [1,2,6–8] mark the physical boundary between stable and unstable composed black hole field configurations. In particular, the amplitude of an external bound-state bosonic field configuration whose proper frequency is characterized by the inequality $\omega_{\text{field}} > \omega_c$ is known to decay in time [4,13], whereas the amplitude of an external bound-state bosonic field configuration whose proper frequency is characterized by the property [see Eqs. (1) and (3)]

$$0 < \omega_{\text{field}} < \min\{\omega_c, \mu\} \tag{4}$$

is known to grow exponentially over time [13–15].

The superradiant instability properties of the composed Kerr-Newman black hole charged massive scalar field system were studied in the interesting work of Furuhashi and Nambu [16]. In particular, it was found that, in the small frequency $M\omega \ll 1$ and small charge-coupling $qQ \ll 1$ regime, the growth rate [17] of the superradiant instabilities is given by the simple expression [16,18]

$$\Im\omega = \frac{\mu^4}{24} a(M^2 - Q^2)(M\mu - qQ)^5 \quad \text{for} \quad \{M\omega \ll 1, M\mu \ll 1, qQ \ll 1\}. \tag{5}$$

Inspecting the relation (5) for the imaginary part of the resonant frequency which characterizes the composed black-hole charged field system, one realizes that, in the small frequency $M\omega \ll 1$ regime, the characteristic growth rate of the superradiant instabilities is a monotonically decreasing function of the dimensionless quantity qQ. That is, for given values $\{M, Q, a\}$ of the black hole physical parameters, $\Im\omega$ is found to be a monotonically decreasing function of the charge-coupling parameter q which characterizes the explosive scalar fields.

The main goal of the present paper is to analyze the instability properties of the composed Kerr-Newman black hole charged massive scalar field system in the regime of large-field frequencies. To this end, we shall study the complex resonance spectrum which characterizes the dynamics of the charged massive scalar fields in the near-extremal charged spinning Kerr-Newman black hole spacetime. In particular, below we shall determine analytically the characteristic growth rates of the superradiant instabilities near the threshold (critical) frequency ω_c [see Eq. (1)] [19]. Interestingly, as we shall explicitly show in the present analysis, the superradiant instability growth rates of the explosive charged massive scalar fields near the critical frequency (1) are characterized by a nontrivial (nonmonotonic) dependence on the dimensionless black hole field charge-coupling parameter qQ. In particular, for given parameters $\{M, Q, a\}$ of the central Kerr-Newman black hole, we shall determine analytically the optimal charge-to-mass ratio q/μ of the explosive scalar field which maximizes the growth rate of the superradiant instabilities in this composed Kerr-Newman black hole charged massive scalar field system.

II. DESCRIPTION OF THE SYSTEM

We shall study analytically the superradiant instability properties of a physical system which is composed of a charged massive scalar field Ψ which is linearly coupled to a charged spinning near-extremal Kerr-Newman black hole. In terms of the familiar Boyer-Lindquist coordinates (t, r, θ, ϕ), the line element which describes the external spacetime of a Kerr-Newman black hole of mass M, electric charge Q, and angular momentum per unit mass $a = J/M$ is given by [11]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2} \times [adt - (r^2 + a^2)d\phi]^2, \tag{6}$$

where $\Delta \equiv r^2 - 2Mr + a^2 + Q^2$ and $\rho^2 \equiv r^2 + a^2\cos^2\theta$. The zeroes of the metric function Δ,

$$r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}, \tag{7}$$

determine the radii of the black hole (event and inner) horizons.

The dynamics of a linearized scalar field of mass μ and charge-coupling constant q in the Kerr-Newman black hole spacetime is governed by the familiar Klein-Gordon wave equation [20,21],

$$[(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2]\Psi = 0, \tag{8}$$

where A_ν is the electromagnetic potential of the charged black hole. It is convenient to decompose the scalar field eigenfunction $\Psi(t, r, \theta, \phi)$ in the form [20–22]

$$\Psi = \sum_{l,m} e^{im\phi} S_{lm}\left(\theta; a\sqrt{\mu^2 - \omega^2}\right) \times R_{lm}(r; M, Q, a, \mu, q, \omega)e^{-i\omega t}, \tag{9}$$

where R_{lm} is the radial part of the scalar eigenfunction and S_{lm} is the angular part of the scalar eigenfunction. Substituting the scalar field decomposition (9) back into the Klein-Gordon wave equation (8) and using the line element (6) of the curved Kerr-Newman black hole spacetime, one obtains [20,21] two coupled ordinary differential equations [see Eqs. (10) and (12) below] of the confluent Heun type [20,21,23–26] for the angular and radial parts of the charged massive scalar eigenfunction.

The angular (spheroidal harmonic) functions $S_{lm}(\theta)$ satisfy the ordinary differential equation [20,21,23–26]

$$\frac{1}{\sin\theta}\frac{d}{\theta}\left(\sin\theta\frac{dS_{lm}}{d\theta}\right) + \left[K_{lm} + a^2(\mu^2 - \omega^2) - a^2(\mu^2 - \omega^2)\cos^2\theta - \frac{m^2}{\sin^2\theta}\right] \times S_{lm} = 0. \tag{10}$$

This differential equation determines the discrete family of angular eigenvalues $\{K_{lm}\}$ which characterize the regular [27] angular eigenfunctions $\{S_{lm}(\theta)\}$ [20,21,23–26]. For later purposes we note that, in the asymptotic $m \gg 1$ regime, the angular eigenvalues of the spheroidal differential equation (10) are characterized by the simple asymptotic behavior [28,29]

$$K_{mm} = m^2[1 + O(m^{-1})] - a^2(\mu^2 - \omega^2). \tag{11}$$

The radial part of the Klein-Gordon wave equation (8) in the Kerr-Newman black hole spacetime is given by [20,21,30]

$$\frac{d}{dr}\left(\Delta\frac{dR_{lm}}{dr}\right)+\left[\frac{H^2}{\Delta}+2ma\omega-\mu^2(r^2+a^2)-K_{lm}\right]R_{lm}=0, \tag{12}$$

where

$$H\equiv\omega(r^2+a^2)-ma-qQr. \tag{13}$$

The differential equation (12), which determines the radial behavior of the charged massive scalar fields in the charged spinning Kerr-Newman black hole spacetime, is supplemented by the physically motivated boundary condition of purely ingoing scalar waves (as measured by a comoving observer) at the outer horizon of the Kerr-Newman black hole [1,2,13,31]:

$$R(r\to r_+)\sim e^{-i(\omega-\omega_c)y}, \tag{14}$$

where the "tortoise" radial coordinate y is defined by the relation $dy/dr=(r^2+a^2)/\Delta$ [32]. In addition, bound-state configurations of the charged massive scalar fields in the Kerr-Newman black hole spacetime are characterized by radial eigenfunctions which, in the small frequency $\omega^2<\mu^2$ regime [see Eq. (3)], decay exponentially fast at spatial infinity [1,2,13]:

$$R(r\to\infty)\sim\frac{1}{r}e^{-\sqrt{\mu^2-\omega^2}r}. \tag{15}$$

The radial differential equation (12), supplemented by the boundary conditions (14) and (15), single out a discrete spectrum of complex resonant frequencies $\{\omega(\mu,q,l,m,M,Q,a;n)\}$ [33] which characterize the dynamics of the charged massive scalar fields in the charged rotating Kerr-Newman black hole spacetime. In particular, resonant frequencies whose imaginary parts are positive are associated with the exponentially growing superradiant instabilities [13–15] which characterize the composed black hole scalar field system [see Eq. (9)]. As we shall show below, for near-extremal Kerr-Newman black holes in the regime $(r_+-r_-)/r_+\ll 1$, the characteristic complex resonance spectrum of the composed Kerr-Newman charged massive scalar field system can be studied analytically in the vicinity of the critical resonant frequency ω_c [see Eq. (1)] [19].

III. THE RESONANCE EQUATION AND ITS REGIME OF VALIDITY

In the present section we shall study the differential equation (12) which determines the spatial behavior of the radial scalar eigenfunctions. In particular, we shall derive a resonance condition [see Eq. (42) below] for the complex eigenfrequencies which characterize the dynamics of the charged massive scalar fields in the spacetime of a near-extremal charged rotating Kerr-Newman black hole.

The resonance equation for the complex resonant frequencies which characterize the dynamics of neutral scalar fields in the spacetime of a neutral near-extremal Kerr black hole was derived in [34]. It is important to emphasize that the analysis presented in [34] is restricted to the regime $M\mu=O(1)$ of moderate field masses. In the present study we shall generalize the analysis of [34] to the regime of charged massive scalar fields propagating in the spacetime of a charged near-extremal Kerr-Newman black hole. In addition, below we shall extend the analysis of [34] to the regime $M\mu\gg 1$ of large-field masses [35].

It is convenient to express the physical quantities which characterize the composed Kerr-Newman black hole linearized charged massive scalar field system in terms of the dimensionless variables [20,21]

$$x\equiv\frac{r-r_+}{r_+};\quad \tau\equiv\frac{r_+-r_-}{r_+};\quad k\equiv 2\omega r_+-qQ;\quad \varpi\equiv\frac{\omega-\omega_c}{2\pi T_{BH}}, \tag{16}$$

where $T_{BH}=(r_+-r_-)/4\pi(r_+^2+a^2)$ is the Bekenstein-Hawking temperature of the charged spinning Kerr-Newman black hole. Substituting (16) into (12), one finds the differential equation

$$x(x+\tau)\frac{d^2R}{dx^2}+(2x+\tau)\frac{dR}{dx}+UR=0 \tag{17}$$

for the radial eigenfunctions of the charged massive scalar fields in the Kerr-Newman black hole spacetime, where

$$U=U(x;\mu,q,\omega,l,m,M,Q,a)=\frac{[\omega r_+x^2+kx+\varpi\tau/2]^2}{x(x+\tau)}-K+2ma\omega-\mu^2[r_+^2(1+x)^2+a^2]. \tag{18}$$

The radial equation (17) can be solved analytically in the two asymptotic regions $x\ll 1$ and $x\gg\max\{\tau,M(\omega_c-\omega)\}$ [34]. Note, in particular, that in the double asymptotic regime [36]

$$\tau\ll 1\quad\text{and}\quad M(\omega_c-\omega)\ll 1, \tag{19}$$

one can use a standard matching procedure in the overlapping region $\max\{\tau,M(\omega_c-\omega)\}\ll x\ll 1$ in order to determine the complex resonant frequencies $\{\omega(\mu,q,l,m,M,Q,a;n)\}$ which characterize the dynamics of the charged massive scalar fields in the charged spinning Kerr-Newman black hole spacetime.

We shall first solve the radial differential equation (17) in the region

$$x\ll 1, \tag{20}$$

in which case one can use the near-horizon approximation $U \to U_{\text{near}} \equiv (kx+\varpi\tau/2)^2/[x(x+\tau)] - K + 2ma\omega - \mu^2(r_+^2+a^2)$ for the effective radial potential in (17). The near-horizon radial solution of (17) which respects the physically motivated boundary condition (14) at the outer horizon of the Kerr-Newman black hole can be expressed in terms of the hypergeometric function [25,34,37]

$$R(x) = x^{-i\frac{\varpi}{2}}\left(\frac{x}{\tau}+1\right)^{i\frac{\varpi}{2}-ik}\,{}_2F_1\left(\frac{1}{2}+i\delta-ik,\frac{1}{2}-i\delta-ik;1-i\varpi;-x/\tau\right), \tag{21}$$

where

$$\delta^2 \equiv -K-\frac{1}{4}+2ma\omega+k^2-\mu^2(r_+^2+a^2). \tag{22}$$

It proves useful to write the near-horizon radial solution (21) in the form (see Eq. 15.3.7 of [25])

$$R(x) = x^{-i\frac{\varpi}{2}}\left(\frac{x}{\tau}+1\right)^{i\frac{\varpi}{2}-ik}\left[\frac{\Gamma(1-i\varpi)\Gamma(2i\delta)}{\Gamma(1/2+i\delta-ik)\Gamma(1/2+i\delta+ik-i\varpi)}\left(\frac{x}{\tau}\right)^{-1/2+i\delta+ik}\right.$$
$$\left.\times\,{}_2F_1\left(\frac{1}{2}-i\delta-ik,\frac{1}{2}-i\delta-ik+i\varpi;1-2i\delta;-\tau/x\right)+(\delta\to-\delta)\right], \tag{23}$$

where the notation $(\delta\to-\delta)$ means "replace δ by $-\delta$ in the preceding term". Using the simple asymptotic behavior (see Eq. 15.1.1 of [25])

$${}_2F_1(a,b;c;z)\to 1 \quad \text{for } \frac{ab}{c}\cdot z\to 0 \tag{24}$$

of the hypergeometric function, one finds from (23) the expression

$$R(x) = \frac{\Gamma(1-i\varpi)\Gamma(2i\delta)\tau^{1/2-i\delta-i\varpi/2}}{\Gamma(1/2+i\delta-ik)\Gamma(1/2+i\delta+ik-i\varpi)}x^{-\frac{1}{2}+i\delta}$$
$$+(\delta\to-\delta) \tag{25}$$

for the radial eigenfunction of the charged massive scalar fields in the intermediate region

$$\tau\times\max(m,\varpi)\ll x\ll 1. \tag{26}$$

We shall next solve the radial differential equation (17) in the region

$$x\gg\max(\tau,\varpi\tau/m), \tag{27}$$

in which case one can replace (17) by

$$x^2\frac{d^2R}{dx^2}+2x\frac{dR}{dx}+U_{\text{far}}R=0, \tag{28}$$

where the effective potential in (28) is given by $U\to U_{\text{far}} = (\omega r_+ x+k)^2 - K + 2ma\omega - \mu^2[r_+^2(1+x)^2+a^2]$. The radial solution of (28) can be expressed in terms of the confluent hypergeometric function [25,34,37]:

$$R(x) = N_1\times(2\epsilon)^{\frac{1}{2}+i\delta}x^{-\frac{1}{2}+i\delta}e^{-\epsilon x}\,{}_1F_1$$
$$\times\left(\frac{1}{2}+i\delta-\kappa,1+2i\delta,2\epsilon x\right)+N_2\times(\delta\to-\delta), \tag{29}$$

where we have used here the dimensionless variables

$$\epsilon\equiv\sqrt{\mu^2-\omega^2}\,r_+; \qquad \kappa\equiv\frac{\omega k r_+-(\mu r_+)^2}{\epsilon}. \tag{30}$$

As we shall show below, the normalization constants $\{N_1,N_2\}$ of the radial solution (29) can be determined analytically by a standard matching procedure. Using the simple asymptotic behavior (see Eq. 13.1.2 of [25])

$${}_1F_1(a,b,z)\to 1 \quad \text{for } \frac{a}{b}\cdot z\to 0 \tag{31}$$

of the confluent hypergeometric function, one finds from (29) the expression

$$R(x) = N_1\times(2\epsilon)^{\frac{1}{2}+i\delta}x^{-\frac{1}{2}+i\delta}+N_2\times(\delta\to-\delta) \tag{32}$$

for the radial eigenfunction of the charged massive scalar fields in the intermediate region

$$\tau\times\max(1,\varpi/m)\ll x\ll m^{-1}. \tag{33}$$

From Eqs. (26) and (33) one learns that, for near-extremal charged spinning Kerr-Newman black holes in the regime $\tau\ll 1$, there is an overlap radial region which is determined by the strong inequalities

$$\tau \times \max(m,\varpi) \ll x_o \ll m^{-1}, \tag{34}$$

in which the expressions (21) and (29) for the radial scalar eigenfunction $R(x)$ are both valid. Note, in particular, that the two expressions (25) and (32) for the radial eigenfunction in the overlap region (34) have the same functional dependence on the dimensionless radial coordinate x. Thus, one can determine the normalization constants N_1 and N_2 of the radial eigenfunction (29) by matching the expressions (25) and (32) in their overlap region (34). This matching procedure yields

$$N_1(\delta)=\frac{\Gamma(1-i\varpi)\Gamma(2i\delta)}{\Gamma(\frac{1}{2}+i\delta-ik)\Gamma(\frac{1}{2}+i\delta+ik-i\varpi)}\tau^{\frac{1}{2}-i\delta-i\frac{\varpi}{2}}(2\epsilon)^{-\frac{1}{2}-i\delta}$$
$$\text{and}\quad N_2(\delta)=N_1(-\delta). \tag{35}$$

We shall now derive the characteristic equation which determines the complex resonant frequencies of the composed Kerr-Newman black hole charged massive scalar field system. We first point out that the radial eigenfunction (29) of the charged massive scalar fields is characterized by the asymptotic behavior (see Eq. 13.5.1 of [25])

$$R(x\to\infty)\to\left[N_1\times(2\epsilon)^{\kappa}\frac{\Gamma(1+2i\delta)}{\Gamma(\frac{1}{2}+i\delta+\kappa)}x^{-1+\kappa}(-1)^{-\frac{1}{2}-i\delta+\kappa}+N_2\times(\delta\to-\delta)\right]e^{-\epsilon x}$$
$$+\left[N_1\times(2\epsilon)^{-\kappa}\frac{\Gamma(1+2i\delta)}{\Gamma(\frac{1}{2}+i\delta-\kappa)}x^{-1-\kappa}+N_2\times(\delta\to-\delta)\right]e^{\epsilon x} \tag{36}$$

at spatial infinity. Taking cognizance of the boundary condition (15), which characterizes the spatial behavior of the bound-state radial scalar eigenfunctions at asymptotic infinity, one realizes that the coefficient of the exploding exponent $e^{\epsilon x}$ in the asymptotic expression (36) must vanish:

$$N_1\times(2\epsilon)^{-\kappa}\frac{\Gamma(1+2i\delta)}{\Gamma(\frac{1}{2}+i\delta-\kappa)}x^{-1-\kappa}+N_2\times(\delta\to-\delta)=0. \tag{37}$$

Substituting into (37) the normalization constants N_1 and N_2 [see Eq. (35)], one finds the resonance equation

$$\left[\frac{\Gamma(-2i\delta)}{\Gamma(2i\delta)}\right]^2\frac{\Gamma(\frac{1}{2}+i\delta-ik)\Gamma(\frac{1}{2}+i\delta-\kappa)\Gamma(\frac{1}{2}+i\delta+ik-i\varpi)}{\Gamma(\frac{1}{2}-i\delta-ik)\Gamma(\frac{1}{2}-i\delta-\kappa)\Gamma(\frac{1}{2}-i\delta+ik-i\varpi)}(2\epsilon\tau)^{2i\delta}=1, \tag{38}$$

which determines the complex resonant frequencies of the charged massive scalar fields in the near-extremal charged rotating Kerr-Newman black hole spacetime.

We note that the resonance equation (38) can be simplified in the regime

$$\tau\ll\frac{\bar{\omega}}{m} \tag{39}$$

of near-extremal Kerr-Newman black holes, where [38]

$$\bar{\omega}\equiv\frac{(r_+^2+a^2)(\omega-\omega_c)}{r_+} \tag{40}$$

is a dimensionless parameter which quantifies the distance between the proper frequency of the charged massive scalar field and the critical frequency (1) [19] for superradiant scattering in the charged rotating Kerr-Newman black hole spacetime. In particular, in the near-extremal regime (39), one can use the approximated relation [25,38]

$$\frac{\Gamma(\frac{1}{2}+i\delta+ik-i\varpi)}{\Gamma(\frac{1}{2}-i\delta+ik-i\varpi)}=(-i\varpi)^{2i\delta}[1+O(m/\varpi)] \tag{41}$$

for the gamma functions that appear in the resonance equation (38). Substituting (41) into (38), one finds the resonance condition

$$\left[\frac{\Gamma(-2i\delta)}{\Gamma(2i\delta)}\right]^2\frac{\Gamma(\frac{1}{2}+i\delta-ik)\Gamma(\frac{1}{2}+i\delta-\kappa)}{\Gamma(\frac{1}{2}-i\delta-ik)\Gamma(\frac{1}{2}-i\delta-\kappa)}(-4i\epsilon\bar{\omega})^{2i\delta}=1. \tag{42}$$

It is worth emphasizing again that the resonance equation (42) is valid in the regime [see Eqs. (16), (34), (39), and (40)]

$$m\tau\ll\bar{\omega}\ll m^{-1}. \tag{43}$$

In the next section, we shall show that, for $\delta\in\mathbb{R}$ [39], the (rather cumbersome) resonance equation (42) yields a remarkably simple expression for the dimensionless ratio $\omega_I/(\omega_R-\omega_c)$, where $\{\omega_R,\omega_I\}$ are, respectively, the real and imaginary parts of the complex resonant frequencies which characterize the dynamics of the charged massive scalar fields in the near-extremal charged spinning Kerr-Newman black hole spacetime.

IV. THE SUPERRADIANT INSTABILITY SPECTRUM OF THE COMPOSED KERR-NEWMAN BLACK HOLE CHARGED MASSIVE SCALAR FIELD SYSTEM

Taking cognizance of the derived resonance equation (42), one finds that the resonant frequencies of the composed Kerr-Newman black hole charged massive scalar field system in the regime (43) can be expressed in the compact form

$$\bar{\omega}=\mathcal{R}\times\mathcal{J}, \tag{44}$$

where [40,41]

$$\mathcal{R}\equiv\frac{e^{-\pi n/\delta}}{4c}\left\{\left[\frac{\Gamma(2i\delta)}{\Gamma(-2i\delta)}\right]^2\frac{\Gamma(\frac{1}{2}-i\delta-\kappa)}{\Gamma(\frac{1}{2}+i\delta-\kappa)}\right\}^{1/2i\delta}\in\mathbb{R} \tag{45}$$

and

$$\mathcal{J}\equiv i\left[\frac{\Gamma(\frac{1}{2}-i\delta-ik)}{\Gamma(\frac{1}{2}+i\delta-ik)}\right]^{1/2i\delta}\in\mathbb{C}. \tag{46}$$

Equations (44), (45), and (46) imply the relations

$$\bar{\omega}_{\rm I}=\mathcal{R}\times\mathcal{J}_{\rm I}\quad\text{and}\quad\bar{\omega}_{\rm R}=\mathcal{R}\times\mathcal{J}_{\rm R}, \tag{47}$$

which, in turn, yield the remarkably simple dimensionless ratio

$$\frac{\omega_{\rm I}}{\omega_{\rm R}-\omega_{\rm c}}=\frac{\mathcal{J}_{\rm I}}{\mathcal{J}_{\rm R}} \tag{48}$$

for the resonant frequencies of the charged massive scalar fields in the near-extremal Kerr-Newman black hole spacetime.

In the next section, we shall study the eikonal large-mass $M\mu\gg1$ regime [42] of the composed Kerr-Newman-black-hole charged massive scalar field system. In particular, below we shall show that the characteristic dimensionless ratio $\omega_{\rm I}/(\omega_{\rm R}-\omega_{\rm c})$ [see Eq. (48)] can be expressed in a remarkably compact form in this large-mass regime.

V. THE EIKONAL LARGE-MASS $M\mu\gg1$ REGIME

In the present section, we shall analyze the asymptotic large-mass regime,

$$M\mu\gg1, \tag{49}$$

of the composed Kerr-Newman black hole charged massive scalar field system. In the asymptotic regime (49), one can use the approximated relation [25,42]

$$\frac{\Gamma(\frac{1}{2}-i\delta-ik)}{\Gamma(\frac{1}{2}+i\delta-ik)}=e^{(2i-\pi)\delta}(k+\delta)^{-i(k+\delta)}(k-\delta)^{i(k-\delta)}\times[1+e^{-2\pi(k-\delta)}][1+O(m^{-1})] \tag{50}$$

for the gamma functions that appear in the expression (46) for $\mathcal{J}$. Substituting (50) into (46), one finds

$$\mathcal{J}=-e(k+\delta)^{-(k+\delta)/2\delta}(k-\delta)^{(k-\delta)/2\delta}[1+e^{-2\pi(k-\delta)}]^{1/2i\delta}, \tag{51}$$

which yields the remarkably simple dimensionless relation [see Eq. (48)] [43]

$$\frac{\omega_{\rm I}}{\omega_{\rm c}-\omega_{\rm R}}=\frac{e^{-2\pi(k-\delta)}}{2\delta} \tag{52}$$

for the characteristic resonant frequencies of the composed Kerr-Newman black hole charged massive scalar field system in the eikonal large-mass regime (49).

As a consistency check, we shall now compare our large-mass result (52) for the resonant frequencies of the composed black-hole-field system with the corresponding large-mass result of Zouros and Eardley [44]. In their highly important work, Zouros and Eardley [44] have performed a WKB analysis for the specific case of neutral scalar fields linearly coupled to a neutral spinning Kerr black hole in the large-mass $M\mu\gg1$ regime. In particular, for the case of near-extremal Kerr black holes in the regime [45]

$$a\simeq M;\qquad l=m\gg1;\qquad\mu\simeq\omega\simeq m\Omega_{\rm H}\simeq m/2M\gg1, \tag{53}$$

Zouros and Eardley [44] have derived the well-known WKB result [46]

$$M\omega_{\rm I}\propto e^{-2\pi(2-\sqrt{2})M\mu}. \tag{54}$$

Note that, for near-extremal Kerr black holes, the specific case (53) corresponds to [see Eqs. (11), (16), and (22)]

$$k=m\quad\text{and}\quad\delta=\frac{m}{\sqrt{2}}+O(1). \tag{55}$$

Substituting (55) into our analytically derived expression (52), one finds the dimensionless ratio

$$\frac{\omega_{\rm I}}{\omega_{\rm c}-\omega_{\rm R}}=\frac{e^{-2\pi(2-\sqrt{2})M\mu}}{2\sqrt{2}M\mu}, \tag{56}$$

a result which is consistent with the important result (54) of Zouros and Eardley [44] for the specific case of neutral scalar fields linearly coupled to a neutral near-extremal spinning Kerr black hole.

VI. THE OPTIMAL CHARGE-TO-MASS RATIO OF THE EXPLOSIVE SCALAR FIELDS

In the present section we shall analyze the functional dependence of the superradiant instability growth rate (52) on the dimensionless ratio q/μ which characterizes the explosive charged massive scalar fields. Taking cognizance of Eqs. (1), (11) [47], (16), and (22), one finds the expression

$$\delta - k = \sqrt{(2\omega_c r_+ - qQ)^2 - (a\omega_c - m)^2 - \mu^2 r_+^2} - (2\omega_c r_+ - qQ) \tag{57}$$

for the exponent of (52) near the superradiant instability threshold (1) [19,48] of the near-extremal Kerr-Newman black holes. From Eq. (57) one immediately learns that the exponent $\delta - k$ is a monotonically decreasing function of the mass parameter μ. Thus, one can maximize the value of the exponent (57) by minimizing (for a given value of the critical field frequency ω_c) the proper mass of the explosive scalar field. In particular, taking cognizance of Eq. (3) one realizes that, for a given value of the critical field frequency ω_c [48], the exponent (57) can be maximized by taking

$$\frac{\mu}{\omega_c} \to 1^+. \tag{58}$$

Substituting Eqs. (1), (2), and (58) into (57), and defining the dimensionless quantities

$$\gamma \equiv \frac{qQ}{m}; \quad s \equiv \frac{a}{r_+}, \tag{59}$$

one finds

$$\delta - k = m \cdot \frac{\sqrt{-s^2(3-s^2)\gamma^2 + 4s(1-s^2)\gamma + 3s^2 - 1} - [2s + (1-s^2)\gamma]}{1+s^2} \tag{60}$$

for the maximally allowed value of the exponent (57) near the superradiant instability threshold (1) of the near-extremal Kerr-Newman black holes.

For a given value of the dimensionless black hole rotation parameter s, the superradiant instability growth rate of the charged massive scalar fields [that is, the value of ω_I; see Eq. (52)] can be maximized by maximizing with respect to γ the expression (60) for the exponent $\delta - k$ [49]. In particular, a simple differentiation of (60) with respect to the dimensionless variable γ yields [50]

$$\max_\gamma\{\delta - k\} = -\frac{m}{s(2+\sqrt{1+s^2})}, \tag{61}$$

where this maximally allowed value of the exponent $\delta - k$ is obtained for

$$\gamma = \gamma^*(s) = \frac{1-s^2}{s(2+\sqrt{1+s^2})}. \tag{62}$$

It is worth noting that the expression (61) for $\max\{\delta - k\}$ is a monotonically increasing function of the dimensionless black hole rotation parameter s. In particular, one finds from (61) $\max\{\delta - k\} = m(1/\sqrt{2} - 1)$ in the $s \to 1$ limit, in agreement with the highly important result (54) of Zouros and Eardley [44] for the specific case of neutral scalar fields linearly coupled to a neutral near-extremal spinning Kerr black hole.

VII. SUMMARY

The superradiant instability properties of the composed Kerr-Newman black hole charged massive scalar field system were studied analytically. In particular, we have analyzed the near-critical [19,48] complex resonance spectrum which characterizes the dynamics of linearized charged massive scalar fields in a near-extremal charged spinning Kerr-Newman black hole spacetime.

Interestingly, it was shown that in the eikonal large-mass regime the superradiant instability growth rates of the explosive charged massive scalar fields are characterized by a nontrivial (nonmonotonic) dependence on the dimensionless black hole field charge-coupling parameter qQ [51]. In particular, for given parameters $\{M, Q, a\}$ of the central near-extremal Kerr-Newman black hole, the superradiant instability growth rate is maximized for [see Eqs. (1), (2), (58), (59), and (62)]

$$(qQ)_{\text{optimal}} = m \cdot \frac{1-s^2}{s(2+\sqrt{1+s^2})} \quad \text{and}$$
$$(M\mu)_{\text{optimal}} = m \cdot \frac{s^2 + \sqrt{1+s^2}}{s(2+\sqrt{1+s^2})\sqrt{1+s^2}}. \tag{63}$$

These relations yield the dimensionless compact expression

$$\left(\frac{q}{\mu}\right)_{\text{optimal}} = \frac{\sqrt{1-s^4}}{s^2 + \sqrt{1+s^2}} \tag{64}$$

for the optimal charge-to-mass ratio of the explosive scalar field which maximizes the growth rate of the superradiant instabilities. Finally, taking cognizance of Eqs. (52), (60), and (62), one finds the large-mass expression

$$\max\left\{\frac{\omega_{\rm I}}{\omega_{\rm c}-\omega_{\rm R}}\right\}=\frac{\sqrt{1+s^2}}{2s\cdot m}\times\exp\left[-\frac{2\pi}{s(2+\sqrt{1+s^2})}\cdot m\right] \tag{65}$$

for the maximum growth rate of the superradiant instabilities in the composed Kerr-Newman black hole charged massive scalar field bomb.

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[30] It is worth emphasizing the fact that, for spinning Kerr-Newman black holes, the ordinary differential equations which determine the angular and radial behaviors of the charged massive scalar fields [see Eqs. (10) and (12), respectively] are coupled by the angular eigenvalues $\{K_{lm}\}$.

[31] For brevity, we shall henceforth omit the harmonic indices $\{l, m\}$ of the scalar field mode.

[32] Note that the near-horizon limit $r \to r_+$ corresponds to $y \to -\infty$.

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[41] Here we have used equation 6.1.23 of [25] to infer the facts that $\Gamma(2i\delta)/\Gamma(-2i\delta) = e^{i\phi_1}$ and $\Gamma(1/2 - i\delta - \kappa)/\Gamma(1/2 + i\delta - \kappa) = e^{i\phi_2}$, where $\{\phi_1, \phi_2\} \in \mathbb{R}$. These relations imply that $\mathcal{R} \in \mathbb{R}$.

[42] Note that the large-mass $M\mu \gg 1$ regime (49) corresponds to the asymptotic $m \gg 1$ regime. This also implies the strong inequalities $\delta \gg 1$ and $k \gg 1$ [see Eqs. (61) and (63) below].

[43] Here we have used the approximated relation $[1 + e^{-2\pi(k-\delta)}]^{1/2i\delta} = 1 - ie^{-2\pi(k-\delta)}/2\delta + O[(e^{-2\pi(k-\delta)}/\delta)^2]$ [25] in the asymptotic $k - \delta \gg 1$ regime [see [42] and Eq. (61) below].

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[48] It is worth emphasizing again that we assume the relation $\omega_R \simeq \omega_c$ [see Eqs. (40) and (43)].

[49] Note that, in the eikonal large-mass $M\mu \gg 1$ regime (which also corresponds to the $m \gg 1$ regime, see [42]), one finds the relation $[e^{-2\pi(k-\delta)}/2\delta]' = [e^{-2\pi(k-\delta)}/2\delta] \times [2\pi(\delta - k)' - \delta'/\delta] = [\pi e^{-2\pi(k-\delta)}/\delta] \times (\delta - k)'[1 + O(m^{-1})]$, where here a prime denotes a derivative with respect to the dimensionless variable γ. Thus, in the eikonal large-mass $M\mu \gg 1$ regime, the location of the maximum of $\delta - k$ corresponds [up to a small correction factor $1 + O(m^{-1})$] to the location of the maximum of $e^{-2\pi(k-\delta)}/2\delta$ [see Eq. (52)].

[50] Note that the exponent of (52) is characterized by the inequality $2\pi(\delta - k) < 0$ in the entire range $s \in (0, 1]$ of the dimensionless black hole rotation parameter (see also [43]).

[51] It is worth emphasizing that, in the small frequency $M\omega \ll 1$ and small charge-coupling $qQ \ll 1$ regime, Furuhashi and Nambu [16] have interestingly found that ω_I is a monotonically decreasing function of the dimensionless black hole field charge-coupling parameter qQ [see Eq. (5)]. Intriguingly, in the present study we have proved that the opposite regime of large $M\omega \simeq M\omega_c \gg 1$ field frequencies is characterized by a nonmonotonic dependence of ω_I on the dimensionless black hole field charge-coupling parameter qQ.