Vacuum and gravitons of relic gravitational waves and the regularization of the spectrum and energy-momentum tensor

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(Received 8 December 2015; published 18 August 2016)

The spectrum of a relic gravitational wave (RGW) contains high-frequency divergences, which should be removed. We present a systematic study of the issue, based on the exact RGW solution that covers the five stages, from inflation to the acceleration, each being a power law expansion. We show that the present RGW consists of vacuum dominating at $f > 10^{11}$ Hz and graviton dominating at $f < 10^{11}$ Hz, respectively. The gravitons are produced by the four cosmic transitions, mostly by the inflation-reheating one. We perform adiabatic regularization to remove vacuum divergences in three schemes: at present, at the end of inflation, and at horizon exit, to the second adiabatic order for the spectrum, and the fourth order for energy density and pressure. In the first scheme, a cutoff is needed to remove graviton divergences. We find that all three schemes yield the spectra of a similar profile, and the primordial spectrum defined far outside horizon during inflation is practically unaffected. We also regularize the gauge-invariant perturbed inflaton and the scalar curvature perturbation by the last two schemes, and find that the scalar spectra, the tensor-scalar ratio, and the consistency relation remain unchanged.

DOI: 10.1103/PhysRevD.94.044033

I. INTRODUCTION

In inflationary cosmology, relic gravitational wave (RGW) is generated during inflation as the traceless-transverse components of metric perturbations [1–16]. After reheating, radiation, matter, and acceleration stages of the expansion, it evolves into a stochastic background in the present Universe. Only slightly affected by some astrophysical processes [17–22] during the evolution, RGW carries unique information about the early Universe besides cosmic microwave background (CMB). Moreover, existing everywhere and all the time and having a very broad spectrum over $(10^{-18}\text{–}10^{11})$ Hz, RGW has been the target for various GW detectors working at different frequency bands, such as LIGO [23], Virgo [24], GEO [25], and KAGRA [26], LISA [27], pulsar timing array (PTA) [28,29], WMAP [30,31], Planck [32], BICEP2 [33,34], and polarized laser beam detectors [35,36].

During inflation, the low-frequency modes of RGW are stretched outside of the horizon and remain constant, $h_k(\tau) = \text{const}$. The modes in the band (10^{-18}–10^{-16} Hz) reenter the horizon around $z \sim 1100$ and leave their imprints on CMB. The polarization spectrum C_l^{BB} in the detection range $l \sim (10\text{–}3000)$ is due to the primordial RGW spectrum [30,32,37–40]. On the other hand, as we shall see, the high-frequency ($f > 10^{11}$ Hz) modes never exit the horizon and decreases as $h_k(\tau) \propto 1/a(\tau)$. These correspond to the vacuum part of RGW, giving a spectrum $\propto f^2$ and leading to UV divergences in the autocorrelation function, the energy density, and pressure. Vacuum divergences also occur in any quantum fields in curved spacetime, such as inflaton fields and scalar metric perturbations. To remove the vacuum divergences, the normal ordering of field operators in the flat spacetime will not be proper in an expanding universe, since certain finite portions of the vacuum do have physical effects. Parker-Fulling's adiabatic regularization with a minimal subtraction rule [41–47] has been developed to deal with the issue and can apply to quantum fields in an expanding universe, including RGW. The vacuum divergences of quantum fields can be efficiently subtracted to a desired adiabatic order, while physically relevant parts of the vacuum are kept. The resulting RGW spectrum after regularization is consequently suppressed in high frequencies, which will serve as the target for the high-frequency GW detectors, such as a polarized, Gaussian laser beam proposed in Refs. [35,36]. For the low range (10^{-18}–10^{11} Hz), the spectrum may be also possibly modified by regularization. To investigate in a precise manner, the structure of RGW as a quantum field in the present accelerating stage needs to be explored in detail.

In the literature on adiabatic regularization of quantum fields during inflation, different schemes and results were put forward [48–58], and there are disagreements on the regularized primordial spectrum and its spectral indices defined at low frequencies. Even doubts arose as to whether adiabatic regularization is proper for removing vacuum divergences. The previous studies considered only the spectrum in the inflation stage, but not in the present stage. Moreover, these studies relied on the slow-roll approximation during inflation [59–64]. Sometimes inconsistent treatments have been

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involved. For instance, in solving the field equation, a slow-roll parameter ϵ was firstly assumed to be a constant, but then it was allowed to vary in calculating the spectral running index. Sometimes the spectrum evaluated at the horizon exit was used in place of the primordial spectrum evaluated at far outside of the horizon. In fact, the two differ drastically in the slope, the latter is the one actually referred to in CMB observations, whereas the former is not. These shortcomings will bring about uncertainties in the resulting spectrum and its regularization.

In this paper, we shall study the spectrum, energy density, and pressure of RGW in the expanding Universe, and investigate the issue of removal of UV divergences of the vacuum of RGW by adiabatic regularization method. To this end, we use the exact solution of RGW that covers the whole course of expansion, from inflation, reheating, radiation, matter, to the present accelerating stage, each stage being described by a power-law scalar factor $a(\tau) \propto \tau^d$ where d is a constant. Using the exact spectrum and the spectral indices n_t and α_t valid at any time and wave number, we show explicitly how the two spectra mentioned above differ drastically and derive a relation between n_t and the slow-roll parameter ϵ. Then we shall explore the structure of RGW as a quantum field in the present stage, decompose it into the vacuum and gravitons, and derive the number of gravitons generated during the cosmic expansions. We identify that the vacuum dominates for $f > 10^{11}$ Hz, gravitons dominate for $f < 10^{11}$ Hz, and both have UV divergences to various extents.

Then, we shall apply the formulation of adiabatic regularization and the minimal-subtraction rule to remove the vacuum divergences of RGW at a generic time. By explicit calculations, we shall show that the second adiabatic order is sufficient for the spectrum of vacuum containing quadratic and logarithmic divergences, whereas the fourth order is needed for the vacuum energy density and pressure containing extra quartic divergences. To achieve a convergent, present RGW spectrum, we shall remove UV divergences, from both vacuum and gravitons. Three schemes of regularization for vacuum divergences will be presented, each at a different time: the present time, the ending of inflation, and the horizon crossing. For the first scheme, we perform adiabatic regularization for the spectrum, energy density, and pressure of the present vacuum, and remove the graviton divergences by a cutoff. In the latter two schemes during inflation, we shall first regularize the spectrum during inflation and let it evolve into the present spectrum, according to the evolution equation. The regularized RGW spectra from the three schemes are all practically similar, except a constant factor in the third scheme, which can be absorbed into the model energy. Finally, in parallel to RGW, both the gauge-invariant perturbed inflaton and the scalar curvature perturbation have exact solutions, and the regularization during inflation are extended to these fields straightforwardly. The regularized spectra are unaffected by adiabatic regularization, so are the tensor-scalar ratio r and the consistency relation.

The paper is organized as follows.

In Sec. II, we give the exact solution of RGW, the exact power spectrum, the spectral indices, valid at any wavelength and any time. The primordial spectrum is examined in detail.

In Sec. III, we analyze the structure of RGW at the present stage and decompose it into vacuum and gravitons. The number density of gravitons is given. The divergent behavior at high frequencies is analyzed for the power spectrum, the spectral energy density, and pressure.

In Sec. IV, we use the adiabatic regularization to remove UV divergences of RGW vacuum. The second adiabatic order regularization is performed on the spectrum, and the fourth order on the energy density and pressure. The formulas are applied to the inflation and accelerating stages.

In Sec. V, three schemes of regularization at different times are presented: at the present time, at the end of inflation, and at the horizon exit.

In Sec. VI, regularization is extended to the gauge-invariant perturbed inflaton and the scalar curvature perturbation during inflation in two schemes.

Section VII gives the conclusions and discussions.

The Appendix gives technical specifications of the exact solution of RGW and the joining condition of the five stages of expansion. We use the unit with $c = \hbar = 1$ in this paper.

II. RELIC GRAVITATIONAL WAVES FROM INFLATION TO THE PRESENT

For a flat Robertson-Walker spacetime, the metric is written as

$$ds^2 = a^2(\tau)[d\tau^2 - (\delta_{ij} + h_{ij})dx^i dx^j], \tag{1}$$

which includes metric perturbations h_{ij} in the synchronous gauge with $h_{00} = h_{0i} = 0$. The tensor perturbation part of h_{ij} is the traceless and transverse RGW, and, to the linear order of metric perturbations, it satisfies the homogeneous wave equation $\Box h^{ij} = 0$. In order to reveal the vacuum structure and graviton content of RGW, in this paper, we take RGW as a quantum field, and expand it as follows:

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \times \sum_{s=+,\times} \overset{s}{\epsilon}_{ij}(k)[a^s_{\mathbf{k}} h^s_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a^{s\dagger}_{\mathbf{k}} h^{s*}_k(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}],$$
$$\mathbf{k} = k\hat{k}, \tag{2}$$

where two polarization tensors satisfy

$$\overset{s}{\epsilon}_{ij}(k)\delta_{ij} = 0, \qquad \overset{s}{\epsilon}_{ij}(k)k^i = 0, \qquad \overset{s}{\epsilon}_{ij}(k)\overset{s'}{\epsilon}_{ij}(k) = \delta_{ss'}, \tag{3}$$

and $a^s_{\mathbf{k}}$ and $a^{s\dagger}_{\mathbf{k}}$ are the annihilation and creation operators of a graviton satisfying the canonical commutation relation

$$[a^s_{\mathbf{k}}, a^{r\dagger}_{\mathbf{k}'}] = \delta_{sr}\delta^3(\mathbf{k} - \mathbf{k}'). \tag{4}$$

For RGW, the two polarization modes h^+_k and $h^\times_k$ are assumed to be independent and statistically equivalent, so that the superscript $s = +, \times$ can be dropped, and the wave equation is

$$h''_k(\tau) + 2\frac{a'(\tau)}{a(\tau)}h'_k(\tau) + k^2 h_k(\tau) = 0. \tag{5}$$

Setting

$$h_k(\tau) = A u_k(\tau)/a(\tau), \tag{6}$$

where A is a normalization constant, the mode u_k satisfies the wave equation

$$u''_k(\tau) + \left[k^2 - \frac{a''(\tau)}{a(\tau)}\right]u_k(\tau) = 0. \tag{7}$$

For each stage of cosmic expansion of the Universe, i.e., inflation, reheating, radiation dominant, matter dominant, and the present accelerating, the scale factor is a power-law form $a(\tau) \propto \tau^d$ where d is a constant [2,3,15], and the exact solution of Eq. (7) is a combination of two Hankel functions,

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\sigma}{2k}}\left[C_2 H^{(1)}_{d-\frac{1}{2}}(\sigma) + C_2 H^{(2)}_{d-\frac{1}{2}}(\sigma)\right], \tag{8}$$

where $\sigma = k\tau$, and C_1, C_2 are coefficients determined by continuity of u_k, u'_k at the transition of two consecutive stages. Thus, we obtain the analytical solution $h_k(\tau)$ for the whole course of evolution [15]. The Appendix gives a detailed account of the coefficients for these five expanding stages and the joining conditions between the adjoining stages. Note that cosmic processes, such as neutrino free streaming [17–19], QCD transition, and e^+e^- annihilation [22] only slightly modify the amplitude of RGW and will be neglected in this study.

In particular, for the inflation stage during which RGW is generated, one has

$$a(\tau) = l_0|\tau|^{1+\beta}, \qquad -\infty < \tau \le \tau_1, \tag{9}$$

where two constants l_0 and β are the parameters of the model and τ_1 is the ending time of inflation [2,15]. The expansion rate is $H = a'/a^2 = -(1+\beta)/l_0|\tau|^{2+\beta}$. In the special case of de Sitter, the inflation index $\beta = -2$, one has $l_0^{-1} = H$. Using observational data WMAP [31] of the scalar spectral index $n_s = 0.9608 \pm 0.0080$, one can infer from the relation $n_s - 1 = 2\beta + 4$ that $\beta \simeq -2.02$. For $\beta \simeq -2$, the expansion of Eq. (9) is quite general and describes a class of inflation models. During inflation, Eq. (7) becomes

$$u''_k + \left[k^2 - \frac{(1+\beta)\beta}{\tau^2}\right]u_k = 0 \tag{10}$$

and has a general solution

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{x}{2k}}[a_1 H^{(1)}_{\beta+\frac{1}{2}}(x) + a_2 H^{(2)}_{\beta+\frac{1}{2}}(x)], \quad -\infty < \tau \le \tau_1, \tag{11}$$

where $x \equiv k|\tau|$, the coefficients a_1 and a_2 are specified by a choice of the initial condition during inflation. Note that $H^{(1)}_{\beta+\frac{1}{2}}(x) = H^{(2)*}_{\beta+\frac{1}{2}}(x)$. We take

$$a_1 = 0, \quad \text{and} \quad a_2 = -ie^{-i\pi\beta/2}, \tag{12}$$

so that the mode is given by

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{x}{2k}}a_2 H^{(2)}_{\beta+\frac{1}{2}}(x), \tag{13}$$

which is the positive-frequency mode in the high frequency limit $k \to \infty$,

$$u_k \to \frac{1}{\sqrt{2k}}e^{-ik\tau}. \tag{14}$$

The solution was equivalently written in terms of Bessel's functions

$$u_k = \sqrt{\frac{x}{2k}}[A_1 J_{\frac{1}{2}+\beta}(x) + A_2 J_{-(\frac{1}{2}+\beta)}(x)], \qquad -\infty < \tau \le \tau_1, \tag{15}$$

with the coefficients $A_1 = -\frac{i}{\cos\beta\pi}\sqrt{\frac{\pi}{2}}e^{i\pi\beta/2}$ and $A_2 = iA_1 e^{-i\pi\beta}$ [15].

We work in the Heisenberg picture, in which RGW is a quantum field evolving in time, whereas Fock space vector of quantum state does not change with time. In addition to the choice of Eq. (13), we assume further that the quantum state during inflation is given by the state vector $|0\rangle$ such that

$$a^s_{\mathbf{k}}|0\rangle = 0, \tag{16}$$

for $s = +, \times$ and all $\mathbf{k}$, i.e., no gravitons are initially present, only the vacuum fluctuations (zero-point energy) of

RGW are present during inflation. The coefficient A in Eq. (6) can be determined by the quantum normalization condition, which requires that, during inflation in each $\mathbf{k}$ mode and each polarization of RGW, there is a zero point energy $\frac{1}{2}\hbar\omega$ in high frequency limit

$$A \equiv \sqrt{32\pi G} = \frac{2}{M_{\rm Pl}}, \tag{17}$$

where $M_{\rm Pl} = 1/\sqrt{8\pi G}$ is the Planck mass. Thus, RGW during inflationary stage is taken to be

$$h_k(\tau) = \frac{\sqrt{32\pi G}}{l_0|\tau|^{1+\beta}}\sqrt{\frac{\pi}{2}}\sqrt{\frac{x}{2k}}(-ie^{-i\pi\beta/2})H^{(2)}_{\beta+\frac{1}{2}}(x),$$
$$-\infty < \tau \le \tau_1. \tag{18}$$

Now, the initial condition during the inflation is fully specified by (16) and (18), which is referred to as the Bunch-Davis vacuum state. This choice will be tested by cosmological observations, such as those via CMB anisotropies and polarizations.

The autocorrelation function of RGW is defined as the expectation value of $h_{ij}h^{ij}$,

$$\langle 0|h^{ij}(\mathbf{x},\tau)h_{ij}(\mathbf{x},\tau)|0\rangle = \frac{1}{(2\pi)^3}\int d^3k(|h_k^+|^2 + |h_k^\times|^2), \tag{19}$$

where Eqs. (2), (3), and (4) have been used. The power spectrum is defined by

$$\int_0^\infty \Delta_t^2(k,\tau)\frac{dk}{k} \equiv \langle 0|h^{ij}(\mathbf{x},\tau)h_{ij}(\mathbf{x},\tau)|0\rangle. \tag{20}$$

So one reads off

$$\Delta_t^2(k,\tau) = 2\frac{k^3}{2\pi^2}|h_k(\tau)|^2, \tag{21}$$

where the factor of 2 is from the polarizations $+$, $\times$. In the literature on GW detections, the characteristic amplitude $h(k,\tau) = \sqrt{\Delta_t^2(k,\tau)}$ is often used [65,66]. The above definition of spectrum can be used for any time τ, from the inflation to the accelerating stage. Notice that the spectrum (21) is evolving in the expanding spacetime, in contrast to the spectrum in the flat spacetime, which is independent of time because of the time translation invariance. Substituting (18) into (21) yields the exact spectrum during inflation

$$\Delta_t^2(k,\tau) = 2\frac{k^3}{2\pi^2 a^2}\frac{4}{M_{\rm Pl}^2}|u_k(\tau)|^2$$
$$= \frac{k^{2(\beta+2)}}{2\pi l_0^2 M_{\rm Pl}^2}x^{-(2\beta+1)}H^{(2)}_{\beta+\frac{1}{2}}(x)H^{(1)}_{\beta+\frac{1}{2}}(x). \tag{22}$$

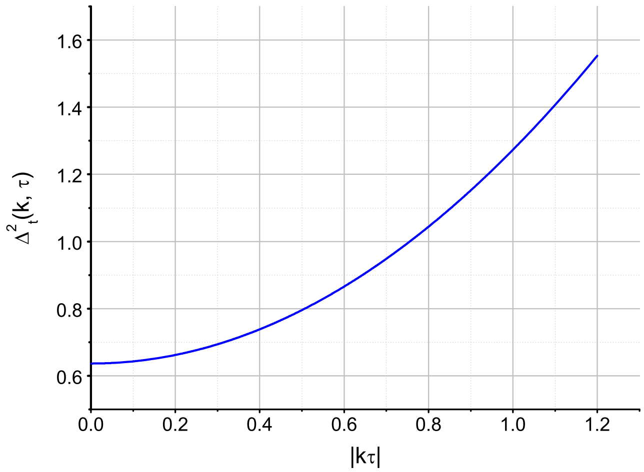

FIG. 1. The shape of $\Delta_t^2(k,\tau)$ in Eq. (22) for $\beta = -2$. It is flat at $k\tau = 0$, but has a steep slope $\propto k^1$ at $k|\tau| = 1$ during inflation.

Figure 1 sketches the shape of $\Delta_t^2(k,\tau)$ as a function of $x = k|\tau|$. The spectral indices follow from (22) accordingly

$$n_t(k,\tau) \equiv \frac{d\ln\Delta_t^2}{d\ln k} = 2\beta + 4 - x\frac{H^{(2)}_{\beta+\frac{3}{2}}(x)}{H^{(2)}_{\beta+\frac{1}{2}}(x)} - x\frac{H^{(1)}_{\beta+\frac{3}{2}}(x)}{H^{(1)}_{\beta+\frac{1}{2}}(x)}, \tag{23}$$

$$\alpha_t(k,\tau) \equiv \frac{d^2\ln\Delta_t^2}{d(\ln k)^2}$$
$$= (2\beta+1)x\frac{H^{(2)}_{\beta+\frac{3}{2}}(x)}{H^{(2)}_{\beta+\frac{1}{2}}(x)} - x^2\left[1 + \left(\frac{H^{(2)}_{\beta+\frac{3}{2}}(x)}{H^{(2)}_{\beta+\frac{1}{2}}(x)}\right)^2\right]$$
$$+ (2\beta+1)x\frac{H^{(1)}_{\beta+\frac{3}{2}}(x)}{H^{(1)}_{\beta+\frac{1}{2}}(x)} - x^2\left[1 + \left(\frac{H^{(1)}_{\beta+\frac{3}{2}}(x)}{H^{(1)}_{\beta+\frac{1}{2}}(x)}\right)^2\right]. \tag{24}$$

For a fixed inflation index β, (23) and (24) in long wavelength limit $x \ll 1$ give

$$n_t \simeq 2\beta + 4 - \frac{2}{2\beta+3}x^2 + O(x^3), \tag{25}$$

$$\alpha_t \simeq -\frac{4}{2\beta+3}x^2 + O(x^3). \tag{26}$$

Figure 2 plots n_t and α_t as functions of x. These results hold for the whole class of inflation models with $a(\tau) \propto |\tau|^{1+\beta}$.

The primordial spectrum is defined far outside the horizon $k \ll 1/|\tau|$ during inflation. In this long wavelength limit, the mode of Eq. (18) reduces to

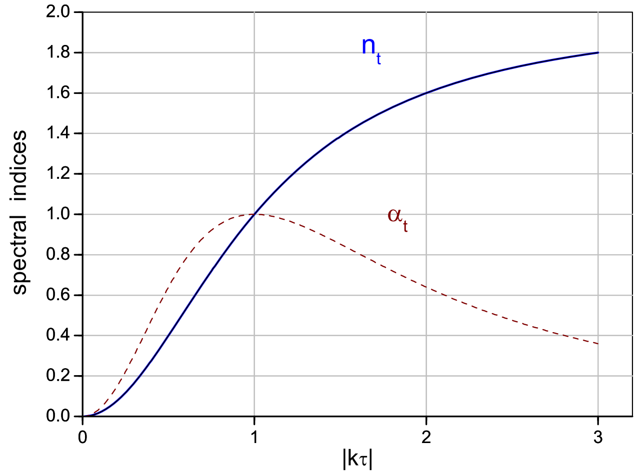

FIG. 2. The spectral indices n_t and α_t defined in (23) and (24) as functions of $k|\tau|$ during inflation. Note that $n_t = \alpha_k = 0$ at $k|\tau| = 0$, but $n_t = \alpha_k = 1$ at $k|\tau| = 1$.

$$h_k(\tau) = \frac{\sqrt{8\pi^2 G}}{l_0} \frac{-ie^{i\beta\pi/2}}{\Gamma(\beta+\frac{3}{2})\cos\beta\pi} \left(\frac{k}{2}\right)^{\beta+\frac{1}{2}} + O(k^{\beta+\frac{3}{2}}), \tag{27}$$

and the primordial spectrum is given by

$$\Delta_t^2(k) \equiv \Delta_t^2(k,\tau)|_{k\ll 1/|\tau|} = a_t^2 \frac{8}{M_{\mathrm{Pl}}^2} \left(\frac{H}{2\pi}\right)^2 k^{2\beta+4} \propto k^{2\beta+4}, \tag{28}$$

with $a_t = \sqrt{\pi}/2^{\beta+1}|\Gamma(\beta+\frac{3}{2})\cos\beta\pi| \simeq 1$. Note that (27) and (28) happen to be independent of τ as a result of the long wavelength limit. The slope of (28) depends on the inflation index via $(2\beta+4)$. In de Sitter case $\beta = -2$ and $a_t = 1$, (28) reduces to a flat spectrum [61]

$$\Delta_t^2(k) = \frac{8}{M_{\mathrm{Pl}}^2} \left(\frac{H}{2\pi}\right)^2 + O(k^2). \tag{29}$$

The primordial spectrum $\Delta_t(k)$ with $\beta = -2.0125$ is shown as the top curve in Fig. 3. The spectral indices in limit $k \to 0$ follow immediately

$$n_t \equiv \left.\frac{d\ln\Delta_t^2}{d\ln k}\right|_{k\to 0} = 2\beta + 4, \tag{30}$$

$$\alpha_t \equiv \left.\frac{d^2\ln\Delta_t^2}{d(\ln k)^2}\right|_{k\to 0} = 0. \tag{31}$$

This value of α_t differs from the result of slow-roll approximation [63,64]. One can introduce the slow-roll parameter

$$\epsilon \equiv -H'/aH^2, \tag{32}$$

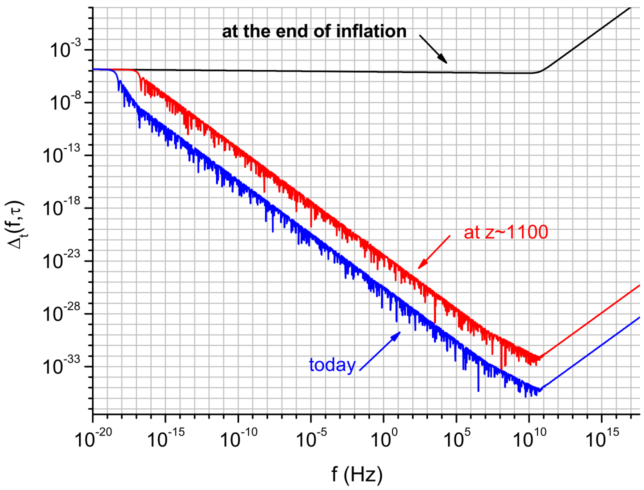

FIG. 3. The spectrum $\Delta_t(f,\tau)$ at three different times: at the end of inflation, at $z \sim 1100$, and at present, respectively. The horizontal axis is the physical frequency $f = k/2\pi a(\tau_H)$ at the present time τ_H. The parameters $\beta = -2.0125$ and $r = 0.12$ are taken for illustration.

whose value is much smaller than 1. Solving Eq. (32) leads to

$$\epsilon = \frac{\beta+2}{\beta+1}. \tag{33}$$

Here ϵ can be positive or negative, depending on β. This generalizes the result $\epsilon > 0$ of a single scalar field model [61]. Plugging Eq. (33) into Eq. (30) yields the following relation:

$$n_t = \frac{-2\epsilon}{1-\epsilon}, \tag{34}$$

which generalizes the result $n_t = -2\epsilon$ of the slow-roll approximation, but reduces to it when high power terms of ϵ are dropped.

In regard to the spectrum and spectral indices, we would like to point out certain inconsistent treatments in the literature. For instance, sometimes the spectrum and spectral indices around $k|\tau| = 1$ were used [61,63,64]. However, here one would have

$$\Delta_t^2(k)|_{k|\tau|\simeq 1} = \frac{8}{M_{\mathrm{Pl}}^2} \left(\frac{H}{2\pi}\right)^2 (2x + O(x-1)^2), \tag{35}$$

$n_t \simeq 1$, and $\alpha_t \simeq 1$, as seen in Figs. 1 and 2, differing drastically from (28), (30), and (31). (This distinction applies also to scalar fields during inflation as will be addressed in Sec. VI.) As far as cosmological observations are concerned, it is incorrect to use $k|\tau| = 1$ in place of $k|\tau| \simeq 0$ for the spectrum and indices. As is known from analytical calculations of CMB anisotropies and polarization [67–69], the power spectra C_l^{XX} located at l are induced by the k modes of metric perturbations in the following manner:

$$C_l^{TT} \propto |h_k(\tau_d)|^2_{k=l/\tau_H}, \qquad C_l^{EE}, C_l^{BB} \propto |\dot{h}_k(\tau_d)|^2_{k=l/\tau_H},$$

where τ_d is the decoupling time corresponding to a redshift $z \sim 1100$ when CMB were formed. Since C_l^{XX} have been observed in a multipole range $l \sim (10\text{–}3000)$ [30,32,38–40], the relevant metric perturbation are those with [68,69]

$$k \sim l/\tau_H \sim l \sim (10\text{–}3000). \tag{36}$$

Among these, the one that entered the horizon exactly at τ_d is given by $k(\tau_d - \tau_m) = 1$, i.e., $k \simeq 26$. The k modes specified by (36) stay far outside the horizon during most of the inflation, for instance, they give $k|\tau_1| \sim (6.4 \times 10^{-29}\text{–}9.6 \times 10^{-26}) \ll 1$ at the end of inflation τ_1. These modes are just described by the formulas of (27) and (28). Hence, we conclude that it is incorrect to use $\Delta_t^2(k)$, n_t, and α_t evaluated at the horizon crossing to substitute for those at far outside the horizon.

The overall amplitude of the primordial spectrum (28) is essentially determined by the expansion rate H of inflation, which in turn is related to the energy density via $H^2 = 8\pi G\rho/3$. In association with observations, the spectrum (28) is often rewritten as [70]

$$\Delta_t(k) = \Delta_R r^{1/2} \left(\frac{k}{k_0}\right)^{\frac{n_t}{2} + \frac{1}{4}\alpha_t \ln(\frac{k}{k_0})}, \tag{37}$$

where k_0 is a pivot conformal wave number corresponding to a physical wave number $k_0/a(\tau_H) = 0.002\ \text{Mpc}^{-1}$, Δ_R is the curvature perturbation determined by observations [71] $\Delta_R^2 = (2.464 \pm 0.072) \times 10^{-9}$, and $r \equiv \Delta_t^2(k_0)/\Delta_R^2(k_0)$ is the tensor-scalar ratio, and $r < 0.12$ by the joint analysis of BICEP2/Keck Array and Planck data [34,72].

The RGW spectrum at present time τ_H follows from Eq. (22):

$$\Delta_t^2(k, \tau_H) = 2\frac{k^3}{2\pi^2 a^2(\tau_H)} \frac{4}{M_{\rm Pl}^2} |u_k(\tau_H)|^2, \tag{38}$$

where the mode $u_k(\tau_H)$ of the present accelerating stage has been obtained and is listed in (39). The present $\Delta_t(k, \tau_H)$ is plotted as the lowest curve in Fig. 3. Notice that $\Delta_t(k, \tau_H)$ is overlapped with the primordial one $\Delta_t(k)$ at the low-frequency end $f < 10^{-18}$ Hz, both being flat there. The wavelength of these modes are longer than horizon, and they remain constant, $h_k \simeq$ const, ever since inflation. At the high-frequency end for $f > 10^{11}$ Hz, the spectrum behaves as $\Delta_t^2(f, \tau_H) \propto f^2$, as shown in Fig. 3. These high-frequency modes have never exited the horizon since inflation, so that their amplitude decreases as $h_k \propto 1/a(\tau)$. This high-frequency behavior will cause the autocorrelation function $\langle 0|h^{ij}h_{ij}|0\rangle$ in Eq. (20) to diverge, an issue to be addressed in Sec. V.

The frequency f at a time τ is related to the comoving wave number k via $f(\tau) = ck/2\pi a(\tau)$. In this paper, we adopt the convention $a(\tau_H) = l_H \simeq 2.8 \times 10^{26}$ m, so that the present frequency is related to k via $f \simeq 1.7 \times 10^{-19} k$ Hz. (see Appendix)

III. DECOMPOSITION OF RGW INTO VACUUM AND GRAVITONS

RGW during inflation has been assumed to be the vacuum state specified by (16) and (18), consisting of vacuum fluctuations. After the subsequent four stages, RGW has evolved into the present accelerating stage with $a(\tau) = l_H|\tau - \tau_a|^{-\gamma}$, where $\gamma \simeq 2.1$ fits the model $\Omega_\Lambda \simeq 0.7$ and $\Omega_m = 1 - \Omega_\Lambda$. The analytical mode is given by

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{s}{2k}}\left[e^{-i\pi\gamma/2}\beta_k H^{(1)}_{-\gamma-\frac{1}{2}}(s) + e^{i\pi\gamma/2}\alpha_k H^{(2)}_{-\gamma-\frac{1}{2}}(s)\right],$$
$$\tau_E < \tau \le \tau_H, \tag{39}$$

where $s \equiv k(\tau - \tau_a)$ and the coefficients β_k, α_k are given by

$$\begin{aligned} e^{-i\pi\gamma/2}\beta_k = \Delta_e^{-1}\Bigg\{ &\sqrt{\frac{z_E}{s_E}}\left[d_1 H^{(1)}_{\frac{3}{2}}(z_E) + d_2 H^{(2)}_{\frac{3}{2}}(z_E)\right] \\ &\times \left[\frac{1}{2\sqrt{s_E}} H^{(2)}_{-\gamma-\frac{1}{2}}(s_E) + \sqrt{s_E} H^{(2)\prime}_{-\gamma-\frac{1}{2}}(s_E)\right] \\ &- H^{(2)}_{-\gamma-\frac{1}{2}}(s_E)\Bigg[\frac{1}{2\sqrt{z_E}}\left(d_1 H^{(1)}_{\frac{3}{2}}(z_E) + d_2 H^{(2)}_{\frac{3}{2}}(z_E)\right) \\ &+ \sqrt{z_E}\left(d_1 H^{(1)\prime}_{\frac{3}{2}}(z_E) + d_2 H^{(2)\prime}_{\frac{3}{2}}(z_E)\right)\Bigg]\Bigg\}, \end{aligned} \tag{40}$$

$$\begin{aligned} e^{i\pi\gamma/2}\alpha_k = \Delta_e^{-1}\Bigg\{ &\sqrt{\frac{z_E}{s_E}}\left[d_1 H^{(1)}_{\frac{3}{2}}(z_E) + d_2 H^{(2)}_{\frac{3}{2}}(z_E)\right] \\ &\times \left[\frac{1}{2\sqrt{s_E}} H^{(1)}_{-\gamma-\frac{1}{2}}(s_E) + \sqrt{s_E} H^{(1)\prime}_{-\gamma-\frac{1}{2}}(s_E)\right] \\ &- H^{(1)}_{-\gamma-\frac{1}{2}}(s_E)\Bigg[\frac{1}{2\sqrt{z_E}}\left(d_1 H^{(1)}_{\frac{3}{2}}(z_E) + d_2 H^{(2)}_{\frac{3}{2}}(z_E)\right) \\ &+ \sqrt{z_E}\left(d_1 H^{(1)\prime}_{\frac{3}{2}}(z_E) + d_2 H^{(2)\prime}_{\frac{3}{2}}(z_E)\right)\Bigg]\Bigg\}, \end{aligned} \tag{41}$$

$$\Delta_e = \sqrt{s_E}\left[H^{(1)}_{-\gamma-\frac{1}{2}}(s_E) H^{(2)\prime}_{-\gamma-\frac{1}{2}}(s_E) - H^{(1)\prime}_{-\gamma-\frac{1}{2}}(s_E) H^{(2)}_{-\gamma-\frac{1}{2}}(s_E)\right], \tag{42}$$

where $s_E = k(\tau_E - \tau_a)$ and $z_E = k(\tau_E - \tau_m)$, and d_1 and d_2 are the coefficients for the precedent matter stage. (See Appendix for details.) In the high-frequency limit $k \to \infty$, β_k, α_k have the following asymptotic expressions:

$$\beta_k = \left(\frac{\beta(\beta+1)}{4x_1^2} - \frac{\beta_s(\beta_s+1)}{4t_1^2}\right)e^{i(x_1+t_1-t_s+y_s-y_2+z_2-z_E+s_E)-i\pi\beta} + \frac{\beta_s(\beta_s+1)}{4t_s^2}e^{i(x_1-t_1+t_s+y_s-y_2+z_2-z_E+s_E)+i\pi\beta} - \frac{1}{2z_2^2}e^{i(x_1-t_1+t_s-y_s+y_2+z_2-z_E+s_E)+i\pi\beta} + \left(\frac{1}{2z_E^2} - \frac{\gamma(\gamma+1)}{4s_E^2}\right)e^{i(x_1-t_1+t_s-y_s+y_2-z_2+z_E+s_E)+i\pi\beta} + \mathcal{O}(k^{-3}) \quad (43)$$

$$\alpha_k = e^{-i(x_1-t_1+t_s-y_s+y_2-z_2+z_E-s_E)+i\pi\beta}\Bigg(1 - i\frac{\beta(\beta+1)}{2x_1} + i\frac{\beta_s(\beta_s+1)}{2t_1} - i\frac{\beta_s(\beta_s+1)}{2t_s} + i\frac{1}{z_2} - i\frac{1}{z_E} + i\frac{\gamma(\gamma+1)}{2s_E} - \frac{\beta^2(\beta+1)^2}{8x_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_s^2} - \frac{1}{2z_2^2} - \frac{1}{2z_E^2} - \frac{\gamma^2(\gamma+1)^2}{s_E^2} + \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_1} - \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_s} + \frac{\beta(\beta+1)}{2x_1z_2} - \frac{\beta(\beta+1)}{2x_1z_E} + \frac{\beta(\beta+1)\gamma(\gamma+1)}{4x_1s_E} + \frac{\beta_s^2(\beta_s+1)^2}{4t_1t_s} - \frac{\beta_s(\beta_s+1)}{2t_1z_2} + \frac{\beta_s(\beta_s+1)}{2t_1z_E} - \frac{\beta_s(\beta_s+1)\gamma(\gamma+1)}{4t_1s_E} + \frac{\beta_s(\beta_s+1)}{2t_sz_2} - \frac{\beta_s(\beta_s+1)}{2t_sz_E} + \frac{\beta_s(\beta_s+1)\gamma(\gamma+1)}{4t_ss_E} + \frac{1}{z_2z_E} - \frac{\gamma(\gamma+1)}{2z_2s_E} + \frac{\gamma(\gamma+1)}{2z_Es_E}\Bigg) + \mathcal{O}(k^{-3}), \quad (44)$$

where $x_1, t_1, t_s, y_s, y_2, \ldots, s_E$ are the time instances of transitions multiplied by the wave number (see Appendix). Analogous to Eq. (13) for inflation, the vacuum mode during the present stage is chosen as

$$v_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{s}{2k}}e^{i\pi\gamma/2}H^{(2)}_{-\gamma-\frac{1}{2}}(s), \qquad \tau_E < \tau \le \tau_H, \quad (45)$$

so that $v_k(\tau) \to \frac{1}{\sqrt{2k}}e^{-ik(\tau-\tau_a)}$ as $k \to \infty$. Thus, in terms of $v_k(\tau)$, Eq. (39) is written as

$$u_k(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau), \quad (46)$$

and α_k and β_k are the Bogolyubov coefficients, satisfying the relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \quad (47)$$

resulting from the commutation relation (4). Starting from vacuum fluctuations described by the positive-frequency mode (13) during inflation, RGW has evolved into a mixture of the positive and negative frequency modes as in Eq. (46) for the present stage. From the field operator h_{ij} in Eq. (2), one sees that the operator for each $\mathbf{k}$ is proportional to

$$a_{\mathbf{k}}u_k + a_{\mathbf{k}}^\dagger u_k^* = A_{\mathbf{k}}v_k + A_{\mathbf{k}}^\dagger v_k^*,$$

where

$$A_{\mathbf{k}} \equiv \alpha_k a_{\mathbf{k}} + \beta_k^* a_{\mathbf{k}}^\dagger$$

is interpreted as the annihilation operator of gravitons of $\mathbf{k}$ for the present stage. Thus, the number density of gravitons in the present stage is

$$N_{\mathbf{k}} = \langle 0|A_{\mathbf{k}}^\dagger A_{\mathbf{k}}|0\rangle = |\beta_k|^2. \quad (48)$$

This result is an application of the theory of particle production in the expanding Universe, developed by Parker [73]. As a function of k, $|\beta_k|^2$ is shown in Fig. 4,

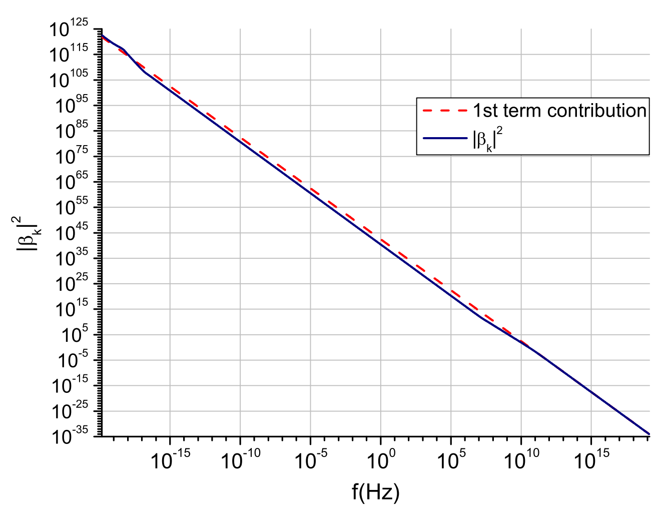

FIG. 4. The number density $|\beta_k|^2$ of gravitons produced in the k mode is shown.

and over the frequency range $f \geq 10^{-18}$ Hz, one has $|\beta_k|^2 \propto k^{-4}$ approximately, as shown in Eq. (43). Moreover, as $k \to \infty$, $\alpha_k \sim 1$, and $\beta_k \propto k^{-2}$, so that u_k of (46) is dominated by the positive frequency mode v_k in the high frequency limit. This confirms the adiabatic theorem [41,73,74], i.e., high frequency modes are essentially unaffected by a slow expansion of the spacetime. Our detailed calculations show that the modes with $f > 10^{11}$ Hz never exit the horizon from inflation up to the present stage.

β_k in Eq. (43) contains terms such as $\frac{\beta(\beta+1)}{4x_1^2} \propto a''/a$. By the Friedmann equation $a''/a = \frac{4\pi G}{3} a^2 T^\mu{}_\mu$, it is revealing to express these in terms of the trace $T^\mu{}_\mu$ of the energy momentum tensor that drives the cosmic expansion. One has

$$\begin{aligned}\beta_k = {} & \frac{a(\tau_1)^2 \pi G}{3k^2}[T^\mu{}_\mu(\tau_1^-) - T^\mu{}_\mu(\tau_1^+)]e^{i(x_1+t_1-t_s+y_s-y_2+z_2-z_E+s_E)-i\pi\beta} \\ & + \frac{a(\tau_s)^2 \pi G}{3k^2}[T^\mu{}_\mu(\tau_s^-) - T^\mu{}_\mu(\tau_s^+)]e^{i(x_1-t_1+t_s+y_s-y_2+z_2-z_E+s_E)+i\pi\beta} \\ & + \frac{a(\tau_2)^2 \pi G}{3k^2}[T^\mu{}_\mu(\tau_2^-) - T^\mu{}_\mu(\tau_2^+)]e^{i(x_1-t_1+t_s-y_s+y_2+z_2-z_E+s_E)+i\pi\beta} \\ & + \frac{a(\tau_E)^2 \pi G}{3k^2}[T^\mu{}_\mu(\tau_E^-) - T^\mu{}_\mu(\tau_E^+)]e^{i(x_1-t_1+t_s-y_s+y_2-z_2+z_E+s_E)+i\pi\beta} + \mathcal{O}(k^{-3}), \end{aligned} \tag{49}$$

where $T^\mu{}_\mu(\tau_1^-)$ is evaluated at the end of inflation, $T^\mu{}_\mu(\tau_1^+)$ at the beginning of reheating, etc. Equation (49) tells that the graviton production is due to the discontinuities at the transitions of the trace of the energy momentum tensor that drives the expansion. In our model, the pressure p is not continuous at the transition points. Furthermore, among the four terms in (49), the first term $\propto 1/(k\tau_1)^2$ by the inflation-reheating transition gives the greatest contribution, other three terms give some modifications. This analytically confirms the conclusion that particle creation at the early stages is of great significance [75]. Our computation shows that the full $|\beta_k|^2$ computed from (e-1) is lower than the square of the first term by 2 orders of magnitude within the range $(10^{-17}$–$10^7)$ Hz.

It is interesting to compare our result with the well-known results of production of scalar particles in RW spacetimes. For a scalar massless field conformally coupled with the curvature, there is no particle production of the scalar field [45,73,74]. This conclusion holds before regularization where one has classically $T^\mu{}_\mu = 0$, i.e., the trace of energy momentum tensor of the scalar field is vanishing, as well as after regularization whereby the trace anomaly $\langle 0|T^\mu{}_\mu|0\rangle_{\rm phys} \neq 0$ appears [45,46,76]. For cases of nonconformal coupling, in general, the trace $T^\mu{}_\mu \neq 0$ and, there are particle productions of the scalar field. However, in our expression (49), $T^\mu{}_\mu$ is that of the background matter content that drives the expansion and may not be the scalar field in Refs. [45,46,76].

Now we analyze the present power spectrum

$$\Delta_t^2(k,\tau_H) = A^2 \frac{k^3}{\pi^2 a^2(\tau_H)} |u_k(\tau_H)|^2, \tag{50}$$

in terms of vacuum and gravitons. By Eq. (46) and the relation in (47), one has the following decomposition:

$$|u_k(\tau)|^2 = |v_k(\tau)|^2 + 2\mathrm{Re}[\alpha_k \beta_k^* v_k(\tau)^2] + 2|\beta_k|^2 |v_k(\tau)|^2, \tag{51}$$

where $|v_k|^2$ is the vacuum term given by (45), and the last two terms containing β_k are due to the gravitons. In high frequency limit, the vacuum term behaves as $|v_k|^2 \propto k^{-1}$, k^{-3}, the cross term as $\mathrm{Re}[\alpha_k \beta_k^* v_k^2] \propto k^{-3}$, and $|\beta_k|^2 |v_k|^2 \propto k^{-5}$, so that the spectrum contains the following quadratic and logarithmic divergences:

$$\Delta_t^2(k,\tau_H) \propto k^2, k^0, \quad \text{for } f > 10^{11} \text{ Hz}, \tag{52}$$

coming from the vacuum and cross terms, which will be removed in later sections. In the range $f < 10^{11}$ Hz, $\Delta_t^2(k,\tau_H)$ is dominated by the graviton terms $\mathrm{Re}[\alpha_k \beta_k^* (v_k)^2] + |\beta_k|^2 |v_k|^2$, both having the same profile, except that $|\beta_k|^2 |v_k|^2$ is smooth, whereas $\mathrm{Re}[\alpha_k \beta_k^* (v_k)^2]$ has extra quick oscillations, caused by the interference of waves between vacuum $\alpha_k v_k$ and gravitons $\beta_k v_k^*$. The slope of the overall profile of $\Delta_t^2(k,\tau_H)$ is $\propto k^{-2+(2\beta+4)}$ in $(10^{-18}$–$10^7)$ Hz, and $\propto k^{-1.5+(\beta+2)}$ in $(10^7$–$10^{11})$ Hz. These features are illustrated in Fig. 5.

The energy momentum tensor of RGW can be also decomposed into vacuum and gravitons. As long as the wavelengths are shorter than the horizon, i.e., $f \geq 10^{-18}$ Hz, the energy-momentum tensor of RGW is well-defined and given by [77–79]

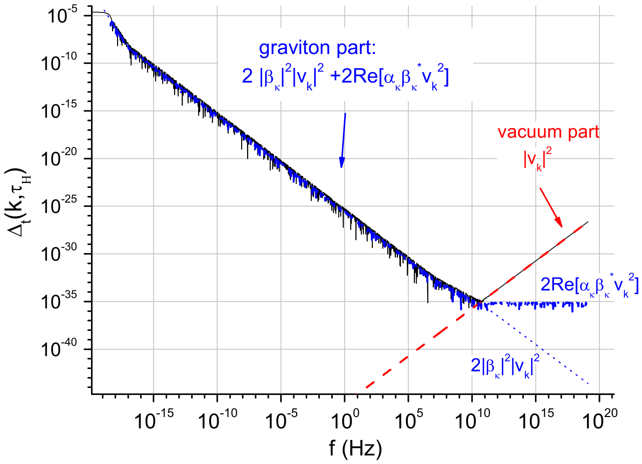

FIG. 5. The present spectrum consists of the vacuum and graviton parts. At $f > 10^{11}$ Hz, the vacuum is dominant and contains both quadratic and log divergences $\Delta_t^2(k) \propto k^2, k^0$, and the gravitons gives only log divergence $\Delta_t^2(k) \propto k^0$.

$$t_{\mu\nu} = \frac{1}{32\pi G}\langle 0|h^{ij}{}_{,\mu}h_{ij,\nu}|0\rangle, \tag{53}$$

the energy density

$$\rho_{gw} = t^0{}_0 = \frac{1}{32\pi G a^2}\langle 0|h'_{ij}h'^{ij}|0\rangle, \tag{54}$$

and the pressure

$$p_{gw} = -\frac{1}{3}t^i{}_i. \tag{55}$$

Substituting (2), (3), and (4) into (54) yields

$$\rho_{gw} = \frac{1}{32\pi G a^2}\int \frac{d^3k}{(2\pi)^3}2|h'_k(\tau)|^2 = \int_0^\infty \rho_k(\tau)\frac{dk}{k}, \tag{56}$$

where the spectral energy density

$$\rho_k(\tau) = 2\frac{k^3}{2\pi^2 a^2}\left|\left(\frac{u_k}{a}\right)'\right|^2 \tag{57}$$

with

$$\left|\left(\frac{u_k}{a}\right)'\right|^2 = \frac{|u'_k|^2}{a^2} + \left(\frac{a'}{a^2}\right)^2|u_k|^2 - \frac{a'}{a^3}(u_k^* u'_k + u_k u'^*_k). \tag{58}$$

The formula (57) holds at any time τ. Note that, in high frequencies, ρ_k is dominated by the first term of (58) and can be approximated by

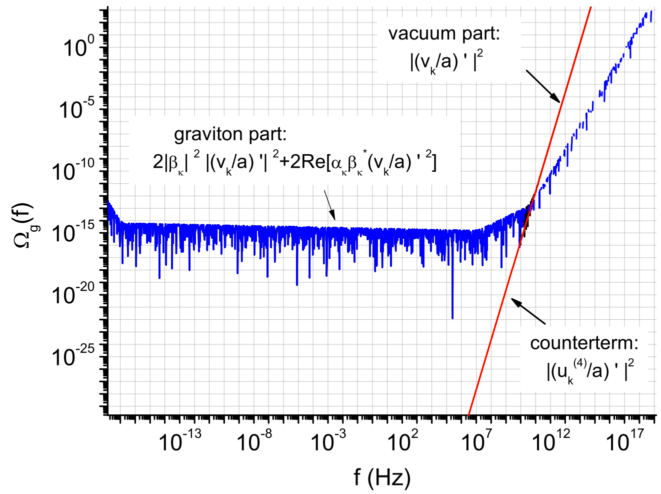

FIG. 6. $\Omega_g(f) = \rho_k(\tau_H)/\rho_c$ at present consists of the vacuum and graviton parts, where ρ_c is the critical density. $\beta = -2.0125$ and $r = 0.12$ are taken.

$$\rho_k(\tau) \simeq \frac{k^3}{\pi^2 a^4}|u_{k'}|^2 \simeq \frac{k^5}{\pi^2 a^4}|u_k|^2 = \frac{1}{32\pi G a^2}k^2\Delta_t^2(k,\tau), \tag{59}$$

which has been often used in literature [2,66,80]. But for regularization later, one should use the full expression (58). Similar to (51), one has the vacuum-graviton decomposition

$$\left|\left(\frac{u_k}{a}\right)'\right|^2 = \left|\left(\frac{v_k}{a}\right)'\right|^2 + 2\mathrm{Re}\left[\alpha_k\beta_k^*\left(\frac{v_k}{a}\right)'^2\right] + 2|\beta_k|^2\left|\left(\frac{v_k}{a}\right)'\right|^2. \tag{60}$$

At $f > 10^{11}$ Hz, the vacuum term $|(\frac{v_k}{a})'|^2 \propto k^1, k^{-1}, k^{-3}$, the cross term gives

$$\mathrm{Re}\left[\alpha_k\beta_k^*\left(\frac{v_k}{a}\right)'^2\right] \propto k^{-1}, k^{-3}, \tag{61}$$

and

$$|\beta_k|^2\left|\left(\frac{v_k}{a}\right)'\right|^2 \propto k^{-3}, \tag{62}$$

so that ρ_k contains quartic, quadratic, and logarithmic divergences

$$\rho_k \propto k^4, \qquad k^2, \qquad k^0, \tag{63}$$

in the integration (56). At $f < 10^{11}$ Hz, the graviton terms $\mathrm{Re}[\alpha_k\beta_k^*(\frac{v_k}{a})'^2] + |\beta_k|^2|(\frac{v_k}{a})'|^2$ dominate, giving $\rho_k \propto k^{2\beta+4}$ in $(10^{-18}\text{–}10^7)$ Hz and $\rho_k \propto k^{0.5+(\beta+2)}$ in $(10^7\text{–}10^{11})$ Hz. These are illustrated in Fig. 6. Similarly, the pressure is [79]

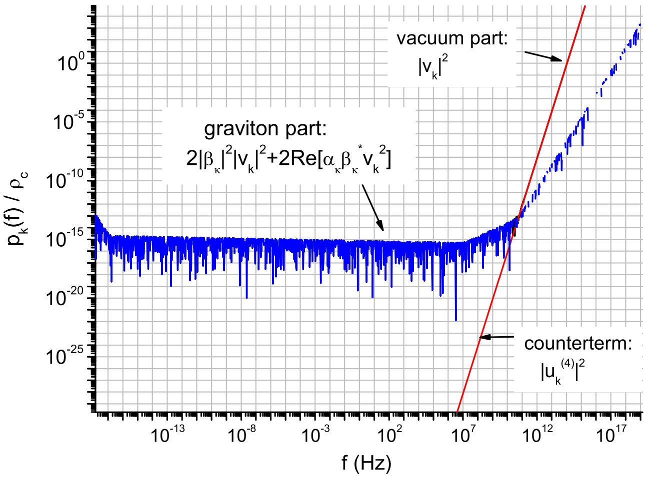

FIG. 7. Spectral pressure $p_k(\tau_H)/\rho_c$ at present consists of the vacuum and graviton parts. $\beta = -2.0125$ and $r = 0.12$.

$$p_{gw} = \frac{1}{96a^2\pi G}\int \frac{d^3k}{(2\pi)^3} 2k^2|h_k|^2 = \int_0^\infty p_k(\tau)\frac{dk}{k}, \quad (64)$$

where

$$p_k(\tau) = \frac{k^5}{3\pi^2 a^4}|u_k(\tau)|^2 \quad (65)$$

is the spectral pressure. By (51), it also has the decomposition

$$p_k(\tau) = \frac{k^5}{3\pi^2 a^4}[|v_k|^2 + 2\mathrm{Re}(\alpha_k\beta_k^* v_k^2) + 2|\beta_k|^2|v_k|^2]. \quad (66)$$

At $f > 10^{11}$ Hz, the vacuum term $|v_k|^2 \propto k^{-1},\ k^{-3},\ k^{-5}$ dominates, the cross term

$$\mathrm{Re}[\alpha_k\beta_k^* v_k^2] \propto k^{-3}, k^{-5}, \quad (67)$$

and

$$|\beta_k|^2|v_k|^2 \propto k^{-5}, \quad (68)$$

so that

$$p_k \propto k^4, k^2, k^0. \quad (69)$$

At $f < 10^{11}$ Hz, the gravitons terms dominate. These are illustrated in Fig. 7. Notice that by (59) and (65) holding for the whole range $f > 10^{-18}$ Hz, there is a relation

$$p_k(\tau) \simeq \frac{1}{3}\rho_k(\tau), \quad (70)$$

i.e., $t^\mu{}_\mu(\tau) = 0$. Thus, ρ_k and p_k have the similar shape, as seen in Figs. 6 and 7.

IV. ADIABATIC REGULARIZATION OF DIVERGENCES OF VACUUM

A. Power spectrum

In field theories, divergences can occur in the expectation values of physical quantities, such as the correlation function $\langle 0|h^{ij}h_{ij}|0\rangle$ in Eq. (20) and the energy-momentum tensor $t_{\mu\nu}$ in Eq. (53) of RGW. In Minkowski spacetime, UV divergences of the vacuum can be removed by the normal ordering of the field operators. However, in curved spacetimes, the normal ordering does apply, and the adiabatic regularization [41–46] suits for removing UV divergences of the vacuum. Since the equation of RGW mode $h_k(\tau)$ has the same form as that of a minimally coupling, massless scalar field, the regularization for the scalar field [41,44,45] can be directly applied to RGW here. From (52), we know that the power spectrum has respective quadratic and logarithmic divergences. By the minimal subtraction rule [41,45], only these two divergent parts are to be removed from the spectrum, and the second adiabatic order subtraction is sufficient, and one should not use the fourth adiabatic order as claimed in Ref. [53]. On the other hand, from (63) and (69), the energy density and pressure contain quartic divergences besides the quadratic and logarithmic ones, so that one should use the fourth adiabatic order as required by the minimal subtraction rule [41,45].

We remark that UV divergences should not be simply dropped out as asserted in Ref. [62]. Moreover, since RGW is regarded as a quantum field, one can not remove UV divergences by applying some smoothing technique, such as a window function, which is often used for classical, stochastic fields.

There is an issue of infrared divergence. The spectrum of (22) in low frequency limit $k \to 0$ behaves as $\Delta_t^2 \propto k^{2\beta+4}$ and will also lead to the infrared divergence in the correlation function. The adiabatic regularization has been developed, aiming at removing the UV divergence of vacuum, not the IR divergence. We shall not discuss the issue in this paper. See Refs. [9,81,82] for further discussions.

To an adiabatic nth order, the mode as a solution to Eq. (10) can be formally written as a general WKB function [41,44,45]

$$u_k^{(n)}(\tau) = \frac{1}{\sqrt{2W^{(n)}(\tau)}}\exp\left[-i\int_{\tau_0}^{\tau} W^{(n)}(\tau')d\tau'\right], \quad (71)$$

where $W^{(n)}(\tau)$ is a function for the adiabatic nth order. For the massless minimally coupled scalar field [44], the zeroth order $W_k^{(0)} = k$, and the second order

$$W_k^{(2)} = \sqrt{k^2 - \frac{a''}{a}}, \quad (72)$$

and the nth adiabatic order [83]

$$W_k^{(n)} = \sqrt{k^2 - \frac{a''}{a} - \frac{1}{2}\left[\frac{W_k^{(n-2)''}}{W_k^{(n-2)}} - \frac{3}{2}\left(\frac{W_k^{(n-2)'}}{W_k^{(n-2)}}\right)^2\right]}. \quad (73)$$

These formulas apply to RGW also. The regularized spectrum is given by [48]

$$\Delta_t^2(k,\tau)_{re} = A^2 \frac{k^3}{\pi^2 a^2}(|u_k(\tau)|^2 - |u_k^{(2)}(\tau)|^2), \quad (74)$$

where

$$|u_k^{(2)}|^2 = \frac{1}{2W_k^{(2)}} = \frac{1}{2k}\left(1 - \frac{a''}{ak^2}\right)^{-\frac{1}{2}} \simeq \frac{1}{2k} + \frac{a''/a}{4k^3} \quad (75)$$

is the counterterm to the second adiabatic order. In principle, the regularization formula (74) can apply at any time τ during expansion. When one chooses different time τ, (74) will give different schemes of regularization. Later in Sec. V, we shall consider three schemes.

As the first example, we apply regularization to the inflation stage. In high-frequency limit $k \to \infty$, the mode $u_k(\tau)$ in Eq. (13) is expanded as

$$\begin{aligned} u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\Bigg(& 1 - i\frac{\beta(\beta+1)}{2k\tau} - \frac{(\beta+2)(\beta+1)\beta(\beta-1)}{8k^2\tau^2} \\ &+ i\frac{(\beta+3)(\beta+2)(\beta+1)\beta(\beta-1)(\beta-2)}{48k^3\tau^3} \\ &+ \frac{(\beta+4)(\beta+3)(\beta+2)(\beta+1)\beta(\beta-1)(\beta-2)(\beta-3)}{384k^4\tau^4}\Bigg) \\ &+ \mathcal{O}(k^{-\frac{11}{2}}), \end{aligned} \quad (76)$$

so that

$$|u_k(\tau)|^2 = \frac{1}{2k}\left(1 + \frac{\beta(\beta+1)}{2k^2\tau^2} + \frac{3(\beta+2)(\beta+1)\beta(\beta-1)}{8k^4\tau^4}\right) + \mathcal{O}(k^{-7}). \quad (77)$$

The first term $1/2k$ is quadratic divergence, corresponding to the usual vacuum fluctuations in Minkowski space. The second term is a logarithmic divergence due to additional vacuum fluctuations in expanding spacetime, which can be written as

$$\frac{\beta(\beta+1)}{4k^2\tau^2} = \frac{R}{24(k/a)^2}, \quad (78)$$

with the scalar curvature $R = 6a''/a^3$. This form agrees with the known result in the R summed and the normal coordinate momentum space methods [45,84–86]. We point out that the logarithmic divergence (78) should not be written as a form $(a'/a)^2/k^2$ of (1.5) in Ref. [51]. These two divergent terms are exactly canceled by the second order adiabatic counterterms $|u_k^{(2)}|^2 = \frac{1}{2k} + \frac{(\beta+1)\beta}{4k^3\tau^2}$ of Eq. (75), giving

$$|u_k|^2 - |u_k^{(2)}|^2 = \frac{3(\beta-1)\beta(\beta+1)(\beta+2)}{16k^5\tau^4} + \mathcal{O}(k^{-7}), \quad (79)$$

which comes from the third term in (77). Thus, the resulting adiabatically regularized spectrum in high frequencies $f > 10^{11}$ Hz is

$$\Delta_t^2(k,\tau)_{re} = A^2 \frac{3(\beta-1)\beta(\beta+1)(\beta+2)}{16\pi^2 a^2 k^2 \tau^4} \propto k^{-2}. \quad (80)$$

We plot (80) for $f > 10^{11}$ Hz in the top part of Fig. 12 for the regularization at the end of inflation τ_1. As for low frequencies $f < 10^{11}$ Hz, the spectrum is less affected by the regularization. When (80) is substituted into (20), it gives a finite contribution to the autocorrelation function from the upper limit of integration. Equation (80) represents the vacuum fluctuations at high frequencies and has definite physical effects. As we shall see in Sec. V, (80) will evolve into the high frequency portion ($f > 10^{11}$ Hz) of the present spectrum and will serve as the target of high-frequency GW detectors [35,36].

We like to clarify two points regarding adiabatic regularization. First, as it stands, the subtraction by the counterterms in (75) applies to the whole frequency range, in contrary to what Ref. [54] suggested only for high frequencies, though its effect on the spectrum is strong for the high k modes and weak for the low k modes. Second, by the the minimal subtraction rule, the above second order regularization is sufficient for the power spectrum [48]. If one tries to do a fourth order adiabatic regularization of the spectrum with the factor $(|u_k|^2 - |u_k^{(2)}|^2)$ in (74) replaced by $(|u_k|^2 - |u_k^{(4)}|^2)$, where the counterterm to the fourth order is defined in (94), the regularized spectrum would be infrared divergent as $\propto k^{-2}$ as $k \to 0$. This is unacceptable. Our calculation confirms that the minimal subtraction rule and a fourth order of adiabatic regularization is incorrect for the power spectrum. This conclusion is just opposite to the claim of Ref. [87].

In the special case of de Sitter inflation with $\beta = -2$, the analytical mode (13) is

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right), \quad (81)$$

and $|u_k(\tau)|^2 = 1/2k + 1/2k^3\tau^2$, which is just equal to the counterterm $|u_k^{(2)}(\tau)|^2$, resulting in a vanishing regularized spectrum $\Delta_T^2(k)_{re} = 0$. This feature of regularization for de Sitter has been pointed out by Parker [48] for a massless scalar field. However, we shall show in Sec. V that regularization at the present time will save the spectrum for the de Sitter.

The next example is for the accelerating stage, in which RGW consists of both vacuum and gravitons. Here consider only the vacuum, whose mode is (45). The calculations are similar to those the inflation stage. One just replaces $(1+\beta)$ by $-\gamma$, and τ by $(\tau-\tau_a)$, in (80), arriving at the regularized vacuum spectrum in high frequencies $k\to\infty$

$$\Delta^2_{\rm vac}(k,\tau)_{re} = A^2\frac{3(\gamma+1)\gamma(\gamma-1)(\gamma-2)}{16\pi^2a^2k^2(\tau-\tau_a)^4}\propto k^{-2}. \quad (82)$$

The divergences of graviton part of the spectrum will addressed in Sec. V. 1.

B. Vacuum energy density and pressure of RGW

The vacuum energy density and pressure of RGW contain quartic, quadratic, and logarithmic divergences as in (63) and (69), which can be removed by the adiabatic regularization to the fourth order [41]. For the spectral energy density, one takes

$$\rho_k(\tau)_{re} = 2\frac{k^3}{2\pi^2a^2}\left(\left|\left(\frac{u_k(\tau)}{a}\right)'\right|^2-\left|\left(\frac{u_k^{(4)}(\tau)}{a}\right)'\right|^2\right), \quad (83)$$

where the counterterm of the fourth adiabatic order is

$$\left|\left(\frac{u_k^{(4)}(\tau)}{a}\right)'\right|^2 = \frac{|u_k^{(4)\prime}|^2}{a^2}+\left(\frac{a'}{a^2}\right)^2|u_k^{(4)}|^2 - \frac{a'}{a^3}(u_k^{(4)*}u_k^{(4)\prime}+u_k^{(4)}u_k^{(4)*\prime}), \quad (84)$$

and the fourth order adiabatic mode is

$$u_k^{(4)}(\tau) = \frac{1}{\sqrt{2W_k^{(4)}(\tau)}}\exp\left[-i\int_{\tau_0}^{\tau}W_k^{(4)}(\tau')d\tau'\right]. \quad (85)$$

Using the formula (73), one calculates (in Refs. [41,43,88] $W_k^{(4)}$ was computed for a massive scalar field),

$$(W_k^{(4)})^2 = k^2-\frac{a''}{a}-\frac{1}{4k^2a^2}\times\left(a''^2-aa''''+2a'a'''-2\frac{a'^2a''}{a}\right), \quad (86)$$

from which follow the terms in (84):

$$\frac{1}{a^2}|u_k^{(4)\prime}|^2 = \frac{W_k^{(4)}}{2a^2}+\frac{(W_k^{(4)\prime})^2}{8(W_k^{(4)})^3a^2} \simeq \frac{k}{2a^2}-\frac{a''/a}{4a^2k}-\frac{1}{16k^3a^4}\times\left(2a''^2-aa''''+2a'a'''-2\frac{a'^2a''}{a}\right) \quad (87)$$

$$\left(\frac{a'}{a^2}\right)^2|u_k^{(4)}|^2 = \left(\frac{a'}{a^2}\right)^2\frac{1}{2W_k^{(4)}}\simeq\left(\frac{a'}{a^2}\right)^2\left(\frac{1}{2k}+\frac{a''/a}{4k^3}\right), \quad (88)$$

$$-\frac{a'}{a^3}(u_k^{(4)*}u_k^{(4)\prime}+u_k^{(4)}u_k^{(4)*\prime}) \simeq \frac{1}{4k^3}\left(\frac{a'^2a''}{a^5}-\frac{a'''a'}{a^4}\right), \quad (89)$$

and the counterterm is

$$\left|\left(\frac{u_k^{(4)}(\tau)}{a}\right)'\right|^2 = \frac{k}{2a^2}+\frac{1}{4a^2k}\left(\frac{2a'^2}{a^2}-\frac{a''}{a}\right) + \frac{1}{8a^2k^3}\left(\frac{5a'^2a''}{a^3}-\frac{a''^2}{a^2}+\frac{a''''}{2a}-\frac{3a'a'''}{a^2}\right). \quad (90)$$

Substituting these into (83) gives the regularized spectral energy density. In a similar fashion for the spectral pressure, one takes

$$p_k(\tau)_{re} = \frac{k^5}{3\pi^2a^4}(|u_k(\tau)|^2-|u_k^{(4)}|^2), \quad (91)$$

where

$$|u_k^{(4)}|^2 = \frac{1}{2W_k^{(4)}}\simeq\frac{1}{2k}+\frac{a''/a}{4k^3}+\frac{1}{16k^5a^2}\times\left(4a''^2-aa''''+2a'a'''-\frac{2a'^2a''}{a}\right) \quad (92)$$

is the fourth order adiabatic counterterm.

The above regularization formulas hold for any time τ. First, apply to the inflation stage, during which the energy density and pressure of RGW have only the vacuum contributions. Using the mode u_k of (13), one has

$$\left|\left(\frac{u_k(\tau)}{a}\right)'\right|^2 = \frac{1}{a^2}\left[\frac{k}{2} + \frac{(\beta+1)(\beta+2)}{4k\tau^2} + \frac{3\beta(\beta+1)(\beta+2)(\beta+3)}{16k^3\tau^4} + \frac{5(\beta-1)\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)}{32k^5\tau^6}\right] + O(k^{-7}),$$

and, by (90), the counterterm is

$$\left|\left(\frac{u_k^{(4)}(\tau)}{a}\right)'\right|^2 = \frac{1}{a^2}\left[\frac{k}{2} + \frac{(\beta+1)(\beta+2)}{4k\tau^2} + \frac{3\beta(\beta+1)(\beta+2)(\beta+3)}{16k^3\tau^4}\right],$$

which just cancels the quartic, quadratic, and logarithmic divergences of $|(u_k(\tau)/a)'|^2$, yielding the regularized spectral energy density in the high-frequency limit

$$\rho_k(\tau)_{re} = \frac{5(\beta-1)\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)}{32\pi^2 a^4 k^2 \tau^6} + O(k^{-4}). \tag{93}$$

For the pressure, similar calculations give

$$|u_k(\tau)|^2 = \frac{1}{2k} + \frac{\beta(\beta+1)}{4k^3\tau^2} + \frac{3(\beta-1)\beta(\beta+1)(\beta+2)}{16k^5\tau^4} + \frac{5(\beta-2)(\beta-1)\beta(\beta+1)(\beta+2)(\beta+3)}{32k^7\tau^6} + O(k^{-8}),$$

and

$$|u_k^{(4)}(\tau)|^2 = \frac{1}{2k} + \frac{\beta(\beta+1)}{4k^3\tau^2} + \frac{3(\beta-1)\beta(\beta+1)(\beta+2)}{16k^5\tau^4}, \tag{94}$$

and the regularized spectral pressure in the high-frequency limit

$$p_k(\tau)_{re} = \frac{5(\beta-2)(\beta-1)\beta(\beta+1)(\beta+2)(\beta+3)}{3\times 32\pi^2 a^4 k^2 \tau^6} + O(k^{-4}). \tag{95}$$

The expressions (93) and (95) hold only at $f > 10^{11}$ Hz. The low frequency $f < 10^{11}$ Hz parts of $\rho_k(\tau)_{re}$ and $p_k(\tau)_{re}$ are less affected by adiabatic regularization. Equations (93) and (95) tell that ρ_{kre}, $p_{kre} \propto k^{-2}$ at high frequencies, and the relation (70) is modified to

$$p_k(\tau)_{re} = \frac{1}{3}\frac{\beta-2}{\beta+4}\rho_k(\tau)_{re}, \tag{96}$$

i.e., there is a trace anomaly of regularized RGW,

$$t^{\mu}{}_{\mu}(\tau)_{re} = \frac{6}{(\beta+4)}\rho_k(\tau)_{re} \neq 0 \tag{97}$$

at high frequencies. This situation of anomaly is similar to what happens to a conformally coupling massless scalar field after regularization [45,46,76,89]. Notice that (93) and (95) give $\rho_{kre} > 0$ and $p_{kre} < 0$ for $\beta < -2$, which is similar to the inflaton field that drives inflation. However, the magnitude of vacuum fluctuations are very small, $\rho_{kre}/\rho \sim (H/M_{\rm Pl})^2 \sim 10^{-16}$, where ρ is the inflation energy scale with $\rho^{1/4} \sim 10^{15}$ Gev. In passing, we notice that a negative pressure of quantum fields at infrared ranges also arises in the context discussed in Ref. [81].

Next apply to the accelerating stage. Consider the vacuum part of the energy density and pressure, i.e., the $v_k(\tau)$ parts of (60) and (66). Just replacing $(\beta+1) \to -\gamma$ in (93) and (95), one obtains the regularized energy density and pressure of the present vacuum at high frequencies $f > 10^{11}$ Hz

$$\rho_k(\tau)_{vre} = \frac{5(\gamma-3)(\gamma-2)(\gamma-1)\gamma(\gamma+1)(\gamma+2)}{32\pi^2 a^4 k^2 (\tau-\tau_a)^6} + O(k^{-4}), \tag{98}$$

$$p_k(\tau)_{vre} = \frac{5(\gamma-2)(\gamma-1)\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{3\times 32\pi^2 a^4 k^2 (\tau-\tau_a)^6}. + O(k^{-4}). \tag{99}$$

The accelerating model can be fitted by $\gamma \simeq 2.1$, (98) and (99) give $\rho_{kre} < 0$ and $p_{kre} > 0$ for $\gamma > 2$. However, as will be seen in the next section, (98) and (99) from the vacuum are overwhelmed by those of gravitons, so that the total spectral energy density and pressure of RGW are positive at high frequencies.

V. REGULARIZATION AT DIFFERENT TIME

As defined in Eq. (21), the spectrum $\Delta_t(k,\tau)$ depends on time. At what time τ should regularization be performed? In literature, there have been disagreements in the context of a scalar inflaton [50,51,54], and the issue also exists for RGW. In the following, we shall regularize in three methods, respectively: at the time of observation, at the end of inflation [54], and at the horizon crossing time for each k mode [50,51]. As it turns out, the three resulting regularized spectra are quite similar in regard to observations.

A. Regularization at the present time

This is the first scheme of regularization. Suppose an observer is to detect RGW now, and it is natural to regularize the spectrum at the present time τ_H of observation during the accelerating stage. To do this, one just sets $\tau = \tau_H$ in the expression (74) of the regularized spectrum

$$\Delta_t^2(k,\tau_H)_{re} = A^2 \frac{k^3}{\pi^2 a(\tau_H)^2}(|u_k(\tau_H)|^2 - |u_k^{(2)}(\tau_H)|^2), \tag{100}$$

where the mode function $u_k(\tau_H)$ is explicitly given by Eq. (39) and the counterterm $|u_k^{(2)}(\tau_H)|^2$ by (75). Using (51) at τ_H, one has

$$|u_k|^2 - |u_k^{(2)}|^2 = (|v_k|^2 - |u_k^{(2)}|^2) + 2\mathrm{Re}[\alpha_k \beta_k^* v_k^2] + 2|\beta_k|^2 |v_k|^2, \tag{101}$$

where $(|v_k|^2 - |u_k^{(2)}|^2) \propto k^{-5}$ is the regularized vacuum part, already known in (79), and the graviton part is unaffected by the adiabatic regularization. The regularized spectrum $\Delta_t(k,\tau_H)_{re}$ of (100) is plotted in Fig. 8 for $\beta = -2.0125$. The nonvanishing spectrum of de Sitter case $\beta = -2$ is plotted in Fig. 9, which, as mentioned earlier, would be vanishing if the regularization is performed during inflation [48].

After adiabatic regularization, the logarithmic divergence due to gravitons of $2\mathrm{Re}[\alpha_k \beta_k^* v_k^2]$ term in (101) still remains, as is indicated by a flat curve at $f > 10^{11}$ Hz in Figs. 8 and 9. It is well-known that the occurrence of this kind of UV divergence is caused by the discontinuity of $a''(\tau)$ between two adjacent stages of the model, thus can be removed by choices of continuous $a''(\tau)$. L. Ford [8]

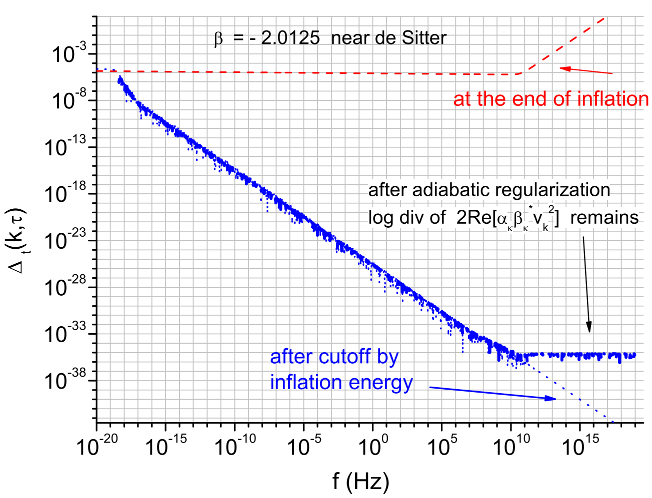

FIG. 8. Regularization at the present time τ_H. The unregularized spectrum at the end of inflation is at the top, and the present spectrum is at the lower part.

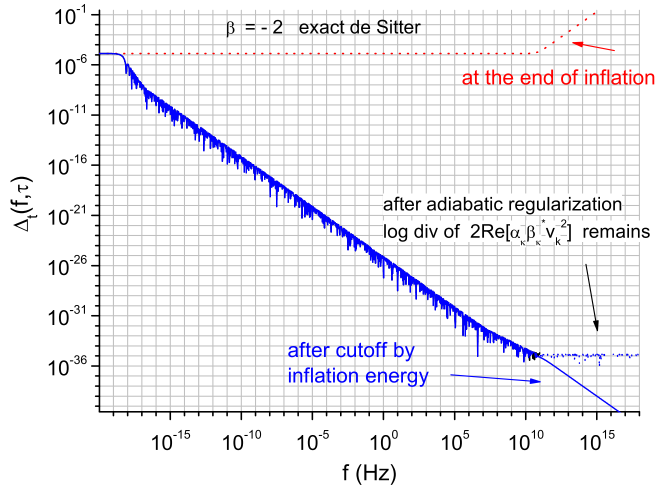

FIG. 9. The spectrum regularized at present is nonvanishing for $\beta = -2$, which would be zero if regularized during inflation [48].

demonstrated this by an explicit example, in which a finite time duration Δt of the transition is assumed, and some smooth $a(\tau)$ with continuous a'' is constructed. These resulted in a graviton number density $\propto \ln(\frac{1}{H\Delta t})$, which is finite. In our model, the discontinuity of a'' corresponds to an abrupt transition with $\Delta t = 0$, so there is no surprise that UV divergences appear in the spectrum, graviton number density, energy density, pressure, etc. To remove this artificial divergence, Ref. [9] proposed that gravitons are not produced with higher energy than the inflation energy scale, say $\sim 10^{16}$ GeV. This yields a cutoff of the logarithmic divergence of gravitons at $f > 10^{11}$ Hz. Here, we adopt this simple treatment. Thus, after adiabatic regularization and cutoff as well, the spectrum becomes convergent, $\Delta_t^2(k,\tau_H)_{re} \propto k^{-2}$ for $f > 10^{11}$ Hz plotted as a dotted line in the lower left part in Figs. 8 and 9, respectively.

For $f < 10^{11}$ Hz, the spectrum is contributed by gravitons and essentially unchanged by regularization and cutoff. In particular, the primordial spectrum defined at the low frequency end $f < 10^{-18}$ Hz remains the same as (28). In a manner similar to (30) and (31), one can also define the regularized spectral indices

$$n_{tre} \equiv \frac{d\ln \Delta_t^2(k)_{re}}{d\ln k}\bigg|_{k\to 0} = 2\beta + 4 \tag{102}$$

and

$$\alpha_{tre} \equiv \frac{d^2 \ln \Delta_t^2(k)_{re}}{d(\ln k)^2}\bigg|_{k\to 0} = 0, \tag{103}$$

both have the same values as those in (30) and (31), respectively.

Now the energy density and pressure at present. The spectral energy density is adiabatically regularized to the fourth order by

$$\rho_k(\tau_H)_{re} = \frac{k^3}{\pi^2 a^2}\left(\left|\left(\frac{v_k}{a}\right)'\right|^2 - \left|\left(\frac{u_k^{(4)}}{a}\right)'\right|^2 + 2\mathrm{Re}\left[\alpha_k\beta_k^*\left(\frac{v_k}{a}\right)'^2\right] + 2|\beta_k|^2\left|\left(\frac{v_k}{a}\right)'\right|^2\right), \tag{104}$$

where $|(\frac{v_k}{a})'|^2 - |(\frac{u_k^{(4)}}{a})'|^2$ is the regularized vacuum part, known in (98) at high frequencies, and the regularized spectral pressure is

$$p_k(\tau_H)_{re} = \frac{k^5}{3\pi^2 a^4}((|v_k|^2 - |u_k^{(4)}|^2) + 2\mathrm{Re}(\alpha_k\beta_k^* v_k^2) + 2|\beta_k|^2|v_k|^2), \tag{105}$$

where $(|v_k|^2 - |u_k^{(4)}|^2)$ is known in (99) at high frequencies. The divergences of ρ_k and p_k due to gravitons have been known in (61), (62), (67), and (68), which, unaffected by regularization, will be cutoff for $f > 10^{11}$ Hz by the same argument of the inflation energy as for the power spectrum. After cutoff, the graviton part of ρ_{kre} and p_{kre} at $f > 10^{11}$ Hz are given by the following leading terms:

$$\begin{aligned}\rho_{kgr} &= 3p_{kgr} \\ &\simeq \frac{k^4}{\pi^2 a^4}\left[\frac{\beta(\beta+1)}{4x_1^2}\left(\frac{(\beta-1)(\beta+2)}{4x_1} + \frac{\beta_s(\beta_s+1)}{4t_1}\right) - \frac{\beta_s(\beta_s+1)}{4t_1^2}\left(\frac{\beta(\beta+1)}{4x_1} + \frac{(\beta_s-1)(\beta_s+2)}{4t_1}\right)\right]^2 \\ &\propto k^{-2},\end{aligned} \tag{106}$$

where $x_1 \equiv k|\tau_1|$ and $t_1 \equiv k|\tau_1 - \tau_p|$. In comparison, this is many orders higher than the present vacuum energy (98). Thus, the total spectral energy after regularization and cutoff is positive for $f > 10^{11}$ Hz. Over the broad range $(10^{-18}\text{–}10^{11})$ Hz, ρ_{kre}, p_{kre} are dominated by gravitons and remain practically unchanged after regularization. The final ρ_{kre}, p_{kre} after regularization and cutoff are plotted in Figs. 10 and 11, respectively.

B. Regularization at the end of inflation

Next, we explore the second scheme of regularization, in which the spectrum of RGW is regularized at the time τ_1, the end of inflation [54]. Moreover, we shall also let the associated, regularized mode $u_k^{re}(\tau)$ evolve subsequently according to its field equation and arrive at the present spectrum at the time τ_H. Since the regularized spectrum vanishes for exact de Sitter inflation, we consider the general case $\beta \neq -2$. We apply the general formula (74) of regularized spectrum at the end of inflation with $\tau = \tau_1$,

$$|u_k^{re}(\tau_1)|^2 = |u_k(\tau_1)|^2 - |u_k^{(2)}(\tau_1)|^2, \tag{107}$$

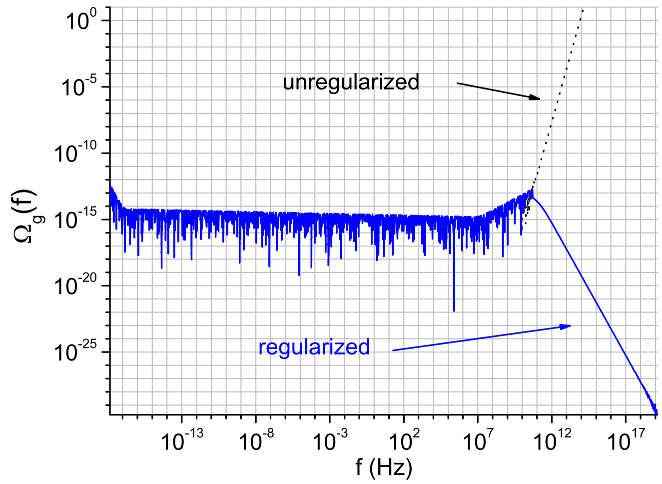

FIG. 10. Regularized spectral energy density $\Omega_g(k)_{re} = \rho_k(\tau_H)_{re}/\rho_c$ at present.

which has been known in Eq. (80). This fixes the amplitude of regularized mode $u_k^{re}(\tau_1)$ at $\tau = \tau_1$ as the initial condition. To determine its phase, assume that $u_k^{re}(\tau_1)$ has the same phase as the unregularized mode $u_k(\tau_1)$, which has the following asymptotic behavior:

$$\begin{aligned} u_k(\tau_1) &\simeq \frac{e^{-ik\tau_1}}{\sqrt{2k}}, \quad \text{for } k \to \infty, \\ u_k(\tau_1) &\simeq \frac{\sqrt{-\pi\tau_1}e^{i(\beta-1)\pi/2}}{2\Gamma(\beta+\frac{3}{2})\cos\beta\pi}\left(\frac{k\tau_1}{2}\right)^{\beta+\frac{1}{2}} + O(k^{\beta+\frac{3}{2}}) \quad \text{for } k \to 0. \end{aligned} \tag{108}$$

So we choose the initial condition at τ_1 to be

$$u_k^{re}(\tau) = e^{i\theta}\sqrt{|u_k(\tau)|^2 - \frac{1}{2k}\left(1 + \frac{\beta(\beta+1)}{2k^2\tau^2}\right)}, \qquad \tau = \tau_1, \tag{109}$$

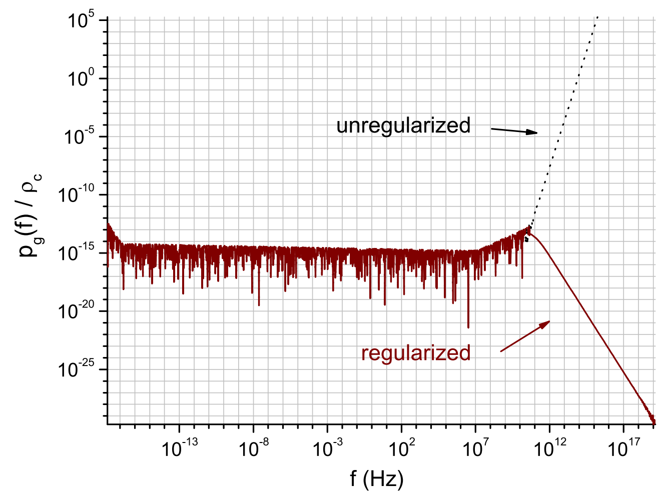

FIG. 11. Regularized spectral pressure $p_k(\tau_H)_{re}/\rho_c$ at present.

where the phase θ is approximately chosen as

$$\theta = \begin{cases} -k\tau & \text{for } k > -\tau_1^{-1} \\ (\beta-1)\pi/2 & \text{for } k < -\tau_1^{-1}, \end{cases} \quad (110)$$

and the time derivative is given by $u_k^{re'}(\tau) = \frac{d}{d\tau} u_k^{re}(\tau)$ at $\tau = \tau_1$. The modes $u_k(\tau)$ before τ_1 during inflation remain the same as that in Eq. (13). The subsequent evolution of $u_k(\tau)$ follows straightforwardly, from reheating, radiation, matter, to the accelerating stage. As has been checked, other choices of θ will give the same outcome of spectrum at present. Although the phase of mode contains information of the quantum state of RGW, in addition to the spectrum [1], the current detectors, such as LIGO, are not capable of detecting such a phase information [90]. The above treatment of the phase is sufficient for our purpose since we are mainly concerned with the spectrum.

The regularized, initial spectrum at the end of inflation, and the present spectrum evolved from the former are plotted in Fig. 12. Notice that the regularized spectrum at present behaves as $\Delta_t^2(k)_{re} \propto f^{-2}$ at high frequencies $f > 10^{11}$ Hz. Thus, all of the UV divergences of the power spectrum have been removed by this scheme of adiabatic regularization, including the quadratic and logarithmic divergences of vacuum and the logarithmic divergence of gravitons as well. No additional cutoff is needed, in contrast to the first scheme in the last Sec. V 1. Comparing with the first scheme, the amplitudes differ by about 20 around $f \sim 10^{11}$ Hz, and by about 1.2 around $f \sim 10^0$ Hz. At the low frequency end $f \sim 10^{-18}$ Hz, the primordial spectrum, the indices n_t, and α_t remain unchanged, as in (28), (30), and (31), respectively. The resulting spectra at present in these two schemes are essentially the same.

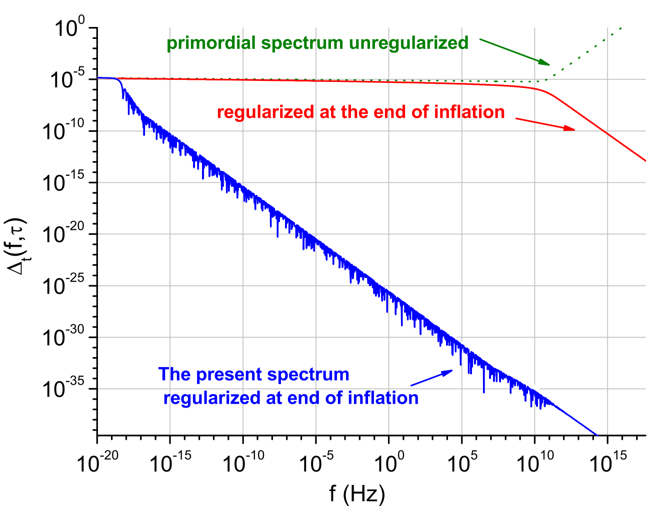

FIG. 12. The spectrum is regularized at the end of inflation, then evolves into the present spectrum, which turns out to be almost the same as that of Fig. 8 regularized at the present time.

C. Regularization at horizon exit

Finally, we explore the regularization at the horizon exit proposed in Refs. [50,51]. Again we consider the case $\beta \neq -2$. The low-frequency modes of RGW exit the horizon during inflation at a time $|\tau_k| = 1/k$, and regularization is performed at τ_k for each mode. So in this scheme the regularization time is not instantaneous but rather at different τ_k for different modes. By the formula of (74) during inflation, one obtains directly

$$\Delta_t^2(k)_{re} = C(\beta)\frac{k^{2\beta+4}}{\pi^2 M_{\rm Pl}^2 l_0^2} \quad \text{for } k < \frac{1}{|\tau_1|}, \quad (111)$$

with $C(\beta) \equiv \pi |H^{(2)}_{\frac{1}{2}+\beta}(1)|^2 - (2+\beta(\beta+1))$. This can be rewritten as

$$\Delta_t^2(k)_{re} = C^2 \Delta_t^2(k) \quad \text{for } k < \frac{1}{|\tau_1|}, \quad (112)$$

where $\Delta_t^2(k)$ is the primordial spectrum in (28) and

$$C^2 \equiv 2^{2\beta+1}\pi^{-1}\Gamma\left(\beta+\frac{3}{2}\right)^2 \cos^2(\beta\pi) C(\beta) \simeq 0.904 \cdot |2+\beta|, \quad (113)$$

depending on β, and $C^2 \simeq 0.01$ for $\beta = -2.0125$. For the modes that have exited the horizon, the subsequent evolution of the regularized modes is given by

$$u_k^{re}(\tau) = C u_k(\tau) \quad \text{for } k < \frac{1}{|\tau_1|}. \quad (114)$$

On the other hand, the high-frequency modes with $k > 1/|\tau_1|$ never exit the horizon during inflation. It is these modes that give rise to the UV divergence of the spectrum. References [50,51] did not give a treatment of these modes. We regularize these modes at the end of inflation $\tau = \tau_1$,

$$u_k^{re}(\tau) = e^{-ik\tau}\sqrt{|u_k(\tau)|^2 - \frac{1}{2k}\left(1+\frac{\beta(\beta+1)}{2k^2\tau^2}\right)} \quad \text{for } k > 1/|\tau_1|, \quad (115)$$

and the time derivative of the modes is given by $u_k^{re'}(\tau) = \frac{d}{d\tau} u_k^{re}(\tau)$ at $\tau = \tau_1$. Thus, the primordial spectrum regularized in the third scheme is

$$\Delta_t^2(\tau_1,k)_{re} = \begin{cases} 2A^2 \frac{k^3}{2\pi^2 a^2} |C u_k(\tau_1)|^2 & \text{for } k < 1/|\tau_1| \\ 2A^2 \frac{k^3}{2\pi^2 a^2} [|u_k(\tau_1)|^2 - |u_k^{(2)}(\tau_1)|^2] & \text{for } k > 1/|\tau_1| \end{cases}, \quad (116)$$

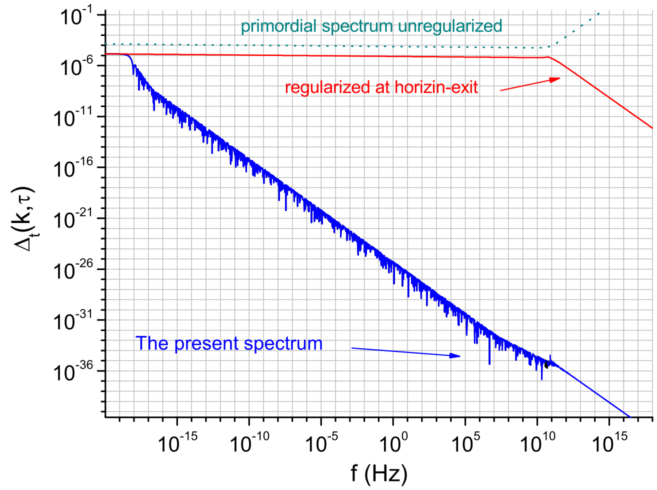

FIG. 13. The spectrum is regularized at the horizon exit, then evolves into the present spectrum, which is similar to that of Figs. 8 and 12, but with a lower, overall amplitude.

as shown in the upper part of Fig. 13. Given the initial condition (115), the subsequent evolution of the high-frequency modes $u_k(\tau)$ is also determined. The regularized, present spectrum $\Delta_t^2(k,\tau_H)_{re}$ is shown in the lower part of Fig. 13. The profile is similar to those in the first and second schemes, but with the amplitude lower by the factor C^2. At the low-frequency end,

$$\Delta_t^2(k,\tau_H)_{re} = C^2\Delta_t^2(k) \quad \text{for } k < 1/|\tau_1|, \tag{117}$$

where $\Delta_t^2(k)$ is the primordial spectrum in (28), and $\Delta_t^2(k) \propto H^2 \propto \rho$, where ρ is the energy density of inflation. Since the amplitude will be eventually fixed by CMB observations as in (37), one just raises ρ of the model by a factor $1/C^2$ to achieve the same amplitude as in the first and second schemes.

Although (117) formally resembles the result of slow-roll approximation in Ref. [50], nevertheless, C^2 is independent of k here. Therefore, the spectral indices n_t and α_t are unaffected by this regularization and remain the same as in Eqs. (30) and (31). We emphasize that no slow-roll approximation is involved in our result.

VI. REGULARIZATION OF PRIMORDIAL SPECTRA OF INFLATON AND SCALAR CURVATURE PERTURBATION

The above result of RGW and regularization for the inflation stage can be extended to the scalar metric perturbation via an inflaton field. The scale factor $a(\tau) \propto |\tau|^{1+\beta}$ in (9) has been taken as a generic inflation model so far. It can be specifically realized by a single scalar inflaton field, also called the power-law inflation [7,91–93]. In this case, the exact solution of perturbed scalar field is the same form as RGW. For inflation driven by a scalar field ϕ, the Friedmann equations are

$$\phi'' + 2\mathcal{H}\phi' + a^2\frac{\partial V}{\partial \phi} = 0, \tag{118}$$

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\phi'^2 + a^2 V\right), \tag{119}$$

where $\mathcal{H} = \frac{a'}{a}$ and V is the potential. For $a(\tau) \propto |\tau|^{1+\beta}$, the solution of Eqs. (118) and (119) is

$$\frac{\phi}{M_{\rm Pl}} = \sqrt{2(1+\beta)(2+\beta)}\,\ln(-\tau) + B,$$
$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2\beta+4}{\beta+1}}\frac{\phi}{M_{\rm Pl}}\right). \tag{120}$$

The slow-roll parameters are defined by

$$\epsilon \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'(\phi)}{V}\right)^2 = \frac{\beta+2}{\beta+1} \qquad \eta \equiv M_{\rm Pl}^2\frac{V''(\phi)}{V} = 2\epsilon. \tag{121}$$

Consider the perturbed scalar field $\delta\phi$, which generally depends on the choice of coordinates. In fact, from $\delta\phi$ and scalar metric perturbations, one can construct a gauge-invariant perturbed scalar field $\overline{\delta\phi}$, satisfying the field equation [94–96]

$$\overline{\delta\phi}'' + 2\mathcal{H}\overline{\delta\phi}' - \nabla^2\overline{\delta\phi} + \left[a^2\frac{\partial^2 V}{\partial\phi^2} - \frac{8\pi G}{a^2}\frac{d}{d\tau}\left(\frac{a^2\phi'^2}{\mathcal{H}}\right)\right]\overline{\delta\phi} = 0, \tag{122}$$

which holds in any coordinates. For $a(\tau) \propto |\tau|^{1+\beta}$, the bracket in Eq. (122) vanishes, so that the equation reduces to

$$w_k'' + \left(k^2 - \frac{a''}{a}\right)w_k = 0, \tag{123}$$

where $\overline{\delta\phi}_k \equiv w_k/a$ for each k mode. Equation (123) is identical to Eq. (10) of RGW, and its solution w_k is the same as u_k in (13). Thus, we obtain the spectrum of $\overline{\delta\phi}$

$$\Delta_{\overline{\delta\phi}}^2(k,\tau) = \frac{k^3}{2\pi^2}|\overline{\delta\phi}_k|^2 = \frac{k^3}{2\pi^2 a^2}|w_k|^2 = \frac{M_{\rm Pl}^2}{8}\Delta_t^2(k,\tau), \tag{124}$$

where $\Delta_t^2(k,\tau)$ is the RGW spectrum in (22). The spectral indices are given $n_{\overline{\delta\phi}} - 1 \equiv \frac{d\ln\Delta_{\overline{\delta\phi}}^2(k)}{d\ln k}|_{k\to 0} = 2\beta+4$, $\alpha_{\overline{\delta\phi}} \equiv \frac{d^2\ln\Delta_{\overline{\delta\phi}}^2(k)}{d(\ln k)^2}|_{k\to 0} = 0$. One is more interested in scalar metric perturbations, which are directly related to observations. Introducing the curvature perturbation $R \equiv \frac{\mathcal{H}}{\phi'}\overline{\delta\phi}$, which is also gauge invariant [97–100], the spectrum of scalar curvature perturbation is given by

$$\Delta_R^2(k,\tau) = \frac{k^3}{2\pi^2}|R_k|^2 = \frac{k^3}{2\pi^2 a^2}\frac{\mathcal{H}^2}{\phi'^2}|w_k|^2 = \frac{1+\beta}{8(2\beta+4)}\Delta_t^2(k,\tau), \quad (125)$$

where $\frac{\mathcal{H}}{\phi'} = \frac{1}{M_{\rm Pl}}\sqrt{\frac{1+\beta}{2\beta+4}}$ is a constant in our model. The primordial spectrum of R relevant to CMB observations is defined at the far outside horizon,

$$\Delta_R^2(k) \equiv \Delta_R^2(k,\tau)|_{k\ll 1/|\tau|} = \frac{1}{2\epsilon M_{\rm Pl}^2} a_t^2\left(\frac{H}{2\pi}\right)^2 k^{2\beta+4}. \quad (126)$$

The scalar spectral index is given by

$$n_s - 1 \equiv \frac{d\ln\Delta_R^2(k)}{d\ln k}|_{k\to 0} = 2\beta + 4 = \frac{-6\epsilon}{1-\epsilon} + \frac{2\eta}{1-\eta/2} \quad (127)$$

differing from the result of slow-roll approximation $n_s = 1 - 6\epsilon + 2\eta$ in Ref. [61], and the scalar running index is

$$\alpha_s \equiv \frac{d^2\ln\Delta_R^2(k)}{d(\ln k)^2}|_{k\to 0} = 0, \quad (128)$$

differing from the result $\alpha_s = -16\epsilon\eta + 24\epsilon^2 + 2\xi^2$ with $\xi^2 \equiv M_{pl}^4\frac{V'V'''}{V^2}$ in slow-roll approximation in Ref. [63]. The tensor-scalar ratio is

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_R^2(k)} = 16\frac{\beta+2}{\beta+1} = 16\epsilon, \quad (129)$$

and the consistency relation is

$$r = \frac{-8n_t}{1-n_t/2}. \quad (130)$$

Both (129) and (130) are valid for the whole relevant range of k. The adiabatic regularization in (74) for RGW is directly adopted here, yielding the regularized scalar curvature spectrum,

$$\Delta_R^2(k,\tau)_{re} = \frac{1+\beta}{2\beta+4}\frac{k^3}{2\pi^2 a^2 M_{\rm Pl}^2}\left(|w_k|^2 - \frac{1}{2k} - \frac{a''/a}{4k^3}\right). \quad (131)$$

For regularization at the end of inflation, similar to Sec. V 2 for RGW, the primordial scalar spectrum, n_s, α_s, and r all remain unchanged. For regularization at the horizon exit, the spectrum becomes

$$\Delta_R^2(k)_{re} = C^2\Delta_R^2(k) \quad \text{for } k \ll 1/|\tau_1|, \quad (132)$$

with C^2 given by Eq. (113), thus n_s, α_s, and r remain unchanged. These results differ from that of the slow-roll approximation in Refs. [50,51].

VII. CONCLUSION AND DISCUSSION

Three aspects of RGW have been studied in this paper: the analytic spectrum and spectral indices, the decomposition of the present RGW into vacuum and gravitons, and the removal of UV divergences of the spectrum, energy density, and pressure, arising from both vacuum and gravitons. Similarly, regularization during inflation is also performed for the gauge-invariant perturbed inflaton and the scalar curvature perturbation.

The analytical $\Delta_t^2(k,\tau)$, n_t, α_t, and r that have been obtained are valid at any time and frequency during inflation. The exact relations involving the indices in (34), (127), and (130) differ from those in the slow-roll approximation and can be tested by future CMB observations. In particular, using the observed spectra C_l^{XX} of CMB in $l \simeq (10\text{–}3000)$, $\Delta_t^2(k,\tau)$ is actually constrained at $k|\tau_1| \sim (10^{-28}\text{–}10^{-26})$, far outside the horizon during inflation. This is where the primordial spectrum (28) is referred. The spectrum at the horizon exit $k|\tau| = 1$ differs drastically from the primordial one.

The present RGW as a quantum field in curved spacetime actually consists of the vacuum and graviton parts. This decomposition sheds light on the structure of the spectrum over the respective frequency ranges. The present vacuum gives quadratic, logarithmic UV divergences to $\Delta_t^2(k,\tau_H)$, and the gravitons give a logarithmic divergence, at the high-frequency end $f > 10^{11}$ Hz. However, at $f < 10^{11}$ Hz, gravitons are dominant, and the current detectors are all operating within this band. In this sense, these detectors are to detect gravitons, not the present vacuum of RGW. The graviton number density $|\beta_k|^2$ is contributed by all four discontinuity points of $a''(\tau)$, among which the inflation-reheating transition is the greatest.

In removing UV divergences of RGW, we have carried out regularization in three schemes: at the present time, at the end of inflation, and at the horizon exit during inflation. The first scheme actually involves two parts. The adiabatic regularization removes only the divergences of present vacuum, and the divergences of gravitons are cut off. The last two schemes remove the vacuum divergences during inflation, the regularized spectrum is then taken as the initial condition and evolves into the present spectrum, which is convergent. Besides, for the spectrum, the second order adiabatic regularization is sufficient to remove UV divergences, and the fourth adiabatic order is not used according to the minimal subtraction rule. In all of these three schemes, the regularized, present spectra are similar, except that the third scheme yields a lower amplitude, which can be raised by a higher inflation energy scale in confronting observational data. At high frequencies $f > 10^{11}$ Hz, the three regularized spectra behave as $\Delta_t^2(k,\tau_H)_{re} \propto k^{-2}$, which can serve as a target for high-frequency GW detectors, such as the polarized, Gaussian laser beam detectors proposed in Refs. [35,36]. At $f < 10^{11}$ Hz, the regularized spectrum remains practically unchanged.

We also calculate the spectral energy density and pressure of RGW. The vacuum ρ_k and p_k contain quartic, quadratic, and logarithmic divergences, and the regularization to fourth adiabatic order is necessary and sufficient to remove them. The regularized vacuum $\rho_{k,re} > 0$ and $p_{kre} < 0$ at high frequencies for inflation with $\beta < -2$, which is implied by the current observations. For the present accelerating stage with $\gamma \simeq 2.1$, the vacuum $\rho_{k,re} < 0$ and $p_{kre} > 0$. The graviton part of ρ_k and p_k at present contain only quadratic and logarithmic divergences, which are cut off by the inflation energy scale, yielding $\rho_{kgr} = 3p_{kgr} \propto k^{-2}$ at $f > 10^{11}$ Hz, greater than those of vacuum by many orders. At $f < 10^{11}$ Hz, ρ_k and p_k are practically unchanged by regularization and cutoff. Hence, the total spectral energy density and pressure, after regularization and cutoff, are dominantly contributed by gravitons over the whole frequency range.

Now we give our assessment of the three schemes of regularization. During the course of cosmic expansion, the actual, physical spectrum is described by the instantaneously regularized $(|u_k(\tau)|^2 - |u_k^{(2)}(\tau)|^2)$ for any instance τ. This instantaneous regularization is closest to the instantaneous normal ordering for quantum field in flat spacetime. The two schemes, at the present time or at the end of inflation, are different demonstrations of this same physical quantity at respective instances. Moreover, as our work has shown, the resulting two spectra for observation at present are nearly the same. So we think that these two schemes are a better choice than the one at the horizon exit, since the latter is not instantaneous but at different times for different k modes. Among the two instantaneous schemes, we would like to remark that, in regard to technical convenience, the first scheme, i.e., regularization at the present time for observation, is a natural choice since it is simpler than the one at the end of inflation.

For the scalar field driving the power-law inflation, the gauge invariant perturbed scalar field has an exact solution which is the same as RGW, so does the scalar curvature perturbation. The regularization is the same as for RGW, in particular, the scalar spectral indices n_s and α_s, tensor-scalar ratio r remain unchanged.

Our study has an advantage that it is based on the exact solutions of RGW for the whole expansion, and of the perturbed inflaton and the scalar curvature perturbation during inflation. Our results have demonstrated, Parker-Fuling's adiabatic regularization and the minimal subtraction rule work perfectly well in removing UV divergences of vacuum, to the second order for the spectrum, to the fourth order for the energy density and pressure, respectively.

ACKNOWLEDGMENTS

Y. Zhang is supported by NSFC Grant No. 11275187, NSFC Grant No. 11421303, SRFDP, and CAS, the Strategic Priority Research Program “The Emergence of Cosmological Structures” of the Chinese Academy of Sciences, Grant No. XDB09000000.

APPENDIX: MODE FUNCTION OF RGW

In this appendix, we list the explicit expressions of mode functions of RGW and the associated coefficients, which are determined analytically from continuous joining of $h_k(\tau)$ and $h'_k(\tau)$ for all five consequent stages of cosmic expansion.

The scale factor $a(\tau)$ in a power-law form is given for all five stages, so that both $a(\tau)$ and its time derivative $a'(\tau)$ are continuous at the four transition points between all five stages. But, a'' is not required to be continuous for simplicity. This simple modeling of $a(\tau)$ works well for our purpose of an exact RGW solution, but the artificial discontinuity of a'' would bring about too much graviton production, as addressed in Sec. V 1.

There is an overall normalization of $a(\tau)$. In this paper, we take $|\tau_H - \tau_a| = 1$. The present Hubble radius $H_0^{-1} = l_H/\gamma = 9.257 \times 10^{28} h^{-1}$ cm with the Hubble parameter $h = 0.69$. This fixes the parameters in the expressions of $a(\tau)$. Furthermore, for the time of the transitions, we use the following cosmological specifications in our computation: The matter-accelerating transition time at a redshift $z \sim 0.347$. The radiation-matter transition taken at $z = 3293$ [30]. The inflation energy scale is taken to 10^{16} Gev, the reheating duration is chosen such that $a(\tau_s)/a(\tau_1) = 300$, so that at the beginning of radiation the energy scale will be 10^{13} Gev. Details of the specifications have been given in Refs. [14,15].

The inflation stage: The scalae factor $a(\tau)$ is in Eq. (9), the mode of RGW is in Eq. (13) as part of the initial condition.

The reheating stage

$$a(\tau) = a_z|\tau - \tau_p|^{1+\beta_s}, \qquad \tau_1 \le \tau \le \tau_s. \tag{A1}$$

As a model parameter, we simply take $\beta_s = -0.3$. The general solution of Eq. (7) during the reheating stage is

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{t}{2k}}\left[b_1(k)H^{(1)}_{\beta_s+\frac{1}{2}}(t) + b_2(k)H^{(2)}_{\beta_s+\frac{1}{2}}(t)\right], \qquad \tau_1 < \tau \le \tau_s, \tag{A2}$$

where $t = k(\tau - \tau_p)$. The two coefficients b_1 and b_2 are determined by the joining condition at τ_1

$$b_1(k) = \Delta_b^{-1}\left\{\sqrt{\frac{x_1}{t_1}}[a_1 H^{(1)}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)}_{\beta+\frac{1}{2}}(x_1)]\left[\frac{1}{2\sqrt{t_1}}H^{(2)}_{\beta_s+\frac{1}{2}}(t_1) + \sqrt{t_1}H^{(2)\prime}_{\beta_s+\frac{1}{2}}(t_1)\right]\right.$$
$$\left. -H^{(2)}_{\beta_s+\frac{1}{2}}(t_1)\left[\frac{1}{2\sqrt{x_1}}\left(a_1 H^{(1)}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)}_{\beta+\frac{1}{2}}(x_1)\right) + \sqrt{x_1}\left(a_1 H^{(1)\prime}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)\prime}_{\beta+\frac{1}{2}}(x_1)\right)\right]\right\}, \tag{A3}$$

$$b_2(k) = \Delta_b^{-1}\left\{\sqrt{\frac{x_1}{t_1}}[a_1 H^{(1)}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)}_{\beta+\frac{1}{2}}(x_1)]\left[\frac{1}{2\sqrt{t_1}}H^{(1)}_{\beta_s+\frac{1}{2}}(t_1) + \sqrt{t_1}H^{(1)\prime}_{\beta_s+\frac{1}{2}}(t_1)\right]\right.$$
$$\left. -H^{(1)}_{\beta_s+\frac{1}{2}}(t_1)\left[\frac{1}{2\sqrt{x_1}}(a_1 H^{(1)}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)}_{\beta+\frac{1}{2}}(x_1)) + \sqrt{x_1}(a_1 H^{(1)\prime}_{\beta+\frac{1}{2}}(x_1) + a_2 H^{(2)\prime}_{\beta+\frac{1}{2}}(x_1))\right]\right\},$$
$$\Delta_b = \sqrt{t_1}[H^{(1)}_{\beta_s+\frac{1}{2}}(t_1)H^{(2)\prime}_{\beta_s+\frac{1}{2}}(t_1) - H^{(1)\prime}_{\beta_s+\frac{1}{2}}(t_1)H^{(2)}_{\beta_s+\frac{1}{2}}(t_1)]. \tag{A4}$$

where $x_1 = k\tau_1$ and $t_1 = k(\tau_1 - \tau_p)$, while the coefficients a_1 and a_2 are given by Eq. (12). In the high frequency limit $k \to \infty$

$$b_1(k) = i\left(\frac{\beta(\beta+1)}{4x_1^2} - \frac{\beta_s(\beta_s+1)}{4t_1^2}\right)e^{-i(x_1+t_1)+i\pi\beta+i\pi\beta_s/2} + \mathcal{O}(k^{-3}) \tag{A5}$$

$$b_2(k) = -ie^{-i(x_1-t_1)+i\pi\beta-i\pi\beta_s/2}\left(1 - i\frac{\beta(\beta+1)}{2x_1} + i\frac{\beta_s(\beta_s+1)}{2t_1} - \frac{\beta^2(\beta+1)^2}{8x_1^2}\right.$$
$$\left. -\frac{\beta_s^2(\beta_s+1)^2}{8t_1^2} + \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1 t_1}\right) + \mathcal{O}(k^{-3}) \tag{A6}$$

The radiation-dominant stage:

$$a(\tau) = a_e(\tau - \tau_e), \qquad \tau_s \le \tau \le \tau_2. \tag{A7}$$

and the mode function is

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{y}{2k}}[c_1(k)H^{(1)}_{\frac{1}{2}}(y) + c_2(k)H^{(2)}_{\frac{1}{2}}(y)],$$
$$\tau_s < \tau \le \tau_2, \tag{A8}$$

where $y = k(\tau - \tau_e)$ and c_1 and c_2 are given by

$$c_1(k) = \Delta_c^{-1}\left\{\sqrt{\frac{t_s}{y_s}}[b_1 H^{(1)}_{\beta_s+\frac{1}{2}}(t_s) + a_2 H^{(2)}_{\beta_s+\frac{1}{2}}(t_s)]\left[\frac{1}{2\sqrt{y_s}}H^{(2)}_{\frac{1}{2}}(y_s) + \sqrt{y_s}H^{(2)\prime}_{\frac{1}{2}}(y_s)\right]\right.$$
$$-H^{(2)}_{\frac{1}{2}}(y_s)\left[\frac{1}{2\sqrt{t_s}}(a_1 H^{(1)}_{\beta_s+\frac{1}{2}}(t_s) + a_2 H^{(2)}_{\beta_s+\frac{1}{2}}(t_s))\right.$$
$$\left.\left. +\sqrt{t_s}\left(a_1 H^{(1)\prime}_{\beta_s+\frac{1}{2}}(t_s) + a_2 H^{(2)\prime}_{\beta_s+\frac{1}{2}}(t_s)\right)\right]\right\}, \tag{A9}$$

$$c_2(k) = \Delta_b^{-1}\Bigg\{\sqrt{\frac{t_s}{y_s}}[a_1H^{(1)}_{\beta_s+\frac{1}{2}}(t_s) + a_2H^{(2)}_{\beta_s+\frac{1}{2}}(t_s)]\left[\frac{1}{2\sqrt{y_s}}H^{(1)}_{\frac{1}{2}}(y_s) + \sqrt{y_s}H^{(1)\prime}_{\frac{1}{2}}(y_s)\right]$$
$$-H^{(1)}_{\frac{1}{2}}(y_s)\left[\frac{1}{2\sqrt{t_s}}(a_1H^{(1)}_{\beta_s+\frac{1}{2}}(t_s) + a_2H^{(2)}_{\beta_s+\frac{1}{2}}(t_s))\right.$$
$$\left.+\sqrt{t_s}(a_1H^{(1)\prime}_{\beta_s+\frac{1}{2}}(t_s) + a_2H^{(2)\prime}_{\beta_s+\frac{1}{2}}(t_s))\right]\Bigg\},$$
$$\Delta_c = \sqrt{y_s}[H^{(1)}_{\frac{1}{2}}(y_s)H^{(2)\prime}_{\frac{1}{2}}(y_s) - H^{(1)\prime}_{\frac{1}{2}}(y_s)H^{(2)}_{\frac{1}{2}}(y_s)], \tag{A10}$$

where $t_s = k(\tau_s - \tau_p)$ and $y_s = k(\tau_s - \tau_e)$. In the high frequency limit $k \to \infty$

$$c_1(k) = i\left(\frac{\beta(\beta+1)}{4x_1^2} - \frac{\beta_s(\beta_s+1)}{4t_1^2}\right)e^{-i(x_1+t_1-t_s+y_s)+i\pi\beta} + i\frac{\beta_s(\beta_s+1)}{4t_s^2}e^{-i(x_1-t_1+t_s+y_s)+i\pi\beta} + \mathcal{O}(k^{-3}), \tag{A11}$$

$$c_2(k) = -ie^{-i(x_1-t_1+t_s-y_s)+i\pi\beta}\Bigg(1 - i\frac{\beta(\beta+1)}{2x_1} + i\frac{\beta_s(\beta_s+1)}{2t_1} - i\frac{\beta_s(\beta_s+1)}{2t_s}$$
$$-\frac{\beta^2(\beta+1)^2}{8x_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_s^2} + \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_1}$$
$$-\frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_s} + \frac{\beta_s^2(\beta_s+1)^2}{4t_1t_s}\Bigg) + \mathcal{O}(k^{-3}). \tag{A12}$$

The matter-dominant stage

$$a(\tau) = a_m(\tau - \tau_m)^2, \qquad \tau_2 \le \tau \le \tau_E. \tag{A13}$$

and the mode function is

$$u_k(\tau) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{z}{2k}}\left[d_1(k)H^{(1)}_{\frac{3}{2}}(z) + d_2(k)H^{(2)}_{\frac{3}{2}}(z)\right], \qquad \tau_2 < \tau \le \tau_E, \tag{A14}$$

where $z = k(\tau - \tau_m)$ and d_1 and d_2 are given by

$$d_1(k) = \Delta_d^{-1}\Bigg\{\sqrt{\frac{y_2}{z_2}}[b_1H^{(1)}_{\frac{1}{2}}(y_2) + a_2H^{(2)}_{\frac{1}{2}}(y_2)]\left[\frac{1}{2\sqrt{y_s}}H^{(2)}_{\frac{3}{2}}(z_2) + \sqrt{z_2}H^{(2)\prime}_{\frac{3}{2}}(z_2)\right]$$
$$-H^{(2)}_{\frac{3}{2}}(z_2)\left[\frac{1}{2\sqrt{y_2}}(a_1H^{(1)}_{\frac{1}{2}}(y_2) + a_2H^{(2)}_{\frac{1}{2}}(y_2)) + \sqrt{y_2}(a_1H^{(1)\prime}_{\frac{1}{2}}(y_2) + a_2H^{(2)\prime}_{\frac{1}{2}}(y_2))\right]\Bigg\}, \tag{A15}$$

$$d_2(k) = \Delta_d^{-1}\Bigg\{\sqrt{\frac{y_2}{z_2}}[a_1H^{(1)}_{\frac{1}{2}}(y_2) + a_2H^{(2)}_{\frac{1}{2}}(y_2)]\left[\frac{1}{2\sqrt{z_2}}H^{(1)}_{\frac{3}{2}}(y_s) + \sqrt{z_2}H^{(1)\prime}_{\frac{3}{2}}(z_2)\right]$$
$$-H^{(1)}_{\frac{3}{2}}(z_2)\left[\frac{1}{2\sqrt{y_2}}(a_1H^{(1)}_{\frac{1}{2}}(y_2) + a_2H^{(2)}_{\frac{1}{2}}(y_2)) + \sqrt{y_2}(a_1H^{(1)\prime}_{\frac{1}{2}}(y_2) + a_2H^{(2)\prime}_{\frac{1}{2}}(y_2))\right]\Bigg\},$$
$$\Delta_d = \sqrt{z_2}[H^{(1)}_{\frac{3}{2}}(z_2)H^{(2)\prime}_{\frac{3}{2}}(z_2) - H^{(1)\prime}_{\frac{3}{2}}(z_2)H^{(2)}_{\frac{3}{2}}(z_2)], \tag{A16}$$

where $z_2 = k(\tau_2 - \tau_m)$ and $y_2 = k(\tau_2 - \tau_e)$. In the high frequency limit $k \to \infty$

$$d_1(k) = -\left(\frac{\beta(\beta+1)}{4x_1^2} - \frac{\beta_s(\beta_s+1)}{4t_1^2}\right)e^{-i(x_1+t_1-t_s+y_s-y_2+z_2)+i\pi\beta} - \frac{\beta_s(\beta_s+1)}{4t_s^2}e^{-i(x_1-t_1+t_s+y_s-y_2+z_2)+i\pi\beta}$$
$$+\frac{1}{2z_2^2}e^{-i(x_1-t_1+t_s-y_s+y_2+z_2)+i\pi\beta} + \mathcal{O}(k^{-3}), \tag{A17}$$

$$d_2(k) = -e^{-i(x_1-t_1+t_s-y_s+y_2-z_2)+i\pi\beta}\left(1 - i\frac{\beta(\beta+1)}{2x_1} + i\frac{\beta_s(\beta_s+1)}{2t_1} - i\frac{\beta_s(\beta_s+1)}{2t_s} + i\frac{1}{z_2} - \frac{\beta^2(\beta+1)^2}{8x_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_1^2} - \frac{\beta_s^2(\beta_s+1)^2}{8t_s^2} - \frac{1}{2z_2^2} + \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_1} - \frac{\beta(\beta+1)\beta_s(\beta_s+1)}{4x_1t_s} + \frac{\beta(\beta+1)}{2x_1z_2} + \frac{\beta_s^2(\beta_s+1)^2}{4t_1t_s} - \frac{\beta_s(\beta_s+1)}{2t_1z_2} + \frac{\beta_s(\beta_s+1)}{2t_sz_2}\right) + \mathcal{O}(k^{-3}). \tag{A18}$$

The accelerating stage up to the present time τ_H

$$a(\tau) = l_H|\tau - \tau_a|^{-\gamma}, \tau_E \le \tau \le \tau_H, \tag{A19}$$

with $\gamma \simeq 2.1$ fits the model $\Omega_\Lambda \simeq 0.7$ and $\Omega_m = 1 - \Omega_\Lambda$. The mode function $u_k(\tau)$ and the coefficients α_k, β_k of this stage are given in (39), (40), and (41).

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