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Fake conformal symmetry in unimodular gravity

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We study Weyl symmetry (local conformal symmetry) in unimodular gravity. It is shown that the Noether currents for both Weyl symmetry and global scale symmetry vanish exactly as in conformally invariant scalar-tensor gravity. We clearly explain why in the class of conformally invariant gravitational theories, the Noether currents vanish by starting with conformally invariant scalar-tensor gravity. Moreover, we comment on both classical and quantum-mechanical equivalences in Einstein's general relativity, conformally invariant scalar-tensor gravity, and the Weyl-transverse gravity. Finally, we discuss the Weyl current in the conformally invariant scalar action and see that it is also vanishing.

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I. INTRODUCTION

One of the biggest mysteries in modern theoretical physics is certainly the problem of the cosmological constant [1]. This problem consists of several facets that must be understood in a proper way. The *old* cosmological constant problem was to understand simply why the vacuum energy density is so small. The vacuum energy density coming from gravitational fluctuations up to the Planck mass scale is larger than that observed experimentally by some 120 orders of magnitude. On the other hand, the *new* cosmological constant problem is to understand why the cosmological constant is not exactly zero and why its energy density is the same order of magnitude as the present matter density.

Among several aspects of the cosmological constant problem—what the author would especially like to understand is the issue of radiative instability of the cosmological constant—is the necessity of fine-tuning the value of the cosmological constant every time the higher-order loop corrections are added in perturbation theory. To resolve this problem, unimodular gravity [2–14] has been put forward where the vacuum energy and *a fortiori* all potential energy are decoupled from gravity since, in the unimodular condition $\sqrt{-g} = 1$, the potential energy cannot couple to gravity at the action level.

However, the world is not so simple since the unimodular condition must be properly implemented via the Lagrange multiplier field in quantum field theories. Radiative corrections then modify the Lagrange multiplier field, which essentially corresponds to the cosmological constant in unimodular gravity, thereby rendering its initial value radiatively unstable.

Thus, if unimodular gravity could provide us with some solution to the issue of radiative instability of the cosmological constant, there should be more symmetry or a still-unknown dynamical mechanism to suppress radiative corrections to the vacuum energy. Actually, there has already appeared such a theory where Weyl symmetry—i.e., local conformal symmetry—is added to the volume preserving diffeomorphisms or, equivalently, the transverse diffeomorphisms (TDiff) of unimodular gravity [15–19]. We will henceforth call this theory that of Weyl-transverse (WTDiff) gravity. Many of the reasons why WTDiff gravity is better than transverse gravity, which is called TDiff gravity, are mentioned in Ref. [18], where, for instance, it is expected that if Weyl symmetry could survive even at the quantum level, this theory would be a finite one, although a one-loop calculation leads to anomalies in the Ward-Takahashi identities.[1]

The purposes of this article are threefold. First, we calculate the Noether current for Weyl symmetry in WTDiff gravity and show that it vanishes identically as in conformally invariant scalar-tensor gravity [20]. Second, we provide simple proof that the Weyl current vanishes in a class of conformally invariant gravitational theories. Finally, we generalize this proof to a conformally invariant scalar matter action.

This paper is organized as follows. In Sec. II, we give two calculations of the Noether current for Weyl symmetry that use the same lines of argument found in Ref. [20]. In Sec. III, we will demonstrate that three kinds of gravitational theories—namely, Einstein's general relativity, conformally invariant scalar-tensor gravity, and WTDiff gravity—are all equivalent, at least classically, and comment on their quantum equivalences as well. In Sec. IV, we will explore why the Noether current made in Sec. II vanishes identically

[1]One reason why we are interested in WTDiff gravity and not the TDiff one is that, in TDiff gravity, $g = \det g_{\mu\nu}$ becomes a dimensionless scalar field, so that any polynomials of g are not excluded by symmetries and are allowed, in principle, to exist in the action. This fact makes it difficult to construct an action consisting of a finite number of terms.

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in this class of conformally invariant gravitational theories. In Sec. V, we consider the Weyl-invariant scalar matter coupling with gravity and see that the Weyl Noether current also vanishes in this case. The final section is devoted to discussions.

II. THE NOETHER CURRENT FOR WEYL SYMMETRY

We will start with the action of WTDiff gravity in unimodular gravity [18,19], which is given by[2]

$$S = \int d^4x \mathcal{L} = \frac{1}{12}\int d^4x |g|^{\frac{1}{4}}\left[R + \frac{3}{32}\frac{1}{|g|^2} g^{\mu\nu}\partial_\mu |g| \partial_\nu |g|\right], \tag{1}$$

where we have confined ourselves to four space-time dimensions since the generalization to a general space-time dimension is straightforward. Moreover, we have selected the coefficient $\frac{1}{12}$ for later convenience. Finally, note that we have defined $g = \det g_{\mu\nu} < 0$.

The action (1) turns out not to be invariant under the full group of diffeomorphisms (Diff), but rather only under the TDiff. Moreover, it is worthwhile to notice that, in spite of the existence of an explicit mass scale (the reduced Planck units, which we have set at $M_p = 1$), this action is also invariant under Weyl transformation. Actually, under the Weyl transformation

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{2}$$

the Lagrangian density in (1) is changed to

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}\partial_\mu\left(|g|^{\frac{1}{4}} g^{\mu\nu}\frac{1}{\Omega}\partial_\nu \Omega\right). \tag{3}$$

Now we will calculate the Noether current for Weyl symmetry by using the Noether procedure [22]. We will closely follow the lines of argument found in Ref. [20]. A general variation of the Lagrangian density in the action (1) reads

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\nu\rho})}\delta(\partial_\mu g_{\nu\rho}) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})}\delta(\partial_\mu \partial_\nu g_{\rho\sigma}). \tag{4}$$

In this expression, let us note that the Lagrangian density at hand includes second-order derivatives of $g_{\mu\nu}$ in the scalar curvature R. Setting $\Omega(x) = e^{-\Lambda(x)}$, the infinitesimal variation $\delta\mathcal{L}$ under Weyl transformation (2) is given by

$$\delta\mathcal{L} = \partial_\mu X_1^\mu, \tag{5}$$

where X_1^μ is defined as

$$X_1^\mu = \frac{1}{2}|g|^{\frac{1}{4}} g^{\mu\nu}\partial_\nu \Lambda. \tag{6}$$

Equation (5) of course indicates that the action (1) is invariant under the Weyl transformation up to a surface term.

Next, using the equations of motion

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho g_{\mu\nu})} - \partial_\rho \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\sigma g_{\mu\nu})}, \tag{7}$$

the variation $\delta\mathcal{L}$ in (4) can be cast to the form

$$\delta\mathcal{L} = \partial_\mu K_1^\mu, \tag{8}$$

where K_1^μ is defined as

$$K_1^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\nu\rho})}\delta g_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})}\partial_\nu \delta g_{\rho\sigma} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})}\delta g_{\rho\sigma}. \tag{9}$$

Using this formula, an explicit calculation yields

$$K_1^\mu = X_1^\mu, \tag{10}$$

thereby giving us the result that the Noether current for Weyl symmetry vanishes identically:

$$J_1^\mu = K_1^\mu - X_1^\mu = 0. \tag{11}$$

Let us note that expressions X_1^μ and K_1^μ are both gauge invariant under the Weyl transformation. This fact will be utilized later when we provide proof in Sec. IV.

As an alternative derivation of the same result, one can also appeal to a more conventional method where the Lagrangian density in (1) does not explicitly involve second-order derivatives of $g_{\mu\nu}$ in the curvature scalar R. To do that, one makes use of the following well-known formula: when one writes the scalar curvature

$$R = R_1 + R_2, \tag{12}$$

the formula takes the form [23]

[2]We follow the notation and conventions by Misner *et al.*'s textbook [21], for instance, the flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$, the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha}$, and the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. The reduced Planck mass is defined as $M_p = \sqrt{\frac{c\hbar}{8\pi G}} = 2.4 \times 10^{18}$ GeV. Through this article, we adopt the reduced Planck units, where we set $c = \hbar = M_p = 1$.

$$R_1 = -2R_2 + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^\mu), \tag{13}$$

where one has defined the following quantities:

$$\begin{aligned} R_1 &= g^{\mu\nu}(\partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho}), \\ R_2 &= g^{\mu\nu}(\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma}) \\ &= g^{\mu\nu}\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} + \frac{1}{2}\Gamma^\rho_{\mu\nu}\partial_\rho g^{\mu\nu}, \\ A^\mu &= g^{\nu\rho}\Gamma^\mu_{\nu\rho} - g^{\mu\nu}\Gamma^\rho_{\nu\rho}. \end{aligned} \tag{14}$$

Here, let us note that R_2 is free of second-order derivatives of $g_{\mu\nu}$, which are now involved in the term including A^μ.

Then we have the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{12}\partial_\mu\left(|g|^{\frac{1}{4}}A^\mu\right), \tag{15}$$

where $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0 = \frac{1}{12}|g|^{\frac{1}{4}}\left[-R_2 + \frac{1}{4}|g|^{-1}A^\mu\partial_\mu|g| + \frac{3}{32}|g|^{-2}g^{\mu\nu}\partial_\mu|g|\partial_\nu|g|\right]. \tag{16}$$

We are now ready to show that the Noether current for Weyl symmetry is also zero by a more conventional method. First of all, let us observe that the variation of $\mathcal{L}$ under the Weyl transformation (2) comes from only the total derivative term

$$\delta\mathcal{L} = \partial_\mu\left(\frac{1}{2}|g|^{\frac{1}{4}}g^{\mu\nu}\partial_\nu\Lambda\right) = \frac{1}{12}\partial_\mu[\delta(|g|^{\frac{1}{4}}A^\mu)]. \tag{17}$$

Total derivative terms are irrelevant to the dynamics so, in what follows, let us focus our attention only on the Lagrangian $\mathcal{L}_0$, which is free of second-order derivatives of $g_{\mu\nu}$.

Second, by an explicit calculation we find that the Lagrangian $\mathcal{L}_0$ is invariant under the Weyl transformation without any surface terms,

$$X_2^\mu = 0. \tag{18}$$

Finally, applying the Noether theorem [22] for $\mathcal{L}_0$, we can derive the following result:

$$K_2^\mu = \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}) = 0. \tag{19}$$

Hence, the Noether current for Weyl symmetry vanishes identically:

$$J_2^\mu = K_2^\mu - X_2^\mu = 0. \tag{20}$$

This result is very similar to that of conformally invariant scalar-tensor gravity [20]. This fact suggests that there might be more universal proof which is independent of the form of actions but reflects only the conformal invariance in this class of Weyl-invariant gravitational theories. In Sec. IV, we will present such proof.

Before closing this section, we should refer to the ambiguity associated with the Noether currents for local Weyl symmetry. Our calculation in this section is based on Noether's first theorem, which is applicable to global symmetries, and the second theorem, which can be applied to local (gauge) symmetries. Of course, the latter case includes the former one as a special case, and both of Noether's theorems give the same result, such that the Noether currents vanish identically. However, we should recall the well-known fact that Noether currents for local (gauge) symmetries always reduce to superpotentials, which leave us with ambiguity. Thus, a more reliable statement which is obtained from our calculation is that the global Weyl symmetry has a vanishing Noether current, and hence no charge and no symmetry generator.

III. CLASSICAL EQUIVALENCE

In this section, we wish to show the classical equivalence among the three kinds of gravitational theories, which are Einstein's general relativity, conformally invariant scalar-tensor gravity, and WTDiff gravity. This equivalence of the three theories will be used in the next section to explain why the Noether current for Weyl symmetry vanishes identically.

To show the equivalence, let us start with the Einstein-Hilbert action of general relativity in four space-time dimensions,

$$S = \frac{1}{12}\int d^4x\sqrt{-g}R. \tag{21}$$

The well-known trick to enlarge gauge symmetries from Diff to WDiff is to introduce the spurion field φ and then construct a Weyl-invariant metric $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$ since, under the Weyl transformation, the spurion field transforms as

$$\varphi \to \varphi' = \Omega^{-1}(x)\varphi. \tag{22}$$

Replacing $g_{\mu\nu}$ with $\hat{g}_{\mu\nu}$ in the Einstein-Hilbert action, one can obtain the action of conformally invariant scalar-tensor gravity [24,25],

$$\begin{aligned} \hat{S} &= \frac{1}{12}\int d^4x\sqrt{-\hat{g}}\hat{R} \\ &= \int d^4x\sqrt{-g}\left[\frac{1}{12}\varphi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \end{aligned} \tag{23}$$

Conversely, beginning with $\hat{S}$, to eliminate the spurion field φ one can take a gauge $\varphi = 1$ for Weyl symmetry, by which

$\hat{S}$ is reduced to the Einstein-Hilbert action S, which is invariant only under Diff. In this sense, Einstein's general relativity is classically equivalent to conformally invariant scalar-tensor gravity.

Next let us show the equivalence between conformally invariant scalar-tensor gravity and WTDiff gravity. In this case, we start with the action $\hat{S}$ of conformally invariant scalar-tensor gravity, then take a different gauge condition, $\varphi^2 = |g|^{-\frac{1}{4}}$, i.e.,

$$\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu} = |g|^{-\frac{1}{4}} g_{\mu\nu}. \tag{24}$$

It is worthwhile to stress that this gauge condition is a gauge condition not for the Weyl transformation but for the longitudinal diffeomorphism. In fact, under the Weyl transformation, $|g|^{-\frac{1}{4}}$ transforms in the same way as the square of the spurion field does:

$$|g|^{-\frac{1}{4}} \to |g'|^{-\frac{1}{4}} = \Omega^{-2}(x)|g|^{-\frac{1}{4}}. \tag{25}$$

Thus, the gauge condition (24) does not break Weyl symmetry, but rather breaks Diff down to TDiff since

$$\hat{g} = \det \hat{g}_{\mu\nu} = -1. \tag{26}$$

Now, substituting the gauge condition (24) into the action (23) of conformally invariant scalar-tensor gravity, it turns out that one arrives at the action (1) of WTDiff gravity. Consequently, via the gauge-fixing procedure, conformally invariant scalar-tensor gravity becomes equivalent to WTDiff gravity, at least at the classical level. To summarize, we have found that the three gravitational theories are classically equivalent via a trick that introduces a Weyl-invariant metric and a gauge-fixing procedure.

An important issue to address is to ensure that the three gravitational theories are equivalent, even at the quantum level. To put it differently, are there some anomalies, specifically, conformal anomaly for Weyl symmetry and a gauge anomaly for the longitudinal diffeomorphism? We believe that there are no such anomalies for the following reasons, even if we lack precise proof in this regard.

In a pioneering work, Englert *et al.* [26] have investigated the local conformal invariance in conformally invariant scalar-tensor gravity. Their result is that, as long as the local conformal symmetry is spontaneously broken, anomalies do not arise. In this context, the spontaneous symmetry breakdown of Weyl symmetry means that the spurion field takes the nonvanishing vacuum expectation value, $\langle\varphi(x)\rangle \neq 0$. Afterwards, this quantum equivalence has been studied from various viewpoints, and an affirmative answer was obtained in [27–32]. This observation could also be applied to the quantum equivalence between conformally invariant scalar-tensor gravity and WTDiff gravity, although in this case the spurion field must take the vacuum expectation value, which is not a constant but a field-dependent value. Anyway, we will need more studies to prove the exact equivalence among the three gravitational theories in the future.

IV. WHY IS THE NOETHER CURRENT VANISHING?

In this section, on the basis of the results obtained in the previous section, we shall provide simple proof that the Noether current for Weyl symmetry in both conformally invariant scalar-tensor gravity and WTDiff gravity vanishes. For simplicity, we will consider the action which includes only first-order derivatives of the metric tensor $g_{\mu\nu}$.

As the starting action, we will take the action $\hat{S}$ of conformally invariant scalar-tensor gravity. As in the case of WTDiff gravity, the action (23) can be rewritten in the first-order derivative form,

$$\hat{S} = \int d^4x \left[\hat{\mathcal{L}}_0 + \frac{1}{12}\partial_\mu(\sqrt{-g}\varphi^2 A^\mu)\right], \tag{27}$$

where $\hat{\mathcal{L}}_0$ is defined by

$$\hat{\mathcal{L}}_0 = \sqrt{-g}\left[-\frac{1}{12}\varphi^2 R_2 - \frac{1}{12}A^\mu\partial_\mu(\varphi^2) + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \tag{28}$$

The total derivative term in $\hat{S}$ plays no role in bulk dynamics, so we will consider $\hat{\mathcal{L}}_0$ from now on.

As shown in Ref. [20], $\hat{\mathcal{L}}_0$ in invariant under the Weyl transfomation without a surface term, from which we have

$$X_0^\mu = 0. \tag{29}$$

Then the Noether theorem [22] gives us

$$K_0^\mu = \frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu\varphi)}\varphi + \frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}). \tag{30}$$

Now we would like to give simpler proof of $K_0^\mu = 0$ without many calculations. The key observation for our proof is to recall that three kinds of gravitational theories are related to each other by a Weyl-invariant metric $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$, taking the differentiation from which we can derive the equation

$$\partial_\mu\hat{g}_{\nu\rho} = 2\varphi\partial_\mu\varphi g_{\nu\rho} + \varphi^2\partial_\mu g_{\nu\rho}. \tag{31}$$

Using this equation, one finds that

$$\frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu\varphi)} = \frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu\hat{g}_{\nu\rho})}2\varphi g_{\nu\rho},$$
$$\frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu g_{\nu\rho})} = \frac{\partial\hat{\mathcal{L}}_0}{\partial(\partial_\mu\hat{g}_{\nu\rho})}\varphi^2. \tag{32}$$

From Eq. (32), Eq. (30) produces the expected result,

$$K_0^\mu = 0. \tag{33}$$

As a result, the Noether current for Weyl symmetry is vanishing:

$$J_0^\mu = K_0^\mu - X_0^\mu = 0. \tag{34}$$

This is simple proof of the vanishing Noether current for Weyl symmetry in conformally invariant scalar-tensor gravity. Since the current is gauge invariant, our proof can be directly applied to any locally conformally invariant gravitational theories, such as WTDiff gravity obtained via the trick $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$.

V. WEYL-INVARIANT MATTER COUPLING

Since there are plenty of matters around us, it is natural to take account of effects of matter fields in the present formalism. In this section, we will show that an introduction of conformal matters does not change the fact that the Weyl current vanishes. As an example, we will work with the WDiff coupling of a real scalar field with gravity, but the generalization to general matter fields is straightforward as long as the matter fields are invariant under the Weyl transformation.

As before, let us first begin with the action of a scalar field ϕ with the potential $V(\phi)$ in curved space-time,

$$S_m = \int d^4x |g|^{\frac{1}{2}} [-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]. \tag{35}$$

Note that this action is manifestly invariant under the full group of Diff. Under the Weyl transformation, the scalar field transforms as

$$\phi \to \phi' = \Omega^{-1}(x)\phi. \tag{36}$$

The trick to enlarging gauge symmetries from Diff to WDiff is to now make both a Weyl-invariant metric $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$ and a Weyl-invariant scalar field $\hat{\phi} = \varphi^{-1}\phi$, then replace the metric and the scalar field in the action (35) with the corresponding Weyl-invariant objects. As a result, the WDiff matter action takes the form

$$\begin{aligned}\hat{S}_m = \int d^4x \hat{\mathcal{L}}_m &= \int d^4x |\hat{g}|^{\frac{1}{2}} [-\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - V(\hat{\phi})] \\ &= \int d^4x |g|^{\frac{1}{2}} \left[-\varphi^2 g^{\mu\nu}\partial_\mu\left(\frac{\phi}{\varphi}\right)\partial_\nu\left(\frac{\phi}{\varphi}\right) \right. \\ &\quad \left. - \varphi^4 V\left(\frac{\phi}{\varphi}\right)\right].\end{aligned} \tag{37}$$

In this section, we shall calculate the Noether current for Weyl symmetry with the two different methods. One method is to calculate the current in the WDiff matter action without gauge-fixing Weyl symmetry like conformally invariant scalar-tensor gravity. The other method is to gauge fix the longitudinal diffeomorphism with the gauge condition, by which the WDiff matter action is reduced to the WTDiff matter one, and then to calculate the Noether current for Weyl symmetry like WTDiff gravity. The Weyl current is a gauge-invariant quantity, so both methods should provide the same result.

First, let us calculate the Noether current for Weyl symmetry on the basis of the WDiff matter action (37). It is easy to see that the action (37) is invariant under the Weyl transformation without a surface term, which implies

$$X_m^\mu = 0. \tag{38}$$

Again, the Noether theorem [22] yields

$$K_m^\mu = \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\phi)}\phi + \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\varphi)}\varphi + \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}). \tag{39}$$

Next, the Weyl-invariant combinations $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$ and $\hat{\phi} = \varphi^{-1}\phi$ give us the relations

$$\begin{aligned}\frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\phi)} &= \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\hat{\phi})}\frac{1}{\varphi}, \\ \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\varphi)} &= \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\hat{g}_{\nu\rho})}2\varphi g_{\nu\rho} - \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\hat{\phi})}\frac{\phi}{\varphi^2}, \\ \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu g_{\nu\rho})} &= \frac{\partial\hat{\mathcal{L}}_m}{\partial(\partial_\mu\hat{g}_{\nu\rho})}\varphi^2.\end{aligned} \tag{40}$$

Using these relations, K_m^μ in (39) becomes zero:

$$K_m^\mu = 0. \tag{41}$$

The Noether current for Weyl symmetry is, therefore, vanishing:

$$J_m^\mu = K_m^\mu - X_m^\mu = 0. \tag{42}$$

This is a general result and, even after fixing the longitudinal diffeomorphism, this result should be valid since the Weyl current is gauge invariant under the Weyl transformation. Indeed, this is the case when we calculate the Weyl current in WTDiff scalar action quickly.

Now let us take the gauge condition (24) for the longitudinal diffeomorphism, which does not break the local conformal symmetry. Inserting the gauge condition

(24) into the WDiff scalar matter action (37) leads to the WTDiff scalar matter action,

$$\hat{S}_m = \int d^4x \hat{\mathcal{L}}_m = \int d^4x \Big[-|g|^{\frac{1}{2}} g^{\mu\nu} \Big(\frac{1}{64} \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{1}{4} \frac{\phi}{|g|} \partial_\mu |g| \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi \Big) - V(|g|^{\frac{1}{8}} \phi) \Big]. \quad (43)$$

Since the action (43) is invariant under the Weyl transformation without a surface term, we have

$$X_m^\mu = 0. \quad (44)$$

The Noether theorem [22] gives us the formula

$$K_m^\mu = \frac{\partial \hat{\mathcal{L}}_m}{\partial(\partial_\mu \phi)} \phi + \frac{\partial \hat{\mathcal{L}}_m}{\partial(\partial_\mu g_{\nu\rho})} (-2 g_{\nu\rho}). \quad (45)$$

It is useful to evaluate each term in (45) separately to see its gauge invariance, whose result is given by

$$\frac{\partial \hat{\mathcal{L}}_m}{\partial(\partial_\mu \phi)} \phi = -\hat{\phi}^2 \hat{g}^{\mu\nu} \partial_\nu \log(\hat{\phi}^2),$$
$$\frac{\partial \hat{\mathcal{L}}_m}{\partial(\partial_\mu g_{\nu\rho})} (-2 g_{\nu\rho}) = \hat{\phi}^2 \hat{g}^{\mu\nu} \partial_\nu \log(\hat{\phi}^2). \quad (46)$$

As promised, each term is manifestly gauge invariant under the Weyl transformation since it is expressed in terms of only gauge-invariant quantities. Adding the two terms in (46), we have

$$K_m^\mu = 0. \quad (47)$$

Thus, the Noether current for Weyl symmetry is certainly vanishing,

$$J_m^\mu = K_m^\mu - X_m^\mu = 0. \quad (48)$$

The results in (42) and (48) both clearly account for the fact that the Noether current for Weyl symmetry is vanishing in the conformally invariant scalar matter action as well.

VI. DISCUSSIONS

In this article, we have explicitly shown that the Noether currents in WTDiff gravity, which is invariant under both the local conformal transformation and the transverse diffeomorphisms, are identically vanishing for both local and global conformal symmetries. Moreover, we have provided simple proof of the vanishing Weyl currents and stressed that all the locally conformally invariant gravitational theories, which are obtained via the trick $\hat{g}_{\mu\nu} = \varphi^2 g_{\mu\nu}$ from the diffeomorphism-invariant gravitational theories, have the vanishing Weyl currents. We have also extended this calculation to conformally invariant scalar matter theory and have shown that the Noether current is vanishing as well.

This result of the vanishing Weyl currents in conformally invariant scalar-tensor gravity and WTDiff gravity is mathematically plausible since these two theories are at least classically equivalent to Einstein's general relativity. In general relativity, there is no conformal invariance, so the Weyl current is trivially zero and it is therefore also vanishing in conformally invariant scalar-tensor gravity and WTDiff gravity. In this sense, the local conformal symmetry existing in conformally invariant scalar-tensor gravity and WTDiff gravity could be called *fake* conformal symmetry [20].

An important issue associated with fake conformal symmetry involves what advantage we have by introducing a spurion field and adding fake conformal symmetry to quantum field theory.[3] One opinion on this issue is that the fake Weyl invariance has no dynamical role and that, at best, a possible calculational device might be achieved [20].[4] However, there is at least one advantage, in the sense that conformally invariant scalar-tensor gravity can be regarded as the more fundamental theory, from which, via a gauge-fixing procedure, both general relativity and WTDiff gravity can be derived in a natural way.

Our opinion on the advantage of fake conformal symmetry is different from that of Ref. [20]. We think that fake Weyl symmetry plays an important role in the cosmological constant problem, particularly at the quantum level. In conventional unimodular gravity, the cosmological constant is generated with quantum corrections but it does not couple to the gravitational field. In this sense, the quantum corrections do not generate the cosmological constant. On the other hand, in WTDiff gravity, there is fake Weyl symmetry, which forbids operators of dimension zero such as the cosmological constant. If in WTDiff gravity the quantum corrections do not generate the cosmological constant as well, we expect that fake Weyl symmetry survives at the quantum level, thereby suppressing radiative corrections. Then fake Weyl symmetry would not give rise to a Weyl anomaly at the quantum level, owing to its "fakeness." Of course, we will require more investigation to check out this interesting conjecture in the future.

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[3] Related models with the fake conformal symmetry are considered in Refs. [33–39].

[4] Similar or related criticisms are seen in [40,41].

[1] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[2] A. Einstein, *The Principle of Relativity* (Dover Publications, New York, 1952).
[3] J. L. Anderson and D. Finkelstein, Am. J. Phys. **39**, 901 (1971).
[4] J. van der Bij, H. van Dam, and Y. J. Ng, Physica (Amsterdam) **116A**, 307 (1982).
[5] W. Buchmuller and N. Dragon, Phys. Lett. B **207**, 292 (1988).
[6] M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989).
[7] W. Buchmuller and N. Dragon, Phys. Lett. B **223**, 313 (1989).
[8] W. G. Unruh, Phys. Rev. D **40**, 1048 (1989).
[9] Y. J. Ng and H. van Dam, J. Math. Phys. (N.Y.) **32**, 1337 (1991).
[10] E. Alvarez and A. F. Faedo, Phys. Rev. D **76**, 064013 (2007).
[11] E. Alvarez, A. F. Faedo, and J. J. Lopez-Villarejo, J. High Energy Phys. 10 (2008) 023.
[12] L. Smolin, Phys. Rev. D **80**, 084003 (2009).
[13] G. F. R. Ellis, H. van Elst, J. Murugan, and J.-P. Uzan, Classical Quantum Gravity **28**, 225007 (2011).
[14] A. Padilla and I. D. Saltas, Eur. Phys. J. C **75**, 561 (2015).
[15] K-I. Izawa, Prog. Theor. Phys. **93**, 615 (1995).
[16] C. Barcelo, R. Carballo-Rubio, and L. J. Garay, Phys. Rev. D **89**, 124019 (2014).
[17] C. Barceló, R. Carballo-Rubio, and L. J. Garay, arXiv:1406.7713.
[18] E. Alvarez, S. Gonzalez-Martin, M. Herrero-Valea, and C. P. Martin, J. High Energy Phys. 08 (2015) 078.
[19] E. Alvarez and S. González-Martín, arXiv:1605.00919.
[20] R. Jackiw and S-Y. Pi, Phys. Rev. D **91**, 067501 (2015).
[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[22] E. Noether, Gott. Nachr. **1918**, 235 (1918) [Transp. Theory Stat. Phys. **1**, 186 (1971)].
[23] Y. Fujii and K-I. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, England, 2003).
[24] P. A. Dirac, Proc. R. Soc. A **333**, 403 (1973).
[25] S. Deser, Ann. Phys. (N.Y.) **59**, 248 (1970).
[26] F. Englert, C. Truffin, and R. Gastmans, Nucl. Phys. **B117**, 407 (1976).
[27] M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671**, 162 (2009).
[28] R. Percacci, New J. Phys. **13**, 125013 (2011).
[29] F. Gretsch and A. Monin, Phys. Rev. D **92**, 045036 (2015).
[30] D. M. Ghilencea, Phys. Rev. D **93**, 105006 (2016).
[31] R. Carballo-Rubio, Phys. Rev. D **91**, 124071 (2015).
[32] D. Benedetti, Gen. Relativ. Gravit. **48**, 68 (2016).
[33] I. Oda, Phys. Rev. D **87**, 065025 (2013).
[34] I. Oda, Phys. Lett. B **724**, 160 (2013).
[35] I. Oda, Adv. Stud. Theor. Phys. **8**, 215 (2014).
[36] I. Oda and T. Tomoyose, J. High Energy Phys. 09 (2014) 165.
[37] I. Oda, arXiv:1602.00851.
[38] I. Oda, arXiv:1602.03478.
[39] I. Oda, arXiv:1603.00112.
[40] M. P. Hertzberg, Phys. Lett. B **745**, 118 (2015).
[41] I. Quiros, arXiv:1405.6668.