Hyperbolicity of physical theories with application to general relativity

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We consider gauge theories from the free evolution point of view, in which initial data satisfying constraints of a theory are given, and because the constraints satisfy a closed evolution system, they remain so. We study a model constrained Hamiltonian theory and identify a particular structure in the equations of motion which we call the standard gauge freedom. The pure gauge subsystem of this model theory is identified, and the manner in which the gauge variables couple to the field equations is presented. We demonstrate that the set of gauge choices that can be coupled to the field equations to obtain a strongly hyperbolic formulation is exactly the set of strongly hyperbolic pure gauges. Consequently we analyze a parametrized family of formulations of general relativity. The generalization of the harmonic gauge formulation to a five parameter family of gauge conditions is obtained.

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I. INTRODUCTION

Field theories often have wavelike, or hyperbolic, degrees of freedom contained somehow in a set of variables, some of which are constrained and some of which, the gauge fields, are not determined by the theory [1]. Physical states are equivalence classes of solutions related by a change of gauge. Solutions to the theory can be understood through properties of the equations of motion, which consist of a mixture of the gauge, constraint and physical quantities. Unraveling this structure in general may be hopeless. But if the gauge is carefully chosen, say by taking the harmonic gauge in general relativity [2], then the full set of equations of motion may be rendered strongly hyperbolic [3,4]. This condition guarantees the existence of a unique solution to the initial value problem that depends continuously on the initial data, at least locally in time. As highlighted in Ref. [5], "Ideally, one would like to exhibit a kind of hyperbolic skeleton of the Einstein equations and a complete characterization of the freedom to fix the gauge from which all hyperbolic reductions should be derivable. Instead, there are at present various different methods available which have been invented to serve specific needs," this *ad hoc* characterization is unsatisfactory. Equations of motion for the gauge choice can be obtained in the absence of any coupling to the theory, which begs the question: what is this skeleton? In other words, what are the set of pure gauges that can be coupled to the theory to form a hyperbolic formulation? Since the basic characterization of a set of partial differential equations can be made in the linear approximation, we may start by directing our efforts there. We thus begin to address these issues in Sec. II for a model linear constrained Hamiltonian system. In Sec. III, we examine conditions under which a formulation of the Hamiltonian theory is strongly hyperbolic. In Sec. IV, we apply our findings to general relativity (GR) with a five parameter family of gauge conditions and obtain the generalization of the harmonic formulation to this family. Finally we conclude in Sec. V.

II. MODEL THEORY WITH GAUGE FREEDOM

Consider the equations of motion for the Hamiltonian density,

$$H = \frac{1}{2}\begin{pmatrix} \partial_i q \\ p \end{pmatrix}^\dagger \begin{pmatrix} V^{ij} & F^{\dagger i} \\ F^j & M^{-1} \end{pmatrix} \begin{pmatrix} \partial_j q \\ p \end{pmatrix} + g_q^\dagger C_{\mathcal{H}} V^{ij}\partial_i\partial_j q + g_p^\dagger C_{\mathcal{M}}{}^i M^{-1}\partial_i p, \tag{1}$$

with canonical positions and momenta (q, p), which we take to be real valued vector fields. Every matrix is real and constant, M^{-1} is invertible, and $F^i = \beta^i I$ for some shift vector β^i, with I the appropriate identity. For *fixed* indices i, j, the matrices V^{ij}, F^i and M^{-1} are square, whereas $C_{\mathcal{H}}$, $C_{\mathcal{M}}{}^i$ may have fewer rows than columns. Without loss of generality, the block matrix in the first line of (1) is symmetric, implying that M^{-1} and V^{ij} are symmetric. Finally, we use $\dagger$ to denote the transposition. Such a Hamiltonian can be obtained from that of GR by linearizing [6] and discarding lower derivatives. The equations of motion are obtained by variation according to Hamilton's equations $\partial_t q = \delta H/\delta p$, and $\partial_t p = -\delta H/\delta q$, giving

$$\partial_t q = M^{-1}p + F^i\partial_i q - M^{-1}C_{\mathcal{M}}{}^{\dagger i}\partial_i g_p,$$
$$\partial_t p = V^{ij}\partial_i\partial_j q + F^i\partial_i p - V^{ij}C_{\mathcal{H}}{}^\dagger\partial_i\partial_j g_q.$$

Variation with respect to the gauge fields (g_q, g_p) reveals the constraints,

$$\mathcal{H} = C_{\mathcal{H}}V^{ij}\partial_i\partial_j q = 0, \qquad \mathcal{M} = C_{\mathcal{M}}{}^i M^{-1}\partial_i p = 0,$$

which we call the Hamiltonian and momentum constraints respectively. We take the constraints to be first class, so that their Poisson bracket with one another returns just combinations of constraints. Insisting on closure of the constraint subsystem, one easily finds that

$$C_{\mathcal{H}} V^{ij} = (A_{\mathcal{H}\mathcal{M}})^{(i} C_{\mathcal{M}}{}^{j)},$$
$$C_{\mathcal{M}}{}^{(i} M^{-1} V^{jk)} C_{\mathcal{H}}^{\dagger} = 0,$$
$$C_{\mathcal{M}}^{(i} M^{-1} V^{jk)} = (A_{\mathcal{M}\mathcal{H}})^{(i} C_{\mathcal{H}} V^{jk)}, \qquad (2)$$

for some matrices $(A_{\mathcal{H}\mathcal{M}})^i$, $(A_{\mathcal{M}\mathcal{H}})^i$ must hold. Index parentheses denote symmetrization. Then, we have

$$\partial_t \mathcal{H} = \beta^i \partial_i \mathcal{H} + (A_{\mathcal{H}\mathcal{M}})^i \partial_i \mathcal{M},$$
$$\partial_t \mathcal{M} = (A_{\mathcal{M}\mathcal{H}})^i \partial_i \mathcal{H} + \beta^i \partial_i \mathcal{M},$$

so the naming convention of the matrices $(A_{\mathcal{H}\mathcal{M}})^i$ and $(A_{\mathcal{M}\mathcal{H}})^i$ is clear. We assume that for every unit spatial vector s^i, the rows of $C_{\mathcal{H}}$ and $C_{\mathcal{M}}{}^s \equiv C_{\mathcal{M}}{}^i s_i$ are contained in the span of the union of the rows of $V = C_{\mathcal{H}} V^{ss}$ and $W = C_{\mathcal{M}}{}^s M^{-1}$, which each have themselves independent rows, and furthermore that the contractions $X = V C_{\mathcal{H}}{}^{\dagger}$ and $Y = W C_{\mathcal{M}}{}^{\dagger s}$ are invertible. We refer to this as *the rank assumption*. Under the rank assumption, it follows that $(A_{\mathcal{M}\mathcal{H}})^i = 0$. To see this, substitute the third expression in (2) into the second. Then, contract with an arbitrary spatial vector on every index, and multiply on the right by X^{-1}.

A. Gauge invariance

The equations of motion are invariant under the transformation,

$$q \to \bar{q} = q - M^{-1} C_{\mathcal{M}}{}^{\dagger i} \partial_i \psi, \qquad g_q \to g_q + \bar{g}_q,$$
$$p \to \bar{p} = p - V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j \theta, \qquad g_p \to g_p + \bar{g}_p, \qquad (3)$$

with the fields θ and ψ, satisfying

$$\partial_t \theta = \beta^i \partial_i \theta + \bar{g}_q,$$
$$\partial_t \psi = (A_{\mathcal{H}\mathcal{M}})^{\dagger i} \partial_i \theta + \beta^i \partial_i \psi + \bar{g}_p, \qquad (4)$$

where $\bar{g}_q$, $\bar{g}_p$ denote the change under (3). Since the change (3) to the canonical variables can be obtained by computing their Poisson bracket with $\theta^{\dagger} \mathcal{H} + \psi^{\dagger} \mathcal{M}$, the constraints are said to generate the gauge transformation. This transformation of the canonical variables leaves the constraints satisfied and the physical state unchanged. We furthermore require that the *field strength* $V^{ij} \partial_i \partial_j q$ and *curl* $\epsilon^i M^{-1} \partial_i p$, defined by some square anti-Hermitian matrices ϵ^i, are invariant under this transformation. In electromagnetism, the field strength is the curl of the magnetic field, or in other words the "Laplace minus grad div A" term. See Ref. [7] for a relevant discussion. Gauge invariance thus gives

$$(A_{\mathcal{H}\mathcal{M}})^{(i} C_{\mathcal{M}}{}^{j)} = C_{\mathcal{H}} V^{ij},$$
$$V^{(ij} M^{-1} C_{\mathcal{M}}{}^{\dagger k)} = 0,$$
$$\epsilon^{(i} M^{-1} V^{jk)} C_{\mathcal{H}}{}^{\dagger} = 0.$$

B. Electric and magnetic degrees of freedom

Without loss of generality, the matrix $C_{\mathcal{H}}$ has linearly independent rows, so we can decompose the potential matrix V^{ij} according to

$$V^{ij} = V^{ij}_{\mathcal{P}} + \hat{C}_{\mathcal{H}}{}^{\dagger} \hat{C}_{\mathcal{H}} V^{ij}, \qquad V^{ij}_{\mathcal{P}} = \perp_{\mathcal{P}} V^{ij}, \qquad (5)$$

with $\hat{C}_{\mathcal{H}} = [C_{\mathcal{H}} C_{\mathcal{H}}{}^{\dagger}]^{-1/2} C_{\mathcal{H}}$ and the projection operator $\perp_{\mathcal{P}}$ defined implicitly by (5). In the absence of Hamiltonian constraints, we assume that $V^{ij} = V^{ij}_{\mathcal{P}}$. The gauge invariant electric and magnetic degrees of freedom are

$$E = V^{ij}_{\mathcal{P}} \partial_i \partial_j q, \qquad B = \epsilon^i M^{-1} \partial_i p.$$

Assuming that we can decompose $V^{ij}_{\mathcal{P}} = \epsilon^{\dagger i} (A_{VB}) \epsilon^j + C_{\mathcal{M}}{}^{\dagger (i} (A_{V\mathcal{M}}) C_{\mathcal{M}}{}^{j)}$ for some Hermitian matrices (A_{VB}) and $(A_{V\mathcal{M}})$, the electric and magnetic degrees of freedom satisfy, up to coupling to the constraints, a closed subsystem. Such fields can be similarly defined in the absence of Hamiltonian constraints. They are not used in the analysis that follows.

C. Closure of the pure gauge subsystem

We call an equation of motion for the gauge fields a gauge choice. Here, we consider only evolution conditions,

$$\partial_t g_q = (A_{g_q g_q})^i \partial_i g_q + (A_{g_q g_p})^i \partial_i g_p + (A_{g_q p}) p,$$
$$\partial_t g_p = (A_{g_p g_q})^i \partial_i g_q + (A_{g_p g_p})^i \partial_i g_p + (A_{g_p q})^i \partial_i q. \qquad (6)$$

We assume that $(A_{g_q p}) = A C_{\mathcal{H}}$ and $(A_{g_p q})^i = B C_{\mathcal{M}}{}^i + C^i C_{\mathcal{H}} M$ for some matrices A, B and C^i, a restriction which can be dropped by altering our arguments slightly. Likewise, we could include $\partial_i q$, π terms in the first and second of these equations respectively. Assume that we are given a solution to the theory. We have already seen that the field equations are invariant under the gauge transformation (3). The pure gauge subsystem (4) is closed by substituting the gauge difference from (3) into (6), taking $g_q \to \bar{g}_q$ and $g_p \to \bar{g}_p$, resulting in

$$\partial_t \bar{g}_q = (A_{g_q g_q})^i \partial_i \bar{g}_q + (A_{g_q g_p})^i \partial_i \bar{g}_p - (A_{g_q p}) V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j \theta,$$
$$\partial_t \bar{g}_p = (A_{g_p g_q})^i \partial_i \bar{g}_q + (A_{g_p g_p})^i \partial_i \bar{g}_p - (A_{g_p q})^{(i} M^{-1} C_{\mathcal{M}}{}^{\dagger j)} \partial_i \partial_j \psi. \qquad (7)$$

Since the pure gauge system (4) and (7) is closed, we may examine the partial differential equations properties without referring to a particular formulation of the theory whatsoever. The key issue we are presently interested in is how, if at all, this system is inherited by the full equations of motion.

D. Free evolution on the expanded phase space

We are free to modify the dynamics of the model theory away from the constraint satisfying hypersurface in phase space, provided that the constraint subsystem remains closed. We define new constraints (Θ, Z) with the same length as (g_q, g_p) respectively. We couple the new constraints to the gauge conditions (4) by a parametrized addition according to

$$\begin{aligned}\partial_t g_q &= (A_{g_q g_q})^i \partial_i g_q + (A_{g_q g_p})^i \partial_i g_p + (A_{g_q p}) p + (A_{g_q \Theta})\Theta,\\ \partial_t g_p &= (A_{g_p g_q})^i \partial_i g_q + (A_{g_p g_p})^i \partial_i g_p + (A_{g_p q})^i \partial_i q\\ &\quad + (A_{g_p Z}) Z.\end{aligned}$$

Likewise, for the equations of motion,

$$\begin{aligned}\partial_t q &= M^{-1} p + F^i \partial_i q - M^{-1} C_{\mathcal{M}}{}^{\dagger i} \partial_i g_p + (A_{q\Theta})\Theta,\\ \partial_t p &= V^{ij} \partial_i \partial_j q + F^i \partial_i p - V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j g_q + (A_{pZ})^i \partial_i Z\\ &\quad + (A_{p\mathcal{H}})\mathcal{H}.\end{aligned}$$

We choose equations of motion for the new constraints,

$$\begin{aligned}\partial_t \Theta &= \beta^i \partial_i \Theta + (A_{\Theta Z})^i \partial_i Z + (A_{\Theta\mathcal{H}})\mathcal{H},\\ \partial_t Z &= (A_{Z\Theta})^i \partial_i \Theta + \beta^i \partial_i Z + (A_{Z\mathcal{M}})\mathcal{M}.\end{aligned}$$

The constraint subsystem is closed by

$$\begin{aligned}\partial_t \mathcal{H} &= (A_{\mathcal{H}\Theta})^{ij} \partial_i \partial_j \Theta + \beta^i \partial_i \mathcal{H} + (A_{\mathcal{H}\mathcal{M}})^i \partial_i \mathcal{M},\\ \partial_t \mathcal{M} &= (A_{\mathcal{M}Z})^{ij} \partial_i \partial_j Z + (A_{\mathcal{M}\mathcal{H}})^i \partial_i \mathcal{H} + \beta^i \partial_i \mathcal{M},\end{aligned}$$

with matrices

$$\begin{aligned}(A_{\mathcal{H}\Theta})^{ij} &= C_{\mathcal{H}} V^{ij} (A_{q\Theta}),\\ (A_{\mathcal{M}Z})^{ij} &= C_{\mathcal{M}}{}^{(i} M^{-1} (A_{pZ})^{j)},\\ (A_{\mathcal{M}\mathcal{H}})^i &= C_{\mathcal{M}}{}^i M^{-1} (A_{p\mathcal{H}}).\end{aligned}$$

One might insist that the whole set of equations of motion are naturally obtained from a Hamiltonian as in Ref. [8]. In that case, the new constraints Θ, Z become the canonical momenta of the gauge fields. On the other hand, one might wonder whether it is sensible to introduce these variables at all. We address this shortly.

E. Natural choice of variables

Next, the rank assumption is used to show that the variables can be appropriately broken up. Note that these are assumptions only on the model theory itself, not on the gauge choice. With these conditions, we can define

$$\begin{aligned}C_\theta &= -X^{-1} C_{\mathcal{H}},\\ C_\psi &= -Y^{-1} C_{\mathcal{M}}{}^s + (A_{\mathcal{H}\mathcal{M}})^{\dagger s} C_\theta [M - C_{\mathcal{M}}{}^{\dagger s} Y^{-1} C_{\mathcal{M}}{}^s],\\ \perp &= I - V^\dagger [V V^\dagger]^{-1} V - W^\dagger [W W^\dagger]^{-1} W,\end{aligned}$$

and the decomposition of $\partial_s q$ and p into gauge, constraint and physical degrees of freedom,

$$\begin{aligned}&\partial_s^2 \theta = C_\theta p + (A_{\theta\Theta})\Theta, \qquad \partial_s^2 \psi = C_\psi \partial_s q + (A_{\psi Z}) Z,\\ &\mathcal{H} = V \partial_s q, \qquad \mathcal{M} = W p,\\ &\partial_s P_q = \perp \partial_s q, \qquad P_p = \perp p, \end{aligned} \tag{8}$$

is invertible. The names on the left-hand sides here serve only to help identify the relationship between the pure gauge and constraints. The labels for the previous projections should now be clear. Multiplication of p by C_θ gives a quantity related to θ from the pure gauge system, at least in the principal symbol, and similarly for $\partial_s q$ and C_ψ. The variables V and W on the other hand extract that part of q and p associated in the principal symbol with the Hamiltonian and momentum constraints. The remainder is associated with the physical degrees of freedom.

F. Principal symbol of a formulation

Once the gauge and constraint addition parameters are fixed, we say that we have a formulation of the theory. The principal symbol of a formulation in the s^i direction is

$$P^s = \begin{pmatrix} P^s_{\mathcal{G}} & P^s_{\mathcal{GC}} & 0 \\ 0 & P^s_{\mathcal{C}} & 0 \\ 0 & 0 & P^s_{\mathcal{P}} \end{pmatrix}. \tag{9}$$

We assume throughout that the constraint addition parameters are annihilated by the projection operator $\perp$. This restriction can also be relaxed. The pure gauge sub-block,

$$P^s_{\mathcal{G}} = \begin{pmatrix} \beta^s & 0 & I & 0 \\ (A_{\mathcal{H}\mathcal{M}})^{\dagger s} & \beta^s & 0 & I \\ -(A_{g_q p}) V^\dagger & 0 & (A_{g_q g_q})^s & (A_{g_q g_p})^s \\ 0 & -(A_{g_p q})^s W^\dagger & (A_{g_p g_q})^s & (A_{g_p g_p})^s \end{pmatrix},$$

is exactly the principal symbol of the pure gauge subsystem described after Eq. (6). The off-diagonal block,

$$P^s_{\mathcal{GC}} = \begin{pmatrix} 0 & (A_{\theta Z}) & (A_{\theta\mathcal{H}}) & 0 \\ (A_{\psi\Theta}) & 0 & 0 & (A_{\psi\mathcal{M}}) \\ (A_\Theta) & 0 & 0 & 0 \\ 0 & (A_Z) & 0 & 0 \end{pmatrix},$$

with submatrices

$$
\begin{aligned}
(A_{\theta Z}) &= (A_{\theta\Theta})(A_{\Theta Z}) + C_\theta (A_{pZ})^s, \\
(A_{\theta\mathcal{H}}) &= (A_{\theta\Theta})(A_{\Theta\mathcal{H}}) - X^{-1} + C_\theta (A_{p\mathcal{H}}), \\
(A_{\psi\Theta}) &= (A_{\psi Z})(A_{\psi\Theta}) - (A_{\mathcal{H}\mathcal{M}})^{\dagger s}(A_{\theta\Theta}) + C_\psi (A_{g_q\Theta}), \\
(A_{\psi\mathcal{M}}) &= (A_{\psi Z})(A_{Z\mathcal{M}}) - Y^{-1} - (A_{\mathcal{H}\mathcal{M}})^{\dagger s} C_\theta C_{\mathcal{M}}{}^{\dagger s} Y^{-1}, \\
(A_{\Theta}) &= (A_{g_q p}) V^\dagger (A_{\theta\Theta}) + (A_{g_q\Theta}), \\
(A_Z) &= (A_{g_p q})^s W^\dagger (A_{\psi Z}) + (A_{g_p Z})
\end{aligned}
$$

parametrizes the coupling of the gauge fields to the constraints. The constraint violating sub-block,

$$
P^s_{\mathcal{C}} = \begin{pmatrix} \beta^s & (A_{\Theta Z})^s & (A_{\Theta\mathcal{H}}) & 0 \\ (A_{Z\Theta})^s & \beta^s & 0 & (A_{Z\mathcal{M}}) \\ (A_{\mathcal{H}\Theta})^{ss} & 0 & \beta^s & (A_{\mathcal{H}\mathcal{M}})^s \\ 0 & (A_{\mathcal{M}Z})^{ss} & (A_{\mathcal{M}\mathcal{H}})^s & \beta^s \end{pmatrix},
$$

is exactly the principal symbol of the constraint subsystem. Finally, the physical sub-block,

$$
P^s_{\mathcal{P}} = \begin{pmatrix} \beta^s & \perp M^{-1} \\ \perp V^{ss} & \beta^s \end{pmatrix},
$$

contains neither constraint addition nor gauge parameters. We see now that introducing the variables Θ and Z gives us greater freedom over both the constraint violating and the off-diagonal sub-blocks. Note, however, that, even without the Θ and Z constraints, we still obtain the upper-block diagonal structure of (9) without any change to the pure-gauge or physical sub-blocks.

G. Strong hyperbolicity

A necessary condition for strong hyperbolicity is that P^s has real eigenvalues and a complete set of eigenvectors for every s^i. Strong hyperbolicity is equivalent to well-posedness, that is, the existence of a unique solution depending continuously on the given data, of the initial value problem [3,4,9]. One linearizes nonlinear and variable coefficient problems about an arbitrary solution and works in the frozen coefficient approximation.

III. BASIC PROPERTIES OF THEORIES WITH THE STANDARD GAUGE FREEDOM

Consider the theory of the previous section. Then, we have the following

Definition: We say that a constrained Hamiltonian system as presented in the previous section has the standard gauge freedom.

Lemma: No formulation of a constrained Hamiltonian system with the standard gauge freedom is strongly hyperbolic if the physical sub-block is not.

Proof.—Obviously, a necessary condition for diagonalizability with real eigenvalues of (9), for any formulation, is that of $P^s_{\mathcal{P}}$. ■

An example of a system in which the physical sub-block fails to satisfy the conditions for strong hyperbolicity would be the initial value problem for GR for a Euclidean metric.

Lemma: A necessary condition for strong hyperbolicity of a formulation is that the pure gauge and constraint violating subsystems are strongly hyperbolic.

Proof.—We need to show that if the matrix (9) is diagonalizable with real eigenvalues then this property holds for the pure gauge and constraint violating sub-blocks. A diagonalizable upper-block triangular matrix has diagonalizable blocks on the diagonal (see Ref. [4], Appendix A, or Ref. [10], Sec. 5). Moreover, the set of eigenvalues of the full matrix is the union of the eigenvalues of the diagonal blocks. The lemma follows. ■

These results show that in applications we can proceed in showing strong hyperbolicity by examining the pure gauge, constraint-violating and physical sub-blocks. If all three are well-behaved and there is additionally sufficient freedom leftover to set the off-diagonal block to zero, we can guarantee strong hyperbolicity of a formulation with a given pure gauge. In the following application to GR, we will see that this is indeed possible for a large class of pure gauges. More generally, given a constrained Hamiltonian system with the standard gauge freedom, we would like to know whether or not every strongly hyperbolic pure gauge can be used to form a strongly hyperbolic formulation, and if not, then which pure gauges are troublesome and why. Presently, we have no sharp condition on this.

IV. APPLICATION TO GR

The Arnowitt-Deser-Misner Hamiltonian [11] for vacuum GR is $H_{\mathrm{ADM}} = -\alpha H + 2\beta^i M_i$, with Hamiltonian and momentum constraints,

$$
H = R - K_{ij}K^{ij} + K^2, \qquad M_i = D^j K_{ij} - D_i K.
$$

The canonical positions and momenta are γ_{ij} and $\pi^{ij} = \sqrt{\gamma}(K^{ij} - \gamma^{ij}K)$ respectively.

A. Gauge freedom in the nonlinear regime

We take the freedom to be to choose coordinates $x^\mu = (t, x^i)$ on spacetime; qualitative features of the model carry over. The constraints are obviously spatially covariant. Given an additional *upper-case* time coordinate T with normal vector N^a such that $N^a = W(n^a + v^a)$, with Lorentz factor W and spatial boost vector v^i, then

$$
\begin{aligned}
{}^{(N)}H &= W^2 H - 2W^2 M_v, \\
\perp \cdot {}^{(N)}M_i &= W M_i + 2W^3 M_v v_i - W^3 H v_i,
\end{aligned}
$$

where $\perp^a_b$ is the projection operator into slices of constant t and subscript v denotes contraction with the velocity v^i. The electric and magnetic parts of the Weyl tensor [12] form a closed subsystem, up to coupling to the constraints, and from the point of view of the lower case observer, the spatial part of the upper-case electric and magnetic parts are

$$\begin{aligned}\perp\cdot{}^{(N)}E_{ij} &= (2W^2-1)E_{ij} - 2W^2E_{v(i}v_{j)} + W^2E_{vv}\gamma_{ij}\\ &\quad + 2W^2\epsilon^k{}_{v(i}B_{j)k},\\ \perp\cdot{}^{(N)}B_{ij} &= W^2B_{ij} - W^2\epsilon^k{}_{ij}E_{kv} - W^2\epsilon^k{}_{vi}E_{jk},\end{aligned}$$

which shows that if the fields vanish in one foliation they vanish in every foliation. We stress that these equations hold without any approximation.

B. Linearized pure gauge subsystem

The linearized pure gauge subsystem is [13]

$$\begin{aligned}\partial_t\theta &= U - \psi_iD^i\alpha + \beta^i\partial_i\theta,\\ \partial_t\psi^i &= V^i + \alpha D^i\theta - \theta D^i\alpha + \mathcal{L}_\beta\psi^i,\end{aligned}\qquad(10)$$

where $\theta = -n_a\Delta[x^a]$, $\psi^i = -\perp^i_a\Delta[x^a]$, $U = \Delta[\alpha]$, and $V^i = \Delta[\beta^i]$. Under an infinitesimal change of gauge, the perturbation to the metric and extrinsic curvature are given by the York equations [14] with $\alpha\to\theta$ and $\beta^i\to\psi^i$,

$$\begin{aligned}\Delta[\gamma_{ij}] &= -2\theta K_{ij} + \mathcal{L}_\psi\gamma_{ij},\\ \Delta[K_{ij}] &= -D_iD_j\theta + \theta[R_{ij} - 2K^k{}_iK_{jk} + K_{ij}K] + \mathcal{L}_\psi K_{ij},\end{aligned}$$

which can be used to close the linearized pure gauge subsystem once we act on the gauge condition with the perturbation operator Δ. The first of these equations can be viewed as arising from taking the spatial part of the condition that $\Delta[g_{ab}] = \nabla_a\Psi_b + \nabla_b\Psi_a$ with $\Psi_a = n_a\theta - \psi_a$. The second can be computed by computing the time derivative of the first and using the definition of the Extrinsic curvature. From the constrained Hamiltonian system point of view, we observe that this result is completely natural. We saw in the discussion after (4) that in the model constrained Hamiltonian system a pure gauge transformation was obtained by computing the Poisson brackets with the constraints. In the case of GR, the situation is the same, except now the complete Hamiltonian is formed of constraints, and therefore the effect of a pure gauge change is given by the full evolution equations which are unique only up to the addition of constraints.

C. Free evolution in the expanded phase space

We expand the phase space by constraints Θ and Z_i and parametrize the equations of motion for the gauge by

$$\begin{aligned}\partial_t\alpha &= -g_1\alpha^2K + g_2\alpha\partial_i\beta^i + \beta^i\partial_i\alpha + 2c_1\alpha^2\Theta,\\ \partial_t\beta^i &= \alpha^2[g_3\gamma^{kl}\gamma^{ij} + g_4\gamma^{il}\gamma^{jk}]\partial_l\gamma_{jk} - g_5\alpha\partial^i\alpha + \beta^j\partial_j\beta^i\\ &\quad + 2\alpha^2c_2Z^i,\end{aligned}\qquad(11)$$

with $g_1 > 0$ and $\bar{g}_3 = 2(g_3 + g_4) > 0$, and for the remaining variables by

$$\begin{aligned}\partial_t\gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta\gamma_{ij} + \frac{1}{3}c_3\alpha\gamma_{ij}\Theta,\\ \partial_tK_{ij} &= -D_iD_j\alpha + \alpha[R_{ij} - 2K^k{}_iK_{jk} + K_{ij}K] + \mathcal{L}_\beta K_{ij}\\ &\quad + 2c_4\alpha\partial_{(i}Z_{j)} + \frac{1}{3}c_5\alpha\gamma_{ij}\partial_kZ^k + \frac{1}{3}c_6\alpha\gamma_{ij}H,\\ \partial_t\Theta &= c_7\alpha H + c_8\alpha\partial_iZ^i + \mathcal{L}_\beta\Theta,\\ \partial_tZ_i &= c_9\alpha M_i + c_{10}\alpha\partial_i\Theta + \mathcal{L}_\beta Z_i.\end{aligned}$$

Strong hyperbolicity for nonlinear and variable coefficient systems is defined, with additional smoothness conditions, by linearizing and working in the high-frequency frozen coefficient approximation [9]. In this approximation, the Hamiltonian density [6] has the structure of (1). Interestingly, in Ref. [9], a strongly hyperbolic formulation of GR is presented, with evolved lapse and shift, without introducing the Θ or Z variables, which shows explicitly that there are some gauge conditions for which the additional freedom afforded by working in the expanded phase space is not required.

D. Relationship with the model theory

The variables of GR map to those of the model theory according to the analogy,

$$\gamma_{ij} = q,\qquad \pi^{ij} = p,\qquad \alpha = g_q,\qquad \beta^i = g_p.$$

The matrices contained in the linearized Hamiltonian for GR are, up to prefactors irrelevant for the decomposition into the natural choice of variables,

$$\begin{aligned}M^{-1\,kl}{}_{mn} &\sim -2\delta^k{}_{(m}\delta^l{}_{n)} + \gamma^{kl}\gamma_{mn},\\ V^{ij\,kl}{}_{mn} &\sim \delta^k{}_{(m}\delta^l{}_{n)}\gamma^{ij} + \delta^i{}_{(m}\delta^j{}_{n)}\gamma^{kl} - \gamma_{mn}\gamma^{ij}\gamma^{kl}\\ &\quad - 2\gamma^{l(i}\delta^{j)}{}_{(m}\delta^k{}_{n)} + \gamma_{mn}\gamma^{l(i}\gamma^{j)k},\\ F^{i\,kl}{}_{mn} &\sim \beta^i\delta^k{}_{(m}\delta^l{}_{n)},\\ C_{\mathcal{H}}{}^{kl} &\sim -\gamma^{kl},\\ C_{\mathcal{M}}{}^{i\,kl}{}_m &\sim \gamma^{ik}\delta^l{}_m - \gamma^{kl}\delta^i{}_m,\end{aligned}$$

where here the expressions γ^{ij} refer to the background metric rather than the canonical momenta q. It is straightforward to see that the rank assumption holds. Notice that, up to nonprincipal terms, the Hamiltonian for the linearized system is of the form (1). This is because the linearized Hamiltonian arises by taking the second variation of the original Hamiltonian and keeping terms quadratic in the perturbation [15]. Crucially, the linearized equations of motion have the same structure in the principal part as those of the original nonlinear system, which guarantees the applicability of our analysis there also.

E. Natural choice of variables

For comparison with the model theory, we give the decomposition of variables used in the following analysis. For the linearized system, we introduce a unit spatial vector s^i and projection operator $q_{ij} = \gamma_{ij} - s_i s_j$, with γ_{ij} the background spatial metric. For the gauge variables, we have

$$\partial_s^2\theta = -K_{ss} + 2\Theta, \qquad \partial_s^2\psi_s = \frac{1}{2}\partial_s\gamma_{ss} + Z_s$$
$$\partial_s^2\psi_A = \partial_s\gamma_{sA} + 2Z_A, \qquad \partial_s\alpha, \partial_s\beta_s, \partial_s\beta_A,$$

where an index s denotes contraction with s^i and upper-case indices A, B are used to denote projection with $q^i{}_j$. For the constraints, instead we have

$$H = -\frac{1}{2}\partial_s\gamma_{qq}, \qquad M_s = -K_{qq},$$
$$M_A = K_{sA}, \qquad \Theta, Z_s, Z_A,$$

where indices qq denote trace with q^{ij}. As in (8), the names on the left-hand sides here are only meant to denote the relationship with the pure gauge and constraint subsystems. Finally, for the physical sub-block, we have

$$\partial_s P^{(q)}_{AB} = \partial_s\gamma^{TF}_{AB}, \qquad P^{(p)}_{AB} = K^{TF}_{AB},$$

as expected, with TF denoting the trace-free part with respect to q_{ij}. In the foregoing three equations, the variables should of course be understood as those of the linearized system.

F. Strong hyperbolicity of the pure gauge subsystem

The principal symbol of the linearized pure gauge subsystem (10) with gauge choice (11), where one must ignore the constraint addition, has eigenvalues $\pm\sqrt{g_3}, \pm v_\pm$, with

$$2v_\pm^2 = g_1 + \bar{g}_3 - g_2 g_5 \pm \sqrt{(g_1 + \bar{g}_3 - g_2 g_5)^2 - 4(g_1 - g_2)\bar{g}_3}.$$

The subsystem is strongly hyperbolic if $g_3 > 0$ and we have either of the following:

(i) $0 \neq g_2 < g_1$ and $g_2 g_5 < g_1 - 2\sqrt{g_1 - g_2}\sqrt{\bar{g}_3} + \bar{g}_3$,

(ii) $g_2 = 0$ and $\bar{g}_3 \neq g_1$ or $g_2 = 0$, $\bar{g}_3 = g_1$ and $g_5 = 1$.

The second clause of case ii is that of generically distinct eigenvalues colliding without loss of diagonalizability.

G. Strong hyperbolicity of the constraint subsystem with vanishing gauge-constraint coupling

Choosing

$$c_1 = g_1, \qquad c_2 = g_3, \qquad c_3 = c_5 = c_6 = 0,$$
$$c_4 = 2c_7 = c_8 = c_9 = 1, \qquad c_{10} = 2\left(1 + \frac{g_4}{g_3}\right) \tag{12}$$

guarantees both that the off-diagonal block of the principal symbol $P^s_{\mathcal{GC}}$ vanishes and that the constraint subsystem is strongly hyperbolic. The eigenvalues of the constraint violating sub-block $P^s_{\mathcal{C}}$ are $\pm\sqrt{c_{10}}$, which are guaranteed to be real inside the class of gauges we are considering, and ± 1 with multiplicity 3.

H. Strong hyperbolicity of physical sub-block

The physical sub-block is diagonalizable with eigenvalues ± 1, at least up to a trivial normalization. Assuming smoothness of the background implies the continuity requirement for strong hyperbolicity in every block.

I. Discussion

The choice (12) is the natural extension of the harmonic gauge formulation [2] to the family of gauge conditions (11). Our results highlight that, despite being very convenient, the harmonic gauge choice should not be viewed as preferred in any particular sense. If a gauge in which the contracted Christoffel symbol is chosen to appear in the shift condition, i.e., when $g_4 = -\frac{1}{2}g_3$, the constraint addition parameters correspond to those of the principal part of the Z4 formulation [16]. Otherwise, it differs in the constraint subsystem but has the principle advantage that possible bad special cases in which hyperbolicity breaks down are avoided by construction. This is because of the block-diagonal structure of the principal symbol, so that, even if speeds of the gauge and constraint subsystems clash, diagonalizability cannot be lost. Furthermore, the constraint subsystem is still propagating (12), in the sense that none of the eigenvalues of the principal symbol are zero, although now they do not always correspond to light speed as in the harmonic gauge formulation. It is thus expected that the combination of (12) with a conformal decomposition will also be of practical use in numerical relativity [17,18]. For an associated discussion of electromagnetism, including a treatment of the initial boundary value problem, see Ref. [7].

V. CONCLUSION

Stimulated by Ref. [5], in which the possibility of identifying *every* hyperbolic formulation of GR was suggested, we identified a particular structure in constrained Hamiltonian equations of motion. We examined how pure gauge is inherited by a formulation of a theory for a large class of gauges. With this structure, the set of strongly hyperbolic pure gauges is exactly those that can be used to form a strongly hyperbolic formulation, in line with the expectation of the physicist. It is expected that the necessity of the pure gauge to be strongly hyperbolic for the construction of a strongly hyperbolic formulation will hold even in larger classes of gauges. On the other hand, coupling the gauge to the rest of the field equations to obtain a strongly hyperbolic formulation may in general be more tricky than what we dealt with here. Extending our results to the stronger notion of symmetric hyperbolicity is not easy because the principal part matrix is not as straightforwardly related to the various subsystems as is the principal symbol. For instance, symmetric hyperbolicty of the full system does not imply symmetric hyperbolicity of the constraint subsystem [10]. However, we do expect that the results can be generalized to include elliptic gauges, at least to some extent. It will furthermore be of interest to treat the initial boundary value problem, especially in GR. For such an analysis, symmetric hyperbolicity is desirable, however. We used our findings to investigate the hyperbolicity of a family of formulations of GR, generalizing [8] to non-Hamiltonian formulations, and obtained the natural generalization of the harmonic gauge formulation to that family. Other open questions in GR include those relating to the long-term existence of solutions to the nonlinear equations with different gauges, as discussed for the harmonic gauge for perturbations around flat space in Ref. [19].

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