

Effective gravitational couplings for cosmological perturbations in generalized Proca theories

Antonio De Felice,¹ Lavinia Heisenberg,² Ryotaro Kase,³ Shinji Mukohyama,^{1,4} Shinji Tsujikawa,³ and Ying-li Zhang^{5,6}

¹*Center for Gravitational Physics, Yukawa Institute for Theoretical Physics,
Kyoto University, 606-8502 Kyoto, Japan*

²*Institute for Theoretical Studies, ETH Zurich, Clausiusstrasse 47, 8092 Zurich, Switzerland*

³*Department of Physics, Faculty of Science, Tokyo University of Science, 1-3,
Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

⁴*Kavli Institute for the Physics and Mathematics of the Universe (WPI),
The University of Tokyo Institutes for Advanced Study, The University of Tokyo,
Kashiwa, Chiba 277-8583, Japan*

⁵*National Astronomy Observatories, Chinese Academy of Science,
Beijing 100012, People's Republic of China*

⁶*Institute of Cosmology and Gravitation, University of Portsmouth,
Portsmouth PO1 3FX, United Kingdom*

(Received 18 May 2016; published 11 August 2016)

We consider the finite interactions of the generalized Proca theory including the sixth-order Lagrangian and derive the full linear perturbation equations of motion on the flat Friedmann-Lemaître-Robertson-Walker background in the presence of a matter perfect fluid. By construction, the propagating degrees of freedom (besides the matter perfect fluid) are two transverse vector perturbations, one longitudinal scalar, and two tensor polarizations. The Lagrangians associated with intrinsic vector modes neither affect the background equations of motion nor the second-order action of tensor perturbations, but they do give rise to nontrivial modifications to the no-ghost condition of vector perturbations and to the propagation speeds of vector and scalar perturbations. We derive the effective gravitational coupling G_{eff} with matter density perturbations under a quasistatic approximation on scales deep inside the sound horizon. We find that the existence of intrinsic vector modes allows a possibility for reducing G_{eff} . In fact, within the parameter space, G_{eff} can be even smaller than the Newton gravitational constant G at the late cosmological epoch, with a peculiar phantom dark energy equation of state (without ghosts). The modifications to the slip parameter η and the evolution of the growth rate $f\sigma_8$ are discussed as well. Thus, dark energy models in the framework of generalized Proca theories can be observationally distinguished from the Λ CDM model according to both cosmic growth and expansion history. Furthermore, we study the evolution of vector perturbations and show that outside the vector sound horizon the perturbations are nearly frozen and start to decay with oscillations after the horizon entry.

DOI: [10.1103/PhysRevD.94.044024](https://doi.org/10.1103/PhysRevD.94.044024)

I. INTRODUCTION

The discovery of a late-time acceleration of the Universe [1] pushed forward an idea that one or more additional degrees of freedom (DOF) to those appearing in the standard model of particle physics may be the origin of dark energy [2]. The simplest example is a minimally coupled scalar field dubbed “quintessence” [3]. The cosmic acceleration can be realized for the scalar field with a slowly varying potential, in which case the dark energy equation of state w_{DE} is larger than -1 . The cosmological constant can be regarded as the nonpropagating limit of quintessence (i.e., vanishing kinetic energy) with $w_{\text{DE}} = -1$. The likelihood analysis based on supernovae type Ia (SN Ia), cosmic microwave background (CMB), and baryon acoustic oscillations (BAO) showed no statistically significant signatures that quintessence is observationally favored over the cosmological constant [4].

There are models of dark energy in which the scalar field ϕ has a nonminimal coupling to the Ricci scalar R with the form $F(\phi)R$, where $F(\phi)$ is a function of ϕ [5]. Brans-Dicke theory [6] with a scalar potential is one of the examples for such modified gravitational theories. For dark energy models in the framework of nonminimally coupled theories it is possible to realize w_{DE} smaller than -1 [7,8] without ghosts. Since the gravitational interaction is also different from that in general relativity (GR), these models leave several interesting observational signatures that can be distinguished from the Λ -cold-dark-matter (Λ CDM) model [9].

The nonminimal coupling $F(\phi)R$ can be extended to contain a derivative coupling in the form $F(\phi, X)R$, where $X = -\partial_\mu\phi\partial^\mu\phi/2$ is the field kinetic energy. In general, unless some counterterms are introduced, such derivative couplings give rise to the equations of motion higher than

second order [10]. The appearance of time derivatives higher than two leads to the so-called Ostrogradski instability [11] with the Hamiltonian unbounded from below. In 1974 Horndeski derived most general scalar-tensor theories with second-order equations of motion [12], which received much attention over the past five years in connection to the problems of dark energy and inflation [13]. A subclass of Horndeski interactions also naturally arises in massive gravity [14]. In scalar-tensor Horndeski theories, there is one scalar propagating DOF besides the two tensor polarizations.

If we consider a vector field A^μ as the source of dark energy, the number of DOF generally increases relative to scalar-tensor Horndeski theories. The massless Maxwell field given by the Lagrangian $\mathcal{L}_F = -F^{\mu\nu}F_{\mu\nu}/4$ (where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$) has two transverse polarizations of the vector mode with a protected $U(1)$ gauge symmetry. The introduction of the vector mass term gives rise to the additional longitudinal propagation of a scalar mode due to the breaking of gauge invariance. In GR with the massive Proca field, there are two transverse and one longitudinal propagating DOF besides the two tensor polarizations.

For the massless gauge-invariant vector field coupled to gravity with Lorentz symmetry, there is a no-go theorem stating that derivative interactions similar to those appearing for covariant Galileons [15] do not arise for a single spin-1 field in any dimensions [16,17] (see also Ref. [18]). This situation is different for massive Proca theories in which the $U(1)$ gauge invariance is explicitly broken. Analogous to scalar-tensor Horndeski theories, it is possible to construct an action of generalized Proca theories with second-order equations of motion having three propagating DOF with two tensor polarizations. The corresponding action has been constructed by using the Levi-Civita tensor to avoid the appearance of time derivatives higher than two. In fact, the analysis based on the Hessian matrix showed that such theories do not propagate extra DOF other than those mentioned above [19]. A subclass of these interactions was also discussed in Ref. [20].

If we impose the condition that the scalar part of the vector field only has terms that do not correspond to trivial total derivative interactions, then the series of the generalized Proca Lagrangian stops at quintic order (\mathcal{L}_5) [19]. By relaxing this condition, it is also possible to construct higher-order derivative interactions associated with the intrinsic vector part [21,22]. The sixth-order Lagrangian \mathcal{L}_6 [22], which contains the double dual Riemann tensor, accommodates an interaction term in the gauge-invariant vector-tensor theories constructed by Horndeski in 1976 [23]. In Ref. [21] the authors derived seventh and higher-order derivative interactions having one longitudinal and two transverse polarizations, but it was later found that they correspond to trivial interactions by virtue of the Cayley-Hamilton theorem. Thus, it suffices to consider the Lagrangians up to sixth order presented in Ref. [22].

Recently, the cosmology in generalized Proca theories up to the quintic Lagrangian \mathcal{L}_5 was studied in Ref. [24] (see also Refs. [25–30] for earlier related works). In such theories, there is a nontrivial branch of the background solutions where the temporal vector component ϕ depends on the Hubble expansion rate H alone. In Ref. [24] the authors proposed a dark energy model in which the solutions finally approach a de Sitter attractor characterized by constant ϕ . The conditions for avoiding ghosts and Laplacian instabilities were generally derived for tensor, vector, and scalar perturbations, which were applied to the proposed dark energy model to search for theoretically consistent parameter spaces. Moreover, there exists viable model parameter spaces in which the propagation speed of tensor perturbations is consistent with the Cherenkov-radiation constraint [31] and the recent detection of gravitational waves [32]. In addition, the cubic and quartic derivative interactions allow the screening of the fifth force mediated by the vector field [33].

In this paper, we extend the analysis of Ref. [24] to include the sixth-order Lagrangian \mathcal{L}_6 as well as the quadratic Lagrangian \mathcal{L}_2 containing the dependence of $X = -A_\mu A^\mu/2$, $F = -F_{\mu\nu}F^{\mu\nu}/4$, and $Y = A^\mu A^\nu F_\mu^\alpha F_{\nu\alpha}$ (which accommodates the terms discussed in Ref. [34]). We derive full linear perturbation equations of motion for tensor, vector, and scalar modes at linear order in the presence of a perfect fluid and then obtain the effective gravitational coupling G_{eff} with matter by employing a quasistatic approximation for perturbations deep inside the sound horizon. We also study the growth rate of matter perturbations and the evolution of gravitational potentials to confront generalized Proca theories with the observations of redshift-space distortions (RSD), CMB, and weak lensing.

The recent observations of RSD [35–37] and cluster counts [38] have shown that the cosmic growth rate is lower than that predicted by the Λ CDM model with σ_8 constrained by the Planck CMB data [39]. This tension reduces with the WMAP bound on σ_8 [40] and the systematic errors of RSD data are still quite large. Hence, in current observations, one cannot conclusively say that weak gravity is really favored over the gravitational law of GR. However, it is of interest to look for the theoretical possibility of realizing weak gravity on cosmological scales. In scalar-tensor Horndeski theories, unless the second-order action of tensor perturbations is modified from GR to a large extent, it is difficult to realize $G_{\text{eff}} < G$ without ghosts due to the presence of attractive scalar-matter couplings [41] (see also Refs. [42–47]). It remains to be seen whether the existence of the vector field can modify this situation. We shall pursue the possibility of weak gravity for a class of dark energy models in generalized Proca theories.

This paper is organized as follows. In Sec. II we obtain the background equations of motion in the presence of a perfect fluid containing the generalized Proca Lagrangian

up to sixth order. In Sec. III we derive the equations of motion for tensor and vector perturbations and identify no-ghost and stability conditions of them in the small-scale limit. In Sec. IV the scalar perturbation equations and the observables associated with large-scale structures, CMB, and weak lensing will be discussed. In Sec. V we analytically obtain the effective gravitational coupling with matter perturbations under the quasistatic approximation and derive a necessary condition for realizing $G_{\text{eff}} < G$. In Sec. VI we study the evolution of observable quantities for dark energy models in a class of generalized Proca theories and discuss how the vector field affects G_{eff} . Section VII is devoted to conclusions.

II. GENERALIZED PROCA THEORIES AND THE BACKGROUND EQUATIONS OF MOTION

We study generalized Proca theories with one longitudinal and two transverse polarizations of a vector field A^μ coupled to gravity. The action of such theories is of the following forms [19,22]:

$$S = \int d^4x \sqrt{-g} (\mathcal{L} + \mathcal{L}_M), \quad \mathcal{L} = \sum_{i=2}^6 \mathcal{L}_i, \quad (2.1)$$

where g is a determinant of the metric tensor $g_{\mu\nu}$, \mathcal{L}_M is a matter Lagrangian, and $\mathcal{L}_{2,3,4,5,6}$ are given by

$$\mathcal{L}_2 = G_2(X, F, Y), \quad (2.2)$$

$$\mathcal{L}_3 = G_3(X) \nabla_\mu A^\mu, \quad (2.3)$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) [(\nabla_\mu A^\mu)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho], \quad (2.4)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(X) G_{\mu\nu} \nabla^\mu A^\nu - \frac{1}{6} G_{5,X}(X) [(\nabla_\mu A^\mu)^3 \\ & - 3 \nabla_\mu A^\mu \nabla_\rho A_\sigma \nabla^\sigma A^\rho + 2 \nabla_\rho A_\sigma \nabla^\rho A^\sigma \nabla^\sigma A^\rho] \\ & - g_5(X) \tilde{F}^{\alpha\mu} \tilde{F}^\beta{}_\mu \nabla_\alpha A_\beta, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \mathcal{L}_6 = & G_6(X) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta \\ & + \frac{1}{2} G_{6,X}(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu, \end{aligned} \quad (2.6)$$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ (and ∇_μ is the covariant derivative operator). The function G_2 depends on the following three quantities:

$$X = -\frac{1}{2} A_\mu A^\mu, \quad (2.7)$$

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.8)$$

$$Y = A^\mu A^\nu F_\mu{}^\alpha F_{\nu\alpha}, \quad (2.9)$$

whereas $G_{3,4,5,6}$ and g_5 are arbitrary functions of X with the notation of partial derivatives as $G_{i,X} \equiv \partial G_i / \partial X$. The vector field is coupled to the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$ through the functions $G_4(X)$ and $G_5(X)$.¹ The $L^{\mu\nu\alpha\beta}$ and $\tilde{F}^{\mu\nu}$ are the double dual Riemann tensor and the dual strength tensor defined, respectively, by

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad (2.10)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor and $R_{\rho\delta\gamma\delta}$ is the Riemann tensor.

In the original Proca theory on the Minkowski background, which corresponds to the functions $G_2(X) = m^2 X$ and $G_{3,4,5,6} = 0$, the $U(1)$ gauge symmetry is explicitly broken due to the nonvanishing mass m of the vector field. In this case, the longitudinal mode arises in addition to the two transverse polarizations. The Lagrangians given above are the generalization of Proca theories coupled to gravity in which the number of propagating DOF remains three besides the two graviton polarizations. The existence of nonminimal couplings in $\mathcal{L}_{4,5,6}$ comes from the demand for keeping the three propagating DOF with second-order equations of motion. The gauge-invariant vector-tensor interaction introduced by Horndeski in 1976 corresponds to the Lagrangian $\mathcal{L} = F + \mathcal{L}_4 + \mathcal{L}_6$ with constant functions G_4 and G_6 [23].

In Ref. [19] there exists a term of the form $f_4(X) (\nabla_\rho A_\sigma \nabla^\rho A^\sigma - \nabla_\rho A_\sigma \nabla^\sigma A^\rho)$ with $f_4(X) = c_2 G_{4,X}$ in the Lagrangian \mathcal{L}_4 , but it can be expressed in terms of X and F as $-2f_4(X)F$. Hence such a term has been absorbed into the Lagrangian \mathcal{L}_2 . The term multiplied by $d_2 G_{5,X}(X)$ in the Lagrangian \mathcal{L}_5 of Ref. [19], which corresponds to an intrinsic vector mode, is now replaced with the last contribution in Eq. (2.5). The function $g_5(X)$ does not need to have a relation with $G_{5,X}(X)$ [21,22], so the prescription in this paper is more general than that of Ref. [19]. Furthermore, we adapt to the same notation as in Ref. [22], which agrees completely with Ref. [21].

In the Lagrangian \mathcal{L}_2 , we have also taken into account the dependence of the quantity Y that can be constructed from A^μ and its derivatives up to first order [19,34]. In principle we can also include the dependence of the term $F^{\mu\nu} \tilde{F}_{\mu\nu}$ in \mathcal{L}_2 . If we impose the parity invariance, however, such a term is irrelevant to the perturbations at linear order. Hence we shall consider the function G_2 depending on the three quantities X, F, Y in this paper.

Let us consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background described with the line element $ds^2 = -dt^2 + a^2(t) dx^2$, where $a(t)$ is the time-dependent

¹It would be interesting to study the consequences of the vector field living on a composite effective metric as it could be for instance the case in massive gravity [46]. This will be studied in a future work.

scale factor. To keep the spatial isotropy of the background, the vector field needs to have a time-dependent temporal component $\phi(t)$ alone, i.e.,

$$A^\mu = (\phi(t), 0, 0, 0). \quad (2.11)$$

For the matter Lagrangian \mathcal{L}_M we consider a perfect fluid with the energy density ρ_M and the isotropic pressure P_M . Assuming that matter is minimally coupled to gravity, we have the continuity equation

$$\dot{\rho}_M + 3H(\rho_M + P_M) = 0, \quad (2.12)$$

where a dot denotes a derivative with respect to t , and $H \equiv \dot{a}/a$ is the expansion rate of the Universe.

Variation of the action (2.1) with respect to $g_{\mu\nu}$ leads to the background equations of motion

$$\begin{aligned} G_2 - G_{2,X}\phi^2 - 3G_{3,X}H\phi^3 + 6G_4H^2 \\ - 6(2G_{4,X} + G_{4,XX}\phi^2)H^2\phi^2 + G_{5,XX}H^3\phi^5 \\ + 5G_{5,X}H^3\phi^3 = \rho_M, \end{aligned} \quad (2.13)$$

$$\begin{aligned} G_2 - \dot{\phi}^2 G_{3,X} + 2G_4(3H^2 + 2\dot{H}) \\ - 2G_{4,X}\phi(3H^2\phi + 2H\dot{\phi} + 2\dot{H}\phi) - 4G_{4,XX}H\dot{\phi}\phi^3 \\ + G_{5,XX}H^2\dot{\phi}\phi^4 \\ + G_{5,X}H\phi^2(2\dot{H}\phi + 2H^2\phi + 3H\dot{\phi}) = -P_M. \end{aligned} \quad (2.14)$$

Varying the action (2.1) with respect to A^μ , it follows that

$$\begin{aligned} \phi(G_{2,X} + 3G_{3,X}H\phi + 6G_{4,X}H^2 + 6G_{4,XX}H^2\phi^2 \\ - 3G_{5,X}H^3\phi - G_{5,XX}H^3\phi^3) = 0. \end{aligned} \quad (2.15)$$

Equations (2.13)–(2.15) are exactly the same as those derived for more specific theories containing the Lagrangians up to \mathcal{L}_5 [24]. Hence the Lagrangian \mathcal{L}_6 and the dependence of F and Y in \mathcal{L}_2 do not affect the background equations. In Eq. (2.15) there exists a branch with $\phi \neq 0$, which gives rise to interesting de Sitter solutions characterized by constant ϕ and H [24].

III. TENSOR AND VECTOR PERTURBATIONS

In what follows we derive the equations of motion for tensor, vector, and scalar perturbations on the flat FLRW background. The discussions about scalar perturbations will be given separately in Sec. IV.

First of all, we decompose temporal and spatial components of the vector field $A^\mu(t, \mathbf{x})$ into the background and perturbed components, as

$$A^0 = \phi(t) + \delta\phi, \quad (3.1)$$

$$A^i = \frac{1}{a^2(t)}\delta^{ij}(\partial_j\chi_V + E_j), \quad (3.2)$$

where the perturbation $\delta\phi$ depends on t and \mathbf{x} . The perturbations χ_V and E_j correspond to the intrinsic scalar and vector parts, respectively, where the latter satisfies the transverse condition $\partial^j E_j = 0$.

As for the matter sector, we consider a single perfect fluid described by the Schutz-Sorkin action [48]:

$$S_M = - \int d^4x [\sqrt{-g}\rho_M(n) + J^\mu(\partial_\mu\ell + \mathcal{A}_1\partial_\mu\mathcal{B}_1 + \mathcal{A}_2\partial_\mu\mathcal{B}_2)], \quad (3.3)$$

where the fluid energy density ρ_M depends on its number density defined by

$$n = \sqrt{\frac{J^\alpha J^\beta g_{\alpha\beta}}{g}}, \quad (3.4)$$

and J^μ is a vector field of weight one, ℓ is a scalar, and $\mathcal{A}_{1,2}$, $\mathcal{B}_{1,2}$ are scalar quantities associated with vector perturbations.

On the FLRW background the temporal component J^0 corresponds to the total fluid number \mathcal{N}_0 , which is constant. From Eq. (3.4) the background number density n_0 reads

$$n_0 = \frac{\mathcal{N}_0}{a^3}. \quad (3.5)$$

The temporal component $\partial_0\ell$ is equivalent to $-\rho_{M,n} \equiv -\partial\rho_M/\partial n$ at the background level, so that the matter action (3.3) reduces to

$$S_M^{(0)} = \int d^4x \sqrt{-g} P_M, \quad P_M = n_0\rho_{M,n} - \rho_M, \quad (3.6)$$

where P_M corresponds to the pressure of the perfect fluid.

The scalar quantities J^0 and ℓ have the perturbations δJ and v , respectively, so they can be written as

$$J^0 = \mathcal{N}_0 + \delta J, \quad (3.7)$$

$$\ell = - \int^t \rho_{M,n} dt' - \rho_{M,n} v, \quad (3.8)$$

where v corresponds to the velocity potential. One can express the spatial components of J^μ in terms of the sum of the scalar part δj and the vector part W_k , as

$$J^i = \frac{1}{a^2} \delta^{ik} (\partial_k \delta j + W_k). \quad (3.9)$$

The vector perturbation W_k obeys the transverse condition $\partial^k W_k = 0$. If we consider the vector field in the form

$W_k = (W_1(t, z), W_2(t, z), 0)$ whose x and y components depend on t and z alone, then it automatically satisfies the transverse condition. For the quantities \mathcal{A}_i and \mathcal{B}_i appearing in Eq. (3.3), the simplest choice keeping the required property of the vector mode is given by [49]

$$\begin{aligned} \mathcal{A}_1 &= \delta\mathcal{A}_1(t, z), & \mathcal{A}_2 &= \delta\mathcal{A}_2(t, z), \\ \mathcal{B}_1 &= x + \delta\mathcal{B}_1(t, z), & \mathcal{B}_2 &= y + \delta\mathcal{B}_2(t, z), \end{aligned} \quad (3.10)$$

where $\delta\mathcal{A}_i$ and $\delta\mathcal{B}_i$ are perturbed quantities.

On taking the variation of the matter action with respect to the field J^μ , we find the fluid normalized four-velocity u_μ as

$$u_\mu \equiv \frac{J_\mu}{n\sqrt{-g}} = \frac{1}{\rho_{M,n}} (\partial_\mu \ell + \mathcal{A}_1 \partial_\mu \mathcal{B}_1 + \mathcal{A}_2 \partial_\mu \mathcal{B}_2), \quad (3.11)$$

whose spatial components, on the FLRW manifold, are split, in terms of scalar and vector perturbations, as

$$u_i = -\partial_i v + v_i, \quad (3.12)$$

where v is the velocity potential given in Eq. (3.8) and v_i is a transverse three-vector satisfying $\partial^i v_i = 0$. From Eqs. (3.10) and (3.11) the intrinsic vector part v_i is related with the linear perturbation $\delta\mathcal{A}_i$, as $\delta\mathcal{A}_i = \rho_{M,n} v_i$. The equation of motion for $\delta\mathcal{A}_i$ follows by varying the second-order action of the vector field with respect to the perturbation $\delta\mathcal{B}_i$.

For the gravity sector, we consider the linearly perturbed line element in the flat gauge [50–53]:

$$\begin{aligned} ds^2 &= -(1 + 2\alpha)dt^2 + 2(\partial_i \chi + V_i)dt dx^i \\ &+ a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \end{aligned} \quad (3.13)$$

where α , χ are scalar metric perturbations, V_i is the vector perturbation obeying the transverse condition $\partial^i V_i = 0$, and h_{ij} is the tensor perturbation satisfying the transverse and traceless conditions $\partial^i h_{ij} = 0$ and $h_i^i = 0$. The temporal and spatial components of gauge transformation vectors are completely fixed under the above gauge choice.

A. Tensor perturbations

We can express the tensor perturbation h_{ij} in terms of the two polarization modes h_+ and h_\times , as $h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$, where e_{ij}^+ and e_{ij}^\times obey the relations $e_{ij}^+(\mathbf{k})e_{ij}^+(-\mathbf{k})^* = 1$, $e_{ij}^\times(\mathbf{k})e_{ij}^\times(-\mathbf{k})^* = 1$, and $e_{ij}^+(\mathbf{k})e_{ij}^\times(-\mathbf{k})^* = 0$ in Fourier space (\mathbf{k} is the comoving wave number). The second-order action for tensor perturbations, which is derived after expanding Eq. (2.1) in h_{ij} up to quadratic order, reads

$$S_T = \sum_{\lambda=+, \times} \int dt d^3x a^3 \frac{q_T}{8} \left[\dot{h}_\lambda^2 - \frac{c_T^2}{a^2} (\partial h_\lambda)^2 \right], \quad (3.14)$$

where

$$q_T = 2G_4 - 2\phi^2 G_{4,X} + H\phi^3 G_{5,X}, \quad (3.15)$$

and

$$c_T^2 = \frac{2G_4 + \phi^2 \dot{\phi} G_{5,X}}{q_T}. \quad (3.16)$$

The quantities q_T and c_T^2 are the same as those derived in Ref. [24], so the Lagrangian \mathcal{L}_6 and the terms F and Y in \mathcal{L}_2 do not affect the dynamics of tensor perturbations. Varying the action (3.14) with respect to h_λ , the tensor perturbation equation of motion in Fourier space is given by

$$\ddot{h}_\lambda + \left(3H + \frac{\dot{q}_T}{q_T} \right) \dot{h}_\lambda + c_T^2 \frac{k^2}{a^2} h_\lambda = 0, \quad (3.17)$$

where $k = |\mathbf{k}|$. The tensor ghost and small-scale Laplacian instabilities are absent for $q_T > 0$ and $c_T^2 > 0$, respectively.

B. Vector perturbations

As we have already mentioned, the vector perturbations E_i , W_i , $\delta\mathcal{A}_i$, and V_i obey the transverse conditions, so the components of these fields can be chosen as $E_i = (E_1(t, z), E_2(t, z), 0)$. On using Eq. (3.10) and expanding the matter action (3.3) up to quadratic order in vector perturbations, the second-order action reads [24]

$$\begin{aligned} (S_M^{(2)})_V &= \int dt d^3x \sum_{i=1}^2 \left[\frac{1}{2a^2 \mathcal{N}_0} \{ \rho_{M,n} (W_i^2 + \mathcal{N}_0^2 V_i^2) \right. \\ &+ \mathcal{N}_0 (2\rho_{M,n} V_i W_i - a^3 \rho_M V_i^2) \} \\ &\left. - \mathcal{N}_0 \delta\mathcal{A}_i \delta\mathcal{B}_i - \frac{1}{a^2} W_i \delta\mathcal{A}_i \right], \end{aligned} \quad (3.18)$$

where the quantities W_i , $\delta\mathcal{A}_i$, $\delta\mathcal{B}_i$ appear only in the matter action (3.18) but not in the quadratic action originating from $\int d^4x \sqrt{-g} \mathcal{L}$.

Varying Eq. (3.18) with respect to W_i , it follows that

$$W_i = \frac{\mathcal{N}_0 (\delta\mathcal{A}_i - \rho_{M,n} V_i)}{\rho_{M,n}} = \mathcal{N}_0 (v_i - V_i). \quad (3.19)$$

On using this relation and varying the matter action with respect to $\delta\mathcal{A}_i$, we obtain

$$\delta\mathcal{A}_i = \rho_{M,n} v_i, \quad \text{where } v_i = V_i - a^2 \delta\mathcal{B}_i. \quad (3.20)$$

Similarly, the variation of the matter action with respect to $\delta\mathcal{B}_i$ gives rise to the conservation equation

$$\rho_{M,n}v_i = \frac{(\rho_M + P_M)}{n_0}v_i = \delta\mathcal{A}_i = C_i, \quad (3.21)$$

where C_i are two constants in time (but may be dependent on k), which are related to the initial conditions for the intrinsic vector modes in the fluid. Therefore, the dynamics of v_i is completely determined as

$$v_i = \frac{\mathcal{N}_0 C_i}{(\rho_M + P_M)a^3}. \quad (3.22)$$

Then, after integrating out the fields W_i and $\delta\mathcal{A}_i$, the resulting second-order matter action reduces to

$$(S_M^{(2)})_V = \int dt d^3x \sum_{i=1}^2 \frac{a}{2} [(\rho_M + P_M)(V_i - a^2 \delta\dot{\mathcal{B}}_i)^2 - \rho_M V_i^2]. \quad (3.23)$$

To expand the action (2.1) up to second order, it is convenient to introduce the following combination:

$$Z_i = E_i + \phi(t)V_i, \quad (3.24)$$

so that $A_i = Z_i$ for vector perturbations. We also introduce the rescaled fields

$$\tilde{V}_i \equiv \frac{1}{a}V_i, \quad \tilde{Z}_i \equiv \frac{1}{a}Z_i. \quad (3.25)$$

Taking into account Eq. (3.23), the full quadratic action for vector perturbation reads

$$\begin{aligned} S_V^{(2)} = \int dt d^3x \sum_{i=1}^2 a^3 & \left[\frac{q_V}{2} \dot{\tilde{Z}}_i^2 - \frac{1}{2a^2} \mathcal{C}_1 (\partial\tilde{Z}_i)^2 - \frac{1}{2} \mathcal{C}_2 \tilde{Z}_i^2 \right. \\ & + \frac{\phi}{2a^2} (2G_{4,X} - G_{5,X}H\phi) \partial\tilde{V}_i \partial\tilde{Z}_i \\ & \left. + \frac{q_T}{4a^2} (\partial\tilde{V}_i)^2 + \frac{1}{2} (\rho_M + P_M) (\tilde{V}_i - a\delta\dot{\mathcal{B}}_i)^2 \right], \quad (3.26) \end{aligned}$$

where

$$q_V = G_{2,F} + 2G_{2,Y}\phi^2 - 4g_5H\phi + 2G_6H^2 + 2G_{6,X}H^2\phi^2, \quad (3.27)$$

$$\mathcal{C}_1 = q_V + 2[G_6\dot{H} - G_{2,Y}\phi^2 - (H\phi - \dot{\phi})(H\phi G_{6,X} - g_5)], \quad (3.28)$$

$$\begin{aligned} \mathcal{C}_2 = 2(2G_{4,X} - H\phi G_{5,X})\dot{H} \\ + (G_{3,X} + 4\phi H G_{4,XX} - G_{5,X}H^2 - \phi^2 G_{5,XX}H^2)\dot{\phi} \\ + 2q_V H^2 + \frac{d}{dt}(q_V H). \quad (3.29) \end{aligned}$$

Since \tilde{V}_i is not coupled with $\dot{\tilde{Z}}_i$, the kinetic term of the field \tilde{Z}_i remains unchanged after the integration of \tilde{V}_i . Hence we need to impose the condition $q_V > 0$ to avoid that \tilde{Z}_i becomes a ghost field. The auxiliary fields $\delta\dot{\mathcal{B}}_i$ acquire a kinetic term which is trivially positive for $q_T > 0$.

It should be noted that the dynamics of the vector perturbations is completely determined by the initial conditions of \tilde{Z}_i and $\dot{\tilde{Z}}_i$ and by the two constants C_i . In fact, in Fourier space, on using Eqs. (3.20) and (3.22), the equations of motion for \tilde{V}_i and \tilde{Z}_i following from Eq. (3.26) are given, respectively, by

$$\begin{aligned} \frac{q_T}{2} \frac{k^2}{a^2} \tilde{V}_i &= -\frac{\mathcal{N}_0 C_i}{a^4} - \frac{\phi}{2} (2G_{4,X} - G_{5,X}H\phi) \frac{k^2}{a^2} \tilde{Z}_i, \quad (3.30) \\ \ddot{\tilde{Z}}_i + \left(3H + \frac{\dot{q}_V}{q_V} \right) \dot{\tilde{Z}}_i \\ &+ \left[\frac{\mathcal{C}_1}{q_V} + \frac{\phi^2}{2q_V q_T} (2G_{4,X} - G_{5,X}H\phi)^2 \right] \frac{k^2}{a^2} \tilde{Z}_i + \frac{\mathcal{C}_2}{q_V} \tilde{Z}_i \\ &= -\frac{\phi}{q_V q_T} (2G_{4,X} - G_{5,X}H\phi) \frac{\mathcal{N}_0 C_i}{a^4}. \quad (3.31) \end{aligned}$$

This shows that there are only two dynamical fields \tilde{Z}_1 and \tilde{Z}_2 and that the matter fields can influence their dynamics only via the term on the rhs of Eq. (3.31), which is independent of the matter equation of state. From Eq. (3.31) we define the mass squared of the vector fields \tilde{Z}_i , as

$$m_V^2 \equiv \frac{\mathcal{C}_2}{q_V}. \quad (3.32)$$

We can easily see, from the expression of \mathcal{C}_2 , that, on the de Sitter solution characterized by $\dot{H} = 0$ and $\dot{\phi} = 0$, m_V^2 reduces to $2H^2$.

The vector propagation speed squared c_V^2 corresponds to the coefficient in front of the $(k^2/a^2)\tilde{Z}_i$ term in Eq. (3.31), i.e.,

$$\begin{aligned} c_V^2 = 1 + \frac{\phi^2 (2G_{4,X} - G_{5,X}H\phi)^2}{2q_T q_V} \\ + \frac{2[G_6\dot{H} - G_{2,Y}\phi^2 - (H\phi - \dot{\phi})(H\phi G_{6,X} - g_5)]}{q_V}, \quad (3.33) \end{aligned}$$

which is required to be positive for the stability on small scales. The Lagrangian \mathcal{L}_6 , the contribution Y to \mathcal{L}_2 , and the g_5 -dependent term in \mathcal{L}_5 affect both q_V and c_V^2 .

In the small-scale limit, the contribution of the matter fields in Eq. (3.30) can be neglected by assuming that the constants C_i are background dominated for large k . In this case we have

$$\tilde{V}_i \simeq -\frac{\phi}{q_T} (2G_{4,X} - G_{5,X} H \phi) \tilde{Z}_i. \quad (3.34)$$

Substituting this relation into Eq. (3.26) and ignoring the effective mass term $m_V^2 \tilde{Z}_i^2$ relative to those containing $(k^2/a^2) \tilde{Z}_i^2$, the second-order action (3.26) in Fourier space reduces to

$$S_V^{(2)} \simeq \int dt d^3x \sum_{i=1}^2 \frac{a^3 q_V}{2} \left(\dot{\tilde{Z}}_i^2 + c_V^2 \frac{k^2}{a^2} \tilde{Z}_i^2 \right). \quad (3.35)$$

Introducing the quantities

$$\mathcal{U}_i = z_V \tilde{Z}_i, \quad z_V = a\sqrt{q_V}, \quad \tau = \int a^{-1} dt, \quad (3.36)$$

the action (3.35) can be expressed as

$$S_V^{(2)} \simeq \int d\tau d^3x \sum_{i=1}^2 \frac{1}{2} \left[\mathcal{U}_i'^2 + c_V^2 k^2 \mathcal{U}_i^2 + \frac{z_V''}{z_V} \mathcal{U}_i^2 \right], \quad (3.37)$$

where a prime represents a derivative with respect to the conformal time τ . Provided the variation of q_V is not significant such that $|\dot{q}_V| \lesssim |Hq_V|$ and $|\ddot{q}_V| \lesssim |H^2 q_V|$, we have that $c_V^2 k^2 \mathcal{U}_i^2 \gg |(z_V''/z_V) \mathcal{U}_i^2|$ for the perturbations deep inside the vector sound horizon ($c_V^2 k^2/a^2 \gg H^2$). Then, the equation of motion for \mathcal{U}_i is given by

$$\mathcal{U}_i'' + c_V^2 k^2 \mathcal{U}_i \simeq 0. \quad (3.38)$$

As long as the frequency $\omega_k = c_V k$ adiabatically changes in time, we have the following WKB solution, which is valid only in the regime $c_V^2 k^2/a^2 > H^2$:

$$\tilde{Z}_i = \frac{\mathcal{U}_i}{z_V} \simeq \frac{1}{a\sqrt{2q_V c_V k}} (\alpha_k e^{-ic_V k \tau} + \beta_k e^{ic_V k \tau}), \quad (3.39)$$

where α_k and β_k are integration constants. Hence, for q_V and c_V slowly varying in time, the perturbation \tilde{Z}_i oscillates with an amplitude decreasing as a^{-1} .

For dark energy models in which the energy density of the temporal vector component comes out at the late cosmological epoch [24] the quantities $G_{4,X}$ and $G_{5,X}$ in Eq. (3.34) are usually small in the radiation and matter eras, so the perturbation \tilde{V}_i should be suppressed. The wave numbers k relevant to the observations of large-scale structures and weak lensing correspond to $k \gg a_0 H_0$ (the lower index ‘‘0’’ represents present values), so unless c_V is not much smaller than 1, the solution (3.39) is valid for such wave numbers from the vector sound horizon entry ($c_V^2 k^2/a^2 = H^2$) to today. This means that, for q_V and c_V^2

adiabatically changing in time, the vector perturbations \tilde{Z}_i tend to be negligible with time.

IV. SCALAR PERTURBATIONS

In this section we derive the equations of motion for scalar perturbations by expanding the action (2.1) up to quadratic order. We also introduce observables associated with measurements of large-scale structures, CMB, and weak lensing.

A. Perturbation equations

First of all, we define the matter perturbation $\delta\rho_M$, as

$$\delta\rho_M = \frac{\rho_{M,n}}{a^3} \delta J = \frac{\rho_M + P_M}{n_0 a^3} \delta J, \quad (4.1)$$

where we used Eq. (3.6) in the second equality. For the expansion of the matter action (3.3) of the scalar mode, we need to consider the perturbation δn of the number density n , as

$$\delta n = \frac{\delta\rho_M}{\rho_{M,n}} - \frac{\mathcal{N}_0^2 (\partial\chi)^2 + 2\mathcal{N}_0 \partial\chi \partial\delta j + (\partial\delta j)^2}{2\mathcal{N}_0 a^5}, \quad (4.2)$$

which is expanded up to quadratic order in scalar perturbations. Then, the second-order matter action of the scalar mode is given by

$$(S_M)_S^{(2)} = \int dt d^3x \left\{ \frac{1}{2a^5 n_0 \rho_{M,n}^2} [\rho_{M,n} (\rho_{M,n}^2 \partial\delta j^2 + 2a^3 n_0 \rho_{M,n}^2 \partial\delta j \partial v + 2a^8 n_0 \rho_{M,n} \dot{v} \delta\rho_M - 6a^8 n_0^2 \rho_{M,nn} H v \delta\rho_M) - a^8 n_0 \rho_{M,nn} \delta\rho_M^2] - a^3 \alpha \delta\rho_M + \frac{\rho_{M,n}}{a^2} \partial\chi \partial\delta j \right\}. \quad (4.3)$$

Varying this action with respect to δj , we obtain

$$\partial\delta j = -a^3 n_0 (\partial v + \partial\chi). \quad (4.4)$$

On account of Eq. (4.4), the perturbation δj appearing in Eq. (4.3) is integrated out.

We introduce the combination

$$\psi = \chi_V + \phi(t)\chi, \quad (4.5)$$

so that $A_i = \partial_i \psi$ for scalar perturbations. On using Eq. (4.3) with the relation (4.4), the second-order action of Eq. (2.1) for scalar perturbations reads

$$\begin{aligned}
S_S^{(2)} = \int dt d^3x a^3 \left\{ -\frac{n_0 \rho_{M,n}}{2} \frac{(\partial v)^2}{a^2} + \left[n_0 \rho_{M,n} \frac{\partial^2 \chi}{a^2} - \dot{\delta\rho}_M - 3H(1 + c_M^2) \delta\rho_M \right] v - \frac{c_M^2}{2n_0 \rho_{M,n}} (\delta\rho_M)^2 \right. \\
- \alpha \delta\rho_M - w_3 \frac{(\partial\alpha)^2}{a^2} + w_4 \alpha^2 - \left[(3Hw_1 - 2w_4) \frac{\delta\phi}{\phi} - w_3 \frac{\partial^2(\delta\phi)}{a^2\phi} - w_3 \frac{\partial^2\dot{\psi}}{a^2\phi} + w_6 \frac{\partial^2\psi}{a^2} \right] \alpha \\
\left. - \frac{w_3}{4} \frac{(\partial\delta\phi)^2}{a^2\phi^2} + w_5 \frac{(\delta\phi)^2}{\phi^2} - \left[\frac{(w_6\phi + w_2)\psi}{2} - \frac{w_3}{2} \dot{\psi} \right] \frac{\partial^2(\delta\phi)}{a^2\phi^2} - \frac{w_3}{4\phi^2} \frac{(\partial\dot{\psi})^2}{a^2} + \frac{w_7}{2} \frac{(\partial\psi)^2}{a^2} + \left(w_1\alpha + \frac{w_2\delta\phi}{\phi} \right) \frac{\partial^2\chi}{a^2} \right\}, \quad (4.6)
\end{aligned}$$

with the shorthand notations

$$\begin{aligned}
w_1 = H^2 \phi^3 (G_{5,X} + \phi^2 G_{5,XX}) \\
- 4H(G_4 + \phi^4 G_{4,XX}) - \phi^3 G_{3,X}, \quad (4.7)
\end{aligned}$$

$$w_2 = w_1 + 2Hq_T, \quad (4.8)$$

$$w_3 = -2\phi^2 q_V, \quad (4.9)$$

$$\begin{aligned}
w_4 = \frac{1}{2} H^3 \phi^3 (9G_{5,X} - \phi^4 G_{5,XXX}) \\
- 3H^2 (2G_4 + 2\phi^2 G_{4,X} + \phi^4 G_{4,XX} - \phi^6 G_{4,XXX}) \\
- \frac{3}{2} H \phi^3 (G_{3,X} - \phi^2 G_{3,XX}) + \frac{1}{2} \phi^4 G_{2,XX}, \quad (4.10)
\end{aligned}$$

$$w_5 = w_4 - \frac{3}{2} H(w_1 + w_2), \quad (4.11)$$

$$\begin{aligned}
w_6 = -\phi [H^2 \phi (G_{5,X} - \phi^2 G_{5,XX}) \\
- 4H(G_{4,X} - \phi^2 G_{4,XX}) + \phi G_{3,X}], \quad (4.12)
\end{aligned}$$

$$\begin{aligned}
w_7 = 2(H\phi G_{5,X} - 2G_{4,X}) \dot{H} + [H^2 (G_{5,X} + \phi^2 G_{5,XX}) \\
- 4H\phi G_{4,XX} - G_{3,X}] \dot{\phi}. \quad (4.13)
\end{aligned}$$

The quantity c_M^2 corresponds to the matter propagation speed squared given by

$$c_M^2 = \frac{n_0 \rho_{M,nn}}{\rho_{M,n}}. \quad (4.14)$$

We note that the terms containing $G_{2,F}$, $G_{2,Y}$, g_5 , G_6 , $G_{6,X}$ appear only in the coefficient w_3 . Hence, the functions $g_5(X)$, $G_6(X)$ as well as $G_2(F, Y)$ lead to modifications to the quadratic action (4.6) through the change of q_V .

Varying the action $S_S^{(2)}$ with respect to α , χ , $\delta\phi$, v , $\partial\psi$, and $\delta\rho_M$, we obtain the following equations of motion in Fourier space respectively:

$$\begin{aligned}
\delta\rho_M - 2w_4\alpha + (3Hw_1 - 2w_4) \frac{\delta\phi}{\phi} \\
+ \frac{k^2}{a^2} (\mathcal{Y} + w_1\chi - w_6\psi) = 0, \quad (4.15)
\end{aligned}$$

$$(\rho_M + P_M)v + w_1\alpha + \frac{w_2}{\phi} \delta\phi = 0, \quad (4.16)$$

$$\begin{aligned}
(3Hw_1 - 2w_4)\alpha - 2w_5 \frac{\delta\phi}{\phi} \\
+ \frac{k^2}{a^2} \left[\frac{1}{2} \mathcal{Y} + w_2\chi - \frac{1}{2} \left(\frac{w_2}{\phi} + w_6 \right) \psi \right] = 0, \quad (4.17)
\end{aligned}$$

$$\dot{\delta\rho}_M + 3H(1 + c_M^2)\delta\rho_M + \frac{k^2}{a^2} (\rho_M + P_M)(\chi + v) = 0, \quad (4.18)$$

$$\begin{aligned}
\dot{\mathcal{Y}} + \left(H - \frac{\dot{\phi}}{\phi} \right) \mathcal{Y} + 2\phi(w_6\alpha + w_7\psi) \\
+ \left(\frac{w_2}{\phi} + w_6 \right) \delta\phi = 0, \quad (4.19)
\end{aligned}$$

$$\dot{v} - 3Hc_M^2 v - c_M^2 \frac{\delta\rho_M}{\rho_M + P_M} - \alpha = 0, \quad (4.20)$$

where

$$\mathcal{Y} \equiv \frac{w_3}{\phi} (\dot{\psi} + \delta\phi + 2\alpha\phi). \quad (4.21)$$

The dynamics of scalar perturbations is known by solving the first-order differential equations (4.18)–(4.21) for $\delta\rho_M$, \mathcal{Y} , v , ψ and the algebraic equations (4.15)–(4.17) for α , χ , $\delta\phi$.

B. Observables associated with nonrelativistic matter

A key observable related with the measurements of large-scale structures and weak lensing is the gauge-invariant matter density contrast δ , defined by

$$\delta \equiv \frac{\delta\rho_M}{\rho_M} + 3H(1 + w_M)v, \quad (4.22)$$

where $w_M \equiv P_M/\rho_M$ is the matter equation of state. We are interested in the evolution of nonrelativistic matter perturbations (dark matter and baryons) satisfying the conditions $w_M = 0$ and $c_M^2 = 0$. In this case, Eqs. (4.18) and (4.20) reduce, respectively, to

$$\dot{\delta} - 3\dot{\mathcal{B}} = -\frac{k^2}{a^2}(\chi + v), \quad (4.23)$$

$$\dot{v} = \alpha, \quad (4.24)$$

where $\mathcal{B} \equiv Hv$.

Taking the time derivative of Eq. (4.23) and using Eq. (4.24), it follows that

$$\ddot{\delta} + 2H\dot{\delta} + \frac{k^2}{a^2}\Psi = 3\dot{\mathcal{B}} + 6H\dot{\mathcal{B}}, \quad (4.25)$$

where Ψ is the gauge-invariant Bardeen gravitational potential defined by [50]

$$\Psi \equiv \alpha + \dot{\chi}. \quad (4.26)$$

The growth of the matter density contrast δ is sourced by the gravitational potential Ψ . We relate Ψ and δ through the modified Poisson equation

$$\frac{k^2}{a^2}\Psi = -4\pi G_{\text{eff}}\rho_M\delta, \quad (4.27)$$

where G_{eff} corresponds to the effective gravitational coupling known by solving the perturbation equations (4.15)–(4.21) for Ψ and δ . To quantify the growth rate of δ , we also define

$$f \equiv \frac{\dot{\delta}}{H\delta}. \quad (4.28)$$

An important observable associated with RSD measurements is the quantity $f\sigma_8$ [54,55], where σ_8 is the amplitude of over-density at the comoving $8h^{-1}\text{Mpc}$ scale (h is the normalized Hubble parameter today $H_0 = 100 h \text{ km sec}^{-1} \text{ Mpc}^{-1}$).

Besides Ψ , we also introduce another gauge-invariant gravitational potential

$$\Phi \equiv H\chi, \quad (4.29)$$

and the gravitational slip parameter

$$\eta \equiv -\frac{\Phi}{\Psi}. \quad (4.30)$$

The effective gravitational potential associated with the deviation of light rays in CMB and weak lensing observations is given by [56]

$$\Phi_{\text{eff}} = \frac{1}{2}(\Psi - \Phi) = \frac{1}{2}(1 + \eta)\Psi, \quad (4.31)$$

which is affected by both Ψ and η .

V. EFFECTIVE GRAVITATIONAL COUPLING FOR MATTER PERTURBATIONS

The comoving wave numbers associated with the galaxy power spectrum for linear perturbations are in the range $0.01 h \text{ Mpc}^{-1} \lesssim k \lesssim 0.2 h \text{ Mpc}^{-1}$ [55], which correspond to $30a_0H_0 \lesssim k \lesssim 600a_0H_0$. To derive analytic expressions of G_{eff} , η , Ψ , Φ on scales relevant to the observations of large-scale structures and weak lensing, we employ the so-called quasistatic approximation for the perturbations inside the sound horizon.

A. Quasistatic approximation on scales deep inside the sound horizon

For the theories with $\mathcal{L}_6 = 0$ and the Lagrangian \mathcal{L}_2 with no Y dependence, the no-ghost and stability conditions of scalar perturbations were derived in Ref. [24] in the small-scale limit. Even for theories with $G_6 \neq 0$ and $\mathcal{L}_2 = G_2(X, F, Y)$, the modifications to scalar perturbations arise only through the change of q_V . In the $k \rightarrow \infty$ limit, the condition for the absence of scalar ghosts is given by

$$Q_S = \frac{a^3 H^2 q_T q_S}{\phi^2 (w_1 - 2w_2)^2} > 0, \quad (5.1)$$

where

$$q_S \equiv 3w_1^2 + 4q_T w_4. \quad (5.2)$$

Since the quantity Q_S does not contain w_3 , the no-ghost condition is not modified relative to the theories studied in Ref. [24]. Besides the matter propagation speed squared (4.14), the propagation speed squared associated with another scalar degree of freedom is given by

$$c_S^2 = \frac{\mu_c}{8H^2 \phi^2 q_T q_V q_S}, \quad (5.3)$$

where

$$\begin{aligned} \mu_c \equiv & [w_6 \phi (w_1 - 2w_2) + w_1 w_2]^2 - w_3 (2w_2^2 \dot{w}_1 - w_1^2 \dot{w}_2) \\ & + 2w_2^2 w_3 (\rho_M + P_M) + \phi (w_1 - 2w_2)^2 w_3 \dot{w}_6 \\ & + w_3 (w_1 - 2w_2) [(H - 2\dot{\phi}/\phi) w_1 w_2 \\ & + (w_1 - 2w_2) \{w_6 (H\phi - \dot{\phi}) + 2w_7 \phi^2\}]. \end{aligned} \quad (5.4)$$

To avoid the small-scale Laplacian instability we require that $c_S^2 > 0$. This translates to $\mu_c > 0$ under the three no-ghost conditions $q_T > 0$, $q_V > 0$, $q_S > 0$. It should be noted that since the expression for c_S^2 contains the term w_3 , compared to the case in Ref. [24], the new Lagrangians \mathcal{L}_6 and $\mathcal{L}_2 = G_2(X, F, Y)$ contribute to the scalar sound speed.

In the following, we employ the quasistatic approximation for the perturbations deep inside the sound horizon ($c_S^2 k^2 / a^2 \gg H^2$) [42,57]. This amounts to picking up the

terms containing k^2/a^2 and $\delta\rho_M$ in Eqs. (4.15)–(4.20); see the Appendix for a more detailed discussion about this approximation. This approximation breaks down for the models in which c_S^2 is very close to 0. In the following we assume that c_S^2 is not very much smaller than 1, in such a way that the condition $c_S^2 k^2/a^2 \gg H^2$ holds for the perturbations relevant to the growth of large-scale structures. We also note that, in some dark energy models like $f(R)$ gravity [8], the mass m of a scalar degree of freedom can be much larger than H in the past. In our generalized Proca theories, we are interested in the mass term $m^2 X$ in the Lagrangian \mathcal{L}_2 with m at most of the order of H_0 [24], so we can consistently ignore its effect for discussing the perturbations deep inside the sound horizon.

In what follows, we focus on nonrelativistic matter satisfying the conditions $P_M = 0$ and $c_M^2 = 0$. Employing the quasistatic approximation mentioned above for Eqs. (4.15) and (4.17), it follows that

$$\delta\rho_M \simeq -\frac{k^2}{a^2}(\mathcal{Y} + w_1\chi - w_6\psi), \quad (5.5)$$

$$\mathcal{Y} \simeq \left(\frac{w_2}{\phi} + w_6\right)\psi - 2w_2\chi. \quad (5.6)$$

Substituting Eq. (5.6) into Eq. (5.5), we have

$$\begin{aligned} \delta\rho_M &\simeq -\frac{k^2}{a^2} \left[(w_1 - 2w_2)\chi + \frac{w_2}{\phi}\psi \right] \\ &= -\frac{k^2}{a^2} \left[\frac{w_1 - 2w_2}{H}\Phi + \frac{w_2}{\phi}\psi \right], \end{aligned} \quad (5.7)$$

where, in the second equality, we expressed χ in terms of Φ . From Eqs. (4.16) and (4.18) we eliminate v and obtain

$$\dot{\delta\rho}_M + 3H\delta\rho_M + \frac{k^2}{a^2} \left(\rho_M\chi - w_1\alpha - \frac{w_2}{\phi}\delta\phi \right) = 0. \quad (5.8)$$

We take the time derivative of Eq. (5.7) and eliminate the terms $\dot{\delta\rho}_M$ and $\delta\rho_M$ in Eq. (5.8). In doing so, we exploit Eq. (5.6) with the definition of \mathcal{Y} given in Eq. (4.21) to remove the $\dot{\psi}$ term. The perturbation $\alpha + \dot{\chi}$ can be expressed in terms of the Bardeen gravitational potential Ψ . This process leads to

$$\phi^2(w_1 - 2w_2)w_3\Psi + \mu_1\Phi + \mu_2\psi \simeq 0, \quad (5.9)$$

where

$$\mu_1 \equiv \frac{\phi^2}{H} [(\dot{w}_1 - 2\dot{w}_2 + Hw_1 - \rho_M)w_3 - 2w_2(w_2 + Hw_3)], \quad (5.10)$$

$$\mu_2 \equiv \phi(w_2^2 + Hw_2w_3 + \dot{w}_2w_3) + w_2(w_6\phi^2 - w_3\dot{\phi}). \quad (5.11)$$

We also take the time derivative of Eq. (5.6) and eliminate the $\dot{\mathcal{Y}}$ and \mathcal{Y} terms in Eq. (4.19). Then, it follows that

$$2\phi^2w_2\Psi + \mu_3\Phi + \mu_4\psi \simeq 0, \quad (5.12)$$

where

$$\mu_3 \equiv \frac{2\phi}{Hw_3}\mu_2, \quad (5.13)$$

$$\begin{aligned} \mu_4 &\equiv -\frac{1}{w_3} [\phi^3(w_6^2 + 2w_3w_7) \\ &\quad + \phi^2(2w_2w_6 + Hw_3w_6 + w_3\dot{w}_6) \\ &\quad + \phi\{w_2^2 + Hw_2w_3 + w_3(\dot{w}_2 - \dot{\phi}w_6)\} - 2\dot{\phi}w_2w_3]. \end{aligned} \quad (5.14)$$

We can solve Eqs. (5.7), (5.9), and (5.12) for Ψ , Φ , and ψ , as

$$\Psi \simeq -\frac{H(\mu_2\mu_3 - \mu_1\mu_4)}{\phi\mu_5} \frac{a^2}{k^2} \rho_M \delta, \quad (5.15)$$

$$\Phi \simeq \frac{\phi H[2w_2\mu_2 - w_3\mu_4(w_1 - 2w_2)]}{\mu_5} \frac{a^2}{k^2} \rho_M \delta, \quad (5.16)$$

$$\psi \simeq \frac{\phi H[w_1w_3\mu_3 - 2w_2(\mu_1 + w_3\mu_3)]}{\mu_5} \frac{a^2}{k^2} \rho_M \delta, \quad (5.17)$$

where

$$\begin{aligned} \mu_5 &\equiv (w_1 - 2w_2)[\phi(w_1 - 2w_2)w_3\mu_4 - 2\phi w_2\mu_2] \\ &\quad + Hw_2[2w_2(\mu_1 + w_3\mu_3) - w_1w_3\mu_3]. \end{aligned} \quad (5.18)$$

Note that we used the approximation $\delta \simeq \delta\rho_M/\rho_M$, which is valid deep inside the sound horizon. From Eqs. (4.27) and (4.30), the effective gravitational coupling and the gravitational slip parameter are given, respectively, by

$$G_{\text{eff}} = \frac{H(\mu_2\mu_3 - \mu_1\mu_4)}{4\pi\phi\mu_5}, \quad (5.19)$$

$$\eta = \frac{\phi^2[2w_2\mu_2 - w_3\mu_4(w_1 - 2w_2)]}{\mu_2\mu_3 - \mu_1\mu_4}. \quad (5.20)$$

Under our approximation scheme, the rhs of Eq. (4.25) is neglected relative to the lhs, so that

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho_M\delta \simeq 0, \quad (5.21)$$

where we used Eq. (4.27). For a given model we can integrate Eq. (4.25) for δ by using the analytic expression

(5.19). In Sec. VI we shall confirm the validity of the above quasistatic approximation for a class of dark energy models in generalized Proca theories.

B. Estimates for G_{eff} and η

We rewrite the effective gravitational coupling (5.19) and the gravitational slip parameter (5.20) in more convenient forms by using physical quantities like q_S and c_S^2 associated with no-ghost and stability conditions (along the similar line performed in Ref. [41] for scalar Horndeski theories). In doing so, we first substitute the relations $w_1 = w_2 - 2Hq_T$ and $w_3 = -2\phi^2 q_V$ into Eq. (5.19) with μ_i given by Eqs. (5.10)–(5.11), (5.13)–(5.14), and (5.18). From the definitions of w_1 , q_T , and w_6 in Eqs. (4.7), (3.15), and (4.12), it follows that

$$G_{3,X} = -\frac{1}{2\phi^3} [w_2 + w_6\phi + 8H\phi^4 G_{4,XX} - 2H^2\phi^3 (G_{5,X} + \phi^2 G_{5,XX})], \quad (5.22)$$

$$G_{4,X} = -\frac{1}{8H\phi^2} (w_2 - w_6\phi - 4H^2\phi^3 G_{5,X}), \quad (5.23)$$

$$G_4 = \frac{1}{8H} (4Hq_T - w_2 + w_6\phi). \quad (5.24)$$

On using these relations with the background equations (2.13)–(2.14), the terms $\rho_M + P_M$ and w_7 can be expressed as

$$\rho_M + P_M = -2q_T \dot{H} - \frac{\dot{\phi}}{\phi} w_2, \quad (5.25)$$

$$w_7 = \frac{1}{2H\phi^3} [(w_2 - w_6\phi)\dot{H}\phi + (w_2 + w_6\phi)H\dot{\phi}]. \quad (5.26)$$

We substitute these relations into Eq. (5.3) and then express \dot{w}_6 with respect to c_S^2 . This allows us to eliminate the \dot{w}_6 term in the expression of G_{eff} (which appears through μ_4). The resulting effective gravitational coupling G_{eff} contains the time derivatives \dot{H} and $\dot{\phi}$. Taking the time derivative of Eq. (2.15) for the branch $\phi \neq 0$, combining it with Eq. (2.14), and eliminating the G_2 and $G_{2,X}$ terms on account of Eqs. (2.13) and (2.15), we can write \dot{H} and $\dot{\phi}$ in terms of w_1 , q_T , and w_4 . Employing the relation (5.2) to express w_4 with respect to q_S , it follows that

$$\dot{H} = \frac{3w_2^2 - q_S}{2q_T q_S} (\rho_M + P_M), \quad (5.27)$$

$$\dot{\phi} = -\frac{3w_2\phi}{q_S} (\rho_M + P_M). \quad (5.28)$$

After setting $P_M = 0$ for nonrelativistic matter, Eqs. (5.19) and (5.20) reduce, respectively, to

$$G_{\text{eff}} = \frac{\xi_2 + \xi_3}{\xi_1}, \quad (5.29)$$

$$\eta = \frac{\xi_4}{\xi_2 + \xi_3}, \quad (5.30)$$

with the shorthand notations

$$\xi_1 = 4\pi\phi^2 (w_2 + 2Hq_T)^2, \quad (5.31)$$

$$\xi_2 = [H(w_2 + 2Hq_T) - \dot{w}_1 + 2\dot{w}_2 + \rho_M]\phi^2 - \frac{w_2^2}{q_V}, \quad (5.32)$$

$$\begin{aligned} \xi_3 = \frac{1}{8H^2\phi^2 q_S^3 q_T c_S^2} & \left[2\phi^2 \{ q_S [w_2 \dot{w}_1 - (w_2 - 2Hq_T) \dot{w}_2] \right. \\ & + \rho_M w_2 [3w_2 (w_2 + 2Hq_T) - q_S] \} \\ & \left. + \frac{q_S}{q_V} w_2 \{ w_2 (w_2 - 2Hq_T) - w_6 \phi (w_2 + 2Hq_T) \} \right]^2, \end{aligned} \quad (5.33)$$

$$\begin{aligned} \xi_4 = \frac{w_2 + 2Hq_T}{4Hq_S^2 q_V q_T c_S^2} & \left[4H^2\phi^2 q_S^2 q_V q_T c_S^2 \right. \\ & + 2\phi^2 q_S q_V w_2 \dot{w}_2 (w_2 - 2Hq_T) + w_2^2 \{ \phi q_S w_6 (w_2 + 2Hq_T) \\ & - w_2 q_S (w_2 - 2Hq_T) - 2\phi^2 q_S q_V \dot{w}_1 \\ & \left. + 2\phi^2 q_V [q_S - 3w_2 (w_2 + 2Hq_T)] \rho_M \} \right]. \end{aligned} \quad (5.34)$$

One can extract useful information from the expressions (5.29) and (5.30). First, the terms proportional to $1/q_V$ in ξ_2 and ξ_3 do not vanish for

$$\begin{aligned} w_2 = -\phi^2 [\phi G_{3,X} + 4H(G_{4,X} + \phi^2 G_{4,XX}) \\ - H^2\phi(3G_{5,X} + \phi^2 G_{5,XX})] \neq 0. \end{aligned} \quad (5.35)$$

If the functions $G_{3,4,5}$ do not have any X dependence, which is the case for GR, then $w_2 = 0$ and hence G_{eff} is not affected by the vector contribution q_V . In such cases we have $w_1 = -4HG_4$ and $q_T = 2G_4$ with constant G_4 , so the quantities (5.31)–(5.34) reduce, respectively, to $\xi_1 = 64\pi G_4^2 H^2 \phi^2$, $\xi_2 = (4G_4 H^2 + 4G_4 \dot{H} + \rho_M)\phi^2$, $\xi_3 = 0$, and $\xi_4 = 4G_4 H^2 \phi^2$. Using the relation $4G_4 \dot{H} = -\rho_M$, which follows from the background equations (2.13)–(2.15), we obtain $G_{\text{eff}} = 1/(16\pi G_4)$ and $\eta = 1$. Since GR corresponds to $G_4 = 1/(16\pi G)$, the effective gravitational coupling reduces to G .

For the theories with $w_2 \neq 0$ the term ξ_3 does not generally vanish, so G_{eff} and η generally differ from G and 1 respectively. Under the three no-ghost and stability conditions $q_S > 0$, $q_T > 0$, and $c_S^2 > 0$, we have that $\xi_3 > 0$. Since ξ_1 is also positive, the presence of the term ξ_3/ξ_1 in Eq. (5.29) increases the gravitational attraction. In

the expression of ξ_2 there exists the term $-w_2^2/q_V$ sourced by the vector sector, which is negative under the no-ghost condition $q_V > 0$. Hence the contribution from the vector sector to ξ_2/ξ_1 works to suppress the gravitational attraction.

In view of the recent tension between the RSD and the Planck data [35–39], we would like to discuss whether the vector field allows the possibility of realizing the gravitational interaction weaker than that in GR. Since ξ_3/ξ_1 is positive, the *necessary* condition for realizing G_{eff} smaller than the Newton gravitational constant G is given by $\xi_2/\xi_1 < G$, i.e.,

$$\begin{aligned} & \phi^2[(w_2 + 2Hq_T)\{H - 4\pi G(w_2 + 2Hq_T)\}] \\ & - \dot{w}_1 + 2\dot{w}_2 + \rho_M < \frac{w_2^2}{q_V}. \end{aligned} \quad (5.36)$$

For the function G_2 containing the standard Maxwell term F , we may write G_2 in the form $G_2 = F + g_2(X, F, Y)$, in which case $q_V = 1 + g_{2,F} + 2g_{2,Y}\phi^2 - 4g_5H\phi + 2G_6H^2 + 2G_{6,X}H^2\phi^2$. If the value of q_V gets smaller than 1 by the existence of functions $g_2(F, Y)$, g_5 , and G_6 , it tends to be easier to satisfy Eq. (5.36). Unlike the case of scalar-tensor Horndeski theories [41] the condition (5.36) does not solely depend on quantities associated with tensor perturbations, so the vector field allows a more flexible possibility for satisfying the necessary condition of weak gravity.

We would like to stress that the condition (5.36) is necessary but not sufficient to realize $G_{\text{eff}} < G$. Even for $\xi_2/\xi_1 < G$, it can happen that the existence of the positive term ξ_3/ξ_1 leads to G_{eff} larger than G . The effect of the vector sector also appears in the expressions of ξ_3 and ξ_4 . In order to see the possibility of G_{eff} smaller than G , we need to compute the three quantities ξ_1 , ξ_2 , and ξ_3 for given models. Note that, for the opposite inequality to that given in Eq. (5.36), G_{eff} is always larger than G .

C. Effective gravitational coupling on the de Sitter background

On the de Sitter fixed point characterized by $\dot{\phi} = 0$ and $\dot{H} = 0$, it is possible to simplify the effective gravitational coupling (5.29) further. Since in this case $\dot{w}_1 = \dot{w}_2 = \dot{w}_6 = 0$, $w_7 = 0$, and $\rho_M = P_M = 0$, the numerator (5.4) of c_S^2 reduces to

$$\begin{aligned} \mu_c &= [w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T)] \\ & \times [w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T) \\ & + 2H\phi^2q_V(w_2 + 2Hq_T)], \end{aligned} \quad (5.37)$$

which is required to be positive to avoid the Laplacian instability. Substituting Eq. (5.3) with Eq. (5.37) into Eq. (5.33), it follows that

$$\xi_3 = \frac{w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T)}{w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T) + 2H\phi^2q_V(w_2 + 2Hq_T)} \frac{w_2^2}{q_V}. \quad (5.38)$$

Under the conditions $\mu_c > 0$ and $q_V > 0$, the quantity ξ_3 is positive. Then the effective gravitational coupling (5.29) reads

$$G_{\text{eff}} = \frac{H(2H\phi^2q_V - w_6\phi - w_2)}{4\pi[2H\phi^2q_V(w_2 + 2Hq_T) + w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T)]}. \quad (5.39)$$

In the weak-coupling limit of vector perturbations ($q_V \rightarrow \infty$), G_{eff} reduces to

$$(G_{\text{eff}})_W = \frac{H}{4\pi(w_2 + 2Hq_T)}, \quad (5.40)$$

whereas, in the strong-coupling limit ($q_V \rightarrow 0$), we have

$$(G_{\text{eff}})_S = \frac{H(w_2 + w_6\phi)}{4\pi[w_6\phi(w_2 + 2Hq_T) - w_2(w_2 - 2Hq_T)]}. \quad (5.41)$$

The difference between $(G_{\text{eff}})_W$ and $(G_{\text{eff}})_S$ is given by

$$\Delta G_{\text{eff}} \equiv (G_{\text{eff}})_W - (G_{\text{eff}})_S = \frac{Hw_2^2}{2\pi[(w_2 + 2Hq_T)\{w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T)\}]}. \quad (5.42)$$

If the condition

$$(w_2 + 2Hq_T)\{w_2(w_2 - 2Hq_T) - w_6\phi(w_2 + 2Hq_T)\} > 0 \quad (5.43)$$

is satisfied, it follows that $(G_{\text{eff}})_W > (G_{\text{eff}})_S$. In this case, the effective gravitational coupling tends to decrease for a stronger coupling of vector modes (i.e., for smaller q_V). In Sec. VI we shall consider a class of generalized Proca theories to see how the change of q_V modifies G_{eff} .

VI. OBSERVABLES IN A CONCRETE DARK ENERGY MODEL

We study the evolution of perturbations associated with the observations of large-scale structures, weak lensing, and CMB for a dark energy model in generalized Proca theories. Let us consider theories given by the functions

$$\begin{aligned} G_2(X, Y, F) &= b_2 X^{p_2} + [1 + g_2(X)]F, \\ G_3(X) &= b_3 X^{p_3}, \quad G_4(X) = \frac{1}{16\pi G} + b_4 X^{p_4}, \\ G_5(X) &= b_5 X^{p_5}, \quad g_5(X) = \tilde{b}_5 X^{q_5}, \\ G_6(X) &= b_6 X^{p_6}, \end{aligned} \quad (6.1)$$

where G is the Newton gravitational constant, $b_{2,3,4,5,6}$, \tilde{b}_5 , $p_{2,3,4,5,6}$, q_5 are constants, and $g_2(X)$ is an arbitrary function of X with $g_2(0) = 0$. Compared to the model studied in Ref. [24], there exist additional functional freedoms in $g_2(X)$ (not necessarily proportional to X^{p_4-1}), $\tilde{b}_5 X^{q_5}$ (not necessarily satisfying $q_5 = p_5 - 1$), and $b_6 X^{p_6}$. The quantity F vanishes on the FLRW background, so they do not affect the background equations of motion. Since the background has $Y = 0$, by adopting the Taylor expansion around $Y = 0$, it is sensible to include a further additional term of the form $\tilde{b}_2 Y X^{q_2}$ in G_2 . However, for simplicity, we do not consider this term in the following. Since the background has a nonvanishing X , we do not adopt the Taylor expansion with respect to X .

A. Cosmological background

For the powers $p_{3,4,5}$ given by

$$\begin{aligned} p_3 &= \frac{1}{2}(p + 2p_2 - 1), \quad p_4 = p + p_2, \\ p_5 &= \frac{1}{2}(3p + 2p_2 - 1), \end{aligned} \quad (6.2)$$

the background solution of the form

$$\phi^p \propto H^{-1} \quad (6.3)$$

can be realized [24], where p is a positive constant. The vector Galileon [19] corresponds to the powers $p = p_2 = 1$. For positive p the temporal vector component ϕ is small in the early cosmological epoch, but it grows with the decrease of H to give rise to the late-time cosmic acceleration. According to the stability analysis around a

late-time de Sitter fixed point, it is always a stable attractor [24].

Since we are interested in the cosmological evolution after the end of the radiation era, we take into account non-relativistic matter alone for the matter Lagrangian \mathcal{L}_M (unlike Ref. [24] in which radiation is also present). We introduce the matter density parameter $\Omega_m = 8\pi G\rho_M/(3H^2)$ and the dimensionless quantities

$$y \equiv \frac{8\pi G b_2 \phi^{2p_2}}{3H^2 2^{p_2}}, \quad \beta_i \equiv \frac{p_i b_i}{2^{p_i - p_2} p_2 b_2} (\phi^p H)^{i-2}, \quad (6.4)$$

where $i = 3, 4, 5$ and the β_i 's are constants from Eq. (6.3). For the branch $\phi \neq 0$ of Eq. (2.15), we have the relation

$$1 + 3\beta_3 + 6(2p + 2p_2 - 1)\beta_4 - (3p + 2p_2)\beta_5 = 0, \quad (6.5)$$

which can be used to express β_3 in terms of β_4 and β_5 .

The dark energy density parameter is given by

$$\begin{aligned} \Omega_{\text{DE}} &= 1 - \Omega_m \\ &= \frac{6p_2^2(2p + 2p_2 - 1)\beta_4 - p_2(p + p_2)(1 + 4p_2\beta_5)}{p_2(p + p_2)} y, \end{aligned} \quad (6.6)$$

which satisfies the differential equation

$$\frac{d\Omega_{\text{DE}}}{d\mathcal{N}} = \frac{3(1+s)\Omega_{\text{DE}}(1-\Omega_{\text{DE}})}{1+s\Omega_{\text{DE}}}, \quad (6.7)$$

where $\mathcal{N} = \ln a$ and $s = p_2/p$. From the matter-dominated fixed point characterized by $\Omega_{\text{DE}} = 0$, the solutions finally approach a de Sitter attractor with $\Omega_{\text{DE}} = 1$. The equation of state of dark energy depends on Ω_{DE} , as

$$w_{\text{DE}} = -\frac{1+s}{1+s\Omega_{\text{DE}}}, \quad (6.8)$$

which evolves from $-1-s$ (matter era) to -1 (de Sitter epoch). The likelihood analysis based on the SN Ia, CMB, and BAO data showed that the constant s is constrained to be $0 \leq s < 0.36$ [58].

B. Evolution of perturbations

The theoretical consistency of the model (6.1) requires that the six quantities $q_T, c_T^2, q_V, c_V^2, q_S, c_S^2$ are positive in the small-scale limit. In Ref. [24] the parameter space consistent with these conditions was discussed for the specific functions $b_6 = 0$, $g_2 = -2c_2 G_{4,X}$, and $g_5 = d_2 G_{5,X}/2$. The generalization to the model (6.1) modifies neither the background equations of motion nor the second-order action of tensor perturbations, but the evolution of vector perturbations is subject to change. The scalar perturbation is also

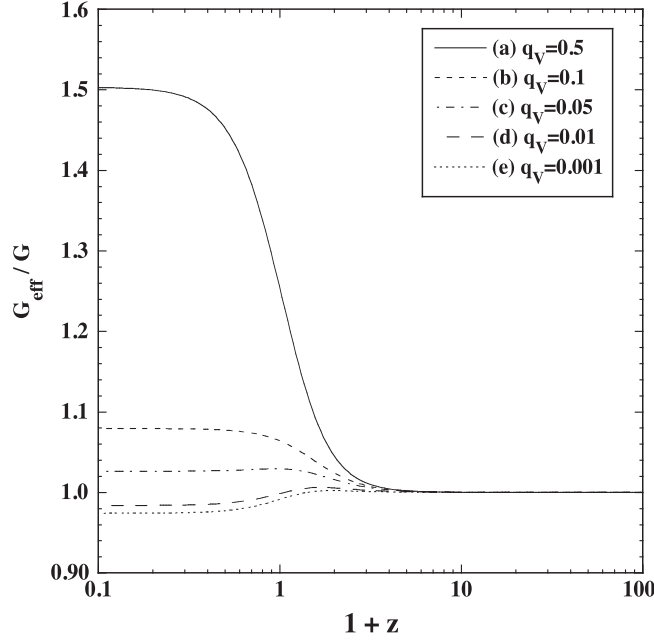


FIG. 1. Evolution of G_{eff}/G for the model parameters $p_2 = 1/2$, $p = p_6 = 5/2$, $g_2 = 0$, $\tilde{b}_5 = 0$, $\beta_4 = 10^{-4}$, $\beta_5 = 0.052$, $\lambda = 1$ with $q_V = 0.5, 0.1, 0.05, 0.01, 0.001$ (from top to bottom). The present epoch (the redshift $z = 0$) is identified as $\Omega_{\text{DE}} = 0.68$.

affected by the new terms of intrinsic vector modes through the change of q_V . In the following, we investigate how the new terms affect the evolution of scalar perturbations and observable quantities. The evolution of vector perturbations is discussed at the end of this section.

For the model given by the functions (6.1), the parameter q_V reads

$$\frac{G_{\text{eff}}}{G} = \frac{(p + p_2)[q_V u^2 - 2p_2 y \{1 - 6\beta_4(2p + 2p_2 - 3) + 2\beta_5(3p + 2p_2 - 3)\}]}{\mathcal{F}_G}, \quad (6.12)$$

where $u = \sqrt{8\pi G}\phi$, and

$$\begin{aligned} \mathcal{F}_G = & q_V u^2 [p + p_2 + 6\beta_4 p_2 y + p_2(p + p_2) \{1 - 6\beta_4(1 + 2p + 2p_2) + 2\beta_5(3 + 3p + 2p_2)\} y] \\ & + 2p_2 y [(p + p_2) \{-1 + 6\beta_4(2p + 2p_2 - 3) + \beta_5(6 - 6p - 4p_2)\} + 6p_2 \{18\beta_4^2(2p + 2p_2 - 1) \\ & - \beta_4[1 + \beta_5(30p + 28p_2 - 6)] + 6\beta_5^2(p + p_2)\} y]. \end{aligned} \quad (6.13)$$

The value of G_{eff} at the de Sitter attractor depends on the parameters p , p_2 , β_4 , β_5 and the quantities y , $q_V u^2$. Let us consider the constant q_V model realized by the nonvanishing Lagrangian \mathcal{L}_6 with

$$p_6 = p, \quad g_2 = 0, \quad \tilde{b}_5 = 0. \quad (6.14)$$

In Fig. 1 we plot the evolution of G_{eff}/G for $p_2 = 1/2$, $p = p_6 = 5/2$, $\beta_4 = 10^{-4}$, and $\beta_5 = 0.052$ versus the

$$q_V = 1 + g_2 - 4\tilde{b}_5 X^{q_5} H\phi + 2b_6(1 + 2p_6)H^2 X^{p_6}, \quad (6.9)$$

where the last term arises for the theories with $b_6 \neq 0$. From Eq. (6.3) the last term of Eq. (6.9) is proportional to $\phi^{2(p_6-p)}$, so it is constant for $p_6 = p$. Depending on the sign of the term $2b_6(1 + 2p_6)$, q_V is either larger or smaller than 1.

The variables w_2 and q_T can be expressed in the following forms:

$$w_2 = -\frac{2^{1-p_2/2}}{\sqrt{24\pi G}} p_2 \phi^{p_2} \sqrt{b_2 y} [1 + 6\beta_4(1 - 2p_2 - 2p) + 2\beta_5(3p + 2p_2)], \quad (6.10)$$

$$q_T = \frac{1}{8\pi G} \left[1 + 6\beta_4 p_2 \left(\frac{1}{p_2 + p} - 2 \right) y + 6\beta_5 p_2 y \right]. \quad (6.11)$$

In the asymptotic past where Ω_{DE} is negligibly small, we have $y \rightarrow 0$ and hence $w_2 \rightarrow 0$ and $q_T \rightarrow 1/(8\pi G)$. This means that, in the early matter era, the quantities ξ_i in Eq. (5.31)–(5.34) are approximately given by $\xi_1 \simeq H^2 \phi^2 / (4\pi G^2)$, $\xi_2 \simeq H^2 \phi^2 / (4\pi G)$, $\xi_3 \simeq 0$, and $\xi_4 \simeq H^2 \phi^2 / (4\pi G)$, respectively, where we used the approximate background equation of motion $\dot{H} \simeq -4\pi G \rho_M$ (neglecting the contribution of dark energy density). Then, in the early matter-dominated epoch, the effective gravitational coupling (5.29) and the slip parameter (5.30) are close to G and 1, respectively.

After the dark energy dominance the quantity w_2 starts to be away from 0, which leads to the deviation of G_{eff} from G . From Eq. (5.39) the effective gravitational coupling on the de Sitter solution is given by

redshift $z = a_0/a - 1$ with five different values of q_V . We choose the negative coefficient $b_2 = -m^2(8\pi G)^{p_2-1}$ (where $m^2 > 0$) with $\lambda \equiv u^p H/m$. For the model parameters given above, all the no-ghost and stability conditions of tensor, vector, and scalar perturbations are consistently satisfied. In Fig. 1 we see that G_{eff} is close to G in the early matter era independent of q_V , but the late-time evolution of G_{eff} is different depending on the values of q_V .

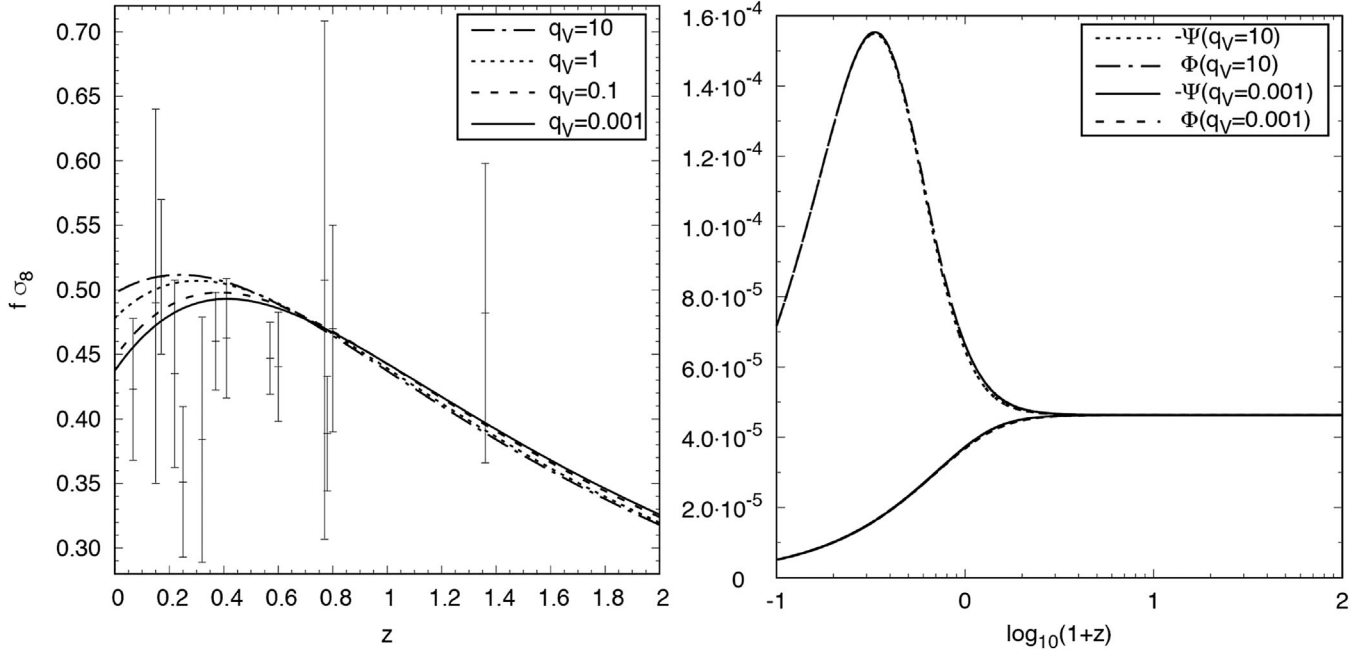


FIG. 2. (Left) Evolution of $f\sigma_8$ for the same model parameters as those used in Fig. 1 with $q_V = 10, 1, 0.1, 0.001$. The initial conditions of perturbations are chosen to match those under the subhorizon approximation discussed in Sec. V with the comoving wave number $k = 230a_0H_0$ and $\sigma_8(z=0) = 0.82$. The black points with error bars correspond to the bounds of $f\sigma_8$ constrained from the data of redshift-space-distortion measurements [59–65]. (Right) Evolution of the gravitational potentials $-\Psi, \Phi$ for $q_V = 10, 0.001$.

For the model parameters used in Fig. 1, the asymptotic values of y and u at the de Sitter attractor are given, respectively, by $y = -0.906$ and $u = 1.252$. The analytic estimation (6.12) shows that, for smaller q_V , the effective gravitational coupling at the de Sitter fixed point decreases, e.g., $G_{\text{eff}}/G = 1.503$ for $q_V = 0.5$ and $G_{\text{eff}}/G = 0.974$ for $q_V = 0.001$. In fact, we have numerically confirmed that the condition (5.43) is satisfied for the model parameters used in Fig. 1. Thus, for q_V close to 0, it is possible to realize G_{eff} smaller than G .

We also numerically computed the quantities ξ_1, ξ_2, ξ_3 and found that the contribution ξ_2/ξ_1 to G_{eff} becomes negative at low redshifts in the numerical simulation of Fig. 1. This is overwhelmed by the positive contribution ξ_3/ξ_1 to G_{eff} , such that G_{eff} stays positive. Thus, the necessary condition (5.36) for realizing $G_{\text{eff}} < G$ is satisfied for all the cases shown in Fig. 1, but we need to evaluate the ξ_3/ξ_1 term for each value of q_V to discuss whether weak gravity is really possible.

In the left panel of Fig. 2, we show the evolution of $f\sigma_8$ for several different values of q_V derived by numerically integrating the perturbation equations (4.15)–(4.20). We choose the comoving wave number $k = 230a_0H_0$, which is within the linear regime of perturbations in the observations of large-scale structures [55]. We recall that, under the quasistatic approximation on scales deep inside the sound horizon ($c_s^2k^2/a^2 \gg H^2$), the matter perturbation obeys Eq. (5.21) with G_{eff} given by Eq. (5.29). In the numerical simulations of Fig. 2 the sound speed squared tends to be

larger for smaller q_V , whose present value is in the range $\mathcal{O}(0.1) < c_s^2 < \mathcal{O}(10^2)$. By solving Eq. (5.21) with Eq. (5.29) numerically, we confirmed that the evolution of δ obtained under the quasi-approximation exhibits very good agreement with the full numerical solutions of Eqs. (4.15)–(4.20). In fact, the theoretical curves of $f\sigma_8$ derived under the quasistatic approximation for the modes $c_s^2k^2/a^2 \gg H^2$ are almost indistinguishable from those obtained by full integrations.

As we see in Fig. 2, the theoretical values of $f\sigma_8$ in low redshifts get smaller for decreasing q_V . This behavior reflects the fact that G_{eff} at the de Sitter fixed point tends to be smaller for q_V closer to 0. In Fig. 2 we also show the observational data constrained from the RSD measurements (including the recent FastSound data [65] measured at the highest redshift $z = 1.4$). To plot the theoretical curves, we have chosen the value $\sigma_8(z=0) = 0.82$ constrained by the recent Planck CMB data [39]. The theoretical prediction is in tension with some of the RSD data, but this property also persists in the Λ CDM model for $\sigma_8(z=0)$ constrained from Planck observations. The tension reduces for smaller $\sigma_8(z=0)$ constrained from the WMAP data [40]. In any case, the present RSD data are not sufficiently accurate to place tight constraints on model parameters of the theory. It is however interesting to note that the models with different values of q_V can be potentially distinguished from each other in future RSD measurements.

In the right panel of Fig. 2 we also plot the evolution of the gravitational potentials for $q_V = 10, 0.001$. As in the

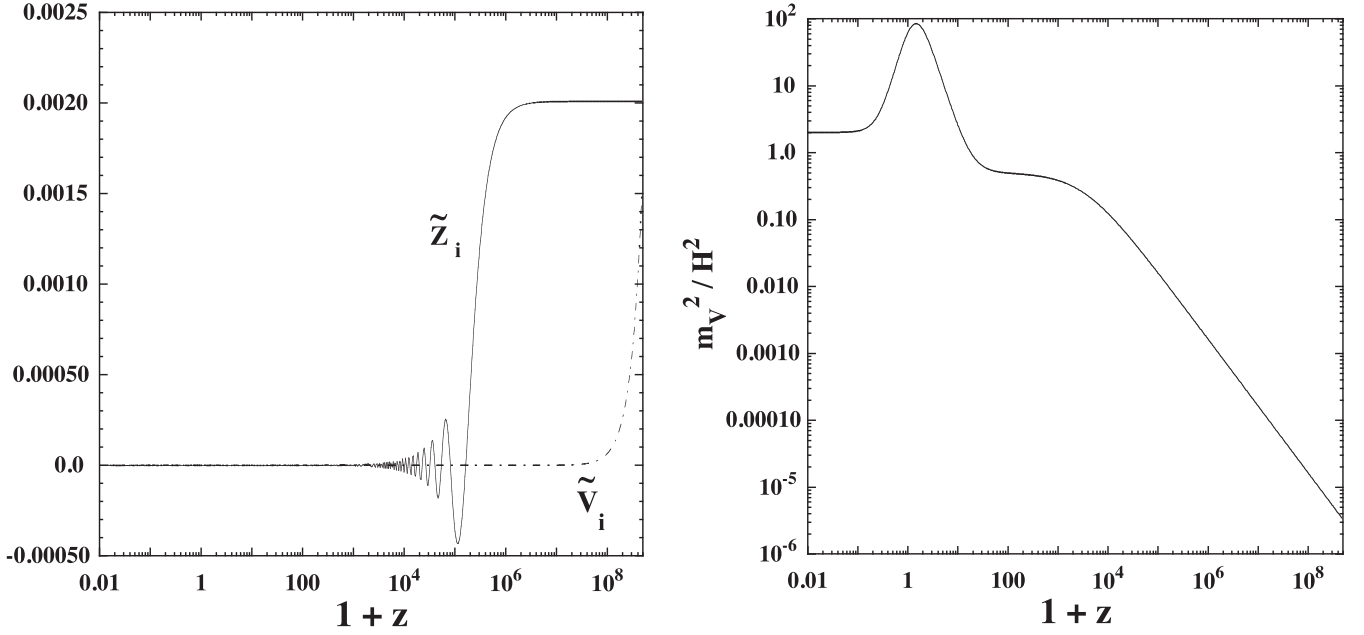


FIG. 3. Evolution of the vector perturbations \tilde{Z}_i (normalized by $1/\sqrt{8\pi G}$) and \tilde{V}_i for the case $q_V = 0.001$ (left) and evolution of the vector mass squared m_V^2 divided by H^2 (right). The model parameters are the same as those used in Fig. 1 with $q_V = 0.001$. The initial conditions are chosen, at the redshift $z = 4.77 \times 10^8$, as $\Omega_{\text{DE}} = 9.74 \times 10^{-38}$, $\Omega_r \equiv 8\pi G \rho_r / (3H^2) = 1 - 6.888 \times 10^{-6}$, $\tilde{Z}_i = 2.0095 \times 10^{-3} / \sqrt{8\pi G}$, $d\tilde{Z}_i/d\mathcal{N} = -10^{-8} / \sqrt{8\pi G}$, and $\tilde{V}_i = 0.0015$ with $v_{i,r} = v_{i,m}$. We choose the comoving wave number to be $k = 230a_0H_0$.

case of GR, both $-\Psi$ and Φ stay nearly constant in the deep matter era with the slip parameter η very close to 1. They start to vary around the end of the matter-dominated epoch, but the difference between $-\Psi$ and Φ is small. Hence the evolution of the effective gravitational potential $-\Phi_{\text{eff}}$ defined by Eq. (4.31) is similar to that of $-\Psi$ and Φ . The deviation of the slip parameter η from 1 is typically insignificant for theoretically consistent model parameters.

In Fig. 2 we see that the gravitational potentials are enhanced for $q_V = 10$ after the onset of cosmic acceleration. This enhancement occurs due to the strong gravitational coupling with $G_{\text{eff}} > G$. On the other hand, for $q_V = 0.001$, both $-\Psi$ and Φ start to decay after the end of the matter era. Thus, it should be possible to distinguish the models with large and small values of q_V from the integrated Sachs-Wolfe effect of CMB observations.

Finally, we discuss the evolution of vector perturbations from the deep radiation era to the de Sitter epoch. For this purpose, we take into account radiation besides nonrelativistic matter in the forms $\rho_M = \rho_r + \rho_m$ and $P_M = \rho_r/3$ with the velocity perturbations $v_{i,r}$, $v_{i,m}$ and solve Eqs. (3.30) and (3.31) numerically. In the left panel of Fig. 3, the evolution of \tilde{Z}_i and \tilde{V}_i is plotted for $q_V = 0.001$ and the wave number $k = 230a_0H_0$. At the initial stage of the radiation era the perturbations are outside the vector sound horizon ($c_V^2 k^2 / a^2 < H^2$), in which regime the dynamical field \tilde{Z}_i is nearly frozen. After the entry of the vector sound horizon, \tilde{Z}_i starts to oscillate with a decreasing amplitude. In this

regime, the evolution of \tilde{Z}_i is well described by the WKB solution given by Eq. (3.39). As we see in Fig. 3, the perturbation \tilde{V}_i does not grow either.

In the right panel of Fig. 3, we show the evolution of the mass squared $m_V^2 = C_2/q_V$ of the dynamical vector field \tilde{Z}_i . The ratio m_V^2/H^2 grows from the radiation era to today and it finally approaches the asymptotic value $m_V^2/H^2 = 2$ at the de Sitter attractor. For small q_V closer to 0, there is a tendency that the mass m_V gets larger than the order of H at low redshifts. In such cases the oscillations of \tilde{Z}_i are also present even for small k , but the amplitude of \tilde{Z}_i does not increase. In summary, there is no growth of \tilde{Z}_i for the dark energy model studied above.

VII. CONCLUSIONS

One promising way to tackle dark energy and cosmological constant problems is to invoke new dynamical degrees of freedom in addition to those appearing in the standard model of particle physics. Modifications in the form of an additional scalar degree of freedom have been mostly studied in the literature. Among them the Galileon and Horndeski interactions received much attention, as the latter are the most general scalar-tensor theories with second-order equations of motion. On the other hand, the presence of a vector degree of freedom can also induce interesting phenomenology besides providing a self-acceleration of the Universe.

In this work, we followed this latter approach and considered the most general vector-tensor interactions in the form of generalized Proca theories with five propagating degrees of freedom, i.e., the two tensor gravitational degrees of freedom and the two transverse and one longitudinal mode of the vector field. To realize some nontrivial cosmological dynamics with a gauge-invariant vector field, one usually needs to introduce spatial components of it at the background level. In our case the $U(1)$ gauge symmetry is explicitly broken, so that the existence of the temporal vector component can lead to interesting cosmological solutions with a late-time de Sitter attractor.

The action of our generalized Proca theories has been constructed in such a way that time derivatives higher than second order do not arise to avoid the Ostrogradski instability. The temporal component ϕ of the vector field, which appears as an auxiliary field, can be entirely expressed in term of the Hubble expansion rate H . The de Sitter solutions, which are relevant to dark energy, can be realized for constant values of ϕ and H . We obtained second-order actions of tensor, vector, and scalar perturbations on top of the general FLRW background in the presence of a matter fluid. This allowed us to derive general conditions for avoiding ghosts and Laplacian instabilities in the small-scale limit.

In difference to the previous analysis, the perturbations coming from the sixth-order Lagrangian \mathcal{L}_6 and the quadratic Lagrangian \mathcal{L}_2 containing the X , F , Y dependence (which preserves the parity invariance) are included as well. The presence of purely vector interactions in \mathcal{L}_2 , g_5 , \mathcal{L}_6 has an important impact on the no-ghost and stability conditions for vector perturbations and on the sound speed of scalar perturbations. To guarantee the absence of any theoretical pathology, we require that six no-ghost and stability conditions are satisfied. This permits us to shrink the allowed parameter space of the theory drastically.

The main goal of this work was to study observational signatures of generalized Proca theories related with linear cosmological perturbations. For this purpose, we derived the full perturbation equations of motion for tensor, vector, and scalar modes and then analytically obtained the effective gravitational coupling G_{eff} with matter density perturbations and the slip parameter η by employing the quasistatic approximation on scales deep inside the sound horizon. In view of the recent tension between the data of redshift-space distortions and CMB, we identified the necessary condition for realizing G_{eff} smaller than the Newton gravitational constant G . One can nicely observe the important impact of intrinsic vector modes on G_{eff} in the quantity q_V associated with the vector no-ghost condition. For smaller q_V there is a tendency that G_{eff} decreases, so the vector field plays an important role to modify the gravitational interaction on cosmological scales relevant to the observations of large-scale structures and weak lensing.

For concreteness, we have considered a class of dark energy models in which the temporal vector component ϕ is of the form $\phi^p \propto H^{-1}$ with $p > 0$. This solution, which has a late-time de Sitter attractor, can be realized for the functions $G_{2,3,4,5,6}$ given by Eq. (6.1) with the powers (6.2). As we see in Fig. 1, it is indeed possible to realize $G_{\text{eff}} < G$ for small q_V , while satisfying six no-ghost and stability conditions. We also numerically integrated the scalar perturbation equations of motion to study the evolution of the growth rate $f\sigma_8$ as well as the gravitational potentials Ψ and Φ . We confirmed that the full numerical results show excellent agreement with those derived under the quasistatic approximation for the perturbations deep inside the sound horizon. As we see in Fig. 2, the evolution of observables is quite different at low redshifts depending on the values of q_V . Since the dark energy equation of state w_{DE} is also smaller than -1 , it is possible to distinguish our model from the Λ CDM model according to both expansion history and cosmic growth.

Concerning the vector perturbations, we have also provided an analytic estimation for the evolution of the transverse vector modes. This analytic estimation has been also confirmed by numerically solving the perturbation equations (3.30) and (3.31) for the model (6.1). The evolution of the vector modes is characterized as follows: far outside the vector sound horizon, the perturbations \tilde{Z}_i are nearly constants. After the horizon entry ($c_V^2 k^2 / a^2 > H^2$), the perturbations start to decay with oscillations. Thus, there is no growth for the dynamical vector fields \tilde{Z}_i .

We have thus shown that generalized Proca theories offer a nice possibility for realizing a dark energy model with peculiar observational signatures. It is of interest to put observational constraints on the allowed parameter space of the model, which we leave for a future work.

ACKNOWLEDGMENTS

We would like to thank M. Motta and F. Piazza for very useful discussions. We are also grateful to T. Okumura for providing us with the recent RSD data. A. D. F. was supported by JSPS KAKENHI Grants No. 16K05348, No. 16H01099. L. H. acknowledges financial support from Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation. R. K. is supported by the Grant-in-Aid for Research Activity Start-up of the JSPS No. 15H06635. The work of S. M. was supported in part by JSPS KAKENHI Grant No. 24540256 and World Premier International Research Center Initiative (WPI), MEXT, Japan. S. T. is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 24540286, No. 16K05359, and MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas ‘‘Cosmic Acceleration’’ (No. 15H05890). Y. Z. is supported by the Strategic Priority Research Program ‘‘The Emergence of

Cosmological Structures” of the Chinese Academy of Sciences, Grant No. XDB09000000.

APPENDIX: SUBHORIZON LIMIT AND QUASISTATIC APPROXIMATION

In this appendix we shall clarify the distinction between the subhorizon limit and the quasistatic approximation.

In the subhorizon approximation we suppose that modes of interest have physical momenta k/a sufficiently higher than the Hubble expansion rate H (but sufficiently lower than the cutoff of the theory under consideration). Let us then introduce a small bookkeeping parameter ϵ ($\ll 1$) so that $Ha/k = \mathcal{O}(\epsilon)$. In the subhorizon limit ($\epsilon \ll 1$) it makes perfect sense to consider a dispersion relation for each propagating mode since there is a clear separation between the scale of the background and that of the perturbation. Assuming that the modes of interest approximately have linear dispersion relations in the subhorizon limit, it is easy to see that a time derivative acted on perturbation variables is of order $H \times \mathcal{O}(\epsilon^{-1})$. With this assignment, we keep the lowest-order part of the quadratic action written in terms of canonically normalized perturbation variables (after eliminating nondynamical variables of course). We can consider this procedure as the subhorizon limit in the context of cosmological perturbations.

In the scalar perturbation sector of the system considered in the present paper, there are two propagating degrees of freedom, one from gravity and the other from dust matter. They follow coupled second-order differential equations. Therefore, a general solution in the scalar sector is a linear combination of four independent modes. This means that we can derive a fourth-order differential equation for one

master variable, e.g., the gauge-invariant density contrast. We can also express all the other (dynamical and non-dynamical) variables, e.g., two gauge-invariant potentials, as linear combinations of the master variable and its derivatives up to third order.

When the sound speed of the degree of freedom from gravity is of order unity, one can easily show that the scalar sector includes two fast modes and two slow modes since the sound speed of dust matter is zero. The two fast modes have the time scale of order a/k , while the two slow modes have the time scale of order $1/H$ ($\gg a/k$).

The quasistatic approximation is nothing but dropping the fast modes and keeping the slow modes in order to describe an adiabatic evolution of the system. In practice we can easily take the quasistatic approximation: we start with the fourth-order differential equation for one master variable and consider that a time derivative acted on the master variable is of order $H \times \mathcal{O}(\epsilon^0)$ [instead of $H \times \mathcal{O}(\epsilon^{-1})$]. By keeping the leading-order contribution in the small- ϵ limit with this new assignment, one obtains a second-order differential equation for the master variable. This is the equation of motion in the quasistatic approximation describing the two slow modes only. From the equation of motion for the gauge-invariant density contrast in the quasistatic approximation, one can easily read off the effective gravitational constant G_{eff} . One can also apply the quasistatic approximation to the expressions of the two gauge-invariant potentials to obtain the Poisson equation and the slip parameter η . The expressions for G_{eff} and η obtained in this way completely agree with those obtained in the main text by a different method.

-
- [1] A. G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998); S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999).
- [2] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006); A. Silvestri and M. Trodden, *Rep. Prog. Phys.* **72**, 096901 (2009); S. Tsujikawa, in *Dark Matter and Dark Energy: A Challenge for Modern Cosmology*, edited by S. Matarrese, M. Colpi, V. Gorini, and U. Moschella (Springer, New York, 2011), p. 331; T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Phys. Rep.* **513**, 1 (2012); A. Joyce, B. Jain, J. Khoury, and M. Trodden, *Phys. Rep.* **568**, 1 (2015); P. Bull *et al.*, *Phys. Dark Univ.* **12**, 56 (2016).
- [3] Y. Fujii, *Phys. Rev. D* **26**, 2580 (1982); L. H. Ford, *Phys. Rev. D* **35**, 2339 (1987); C. Wetterich, *Nucl. Phys.* **B302**, 668 (1988); B. Ratra and P. J. E. Peebles, *Phys. Rev. D* **37**, 3406 (1988); T. Chiba, N. Sugiyama, and T. Nakamura, *Mon. Not. R. Astron. Soc.* **289**, L5 (1997); P. G. Ferreira and M. Joyce, *Phys. Rev. Lett.* **79**, 4740 (1997); R. R. Caldwell, R. Dave, and P. J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998).
- [4] T. Chiba, A. De Felice, and S. Tsujikawa, *Phys. Rev. D* **87**, 083505 (2013); S. Tsujikawa, *Classical Quantum Gravity* **30**, 214003 (2013).
- [5] L. Amendola, *Phys. Rev. D* **60**, 043501 (1999); J. P. Uzan, *Phys. Rev. D* **59**, 123510 (1999); T. Chiba, *Phys. Rev. D* **60**, 083508 (1999); N. Bartolo and M. Pietroni, *Phys. Rev. D* **61**, 023518 (1999); F. Perrotta, C. Baccigalupi, and S. Matarrese, *Phys. Rev. D* **61**, 023507 (1999).
- [6] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [7] J. Martin, C. Schmid, and J. P. Uzan, *Phys. Rev. Lett.* **96**, 061303 (2006); R. Gannouji, D. Polarski, A. Ranquet, and A. A. Starobinsky, *J. Cosmol. Astropart. Phys.* **09** (2006) 016; L. Perivolaropoulos, *J. Cosmol. Astropart. Phys.* **10** (2005) 001.
- [8] L. Amendola and S. Tsujikawa, *Phys. Lett. B* **660**, 125 (2008); W. Hu and I. Sawicki, *Phys. Rev. D* **76**, 064004

- (2007); A. A. Starobinsky, *JETP Lett.* **86**, 157 (2007); S. Tsujikawa, *Phys. Rev. D* **77**, 023507 (2008); H. Motohashi, A. A. Starobinsky, and J. Yokoyama, *Prog. Theor. Phys.* **123**, 887 (2010).
- [9] S. M. Carroll, I. Sawicki, A. Silvestri, and M. Trodden, *New J. Phys.* **8**, 323 (2006); R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, *Phys. Rev. D* **75**, 064020 (2007); S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol, and J. Yokoyama, *Phys. Rev. D* **77**, 103009 (2008); R. Gannouji, B. Moraes, D. F. Mota, D. Polarski, S. Tsujikawa, and H. A. Winther, *Phys. Rev. D* **82**, 124006 (2010).
- [10] C. Deffayet, G. Esposito-Farese, and A. Vikman, *Phys. Rev. D* **79**, 084003 (2009); C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **80**, 064015 (2009).
- [11] M. V. Ostrogradski, *Mem. Acad. St. Petersburg VI* **4**, 385 (1850).
- [12] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [13] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, *Phys. Rev. D* **84**, 064039 (2011); T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Prog. Theor. Phys.* **126**, 511 (2011); C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, *Phys. Rev. Lett.* **108**, 051101 (2012).
- [14] C. de Rham and L. Heisenberg, *Phys. Rev. D* **84**, 043503 (2011); L. Heisenberg, R. Kimura, and K. Yamamoto, *Phys. Rev. D* **89**, 103008 (2014).
- [15] C. Deffayet, G. Esposito-Farese, and A. Vikman, *Phys. Rev. D* **79**, 084003 (2009); C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **80**, 064015 (2009).
- [16] C. Deffayet, A. E. Gumrukcuoglu, S. Mukohyama, and Y. Wang, *J. High Energy Phys.* **04** (2014) 082.
- [17] C. Deffayet, S. Mukohyama, and V. Sivanesan, *Phys. Rev. D* **93**, 085027 (2016).
- [18] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **82**, 061501 (2010).
- [19] L. Heisenberg, *J. Cosmol. Astropart. Phys.* **05** (2014) 015.
- [20] G. Tasinato, *J. High Energy Phys.* **04** (2014) 067; *Classical Quantum Gravity* **31**, 225004 (2014).
- [21] E. Allys, P. Peter, and Y. Rodriguez, *J. Cosmol. Astropart. Phys.* **02** (2016) 004.
- [22] J. B. Jimenez and L. Heisenberg, *Phys. Lett. B* **757**, 405 (2016).
- [23] G. W. Horndeski, *J. Math. Phys. (N.Y.)* **17**, 1980 (1976).
- [24] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y. I. Zhang, *J. Cosmol. Astropart. Phys.* **06** (2016) 048.
- [25] J. D. Barrow, M. Thorsrud, and K. Yamamoto, *J. High Energy Phys.* **02** (2013) 146.
- [26] J. B. Jimenez, R. Durrer, L. Heisenberg, and M. Thorsrud, *J. Cosmol. Astropart. Phys.* **10** (2013) 064.
- [27] G. Tasinato, K. Koyama, and N. Khosravi, *J. Cosmol. Astropart. Phys.* **11** (2013) 037.
- [28] M. Hull, K. Koyama, and G. Tasinato, *J. High Energy Phys.* **03** (2015) 154; *Phys. Rev. D* **93**, 064012 (2016).
- [29] W. Li, [arXiv:1508.03247](https://arxiv.org/abs/1508.03247).
- [30] J. B. Jimenez and T. S. Koivisto, *Classical Quantum Gravity* **31**, 135002 (2014); J. B. Jimenez, L. Heisenberg, and T. S. Koivisto, *J. Cosmol. Astropart. Phys.* **04** (2016) 046.
- [31] G. D. Moore and A. E. Nelson, *J. High Energy Phys.* **09** (2001) 023; R. Kimura and K. Yamamoto, *J. Cosmol. Astropart. Phys.* **07** (2012) 050.
- [32] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), *Phys. Rev. Lett.* **116**, 061102 (2016); D. Blas, M. M. Ivanov, I. Sawicki, and S. Sibiryakov, *Pis'ma Zh. Eksp. Teor. Fiz.* **103**, 708 (2016).
- [33] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y. I. Zhang, and G. B. Zhao, *Phys. Rev. D* **93**, 104016 (2016).
- [34] P. Fleury, J. P. B. Almeida, C. Pitrou, and J. P. Uzan, *J. Cosmol. Astropart. Phys.* **11** (2014) 043.
- [35] F. Beutler *et al.* (BOSS Collaboration), *Mon. Not. R. Astron. Soc.* **443**, 1065 (2014).
- [36] L. Samushia *et al.*, *Mon. Not. R. Astron. Soc.* **439**, 3504 (2014).
- [37] E. Macaulay, I. K. Wehus, and H. K. Eriksen, *Phys. Rev. Lett.* **111**, 161301 (2013).
- [38] A. Vikhlinin *et al.*, *Astrophys. J.* **692**, 1060 (2009).
- [39] P. A. R. Ade *et al.* (Planck Collaboration), [arXiv:1502.01589](https://arxiv.org/abs/1502.01589).
- [40] G. Hinshaw *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **208**, 19 (2013).
- [41] S. Tsujikawa, *Phys. Rev. D* **92**, 044029 (2015).
- [42] A. De Felice, T. Kobayashi, and S. Tsujikawa, *Phys. Lett. B* **706**, 123 (2011).
- [43] L. Amendola, M. Kunz, M. Motta, I. D. Saltas, and I. Sawicki, *Phys. Rev. D* **87**, 023501 (2013).
- [44] E. Bellini and I. Sawicki, *J. Cosmol. Astropart. Phys.* **07** (2014) 050.
- [45] L. Perenon, F. Piazza, C. Marinoni, and L. Hui, *J. Cosmol. Astropart. Phys.* **11** (2015) 029.
- [46] C. de Rham, L. Heisenberg, and R. H. Ribeiro, *Classical Quantum Gravity* **32**, 035022 (2015).
- [47] C. de Rham, L. Heisenberg, and R. H. Ribeiro, *Phys. Rev. D* **90**, 124042 (2014); L. Heisenberg, *Classical Quantum Gravity* **32**, 105011 (2015); *J. Cosmol. Astropart. Phys.* **11** (2015) 005.
- [48] B. F. Schutz and R. Sorkin, *Ann. Phys. (N.Y.)* **107**, 1 (1977).
- [49] A. De Felice, J. M. Gerard, and T. Suyama, *Phys. Rev. D* **81**, 063527 (2010).
- [50] J. M. Bardeen, *Phys. Rev. D* **22**, 1882 (1980).
- [51] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1 (1984).
- [52] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
- [53] B. A. Bassett, S. Tsujikawa, and D. Wands, *Rev. Mod. Phys.* **78**, 537 (2006).
- [54] N. Kaiser, *Mon. Not. R. Astron. Soc.* **227**, 1 (1987).
- [55] M. Tegmark *et al.* (SDSS Collaboration), *Astrophys. J.* **606**, 702 (2004).
- [56] L. Amendola, M. Kunz, and D. Sapone, *J. Cosmol. Astropart. Phys.* **04** (2008) 013.
- [57] B. Boisseau, G. Esposito-Farese, D. Polarski, and A. A. Starobinsky, *Phys. Rev. Lett.* **85**, 2236 (2000); S. Tsujikawa, *Phys. Rev. D* **76**, 023514 (2007); S. Nesseris, *Phys. Rev. D* **79**, 044015 (2009); A. De Felice, S. Mukohyama, and S. Tsujikawa, *Phys. Rev. D* **82**, 023524 (2010); A. De Felice, R. Kase, and S. Tsujikawa, *Phys. Rev. D* **83**, 043515 (2011).
- [58] A. De Felice and S. Tsujikawa, *J. Cosmol. Astropart. Phys.* **03** (2012) 025.
- [59] W. J. Percival *et al.* (2dFGRS Collaboration), *Mon. Not. R. Astron. Soc.* **353**, 1201 (2004).

- [60] F. Beutler, C. Blake, M. Colless, D. H. Jones, L. Staveley-Smith, G. B. Poole, L. Campbell, Q. Parker, W. Saunders, and F. Watson, *Mon. Not. R. Astron. Soc.* **423**, 3430 (2012).
- [61] C. Blake *et al.*, *Mon. Not. R. Astron. Soc.* **415**, 2876 (2011).
- [62] L. Samushia, W. J. Percival, and A. Raccanelli, *Mon. Not. R. Astron. Soc.* **420**, 2102 (2012).
- [63] B. A. Reid *et al.*, *Mon. Not. R. Astron. Soc.* **426**, 2719 (2012).
- [64] S. de la Torre *et al.*, *Astron. Astrophys.* **557**, A54 (2013).
- [65] T. Okumura *et al.*, *Publ. Astron. Soc. Jpn.* **68**, 38 (2016).