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Primordial magnetic field and kinetic theory with Berry curvature

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We study the generation of a magnetic field in primordial plasma of standard model particles at a temperature $T > 80$ TeV—much higher than the electroweak scale. It is assumed that there is an excess number of right-handed electrons compared to left-handed positrons in the plasma. Using the Berry-curvature modified kinetic theory to incorporate the effect of the Abelian anomaly, we show that this chiral imbalance leads to the generation of a hypermagnetic field in the plasma in both the collision dominated and collisionless regimes. It is shown that, in the collision dominated regime, the chiral-vorticity effect can generate finite vorticity in the plasma together with the magnetic field. Typical strength of the generated magnetic field is 10^{27} G at $T \sim 80$ TeV with the length scale $10^5/T$, whereas the Hubble length scale is $10^{13}/T$. Furthermore, the instability can also generate a magnetic field of the order 10^{31} G at a typical length scale $10/T$. But there may not be any vorticity generation in this regime. We show that the estimated values of the magnetic field are consistent with the bounds obtained from current observations.

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I. INTRODUCTION

There is a strong possibility that the observed magnetic fields in galaxies and in the intergalactic medium could be due to a process in the very early Universe. Understanding the origin and dynamics of the primordial magnetic field is one of the most intriguing problems of the cosmology (see the recent reviews [1–3]). It should be noted here that there still exists a possibility that the fields may not be of primordial origin but might be created during the gravitational collapse of galaxies [4,5]. In this work we are interested in the primordial origin of the magnetic fields. There exist several models describing the generation of primordial magnetic fields in terms of cosmological defects [6–8]: phase transitions [9–12], inflation [13,14], the electroweak Abelian anomaly [15,16], string cosmology [17,18], temporary electric charge nonconservation [19], the trace anomaly [20], and breaking gauge invariance [21]. In a recent work [22], it was shown that a process like Biermann battery can play a role in generating the primordial magnetic field just after the recombination era.

In recent times there has been considerable interest in studying the role of the quantum chiral anomaly in generation of the primordial magnetic field [23–25]. In Ref. [15] (see also [16]), it was argued that there can be more right-handed electrons over left-handed positrons due to a process in the early Universe at temperatures T very much higher than the electroweak phase transition (EWPT) scale ($\sim$100 GeV). Their number is effectively conserved at energy scales significantly above the electroweak phase transitions and this allows one to introduce the chiral chemical potentials $\mu_R(\mu_L)$. At temperatures lower than $T_R \sim 80$ TeV, processes related to electron chirality flipping may dominate over the Hubble expansion rate and the chiral chemical potentials are not defined [26–28]. Furthermore, the right-handed current is not conserved due to the Abelian anomaly in the standard model (SM), and it satisfies the following equation:

$$\partial_\mu \mathcal{J}^\mu_R = -\frac{g'^2 y_R^2}{64\pi^2}\mathcal{Y}^{\mu\nu}\tilde{\mathcal{Y}}^{\mu\nu} = -\frac{g'^2}{4\pi^2}\boldsymbol{\mathcal{E}}_Y\cdot\boldsymbol{B}_Y. \qquad (1)$$

Here, $\mathcal{Y}^{\mu\nu} = \partial^\mu Y^\nu - \partial^\nu Y^\mu$ is the field tensor associated with the hypercharge gauge field Y^μ, and $\tilde{\mathcal{Y}}^{\mu\nu} = 1/2\epsilon^{\mu\nu\rho\lambda}\mathcal{Y}_{\rho\lambda}$. $\boldsymbol{\mathcal{E}}_Y$ and $\boldsymbol{\mathcal{B}}_Y$ denote hyperelectric and hypermagnetic fields, respectively. Furthermore, g' indicates associate gauge coupling, and $y_R = -2$ represents the hypercharge of the right electrons. The right-hand sides of the first and second equality signs are related with the Chern-Simons number $n_{\rm CS}$:

$$n_{\rm CS} = -\frac{g'^2}{32\pi^2}\int d^3x\boldsymbol{\mathcal{B}}_Y\cdot\boldsymbol{\mathcal{Y}}. \qquad (2)$$

The anomaly equation (1) relates the change in the right-handed electron density to the variation of the topological (Chern-Simons or helicity) charge of the gauge fields. It was shown in Ref. [29] that the Chern-Simons term contributes, in the effective standard model Lagrangian of the field Y_μ, by the polarization effect through the nonzero mean pseudovector current $\mathcal{J}_{j5} = g'^2 y_R^2/2\langle\bar{e}_R\gamma_j\gamma_5 e_R\rangle = -g'^2 y_R^2\mu_{eR}\mathcal{B}_j/4\pi^2$, and the effective Lagrangian for gauge field Y_μ in the SM is [29,30]

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$$L_Y = -\frac{1}{4}\mathcal{Y}_{\mu\nu}\mathcal{Y}^{\mu\nu} - \mathcal{J}_\mu Y^\mu - \frac{g'^2 y_R^2 \mu_{eR}}{4\pi^2}\boldsymbol{\mathcal{B}}_Y \cdot \boldsymbol{Y}. \quad (3)$$

If these hypermagnetic fields survive at the time of the EWPT, they will produce ordinary magnetic fields due to the electroweak mixing $\mathcal{A}_\mu = \cos\theta_w Y_\mu$. Where Y_μ is the massless $U(1)_Y$ Abelian gauge hypercharge field. It was shown that the chiral imbalance in the early Universe could give rise to a magnetic field $B \sim 10^{22}$ G at the temperature $T \sim 100$ GeV with a typical inhomogeneity scale $\sim 10^6/T$ [15]. In this work the authors studied the Maxwell equations with the Chern-Simons term and a kinetic equation consistent with Eq. (1). It was found that the transverse modes could become unstable and give rise to the hypercharge magnetic field [31,32]. In Ref. [23] the authors used magnetohydrodynamics in the presence of chiral asymmetry to study the evolution of the magnetic field. They showed that the chiral-magnetic [31,33,34] and chiral-vorticity effects [35] can play a significant role in the generation and dynamics of the primordial magnetic field. Furthermore, it was demonstrated in Ref. [25] that evolution of the primordial magnetic field is strongly influenced by the chiral anomaly, even at a temperature as low as 10 MeV. It was shown that an isotropic and translationally invariant initial state of the standard model plasma in thermal equilibrium can become unstable in the presence of global charges [24]. The most general form of the polarization operator Π^{ij} can be written

$$\Pi^{ij}(\mathbf{k}) = (k^2\delta^{ij} - k^i k^j)\Pi_1(k^2) + i\epsilon^{ijk}k^k\Pi_2(k^2), \quad (4)$$

where $\mathbf{k}$ is a wave vector and $k^2 = |\mathbf{k}|^2$. This equation satisfies the transversality condition $k_i\Pi^{ij} = 0$. It should be noted here that the Chern-Simons term is $\propto \boldsymbol{Y}\cdot\boldsymbol{\partial Y}$, whereas the kinetic term is $\propto (\boldsymbol{\partial Y})^2$, and therefore the Chern-Simons term can dominate over large length scales. Thus, a nonzero value of Π_2 when $k \to 0$, implying the presence of a Chern-Simons term in the expression for free energy. Using the field theoretic framework in [24], it was shown that, for a sufficiently small $k < \Pi_2(k^2)/\Pi_1(k^2)$, the polarization tensor Π^{ij} has a negative eigenvalue, and the corresponding eigenmode provides instability.

Recently, there has been an interesting development in incorporating the parity-violating effects into a kinetic theory formalism (see Refs. [31,36–38]). In this approach the kinetic (Vlasov) equation is modified by including the Berry-curvature term which takes into account chirality of the particles. The modified kinetic equation is consistent with the anomaly equation (1). Incorporation of the parity-odd physics in kinetic theory leads to a redefinition of the Poisson brackets which includes a contribution from the Berry connection. Confidence that the new kinetic equation captures the proper physics stems from the fact that the equation is consistent with the anomaly equation (1) and that it also reproduces some of the known results obtained using the quantum field theory with the parity-odd interaction [39]. In fact, the "classical" kinetic equation can reproduce—in the leading order in the hard dense loop approximation—the parity-odd correlation of the underlying quantum field theory [30,40]. The modified kinetic equation can also be derived from the Dirac Hamiltonian by performing a semiclassical Foldy-Wouthuysen diagonalization [41,42]. The modified kinetic equation can be applied to both the high density and the high temperature regime [41]. Furthermore, in Ref. [39] normal modes of chiral plasma were analyzed using the modified kinetic theory in the context of heavy-ion collisions. In that work the authors found that, in the collisionless limit, the transverse branch of the dispersion relation could become unstable with a typical wave number $k \sim \alpha'\mu/\pi$, where α' is the coupling constant and μ refers to the chiral chemical potential.

Here, we would like to note that the authors in Ref. [15] have used a heuristically written kinetic equation which is consistent with Eq. (1) to study generation of the primordial magnetic field. In addition, authors have used an expression for the current by incorporating standard electric resistivity and the chiral-magnetic effect. The chiral-vorticity effect was not considered. It should be emphasized that the forms of the kinetic equation used in Ref. [15] and in the Berry-curvature modified theory [37] are very different. As both the approaches describe the same physics, it would be interesting to see under what conditions they give similar predictions. Keeping the above discussion in mind, we believe that it would be highly useful to consider the problem of generating a primordial magnetic field in the presence of the Abelian anomaly by using the Berry-curvature modified kinetic theory. In this work we incorporate the effect of collisions in the modified kinetic theory and derive expressions for the electric and magnetic resistivities. The new kinetic framework also allows us to calculate the generation of the primordial magnetic field and the vorticity. Further on in our calculation, we considered an isotropic and homogeneous initial state of the particle distribution function. The magnetic field is generated by the unstable transverse modes in the presence of chiral charges (Q_5). This can be seen by integrating Eq. (1) over space. One can write $\partial_0(Q_5 + \frac{\alpha'}{\pi}\mathcal{H}) = 0$, where $Q_5 = \int j^0 d^3x$ and the helicity $\mathcal{H} = \frac{1}{V}\int d^3x(\boldsymbol{Y}\cdot\boldsymbol{\mathcal{B}_Y})$. The finite helicity state can be created even if the initial state has $\mathcal{H} = 0$, but $Q_5 \neq 0$. Thus, the magnetogenesis by net nonzero chiral charges may not require any preexisting seed field.

This manuscript is organized into three forthcoming sections. In Sec. II we briefly state the $(3+1)$ formalism of MacDonald and Thorne [43] and the kinetic theory with the Berry curvature. In Sec. III we apply this formalism to a study of the primordial magnetic field generation in the presence of chiral asymmetry. We also calculate the vorticity generation in the plasma due to the chiral imbalance. Section IV contains our results and a brief discussion. We show that our estimated value of the peak magnetic field actually falls within the constraints obtained from current observations.

II. BASIC FORMALISM

A. Maxwell's equations in the expanding Universe

In this work we shall study the generation of a primordial magnetic field at the time when the temperature of the Universe was much higher than $T_R \sim 80$ TeV (much higher than the EWPT temperature). We intend to solve the coupled system of the modified kinetic and the Maxwell's equations in the expanding Universe background. Here, we note that we ignore the fluctuations in the metric due to the matter perturbation. For this, one needs to write the underlying equations in a general covariant form. Interestingly, the techniques developed in Refs. [44–47] allow one to write the system of kinetic and Maxwell's equations in the expanding background in a form that looks similar to their flat space-time form. In this formalism the well developed intuition and techniques of the flat space-time plasma physics can be exploited to study the problem at hand. This can be accomplished by choosing a particular set of fiducial observers [44] at each point of space-time at which all of the physical quantities, including hyperelectric and magnetic fields, are measured. A line element for the expanding background can be written using the Friedmann-Lemaître-Robertson-Walker metric as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{5}$$

where x, y, and z represent comoving coordinates. Here, t is the proper time seen by observers at a fixed x, y, and z and $a(t)$ is the scale factor. One can introduce the conformal time η using the definition $\eta = \int dt/a^2(t)$ to write this metric as

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)). \tag{6}$$

The (hyper)electric $\boldsymbol{E}_{\rm phy}$, the (hyper)magnetic $\boldsymbol{B}_{\rm phy}$, and the current density $\boldsymbol{J}_{\rm phy}$ are related to the corresponding fiducial quantities by transformations: $\boldsymbol{\mathcal{E}} = a^2\boldsymbol{E}_{\rm phy}$, $\boldsymbol{\mathcal{B}} = a^2\boldsymbol{B}_{\rm phy}$, $\mathcal{J} = a^3\boldsymbol{J}_{\rm phy}$. One can now write the Maxwell's equations in the fiducial frame as

$$\frac{\partial\boldsymbol{\mathcal{B}}}{\partial\eta} + \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} = 0, \tag{7}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}} = 4\pi\rho_e, \tag{8}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{B}} = 0, \tag{9}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} = 4\pi\mathcal{J} + \frac{\partial\boldsymbol{\mathcal{E}}}{\partial\eta}, \tag{10}$$

where $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{E}}$, ρ_e, and $\mathcal{J}$ are, respectively, the magnetic field, the electric field, the charge density, and the current density seen by the fiducial observer.

B. Kinetic theory with Berry curvature

The charge ρ_e and current $\mathcal{J}$ in the Maxwell's equations (7)–(10) can be calculated using the Berry-curvature modified kinetic equation, which is also consistent with the quantum anomaly equation. The modified kinetic equation is given by

$$\frac{\partial f}{\partial\eta} + \frac{1}{1+e\Omega_{\boldsymbol{p}}\cdot\boldsymbol{\mathcal{B}}}\left[(e\tilde{\boldsymbol{\mathcal{E}}} + e\tilde{\boldsymbol{v}}\times\boldsymbol{\mathcal{B}} + e^2(\tilde{\boldsymbol{\mathcal{E}}}\cdot\mathcal{B})\Omega_{\boldsymbol{p}})\cdot\frac{\partial f}{\partial\boldsymbol{p}} + (\tilde{\boldsymbol{v}} + e\tilde{\boldsymbol{E}}\times\Omega_{\boldsymbol{p}} + e(\tilde{\boldsymbol{v}}\cdot\Omega_p)\boldsymbol{\mathcal{B}})\cdot\frac{\partial f}{\partial\boldsymbol{r}}\right] = \left(\frac{\partial f}{\partial\eta}\right)_{\rm coll}, \tag{11}$$

where e is the charge of the particles and has relation with the electroweak (EW) mixing angle θ_w: $e = g'\cos\theta_w$. Also, the g' is related with the $U(1)_Y$ gauge coupling constant α' as $\alpha' = g'^2\cos^2\theta_w/4\pi$. $\tilde{\boldsymbol{v}} = \partial\epsilon_{\boldsymbol{p}}/\partial\boldsymbol{p} = \boldsymbol{v}$, and $e\tilde{\boldsymbol{\mathcal{E}}} = e\boldsymbol{\mathcal{E}} - \partial\epsilon_{\boldsymbol{p}}/\partial\boldsymbol{r}$. $\Omega_p = \pm\boldsymbol{p}/(2p^3)$ is the Berry curvature. $\epsilon_{\boldsymbol{p}}$ is defined as $\epsilon_{\boldsymbol{p}} = p(1 - e\boldsymbol{\mathcal{B}}\cdot\Omega_{\boldsymbol{p}})$, with $p = |\boldsymbol{p}|$. The positive sign corresponds to right-handed fermions, whereas the negative sign is for left-handed ones. In the absence of a Berry correction, i.e., $\Omega_{\boldsymbol{p}} = 0$, the above equation reduces to the Vlasov equation when the collision term on the right-hand side of Eq. (11) is absent.

As we have already stated, we are interested in the temperature regime $T > T_R \gg T_{\rm EW}$. At these temperatures, electrons are massless. The only process that can change electron chirality is its Yukawa interaction with the Higgs boson. However, at this temperature this interaction is not strong enough to alter electron chirality. It is important here to note that, for a temperature smaller than T_R, electron mass plays a major role in left-right asymmetry. Recently, in Ref. [48], it was shown that at a temperature of the order of MeV, the mass of the electron plays an important role in determining the magnetic properties of the protoneutron star by suppressing the chiral charge density during the core collapse of the supernova. However, for the present case we ignore the electron mass by considering only the $T > T_R$ regime. Thus, we write the particle number density modified by the Berry term as

$$N = \int\frac{d^3p}{(2\pi)^3}(1 + e\boldsymbol{\mathcal{B}}.\Omega_{\boldsymbol{p}})f. \tag{12}$$

The above equation (11) can be converted to the following form by multiplying it by $(1 + e\boldsymbol{\mathcal{B}}\cdot\Omega_{\boldsymbol{p}})$ and integrating over p:

$$\frac{\partial N}{\partial \eta} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{J}} = -e^2 \int \frac{d^3p}{(2\pi)^3}\left(\Omega_p \cdot \frac{\partial f}{\partial \boldsymbol{p}}\right)(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{B}}). \quad (13)$$

In Eq. (13) $\mathcal{J}$ is the total current and is defined as $\mathcal{J} = \Sigma_a \mathcal{J}_a$. Here, index a denotes the current contribution from different species of the fermion, e.g., right-left particles and their antiparticles. $\mathcal{J}_a$ is defined as

$$\mathcal{J}_a = -e^a \int \frac{d^3p}{(2\pi)^3}\left[\epsilon_p^a \frac{\partial f_a}{\partial \boldsymbol{p}} + e^a\left(\Omega_{\boldsymbol{p}}^a \cdot \frac{\partial f_a}{\partial \boldsymbol{p}}\right)\epsilon_{\boldsymbol{p}}^a \mathcal{B} + \epsilon_{\boldsymbol{p}} \Omega_{\boldsymbol{p}}^{\boldsymbol{a}} \times \frac{\partial f_a}{\partial \boldsymbol{r}}\right] + e^a(\boldsymbol{\mathcal{E}} \times \boldsymbol{\sigma}^a), \quad (14)$$

where $\sigma^a = \int d^3p/(2\pi)^3 \Omega_{\boldsymbol{p}}^a f_a$ and $\epsilon_{\boldsymbol{p}}^a = p(1 - e^a \boldsymbol{\mathcal{B}}\cdot\Omega_{\boldsymbol{p}}^a)$, with $p = |\boldsymbol{p}|$. Depending on the species, the charge e, the energy of the particles ϵ_p, the Berry curvature $\Omega_{\boldsymbol{p}}$, and the form of the distribution function f changes. For the right-handed particle, $a = R$ with a hypercharge e, for the right-handed antiparticle, $a = \bar{R}$ with a charge of $-e$, etc. It is clear from Eq. (13) that, in the presence of external electric and magnetic fields, the chiral current is no longer conserved. The first term in Eq. (14) is the usual current equivalent to the kinetic theory and the remaining second and third terms are the current contribution by the Berry correction. The last term results from the anomalous Hall effect and it vanishes for a spherically symmetric distribution function. If we follow the power counting scheme used in [31], i.e., $\boldsymbol{Y_\mu} = \mathcal{O}(\epsilon)$, $\partial_r = \mathcal{O}(\delta)$ [where $\boldsymbol{Y_\mu}$ represents the $U(1)_Y$ gauge field] and considering only those terms of the order of $\mathcal{O}(\epsilon\delta)$ in Eq. (11), we have

$$\left(\frac{\partial}{\partial \eta} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) f_a + \left(e^a \boldsymbol{\mathcal{E}} + e^a(\boldsymbol{v}\times\boldsymbol{\mathcal{B}}) - \frac{\partial \epsilon_p^a}{\partial \boldsymbol{r}}\right)\cdot \frac{\partial f_a}{\partial \boldsymbol{p}} = \left(\frac{\partial f_a}{\partial \eta}\right)_{\text{coll}}, \quad (15)$$

where we have taken $\boldsymbol{v} = \boldsymbol{p}/p$. In the subsequent discussion, we shall apply this equation to a study of the evolution of the primordial magnetic field.

C. Current and polarization tensor for chiral plasma

Here, we assume that the plasma of the standard particles is in a state of "thermal equilibrium" at a temperature $T > T_R$, and at these temperatures the masses of the plasma particles can be ignored. We also assume that a left-right asymmetry exists and that there is no large-scale electromagnetic field. Thus, the equilibrium plasma is considered to be in a homogeneous and isotropic state, which is similar to the assumptions made in Ref. [15,24]. For a homogeneous and isotropic conducting plasma in thermal equilibrium, the distribution function for different species is

$$f_{0a}(p) = \frac{1}{\exp(\frac{\epsilon_p^0 - \mu_a}{T}) + 1}. \quad (16)$$

If δf_R and $\delta f_{\bar{R}}$ are fluctuations in the distribution functions of the right electron and the right antiparticles around their equilibrium distribution. Then we can write the perturbed distribution functions as

$$f_R(\boldsymbol{r},\boldsymbol{p},\eta) = f_{0R}(p) + \delta f_R(\boldsymbol{r},\boldsymbol{p},\eta), \quad (17)$$

$$f_{\bar{R}}(\boldsymbol{r},\boldsymbol{p},\eta) = f_{0\bar{R}}(p) + \delta f_{\bar{R}}(\boldsymbol{r},\boldsymbol{p},\eta). \quad (18)$$

Subtracting the equation for $a = \bar{R}$ from $a = R$ using Eq. (15), one can write

$$\left(\frac{\partial}{\partial \eta} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) f(\boldsymbol{r},\boldsymbol{p},\eta) + ep\frac{\partial(\boldsymbol{\mathcal{B}}\cdot\Omega_{\boldsymbol{p}})}{\partial \boldsymbol{r}}\cdot\frac{\partial f_0}{\partial \boldsymbol{p}} + (\boldsymbol{\mathcal{E}}.\boldsymbol{v})\frac{df_0}{dp} = \left(\frac{\partial f(\boldsymbol{r},\boldsymbol{p},\eta)}{\partial \eta}\right)_{\text{coll}}, \quad (19)$$

where $f(\boldsymbol{r},\boldsymbol{p},\eta) = (f_R - f_{\bar{R}})$ and $f_0 = f_{0R} + f_{0\bar{R}}$. Here, we have used $\frac{\partial f^0}{\partial \boldsymbol{p}} = \boldsymbol{v}\frac{df^0}{dp}$. This equation relates the fluctuations of the distribution functions of the charged particles to the induced gauge field fluctuations. The gauge field fluctuations can be seen from the Maxwell's electromagnetic equations (7)–(10). Under the relaxation time approximation, the collision term can be written as $(\partial f_a/\partial\eta)_{\text{coll}} \approx -\nu_c(f_a - f_{0a})$ (one can also look at some studies of chiral kinetic theory with collisions in Refs. [49,50]). Next, we take the Fourier transform of all of the perturbed quantities, namely, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$, and $f(\boldsymbol{r},\boldsymbol{p},\eta)$, by considering the spatiotemporal variation of these quantities as $\exp[-i(\omega\eta - \boldsymbol{k}\cdot\boldsymbol{r})]$. Then, using Eq. (19), one can get

$$f_{\boldsymbol{k},\omega} = \frac{-e[(\boldsymbol{v}\cdot\boldsymbol{\mathcal{E}_k}) + \frac{i}{2p}(\boldsymbol{v}\cdot\boldsymbol{\mathcal{B}_k})(\boldsymbol{k}\cdot\boldsymbol{v})]\frac{df_0}{dp}}{i(\boldsymbol{k}\cdot\boldsymbol{v} - \omega - i\nu_c)}. \quad (20)$$

Thus, the current contribution for the right-handed particle and the right-handed antiparticles in terms of a mode function can be written using Eq. (14) as (ignoring the anomalous Hall current part)

$$\mathcal{J}_{\boldsymbol{k}R} = e\int \frac{d^3p}{(2\pi)^3}\left[\left\{\boldsymbol{v} - \frac{i}{2p}(\boldsymbol{v}\times\boldsymbol{k})\right\} f_{\boldsymbol{k}\omega R} - \frac{e}{2p^2}\{\boldsymbol{\mathcal{B}_k} - \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\mathcal{B}_k})\} f_0 + \frac{e}{2p}\boldsymbol{\mathcal{B}_k}\frac{df_0}{dp}\right]. \quad (21)$$

In a similar way, we can get current contributions from the left-handed particle and the left-handed antiparticle. Thus, we can obtain the total current by adding the contributions from both the left- and right-handed particles and antiparticles by putting perturbations $f_{k\omega}$ for all species into Eq. (21) and adding them:

$$\mathcal{J}_{\boldsymbol{k}} = -m_D^2 \int \frac{d\Omega}{4\pi} \frac{\boldsymbol{v}(\boldsymbol{v}.\boldsymbol{\mathcal{E}}_{\boldsymbol{k}})}{i(\boldsymbol{k}.\boldsymbol{v} - \omega - i\nu_c)} - \frac{h_D^2}{2} \int \frac{d\Omega}{4\pi} \{\boldsymbol{\mathcal{B}}_k - \boldsymbol{v}(\boldsymbol{v}.\boldsymbol{\mathcal{B}}_{\boldsymbol{k}})\} - \frac{ig_D^2}{4} \int \frac{d\Omega}{4\pi} \frac{(\boldsymbol{v}\times\boldsymbol{k})(\boldsymbol{v}.\boldsymbol{\mathcal{B}}_{\boldsymbol{k}})(\boldsymbol{k}.\boldsymbol{v})}{(\boldsymbol{k}.\boldsymbol{v} - \omega - i\nu_c)} - \frac{c_D^2}{2} \int \frac{d\Omega}{4\pi} \left\{ \frac{\boldsymbol{v}(\boldsymbol{v}.\boldsymbol{\mathcal{B}}_{\boldsymbol{k}})(\boldsymbol{k}.\boldsymbol{v}) - (\boldsymbol{v}\times\boldsymbol{k})(\boldsymbol{v}.\boldsymbol{\mathcal{E}}_{\boldsymbol{k}})}{(\boldsymbol{k}.\boldsymbol{v} - \omega - i\nu_c)} + \boldsymbol{\mathcal{B}}_k \right\}. \tag{22}$$

Here, Ω represents the angular integrals. In Eq. (22), we have defined $m_D^2 = e^2 \int \frac{p^2 dp}{2\pi^2} \frac{d}{dp} (f_{0R} + f_{0\bar{R}} + f_{0L} + f_{0\bar{L}})$, $c_D^2 = e^2 \int \frac{p dp}{2\pi^2} \frac{d}{dp} (f_{0R} - f_{0\bar{R}} - f_{0L} + f_{0\bar{L}})$, $g_D^2 = e^2 \int \frac{dp}{2\pi^2} \frac{d}{dp} \times (f_{0R} + f_{0\bar{R}} + f_{0L} + f_{0\bar{L}})$, and $h_D^2 = e^2 \int \frac{dp}{2\pi^2} (f_{0R} - f_{0\bar{R}} - f_{0L} + f_{0\bar{L}})$.

Expression for the polarization tensor Π^{ij} can be obtained from Eq. (22) by writing the total current in the following form: $\mathcal{J}_k^i = \Pi^{ij}(k) Y_j(k)$, using $\boldsymbol{\mathcal{E}}_{\boldsymbol{k}} = -i\omega \boldsymbol{Y}_{\boldsymbol{k}}$ and $\boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = i(\boldsymbol{k} \times \boldsymbol{Y}_{\boldsymbol{k}})$. One can express Π^{ij} in terms of the longitudinal $P_L^{ij} = k^i k^j / k^2$, transverse $P_T^{ij} = (\delta^{ij} - k^i k^j / k^2)$, and axial $P_A^{ij} = i\epsilon^{ijk} k^k$ projection operators as $\Pi^{ik} = \Pi_L P_L^{ik} + \Pi_T P_T^{ik} + \Pi_A P_A^{ik}$. After performing the angular integrations in Eq. (22), one obtains Π_L, Π_T, and Π_A as given below:

$$\Pi_L = -m_D^2 \frac{\omega\omega'}{k^2} [1 - \omega' L(k)], \tag{23}$$

$$\Pi_T = m_D^2 \frac{\omega\omega'}{k^2} \left[1 + \frac{k^2 - \omega'^2}{\omega'} L(k) \right], \tag{24}$$

$$\Pi_A = -\frac{h_D^2}{2} \left[1 - \omega \left(1 - \frac{\omega'^2}{k^2} \right) L(k) - \frac{\omega'\omega}{k^2} \right], \tag{25}$$

where $L(k) = \frac{1}{2k} \ln(\frac{\omega' + k}{\omega' - k})$ and $\omega' = \omega + i\nu_c$. Also, $m_D^2 = 4\pi\alpha' (\frac{T^2}{3} + \frac{\mu_R^2 + \mu_L^2}{2\pi^2})$ and $h_D^2 = \frac{2\alpha' \Delta\mu}{\pi}$. We have defined $\Delta\mu = (\mu_R - \mu_L)$. In the above integrals, we have replaced e by α' using the relation $e^2 = 4\pi\alpha'$. First, consider the case where $\nu_c = 0$. In the limit $\omega \to 0$, Π_L and Π_T vanish and the parity-odd part of the polarization tensor $\Pi_A = h_D^2/2 \approx \alpha' \Delta\mu/\pi$. Here, it should be noted that Π_A does not get thermal correction. This could be due to the fact that the origin of the Π_A term is related to the axial anomaly, and it is well known that the anomaly does not receive any thermal correction [51–53]. This form of Π_A is similar to the result obtained in [24] using quantum field theoretic arguments at $T \leq 40$ GeV. However, in the kinetic theory approach presented here, no such assumption is made. Normal modes for the plasma can be obtained by using expressions for Π_L, Π_T, and Π_A. Using the equation $\partial_\nu F^{\mu\nu} = -4\pi \mathcal{J}^\mu$, we can write the following relation:

$$[M^{-1}]^{ij} Y_j(k) = -4\pi \mathcal{J}_k^i, \tag{26}$$

where $[M^{-1}]^{ik} = [(k^2 - \omega^2)\delta^{ik} - k^i k^k + \Pi^{ik}]$. Dispersion relations can be obtained from the poles of $[M^{-1}]^{ik}$, which are as given below:

$$\omega^2 = \Pi_L,$$
$$\omega^2 = k^2 + \Pi_T(k) \pm k\Pi_A.$$

One can study the normal modes of the chiral plasma and instabilities using these dispersion relations. However, it is more instructive to study the dynamical evolution of the magnetic field by explicitly writing time dependent Maxwell equations.

III. GENERATION OF THE PRIMORDIAL MAGNETIC FIELD AND VORTICITY

Plasmas with chirality imbalance are known to have instabilities that can generate magnetic fields in two different regimes: (i) collision dominated, k, $\omega \ll \nu_c$ [15], and (ii) the collisionless case, i.e., $\nu_c = 0$ [39]. In this section we analyze how the magnetic fields evolve in the plasma due to these instabilities, within the modified kinetic theory framework. Expression for the total current described by Eq. (22) can be written as $\mathcal{J}_k^i = \sigma_E^{ij} \mathcal{E}_k{}^j + \sigma_B^{ij} \mathcal{B}_k^j$, where σ_E^{ij} and σ_B^{ij} are electrical and magnetic conductivities. The integrals involved in Eq. (22) are rather easy to evaluate in the limit k, $\omega \ll \nu_c$, and one can write the expression for σ_E^{ij} and σ_B^{ij} as

$$\sigma_E^{ij} \approx \left(\frac{m_D^2}{3\nu_c} \delta^{ij} + \frac{i}{3\nu_c} \frac{\alpha' \Delta\mu}{\pi} \epsilon^{ijl} k^l \right), \tag{27}$$

$$\sigma_B^{ij} \approx -\frac{4}{3} \frac{\alpha' \Delta\mu}{\pi} \delta^{ij}, \tag{28}$$

Here, we would like to note that the Berry-curvature correction in the kinetic equation gives us an additional contribution in the expression for σ_E^{ij} which was not incorporated in Ref. [15]. The first term is the usual dissipative part of the electric current and it contributes to the Joules dissipation. The second term results from the chiral imbalance and it does not give any contribution to the Joules heating. As we shall demonstrate later, this term is responsible for the vorticity current [33]. One can write the total current as $\mathcal{J}_{\boldsymbol{k}}^i = \sigma_E^{ij} \mathcal{E}_{\boldsymbol{k}}^j + \sigma_B^{ij} \mathcal{B}_{\boldsymbol{k}}^j$ and the Maxwell's equation as $i(\boldsymbol{k} \times \boldsymbol{\mathcal{B}}_{\boldsymbol{k}})^i = 4\pi \mathcal{J}_{\boldsymbol{k}}^i$. Here, we have dropped the displacement current term (this is valid when $\frac{\omega}{4\pi\sigma} \ll 1$).

Next, by taking a vector product of $\boldsymbol{k}$ with the above Maxwell equation, one obtains (using Maxwell's equations and after some simplification)

$$\frac{\partial \boldsymbol{\mathcal{B}}_k}{\partial \eta} + \left(\frac{3\nu_c}{4\pi m_d^2}\right) k^2 \boldsymbol{\mathcal{B}}_k - \left(\frac{\alpha' \Delta\mu}{\pi m_D^2}\right)(\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{\mathcal{E}}_{\boldsymbol{k}})) + i\frac{4\alpha' \nu_c \Delta\mu}{\pi m_D^2}(\boldsymbol{k} \times \boldsymbol{\mathcal{B}}_{\boldsymbol{k}}) = 0. \tag{29}$$

This is the magnetic diffusivity equation for the chiral plasma. By replacing $(\boldsymbol{k} \times \boldsymbol{\mathcal{E}}_{\boldsymbol{k}})$ by $-\frac{1}{i}\frac{\partial \boldsymbol{\mathcal{B}}_k}{\partial \eta}$ in Eq. (29), we can solve this equation without a loss of generality by considering the propagation vector $\boldsymbol{k}$ in the z direction and the magnetic field having components perpendicular to the z axis. After defining two new variables, $\tilde{\mathcal{B}}_k = (\mathcal{B}_k^1 + i\mathcal{B}_k^2)$ and $\tilde{\mathcal{B}}_k' = (\mathcal{B}_k^1 - i\mathcal{B}_k^2)$, one can rewrite Eq. (29) as

$$\frac{\partial \tilde{\mathcal{B}}_k}{\partial \eta} + \left[\frac{\left(\frac{3\nu_c}{4\pi m_d^2}\right)k^2 - \left(\frac{4\alpha' \nu_c \Delta\mu}{\pi m_D^2}\right)k}{\left(1 + \frac{\alpha' \Delta\mu k}{\pi m_D^2}\right)}\right]\tilde{\mathcal{B}}_k = 0, \tag{30}$$

$$\frac{\partial \tilde{\mathcal{B}}_k'}{\partial \eta} + \left[\frac{\left(\frac{3\nu_c}{4\pi m_d^2}\right)k^2 + \left(\frac{4\alpha' \nu_c \Delta\mu}{\pi m_D^2}\right)k}{\left(1 - \frac{\alpha' \Delta\mu k}{\pi m_D^2}\right)}\right]\tilde{\mathcal{B}}_k' = 0. \tag{31}$$

Thus, the magnetic field vector $\mathcal{B}$ can be decomposed into these new variables, $\tilde{\mathcal{B}}_k$ and $\tilde{\mathcal{B}}_k'$, having definite helicity (or circular polarization). The effect of Ohmic decay is already there in the above equations due to the inclusion of collision frequency ν_c. It should be noted here that, if $\alpha' \Delta\mu k/\pi m_D^2 \ll 1$, Eq. (30) is similar to the magnetic field evolution equation considered in Ref. [15]. In this limit, Eq. (31) will give a purely damping mode. In this case, the dispersion relation will be

$$i\omega = \frac{3\nu_c}{4\pi m_d^2}k^2 - \frac{4\alpha' \nu_c \Delta\mu}{\pi m_D^2}k. \tag{32}$$

In the Appendix we show that the dispersion relation we have found here using kinetic theory matches the dispersion relation obtained in [39].

The instability can also occur in the collisionless regime ($\nu_c = 0$) [39]. In the quasistatic limit, i.e., $\omega \ll k$, one can define the electric conductivity as $\sigma_E^{ij} \approx \pi(m_D^2/2k)\delta^{ij}$ and the magnetic conductivity as $\sigma_B^{ij} \approx (h_D^2/2)\delta^{ij}$. Here, it should be noted that the above conductivities do not depend upon the collision frequency. Similar to the previous case, one can take the propagation vector in the z direction and then consider the components of the magnetic field in the direction perpendicular to the z axis. One can write a set of decoupled equations describing the evolution of the magnetic field using the variables $\tilde{\mathcal{B}}_k$ and $\tilde{\mathcal{B}}_k{}'$ as

$$\frac{\partial \tilde{\mathcal{B}}_k}{\partial \eta} + \left[\frac{k^2 - \frac{4\alpha' \Delta\mu k}{3}}{\frac{\pi m_D^2}{2k}}\right]\tilde{\mathcal{B}}_k = 0, \tag{33}$$

$$\frac{\partial \tilde{\mathcal{B}}_k'}{\partial \eta} + \left[\frac{k^2 + \frac{4\alpha' \Delta\mu k}{3}}{\frac{\pi m_D^2}{2k}}\right]\tilde{\mathcal{B}}_k{}' = 0. \tag{34}$$

Here, we note that, if one replaces $\partial/\partial\eta$ by $-i\omega$, Eqs. (33) and (34) give the same dispersion relation for the instability as discussed in Ref. [39].

A. Vorticity generated from chiral imbalance in the plasma

It would be interesting to see whether the instabilities arising due to chiral imbalance can lead to vorticity generation in the plasma. In order to study the vorticity of the plasma, we define the average velocity as

$$\langle \boldsymbol{v} \rangle = \frac{1}{\bar{n}}\int \frac{d^3p}{(2\pi)^3}\boldsymbol{v}(\delta f_R - \delta f_{\bar{R}} + \delta f_L - \delta f_{\bar{L}}). \tag{35}$$

Here, we have used the perturbed distribution function in the numerator of the above equation, which is due to the fact that the equilibrium distribution function is assumed to homogeneous and isotropic and therefore will not contribute to vorticity dynamics. The denominator is the total number density and is defined as (in Ref. [54], p. 63)

$$\bar{n} = n_{\text{particle}} - n_{\text{antiparticle}} = d_f \int_0^\infty \frac{d^3p}{(2\pi)^3}\left(\frac{1}{1+\exp(\frac{p-\mu}{T})} - \frac{1}{1+\exp(\frac{p+\mu}{T})}\right), \tag{36}$$

which, in the case of chiral plasma, gives $\bar{n} = \frac{2}{3}T^2(\mu_R + \mu_L)$. We consider the k, $\omega \ll \nu_c$ regime; in this case, the perturbed distribution function for, say, the right-handed particles can be written as

$$\delta f_{\boldsymbol{k},\omega R} = -\frac{e}{\nu_c}\left[(\boldsymbol{v}\cdot\boldsymbol{\mathcal{E}}_{\boldsymbol{k}}) + \frac{i}{2p}(\boldsymbol{v}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{k}})(\boldsymbol{k}\cdot\boldsymbol{v})\right]\frac{df_{0R}}{dp}. \tag{37}$$

If we add the contribution for all of the particles species and their antiparticles, one can write the numerator in Eq. (35) as

$$\sqrt{\frac{\alpha'}{\pi^3}}\frac{1}{\nu_c}\left(\frac{T^2}{3} + \frac{\mu_R^2 + \mu_L^2}{2\pi^2}\right)\boldsymbol{\mathcal{E}}_k \approx \sqrt{\frac{\alpha'}{\pi^3}}\frac{T^2}{3\nu_c}\boldsymbol{\mathcal{E}}_k. \tag{38}$$

Above, we have neglected $\frac{3}{2\pi^2}\frac{\mu_R^2+\mu_L^2}{T^2}$ in comparison to one, as the values of μ_R/T and μ_L/T are very small [$O(10^{-4})$]. One can write the average velocity as follows:

$$\langle \boldsymbol{v}_k \rangle = \sqrt{\frac{\alpha'}{\pi^3}} \frac{1}{2\nu_c} \frac{1}{\mu_R + \mu_L} \boldsymbol{\mathcal{E}}_k. \tag{39}$$

The vorticity can now be obtained by taking the curl of Eq. (39) and assuming that the chemical potentials and the temperature are constant in space and time:

$$\langle \boldsymbol{\omega}_k \rangle = i\sqrt{\frac{\alpha'}{\pi^3}} \frac{1}{2\nu_c} \frac{1}{\mu_R + \mu_L} (\boldsymbol{k} \times \boldsymbol{\mathcal{E}}_k). \tag{40}$$

One can find the contribution of the vorticity to the total current from Eq. (27) and $\mathcal{J}^i_{k\omega} = \sigma_E^{ij} \mathcal{E}_k{}^j + \sigma_B^{ij} \mathcal{B}^j_k$. By using Eq. (40), the vorticity current can be written as

$$\boldsymbol{\mathcal{J}}_\omega \approx -\sqrt{\frac{4\pi\alpha'}{9}} (\mu_R^2 - \mu_L^2) \boldsymbol{\omega} = \xi \boldsymbol{\omega}. \tag{41}$$

Thus, in the absence of any chiral imbalance, there is no vorticity current. Here, we note that our definition agrees with Ref. [55]. In the Appendix we demonstrate that our kinetic theory is also consistent with the second law of thermodynamics. Furthermore, using Eq. (40), one can eliminate $(\boldsymbol{k} \times \boldsymbol{\mathcal{E}}_{\boldsymbol{k}})$ in Eq. (29) and obtain

$$\frac{\partial \boldsymbol{\mathcal{B}}_k}{\partial \eta} + \frac{3\nu_c}{4\pi m_d^2} k^2 \boldsymbol{\mathcal{B}}_k + i\frac{\sqrt{4\pi\alpha'}\nu_c}{m_D^2} (\mu_R^2 - \mu_L^2)(\boldsymbol{k} \times \boldsymbol{\omega}_{\boldsymbol{k}}) + i\left(\frac{4\alpha'\nu_c}{\pi m_D^2}\right)(\mu_R - \mu_L)(\boldsymbol{k} \times \boldsymbol{\mathcal{B}}_{\boldsymbol{k}}) = 0. \tag{42}$$

In this equation, the second term is the usual diffusivity term. However, the third and fourth terms are additional terms, which, respectively, represent the vorticity and chiral-magnetic effects on the chiral plasma. Therefore, Eq. (29) actually contains terms resulting from vorticity and the magnetic effect. The saturated state of the instability can be studied by setting $\partial_\eta \boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = 0$ in Eq. (42). After taking a dot product of Eq. (42) with fluid velocity $\boldsymbol{v}_{\boldsymbol{k}}$ after setting $\partial_\eta \boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = 0$, one can obtain

$$\left(\boldsymbol{\omega}_{\boldsymbol{k}} - i\frac{16T\delta}{3}\boldsymbol{v}_k\right) \cdot \boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = 0. \tag{43}$$

Here, we have defined $\delta = \alpha'(\mu_R - \mu_L)/T$. We can write an expression for the magnetic field which satisfies the above equation (43) as

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = g(\boldsymbol{k})\boldsymbol{k} \times \left[\boldsymbol{\omega}_{\boldsymbol{k}} - i\frac{16T\delta}{3}\boldsymbol{v}_k\right]. \tag{44}$$

Where $g(k)$ is any general function, which can be determined by substituting the above expression for the magnetic field into Eq. (42) in a case of steady state. In a very large length scale, i.e., $\boldsymbol{k} \to 0$,

$$g(\boldsymbol{k}) = -\frac{3}{32}\sqrt{\frac{\pi^3}{\alpha'^3}} \frac{\mu_R^2 - \mu_L^2}{(\mu_R - \mu_L)^2}. \tag{45}$$

Thus, for a very large length scale $\boldsymbol{k} \to 0$, the magnetic field in the steady state is

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = -i\sqrt{\frac{\pi^3}{4\alpha'}} \frac{\mu_R^2 - \mu_L^2}{(\mu_R - \mu_L)} \boldsymbol{\omega}_{\boldsymbol{k}}. \tag{46}$$

This equation relates the vorticity generated during the instability to the magnetic field in the steady state.

However, in the collisionless regime ($\omega \ll k$ and $\nu_c = 0$), one can have an instability described by Eq. (33) with the typical scales $k \sim \alpha'\Delta\mu$ and $|\omega| \sim \alpha'^2 T\delta$ [39]. Using the expression for electric and magnetic conductivities for modes in this regime, one can write the magnetic diffusivity equation as

$$\frac{\partial \boldsymbol{\mathcal{B}}_{\boldsymbol{k}}}{\partial \eta} + \frac{k^2}{4\pi\sigma_1} \boldsymbol{\mathcal{B}}_{\boldsymbol{k}} - i\frac{T\delta}{\pi\sigma_1}(\boldsymbol{k} \times \boldsymbol{\mathcal{B}}_{\boldsymbol{k}}) = 0, \tag{47}$$

where $\sigma_1 = \pi m_D^2/2k$. Here, it should be noted that, unlike Eq. (42), the above equation does not have a vorticity term. The last term on the left-hand side arises from the chiral-magnetic effect. In the steady state ($\partial_\eta \boldsymbol{\mathcal{B}}_{\boldsymbol{k}} = 0$), one can get $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = (4T\delta)\boldsymbol{\mathcal{B}}$. This equation resembles the case of a magnetic field in a force free configuration of the conventional plasma where the plasma pressure is assumed to be negligible in comparison to the magnetic pressure [56]. However, for our case no such assumption about the plasma pressure is required.

IV. RESULTS AND DISCUSSION

In the previous sections, we applied the modified kinetic theory in the presence of chiral imbalance and obtained equations for the magnetic field generation for both the collision dominated and the collisonless regime. The instability can lead to generation of the magnetic field at the cost of the chiral imbalance. This can be seen from the anomaly equation $n_L - n_R + 2\alpha'\mathcal{H} = \text{constant}$ above $T > 80$ TeV, where $n_{L,R} = \frac{\mu_{L,R}T^2}{6}$ and $\mathcal{H}$ is the magnetic helicity, defined as

$$\mathcal{H} = \frac{1}{V}\int d^3x(\boldsymbol{Y} \cdot \boldsymbol{\mathcal{B}}_{\boldsymbol{Y}}). \tag{48}$$

One can estimate the strength of the generated magnetic field as follows. From Eqs. (1) and (2), one can notice that the right-handed electron number density n_R changes with the Chern-Simons number n_{cs} of the hypercharge field configuration, as $\Delta n_R = \frac{1}{2} y_R^2 n_{cs}$. Here, $n_{\rm CS} \approx \frac{g^2}{16\pi^2} kY^2$ and $\Delta n_R = \mu_R T^2 = \frac{88}{783}\delta T^3$ [54]. From this, one can estimate the magnitude of the generated physical magnetic field as

$$B_Y^{\rm phy} \approx \left[\frac{\pi^2 k\delta}{g^2\alpha' T}\right]^{\frac{1}{2}} T^2, \tag{49}$$

where we have used $B_Y^{\text{phy}} \sim kY$ and k^{-1} is the physical length scale, which is related to the comoving length by $k_{\text{phy}}^{-1} = (a/k_c)^{-1}$.

Now consider the regime $\omega, k \ll \nu_c$, where the dynamics for the magnetic field is described by Eqs. (30) and (31). Equation (30) clearly gives unstable modes for which $(\frac{T\delta}{\pi m_D^2})k < 1$ is satisfied. However, Eq. (31) gives a purely damping mode if the condition $(\frac{T\delta}{\pi m_D^2})k \ll 1$ is satisfied. One can rewrite this condition as $(\frac{T\delta}{3\pi\sigma\nu_c})k \sim (\frac{10^{-2}}{3\pi}\frac{\delta}{\nu_c})k \ll 1$. Here, we have used $m_D^2 = 3\nu_c\sigma$ with $\nu_c \sim \alpha'^2 \ln(\frac{1}{\alpha'})T$ [57] and $\sigma = 100T$. Thus, for $k \ll \nu_c$ and $\delta \ll 1$, Eq. (31) can only give purely damped modes. For these values of k and δ, Eq. (30) assumes a form similar to the equation for the magnetic field dynamics considered in Ref. [15]. If one replaces $\frac{\partial}{\partial\eta}$ by $-i\omega$ in Eq. (30), the dispersion relation for the unstable modes can be obtained. The fastest growth of the perturbation occurs for $k_{\max 1} \sim \frac{8T\delta}{3}$, and the maximum growth rate can be found to be $\Gamma_1 \sim \frac{16}{3\pi}\frac{T^2\delta^2}{m_D^2}\nu_c$. Here, we note that our $k_{\max 1}$ differs by a numerical factor from the value of k where the peak in the magnetic energy is calculated using chiral magnetohydrodynamics [23]. For $\delta \sim 10^{-6}$ and $\alpha' \sim 10^{-2}$, one can show that $\frac{k_{\max 1}}{\nu_c} \ll 1$ and $\frac{\Gamma_1}{\nu_c} \ll 1$ is satisfied. For these values of $k_{\max 1}$, α', and δ, one can estimate the magnitude of the generated magnetic field using Eq. (49). We find $B \sim 10^{26}$ G for $\alpha' \sim 10^{-2}$ and the typical length scale $\lambda \sim 10^5/T$. Here, we would like to note that the typical Hubble length scale is $\sim 10^{13}/T$, which is much larger than the typical length scale of instability. Our estimate of magnetic field strength B in the collision dominated regime broadly agrees with Ref. [15]. Here, we note again that Eq. (30) includes the effect of Ohmic decay due to the presence of the collision term. Our analysis shows that Ohmic decay is not important for instability. Furthermore, we have shown that the chiral instability can also lead to the generation of vorticity in the collision dominated regime. The typical length scale for vorticity is similar to that of the magnetic field. From Eq. (46), we take the magnitude of the vorticity to be $\omega_v \sim 10^{-4}B/T$.

Next, we analyze the chiral instability in collisionless regime $\nu_c \ll \omega \ll k$, considering Eqs. (33) and (34). Here, one finds the wave number $k_{\max 2} = \frac{8\delta T}{9}$, at which the maximum growth rate $\Gamma_2 = \frac{1}{2\pi}\frac{T^3\delta^3}{m_D^2}$ can occur. Now, $\frac{k_{\max 2}}{\nu_c} = \frac{8}{9}\frac{\delta}{\alpha^2} \ll 1$ and $\frac{\Gamma_2}{\nu_c} \sim \frac{3\delta^3}{8\pi^2\alpha^3\ln(1/\alpha)} \ll 1$, which puts constraints on the allowed values of δ. For $\delta \sim 10^{-1}$, $\alpha \sim 10^{-2}$, and $T \sim T_R$, one can estimate the magnitude of the magnetic field to be 10^{31} G. The typical length scale for the magnetic field is $\lambda_2 \sim 10/T$, which is much smaller than the length scale in the collision dominated case. This is to be expected, as the typical length scale associated with kinetic theory is smaller than the hydrodynamical case (related to the collision dominated regime).

The upper and lower bounds on the present observed magnetic field strength from the Planck 2015 results [58], and blazars [59,60] are between 10^{-17} G and 10^{-9} G. However, recently, in Ref. [61], it was shown that, if the magnetic field is helical and was created before the electroweak phase transition, then it can produce some baryon asymmetry. This can put more stringent bounds on the magnetic field (10^{-14} G–10^{-12} G). Since the magnetic fields and the plasma evolutions are coupled, the produced magnetic field may not evolve adiabatically—i.e., like $a(\eta)^{-2}$—due to plasma processes like turbulence. Similarly, the magnetic correlation length $\lambda_B \propto k_{\max}^{-1}$ may not be proportional to $a(\eta)$. Typical values of λ_B for the collision dominated and collisionless cases in our case are $10^5/T$ and $10/T$, respectively. The length scale of turbulence can be written as $\lambda_T \approx \frac{B_p}{\sqrt{\epsilon^{ch}+p^{ch}}}\eta \sim \frac{B_p}{\sqrt{\epsilon^{ch}+p^{ch}}}H^{-1}$, where B_p is the physical value of the magnetic field and ϵ^{ch} and p^{ch} are, respectively, the energy and pressure densities of the charged particles. $g_*^{ch}(T)$ and $g_*^{\text{total}}(T)$ are the number of degrees of freedom of the $U(1)$ charged particles in the thermal bath. For $\lambda_B \gg \lambda_T$, the effect of turbulence can be negligible. However, the maximum value of the magnetic field (for $\nu_c = 0$) is about 10^{31} G in our case, and this gives $\lambda_T \approx 10^6/T$. Thus, we have $\lambda_B \ll \lambda_T$ and, following Ref. [61], we assume that the generated magnetic fields will undergo an inverse cascade soon after their generation. One can relate B_p and λ_B, which undergo the process of inverse cascading with the present day values of magnetic field B_0 and the correlation length λ_0 using the following equations [61]:

$$B_P^{\text{IC}}(T) \simeq 9.3 \times 10^9 \text{ G}\left(\frac{T}{10^2 \text{ GeV}}\right)^{7/3}\left(\frac{B_0}{10^{-14} \text{ G}}\right)^{2/3} \times \left(\frac{\lambda_0}{10^2 \text{ pc}}\right)^{1/3}\mathcal{G}_B(T), \tag{50}$$

$$\lambda_B^{\text{IC}}(T) \simeq 2.4 \times 10^{-29} \text{ Mpc}\left(\frac{T}{10^2 \text{ GeV}}\right)^{-5/3}\left(\frac{B_0}{10^{-14} \text{ G}}\right)^{2/3} \times \left(\frac{\lambda_0}{1 \text{ pc}}\right)\mathcal{G}_\lambda(T), \tag{51}$$

where $\mathcal{G}_B(T) = (g_*^{\text{total}}(T)/106.75)^{1/6}(g_*^{ch}(T)/82.75)^{1/6}\times (g_{*s}(T)/106.75)^{1/3}$ and $\mathcal{G}_\lambda(T) = (g_*^{\text{total}}(T)/106.75)^{-1/3} (g_*^{ch}(T)/82.75)^{-1/3}(g_{*s}(T)/106.75)^{1/3}$. From these equations, one can see that, for the collision dominated case, $B_p \simeq 10^{26}$ G can be achieved when $B_0 \simeq 10^{-12}$ G and $\lambda_0 \simeq 100$ Kpc. However, in the collisionless regime, a value of $B_0 \simeq 10^{-11}$ G and $\lambda_0 \simeq 1$ Mpc at temperature $T = 80$ TeV gives the values that we have found in our estimates for the peak value of the magnetic field. Thus, the values of the magnetic field and the correlation length scale estimated by us can be consistent with the current bounds obtained from cosmic microwave background (CMB) observation and necessary for current observed baryon assymmetry. Since the value of B_p and λ_B for the collision dominated case are similar to that given in Ref. [62], we believe that they are also consistent with big bang nucleosynthesis constraints.

In conclusion, we have studied the generation of a magnetic field due to the anomaly in primordial plasma consisting of standard model particles. We have applied the Berry-curvature modified kinetic theory to the study of this problem. The effect of collision in the kinetic equation had been incorporated using the relaxation time approximation. We have found that chiral instability can occur in the presence of dissipation in both collision dominated and collisionless regimes. We have found in the collision dominated case that chiral instability can produce a magnetic field of the order of 10^{27} G, with the typical length scale being $10^5/T$. These results are in broad agreement with Ref. [15]. However, in that work the authors used a heuristic kinetic equation, and the collision term was not explicitly written in the kinetic equation. However, the expression for the total current included the Ohm's law. We have obtained expressions for electric and magnetic conductivities using the modified kinetic theory. We have found that the expression for electric conductivity in chiral plasma has a nondissipative term in addition to the standard Ohmic term. It has been shown that this new term is related to the vorticity current term found in chiral magnetohydrodynamics [63]. Furthermore, we have also studied chiral instability in the collisionless regime. It has been shown that, in this regime, a magnetic field of strength 10^{31} G can be generated at the length scale $10/T$. These length scales are much smaller than the length scale of the magnetic field in the collision dominated regime. Furthermore, the obtained values for the magnetic-field strength and the length scale have been shown to be consistent with the recent constraints from CMB data. We have also shown that in the collision dominated regime results of kinetic theory agree with the hydrodynamic treatment.

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APPENDIX: SECOND LAW OF THERMODYNAMICS AND DISPERSION RELATION FROM HYDRODYNAMICS FOR CHIRAL PLASMA

In Ref. [55] it was shown that parity-violating hydrodynamics violate the second law of thermodynamics, $\partial_\alpha s^\alpha \geq 0$, where s^α is the entropy current density, unless certain constraints on the transport coefficients are imposed. Therefore, our results in the collision dominated regime should be in agreement with Ref. [55]. The most general equations for U(1) and entropy currents can be written as [55]

$$\nu^\alpha = \nu'^\alpha + \xi(\mu, T)\omega^\alpha + \xi_B(\mu, T)\mathcal{B}^\alpha, \tag{A1}$$

$$s^\alpha = s'^\alpha + D(\mu, T)\omega^\alpha + D_B(\mu, T)\mathcal{B}^\alpha, \tag{A2}$$

where $\nu'^\alpha = \rho U^\alpha + \sigma\mathcal{E}^\alpha$ and $s'^\alpha = sU^\alpha - \frac{\mu}{T}\nu^\alpha$, with ρ and U^α being the charge density and the hydrodynamic four-velocity, respectively. Entropy density can be found using the thermodynamic relation $sT + \mu\rho = (\varepsilon + p)$, where ε denotes the energy density and p denotes the pressure. In the collision dominated limit there is no charge separation in the plasma, and one can regard the total charge density as zero. Using the energy-momentum tensor $T^{\alpha\beta} = (\varepsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}$, one can write the equation of motion as $\partial_\alpha T^{\alpha\beta} = F^{\beta\gamma} j_\gamma$ and the divergence of the entropy current as

$$\partial_\alpha(s^\alpha - D\omega^\alpha - D_B B^\alpha) = -(\nu^\alpha - \xi\omega^\alpha - \xi_B B^\alpha) \times \left(\partial_\alpha \frac{\mu}{T} - \frac{E_\alpha}{T}\right) - C\frac{\mu}{T}E_\alpha B^\alpha. \tag{A3}$$

According to Ref. [55], the second law of thermodynamic is satisfied if the following four equations are satisfied:

$$\partial_\alpha D - \frac{2\partial_\alpha p}{\varepsilon + p}D - \xi\partial_\alpha\frac{\mu}{T} = 0, \tag{A4}$$

$$\partial_\alpha D_B - \frac{\partial_\alpha p}{\varepsilon + p}D_B - \xi_B\partial_\alpha\frac{\mu}{T} = 0, \tag{A5}$$

$$-2D_B + \frac{\xi}{T} = 0, \tag{A6}$$

$$\frac{\xi_B}{T} - C\frac{\mu}{T} = 0. \tag{A7}$$

In Ref. [55] these equations are solved and one can know the dependence of ξ, ξ_B, D, and D_B on μ and T up to an arbitrary constant. Next, we assume the perturbation scheme considered for the kinetic approach. With no background field tensor, one can write $F^{\alpha\beta} = \delta F^{\alpha\beta}$. Also, the energy density ε, the pressure p, and the flow velocity U^α can be written in this scheme as $\varepsilon = \varepsilon_0 + \delta\varepsilon$, $p = p_0 + \delta p$, and $U^\alpha = U_0^\alpha + \delta u^\alpha$, respectively, where all quantities with the subscript 0 are background values. The background and perturbed velocities are defined as $U_0^\alpha = (1, 0, 0, 0)$ and $\delta u^\alpha = (0, \delta\mathbf{u})$, respectively. Here, $\mathbf{u}$ is the three flow velocity. We assume that the background is homogeneous and isotropic. The equation of motion for the background gives $\partial_0\varepsilon_0 = 0$ & $\partial_0 p_0 = 0$. Since the ε_0 and p_0 are functions of temperature and chemical potential, we regard the background temperature as constant. In this scheme the zeroth and ith components of the equation of motion can be written as

$$\partial_0\delta\varepsilon + (\varepsilon_0 + p_0)\partial_i\delta u^i = 0, \tag{A8}$$

$$(\varepsilon_0 + p_0)\partial_0\delta u^i + g^{ij}\partial_j\delta p = 0. \tag{A9}$$

Ignoring the time derivative term in Maxwell's equation, one can write $\nabla \times \delta\boldsymbol{B} = 4\pi\delta\boldsymbol{j}$, and by using the expression for the perturbed current, one can obtain the following dispersion relation:

$$i\omega = \frac{k^2}{4\pi\sigma_0} \pm \frac{\xi_{0B}}{\sigma_0}k, \tag{A10}$$

where we have used $\boldsymbol{k} = k\hat{z}$ and $\delta\mathbf{B}_k = \delta\mathbf{B}_{kx}\hat{x} + \delta\mathbf{B}_{ky}\hat{y}$. It should be noted that Eq. (A10) matches the dispersion relation obtained by the kinetic theory approach [Eq. (32)].

We first emphasize that there is no current in the background, and therefore the transport coefficients that appear in the expression for the perturbed current depend only on the background temperature and the chemical potentials. Now consider Eqs. (A4)–(A7), which, for the background quantities, can be described by the following two equations:

$$-2D_{B0} + \frac{\xi_0}{T_0} = 0, \tag{A11}$$

$$\frac{\xi_{B0}}{T_0} - C\frac{\mu_0}{T_0} = 0. \tag{A12}$$

These equations are satisfied by each species considered. Thus, one can write $\xi_0 = \xi_{R0} + \xi_{L0}$ and $\xi_{B0} = \xi_{BR0} + \xi_{BL0}$. Using the expression for ξ_0 from the kinetic equation, one can calculate D_{B0} from Eq. (A11). It agrees with the expression obtained in Ref. [55] and the expression obtained for ξ_B using kinetic theory is also in agreement with it. Thus, we have shown that the modified kinetic theory respects the constraint implied by the second law of thermodynamics.

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